Topological Data Analysis

12 December 2019

11 Proof of the Stability Theorem

11.1 Contiguous simplicial maps

If f and g are simplicial maps from an abstract simplicial complex K to another abstract simplicial complex L, we say that f and g are *contiguous* if, for each face $\sigma = \{v_{i_0}, \dots, v_{i_n}\}$ of K, the points

$$f(v_{i_0}), \dots, f(v_{i_n}), g(v_{i_0}), \dots, g(v_{i_n})$$

(which need not be distinct) form a face of L. Hence $f(\sigma)$ and $g(\sigma)$ are faces of L contained in some common larger face of L. From this fact it follows that, if f and g are contiguous, then they yield homotopic maps $|K| \to |L|$ on the geometric realizations of K and L. Consequently, they induce the same homomorphisms $H_n(K) \to H_n(L)$ for all n.

11.2 Stability for point clouds

In what follows, homology is meant with coefficients in any field, which is not specified. Suppose given two point clouds X and Y in $\mathbb{R}^N$ for some N. We consider their Vietoris–Rips complexes $R_t(X)$ and $R_t(Y)$ for each $t > 0$, and the corresponding persistence modules $V_t(X) = H_*(R_t(X))$ and $V_t(Y) = H_*(R_t(Y))$, where $R_t(X) = \emptyset$ and $R_t(Y) = \emptyset$ for $t \leq 0$.

Our goal in this section is to prove the inequality

$$\frac{1}{2}\, d_{\text{int}}(V(X), V(Y)) \leq d_{GH}(X, Y),$$

where d_{int} is the interleaving distance and d_{GH} is the Gromov–Hausdorff distance. Recall that

$$d_{\text{int}}(V(X), V(Y)) = \inf\{\delta > 0 \mid V(X) \text{ and } V(Y) \text{ are } \delta\text{-interleaved}\}.$$

Thus, we need to show that $V(X)$ and $V(Y)$ are δ-interleaved for $\delta = 2d_{GH}(X, Y)$.

Recall also that

$$d_{GH}(X, Y) = \frac{1}{2} \inf\{\operatorname{dis}(C) \mid C \subseteq X \times Y\},$$

where the infimum is taken over all correspondences $C \subseteq X \times Y$, and $\operatorname{dis}(C)$ denotes the distortion of C. Since C is a finite set, if $\delta = 2d_{GH}(X, Y)$ then there is some correspondence $C \subseteq X \times Y$ with $\delta = \operatorname{dis}(C)$.

A function $f\colon X \to Y$ is called *subordinate* to C if $\{(x, f(x)) \mid x \in X\} \subseteq C$. If $f\colon X \to Y$ is any function subordinate to C, then, for each face $\sigma = \{v_{i_0}, \dots, v_{i_n}\}$

of $R_t(X)$, we have that $\operatorname{dis}(C) \geq |d(v_{i_k}, v_{i_\ell}) - d(f(v_{i_k}), f(v_{i_\ell}))|$ for all k and ℓ, from which we infer that

$$d(f(v_{i_k}), f(v_{i_\ell})) \leq d(v_{i_k}, v_{i_\ell}) + \operatorname{dis}(C) \leq t + \delta$$

for all k and ℓ, and this implies that $f(\sigma)$ is a face of $R_{t+\delta}(Y)$. Hence f induces a simplicial map $f_t\colon R_t(X) \to R_{t+\delta}(Y)$ for each $t \in \mathbb{R}$. Let

$$F_t\colon V_t(X) \longrightarrow V_{t+\delta}(Y) = V(Y)[\delta]_t$$

be the linear map induced by f_t in homology.

Similarly, we may choose any function $g\colon Y \to X$ subordinate to C^{-1}, and we obtain a simplicial map $g_t\colon R_t(Y) \to R_{t+\delta}(X)$ for every t and hence a linear map

$$G_t\colon V_t(Y) \longrightarrow V_{t+\delta}(X) = V(X)[\delta]_t$$

for each t. In fact the collections $\{F_t\}$ and $\{G_t\}$ define morphisms of persistent modules $F\colon V(X) \to V(Y)[\delta]$ and $G\colon V(Y) \to V(X)[\delta]$.

We next check that $V(X)$ and $V(Y)$ are δ-interleaved by means of F and G. The shift morphism

$$\sigma_{2\delta}\colon V(X) \longrightarrow V(X)[2\delta]$$

is induced in homology by the inclusions $R_t(X) \subseteq R_{t+2\delta}(X)$. Hence it is enough to prove that $(g \circ f)_t$ is contiguous to the inclusion map $R_t(X) \to R_{t+2\delta}(X)$ for each t. For this, pick any face $\{v_{i_0}, \ldots, v_{i_n}\}$ of $R_t(X)$. Then $(v_{i_\ell}, f(v_{i_\ell})) \in C$ and $(g(f(v_{i_k})), f(v_{i_k})) \in C$ for all k, ℓ, and this implies that

$$d(g(f(v_{i_k})), v_{i_\ell}) \leq d(f(v_{i_k}), f(v_{i_\ell})) + \delta.$$

Next, we use that $(v_{i_k}, f(v_{i_k})) \in C$ and $(v_{i_\ell}, f(v_{i_\ell})) \in C$ to infer that

$$d(f(v_{i_k}), f(v_{i_\ell})) + \delta \leq d(v_{i_k}, v_{i_\ell}) + 2\delta \leq t + 2\delta.$$

Similarly, using that $(g(f(v_{i_k})), f(v_{i_k})) \in C$ and $(g(f(v_{i_\ell})), f(v_{i_\ell})) \in C$ for all k, ℓ, we obtain that

$$d(g(f(v_{i_k})), g(f(v_{i_\ell}))) \leq d(f(v_{i_k}), f(v_{i_\ell})) + \delta \leq d(v_{i_k}, v_{i_\ell}) + 2\delta \leq t + 2\delta.$$

This proves that the points

$$g(f(v_{i_0})), \ldots, g(f(v_{i_n})), v_{i_0}, \ldots, v_{i_n}$$

form a face of $R_{t+2\delta}(X)$, as needed. The argument with $f \circ g$ is analogous.

References: This proof has been extracted from the original source [F. Chazal, V. de Silva, S. Oudot, Persistence stability for geometric complexes, *Geom. Dedicata*, 173 (2014), 193–214]. It can also be found as Theorem 1.5.4 in [L. Polterovich, D. Rosen, K. Samvelyan, J. Zhang, Topological Persistence in Geometry and Analysis, arXiv:1904.04044 (2019)].

Longer exercise

(1) Prove that contiguous simplicial maps induce the same homomorphisms in homology. For this, see Theorem 12.5 in [J. R. Munkres, Elements of Algebraic Topology, Addison-Wesley, Menlo Park, 1984].