Longer Exercises

Exercise 1 (Exercise 3 Lecture 1)**.** Find and discuss the definition of the *alpha complex* $A_\varepsilon(X)$ of a point cloud X, including examples.

Solution. We need the concept of *union of balls.* Let X be a finite set of points in $\mathbb{R}^n$ and let $r > 0$ be a real number. For each $p \in X$, let $B_p(r)$ be the closed ball with center p and radius r. Then,

$$\text{Union}(r) = \{x \in \mathbb{R}^n \mid \exists p \in X \ s.t. \ d(x,p) \leq r\}.$$

Then, let

$$R_p(r) := B_p(r) \cap V_p,$$

where V_p is the Voronoi cell of point p. Then, the *alpha complex* is isomorphic to the nerve of the cover $\{R_p(r) \mid p \in X\}$. That is,

$$A_r(X) := \{\sigma \subseteq X \mid \bigcap_{p\in\sigma} R_p(r) \neq \varnothing\}.$$

Here is an example of an alpha complex taken from `https://www2.cs.duke.edu/courses/fall06/cps296.1/Lectures/sec-III-4.pdf`:

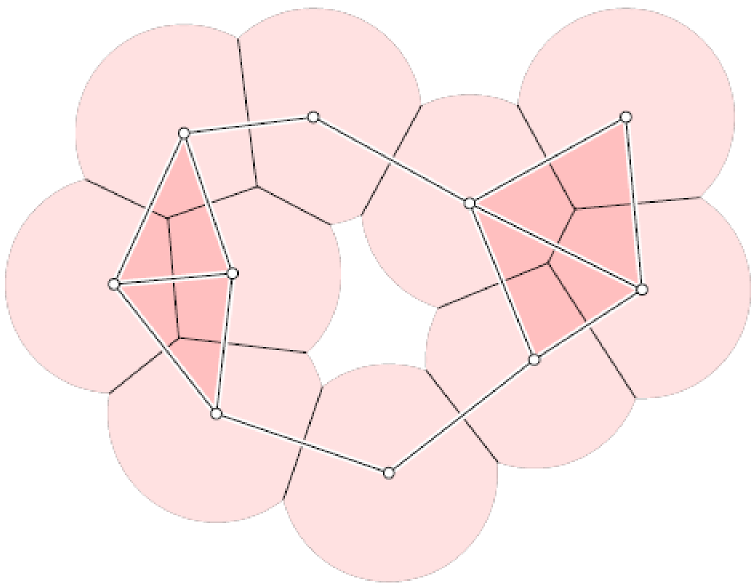


Figure 1: The union of disks is decomposed into convex regions by the Voronoi cells. The corresponding alpha complex is superimposed.

Some remarks:

- Since $B_p(r)$ and V_p are both closed and convex, so is $R_p(r)$.
- Since $R_p(r) \subseteq B_p(r)$, we have $A_r(X) \subseteq C_r(X)$.
- Since $R_p(r)$ are closed and convex and together they conver $\text{Union}(r)$, the Nerve Theorem implies that $\text{Union}(r)$ and $A_r(X)$ have the same homotopy type.

A more general approach is to weight the balls B_p and give them a radius $r_p = \sqrt{w_p}$, where w_p is the weight for point $p \in X$. Here is an image from the same pdf:

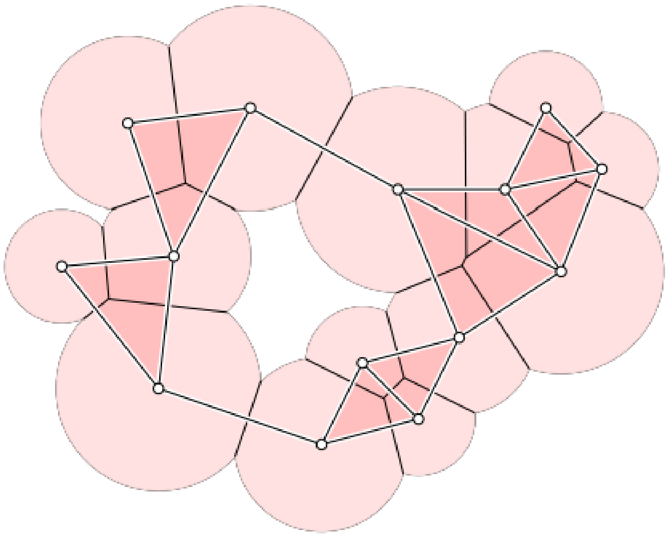


Figure 2: Convex decomposition of a union of disks and the weighted alpha complex superimposed.

