Topological Data Analysis

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4 Classification of persistence modules

4.1 Persistence modules

Fix any field $\mathbb{F}$. A *persistence module* over $\mathbb{F}$ is a pair (V, π) where $V = \{V_t\}_{t\in\mathbb{R}}$ is a collection of $\mathbb{F}$-vector spaces of finite dimension and π is a collection of $\mathbb{F}$-linear maps $\pi_{s,t}\colon V_s \to V_t$ for $s \le t$, such that the following conditions hold:

(a) (Persistence) $\pi_{s,t} \circ \pi_{r,s} = \pi_{r,t}$ if $r \le s \le t$.

(b) (Finite type) There is a finite set $A = \{a_0, \dots, a_k\} \subset \mathbb{R}$ such that:

  (i) For all $x \in \mathbb{R} \smallsetminus A$ there is a neighbourhood U of x such that $\pi_{s,t}$ is an isomorphism for all $s \le t$ in U.

  (ii) For every $a \in A$ there is an $\varepsilon > 0$ so that if $a \le t < a + \varepsilon$ then $\pi_{a,t}$ is an isomorphism and if $a - \varepsilon < s < a$ then $\pi_{s,a}$ is not an isomorphism.

(c) (Zero origin) $V_t = \{0\}$ for $t < a_0$, assuming that $a_0 < \cdots < a_k$.

It follows from these conditions that $\pi_{t,t} = \mathrm{id}$ for all $t \in \mathbb{R}$, and $\pi_{s,t}$ is an isomorphism if $a_k \le s \le t$. We write V_∞ to denote V_t for $t \ge a_k$; thus V_∞ is the direct limit of (V, π) viewed as a directed diagram.

The set A is called the *spectrum* of (V, π) and its elements are *spectral points*.

If X is a point cloud in $\mathbb{R}^N$ for some N and $R_t(X)$ denotes the Vietoris–Rips complex associated with X for each value of $t > 0$, then

$$V_t = H_*(R_t(X)) = \oplus_{i=0}^{\infty} H_i(R_t(X)) \text{ if } t > 0 \text{ and } V_t = 0 \text{ for } t \le 0$$

defines a persistence module, with $\pi_{s,t}$ the homomorphisms induced in homology by the inclusions $R_s(X) \subseteq R_t(X)$, where we mean $R_s(X) = \emptyset$ if $s \le 0$. This persistence module is the *Vietoris–Rips module* of X.

4.2 Normal form and barcodes

A *morphism* $f\colon (V, \pi) \to (V', \pi')$ of persistence modules over a field $\mathbb{F}$ is a collection of $\mathbb{F}$-linear maps $f_t\colon V_t \to V'_t$ such that

$$f_t \circ \pi_{s,t} = \pi'_{s,t} \circ f_s$$

whenever $s \le t$. A morphism of persistence modules is an *isomorphism* if it has a two-sided inverse, that is, $g\colon (V', \pi') \to (V, \pi)$ with $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. Then it follows that f_t is an isomorphism for every t.

For every interval $I = [a, b) \subset \mathbb{R}$ with $a < b$ or $I = [a, \infty)$, define a persistence module $\mathbb{F}(I)$ as follows:

$$\mathbb{F}(I)_t = \begin{cases} \mathbb{F} & \text{if } t \in I \\ 0 & \text{otherwise,} \end{cases}$$

with $\pi_{s,t} = \mathrm{id}$ if $s, t \in I$ and $\pi_{s,t} = 0$ otherwise. Such persistence modules are called *interval modules*. Their spectrum is $\{a, b\}$ if $I = [a, b)$ or $\{a\}$ if $I = [a, \infty)$.

If (V, π) and (V', π') are persistence modules, their *direct sum* is the persistence module (W, θ) with $W_t = V_t \oplus V'_t$ for all t and $\theta_{s,t} = \pi_{s,t} \oplus \pi'_{s,t}$ for all s, t.

We denote, for every positive integer m,

$$\mathbb{F}(I)^m = \mathbb{F}(I) \oplus \overset{m}{\cdots} \oplus \mathbb{F}(I),$$

so $\mathbb{F}(I)^m$ also becomes a persistence module.

Theorem 4.1 (Normal Form Theorem)**.** *For every persistence module (V, π) there is a finite collection of intervals $\{I_i\}_{i=1}^N$ with $I_i = [a_i, b_i)$ or $I_i = [a_i, \infty)$ for each i, such that $I_i \neq I_j$ if $i \neq j$, and there is an isomorphism of persistence modules*

$$V \cong \bigoplus_{i=1}^{N} \mathbb{F}(I_i)^{m_i}$$

where $m_1, \ldots, m_N$ are positive integers.

Therefore we may represent each persistence module (V, π) by means of a *barcode* whose horizontal segments are the intervals $\{I_i\}_{i=1}^N$ with multiplicities m_i given by Theorem 4.1.

4.3 Shift action

Let (V, π) be a persistence module with spectrum $A = \{a_0, \ldots, a_k\}$ for $a_0 < \cdots < a_k$. Then the polynomial ring $\mathbb{F}[t]$ acts on the vector space $V_* = V_{a_0} \oplus \cdots \oplus V_{a_k}$ by

$$t \cdot v = \pi_{a_i, a_{i+1}}(v) \text{ if } v \in V_{a_i} \text{ with } i < k, \text{ and } t \cdot v = v \text{ if } v \in V_{a_k}.$$

In this way, V_* becomes an $\mathbb{N}$-graded $\mathbb{F}[t]$-module, with V_{a_i} in degree i and $V_\infty = V_{a_k}$ in all degrees bigger than or equal to k.

For a graded $\mathbb{F}[t]$-module M_*, we denote by $(\Sigma M)_*$ the upwards shifted graded module, that is, $(\Sigma M)_i = M_{i-1}$ for $i \geq 1$, and $(\Sigma M)_0 = \{0\}$. The notation Σ is borrowed from the suspension operator in Topology and Homological Algebra.

Theorem 4.2 (Structure Theorem)**.** *Let M_* be a finitely generated $\mathbb{N}$-graded module over the polynomial ring $\mathbb{F}[t]$, where $\mathbb{F}$ is a field. Then*

$$M_* \cong \bigoplus_{i=1}^{n} \Sigma^{p_i}\, \mathbb{F}[t] \;\oplus\; \left(\bigoplus_{j=1}^{m} \Sigma^{q_j}\, \mathbb{F}[t]/(t^{r_j}) \right)$$

for some collections of integers $p_i \geq 0$, $q_j \geq 0$ and $r_j \geq 1$. Moreover, this decomposition is unique up to a permutation of summands.

For a persistence module (V, π) and the associated graded $\mathbb{F}[t]$-module V_*, a vector $u \in V_*$ corresponding to the first summand in Theorem 4.2 is seen in the barcode of (V, π) as the origin of an infinite ray starting at a value $a \in A$ with V_a in degree p_i, and a vector $v \in V_*$ corresponding to the second summand is seen as the origin of a segment that starts at $a \in A$ with V_a in degree q_j and ends at $b \in A$ with V_b in degree $q_j + r_j$.

Short exercise

(1) Prove that two isomorphic persistence modules have the same spectrum.

Longer exercise

(1) Infer the Normal Form Theorem from the Structure Theorem for finitely generated graded modules over the ring $\mathbb{F}[t]$. Useful references are [A. Zomorodian, G. Carlsson, Computing persistent homology, *Disc. Comput. Geom.* 33 (2005), 247–274] and [L. Polterovich, D. Rosen, K. Samvelyan, J. Zhang, Topological persistence in geometry and analysis, arXiv:1904.04044 (2019)].