Topological Data Analysis

5 December 2019

9 Further about Gromov–Hausdorff distance

Recall that, for X and Y nonempty compact metric spaces, the *Gromov–Hausdorff distance* between X and Y is defined as

$$d_{GH}(X,Y) = \inf\{d_H(f(X), g(Y)) \mid f\colon X \hookrightarrow M,\ g\colon Y \hookrightarrow M\},$$

where the infimum is taken over all isometric embeddings $f\colon X \hookrightarrow M$, $g\colon Y \hookrightarrow M$ into some common metric space M.

Recall also that a *correspondence* between X and Y is a surjective multivalued map, and the *distortion* of a correspondence $C \subseteq X \times Y$ is defined as

$$\operatorname{dis}(C) = \max\{|d_X(x,x') - d_Y(y,y')| \,:\, (x,y),(x',y') \in C\}.$$

The Gromov–Hausdorff distance between X and Y can be computed as follows:

$$d_{GH}(X,Y) = \frac{1}{2} \inf\{\operatorname{dis}(C) \mid C \subseteq X \times Y\},$$

where the infimum is taken over all correspondences between X and Y.

For example, the Gromov–Hausdorff distance between a point and the set of vertices of an equilateral triangle of side 1 is equal to 0.5.

Short exercise

(1) What is the Gromov–Hausdorff distance between a point and the vertices of a standard n-simplex for $n \geq 1$?

Longer exercises

(1) Use length formulas in the hyperbolic upper half-plane or the Poincaré disk of curvature -1 to prove that the distance between the center of an equilateral geodesic triangle of side s and any of its vertices is smaller than $s\sqrt{3}/3$.

(2) Search the Internet to find out software or efficient algorithms to compute Gromov–Hausdorff distances between point clouds.