Also, we can give a filtration of this more general alpha complex by letting $r_p = \sqrt{w_p + r^2}$, where r will be the free parameter for the filtration to exist.
To finish this exercise, let us give an example of an alpha complex. We are going to compute the alpha complex of the point cloud given in the short exercise of Lecture 3:

$$X = \{(0,0),(1,2),(2,1),(4,3),(4,-3)\}.$$

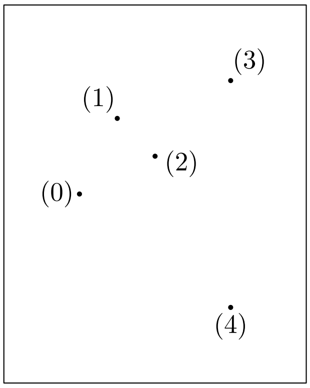
This will allow us to compare the alpha complex to the Vietoris-Rips complex that was computed in there. Using the following code:

```
import gudhi as gd
X = [[0,0], [1,2], [2,1], [4,3], [4,-3]]
alpha_complex = gd.AlphaComplex(points=X)
simplex_tree = alpha_complex.create_simplex_tree()
fmt = '%s -> %.2f'
for filtered_value in simplex_tree.get_filtration():
    print(fmt % tuple(filtered_value))
```

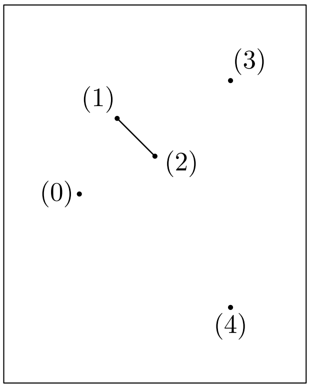
we obtain the filtration $A_\varepsilon(X)$, which is given as

```
[0] -> 0.00
[1] -> 0.00
[2] -> 0.00
[3] -> 0.00
[4] -> 0.00
[1, 2] -> 0.50
[0, 1] -> 1.25
[0, 2] -> 1.25
[0, 1, 2] -> 1.39
[2, 3] -> 2.00
[1, 3] -> 2.50
[1, 2, 3] -> 2.50
[2, 4] -> 5.00
[0, 4] -> 6.25
[0, 2, 4] -> 6.25
[3, 4] -> 10.00
[2, 3, 4] -> 10.00
```

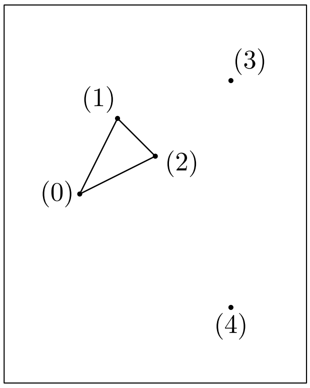
So we can draw it as:



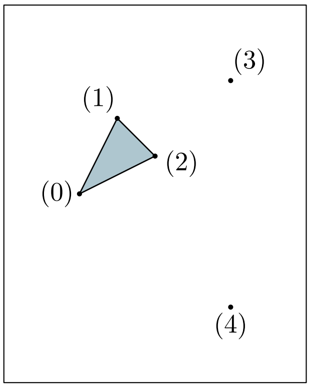
$\varepsilon < 0.5$



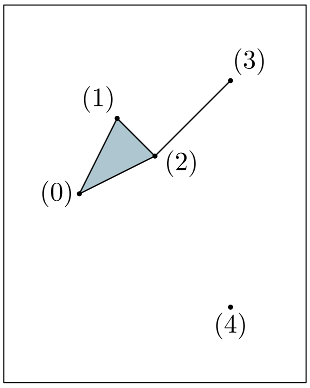
$0.5 \leq \varepsilon < 1.25$



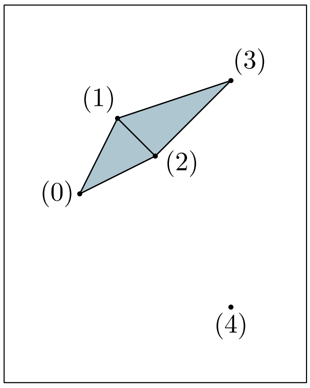
$1.25 \leq \varepsilon < 1.39$



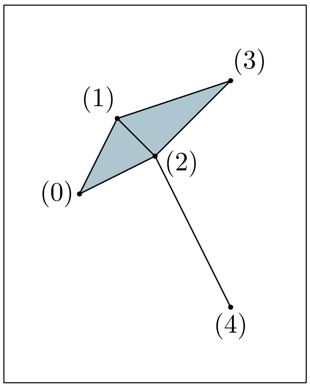
$1.39 \leq \varepsilon < 2$



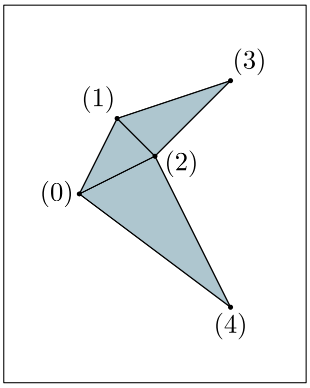
$2 \leq \varepsilon < 2.5$



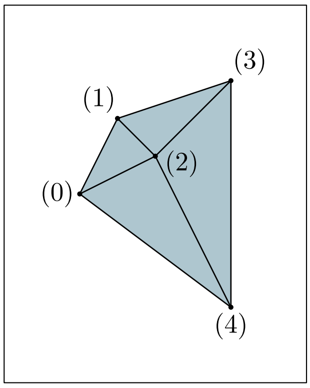
$2.5 \leq \varepsilon < 5$



$5 \leq \varepsilon < 6.25$

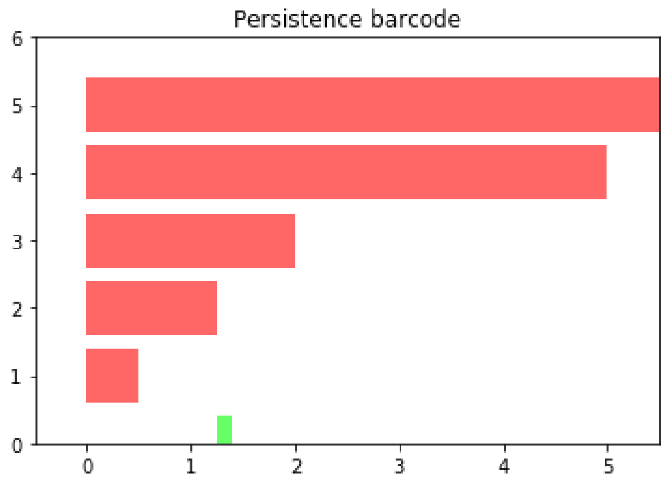


$6.25 \leq \varepsilon < 10$



$\varepsilon \geq 10$

Finally the persistence barcode is the following:



□

Exercise 2 (Exercise 3 Lecture 6)**.** Let K be the abstract simplicial complex whose maximal faces are

$$(124)\ (125)\ (135)\ (136)\ (146)\ (234)\ (236)\ (256)\ (345)\ (456).$$

(*a*) Compute the homology groups $H_i(K;\mathbb{Z})$ using

`GUDHI` (`http://gudhi.gforge.inria.fr`),

`Dyonisus` (`https://mrzv.org/software/dionysus2/`),

or any other software of your preference.

(*b*) Find a generator of $H_1(K;\mathbb{Z})$.

(*c*) Draw a picture showing that the geometric realisation of K is homeomorphic to the real projective plane.

Solution. □

(*a*) We are going to use `SAGE` for this, since it will be easier.

```
S = SimplicialComplex(maximal_faces=[[1,2,4], [1,2,5], [1,3,5], [1,3,6],\
    [1,4,6], [2,3,4], [2,3,6], [2,5,6], [3,4,5], [4,5,6]], is_mutable=False)
CC = S.chain_complex(base_ring=ZZ)
CC.homology()
```

This outputs

```
{0: Z, 1: C2, 2: 0}
```

Hence,

$$H_0(K;\mathbb{Z}) = \mathbb{Z}, \quad H_1(K;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}, \quad H_i(K;\mathbb{Z}) = 0,\ \forall i \geq 2.$$

(*b*) We denote by $C_*(K)$ the simplicial chain complex of K. Then,

$$C_2(K) = \mathbb{Z}^{10}, \quad C_1(K) = \mathbb{Z}^{15}, \quad C_0(K) = \mathbb{Z}^6,$$

where C_2 is spanned by the maximal faces of K, C_1 is spanned by the faces (ij), for $1 \leq i < j \leq 6$, and C_0 is spanned by the vertices (i), for $1 \leq i \leq 6$.

Then we have the boundary operators,

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

whose matices are:

$$\partial_1 = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

As usual, we let $B_1(K) = \mathrm{Im}(\partial_2)$ and $Z_1(K) = \mathrm{Ker}(\partial_1)$. Then, $H_1(K;\mathbb{Z}) = B_1/Z_1$.
By column reducing ∂_1, we obtain a basis for $Z_1(K)$, namely:

$$b_1 = (23) - (13) + (12)$$

$$b_2 = (24) - (14) + (12)$$

$$b_3 = (25) - (15) + (12)$$

$$b_4 = (26) - (16) + (12)$$

$$b_5 = (34) - (14) + (13)$$

$$b_6 = (35) - (15) + (13)$$

$$b_7 = (36) - (16) + (13)$$

$$b_8 = (45) - (15) + (14)$$

$$b_9 = (46) - (16) + (14)$$

$$b_{10} = (56) - (16) + (15)$$

Now, $\mathrm{rank}(\partial_2) = 10$, hence $B_1(K)$ is spanned by $\partial_2(\sigma)$, where σ are the generators of $C_2(K)$. We can also write them in terms of the basis for Z_1.

$$\partial_2(124) = (24) - (14) + (12) = b_2$$

$$\partial_2(125) = (25) - (15) + (12) = b_3$$

$$\partial_2(135) = (35) - (15) + (13) = b_6$$

$$\partial_2(136) = (36) - (16) + (13) = b_7$$

$$\partial_2(146) = (46) - (16) + (14) = b_9$$

$$\partial_2(234) = (34) - (24) + (23) = b_5 - b_2 + b_1$$

$$\partial_2(236) = (36) - (26) + (23) = b_7 - b_4 + b_1$$

$$\partial_2(256) = (56) - (26) + (25) = b_{10} - b_4 + b_3$$

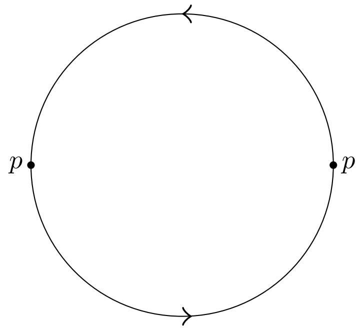
$$\partial_2(345) = (45) - (35) + (34) = b_8 - b_6 + b_5$$
$$\partial_2(456) = (56) - (46) + (45) = b_{10} - b_9 + b_8$$

Hence, $H_1(K;\mathbb{Z}) = \langle b_1, b_4, b_5, b_8, b_{10} \mid b_1 + b_5 = b_1 + b_4 = b_{10} - b_4 = b_5 + b_8 = b_8 + b_{10} = 0\rangle$. This can be further reduced, using the relations, to $b_5 = -b_1$, $b_4 = b_1$, $b_{10} = b_4 = b_1$, $b_8 = -b_5 = b_1$ and $b_8 + b_10 = 0$ implies $2b_1 = 0$. Hence,

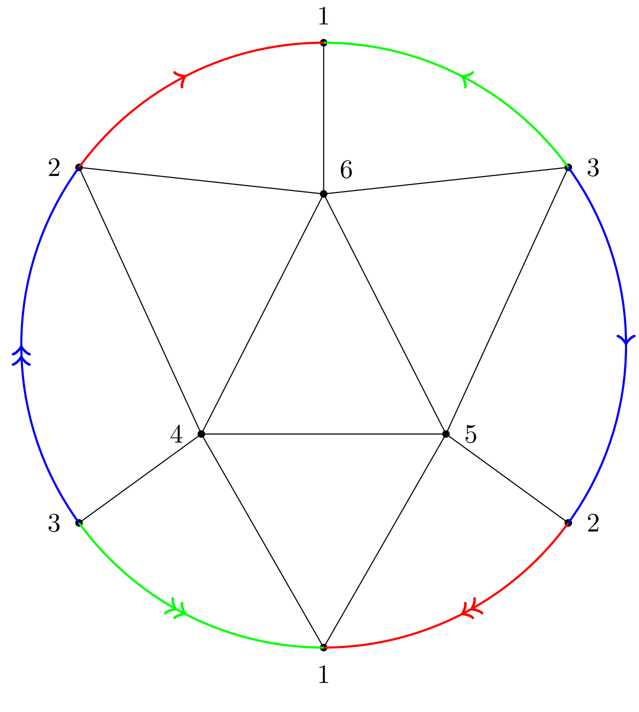
$$H_1(K;\mathbb{Z}) = \langle b_1 \mid 2b_1 = 0\rangle \cong \mathbb{Z}/2\mathbb{Z}\langle b_1\rangle.$$

Therefore, we have found our generator.

(c) Let us draw the usual representation of $\mathbb{R}P^2$:



Then, the triangulation above can be drawn as follows:



The path that is highlighted is the generator of $H_1(K;\mathbb{Z})$, $b_1 = (23) - (13) + (12)$. Notice that the colored paths must be identified together and the arrow indicates the generator b_1 and also that doing it twice returns you to the same point (i.e. $2b_1 = 0$). Finally, as a last comment, the green path (31) is reversed because there is a minus sign in b_1.