# Topological Data Analysis

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## 6 Conventions about landscapes

Let $(V, \pi)$ be a persistence module over a field $\mathbb{F}$ with spectrum $A = \{a_0, \dots, a_m\}$ where $a_0 < \cdots < a_m$. We defined the *landscape functions* $\{\lambda_k \colon \mathbb{R} \to \mathbb{R}\}_{k \geq 1}$ associated with $(V, \pi)$ as follows. For each $t \in \mathbb{R}$ and $k \geq 1$, the value $\lambda_k(t)$ is

$$\sup\{x \geq 0 \mid \beta_{t-x,\,t+x} \geq k\} = \operatorname{kmax}\{\Lambda_{(b_i,d_i)}(t)\}_{i=1}^{N} \tag{6.1}$$

where kmax returns the $k$th largest value in a given set of real numbers, counted with their multiplicities, and $\{(b_i, d_i)\}_{i=1}^{N}$ is the set of points (with their respective multiplicities) in the persistence diagram of $(V, \pi)$. In (6.1) we use the notation

$$\beta_{t-x,\,t+x} = \dim_{\mathbb{F}} \operatorname{Im}(\pi_{t-x,\,t+x})$$

$$\Lambda_{(b,d)}(t) = \max\{0, \min\{t - b, d - t\}\}.$$

We next prove that the two expressions in (6.1) are indeed equal. For this, some conventions are needed.

- Each ray $[b_i, \infty)$ in the barcode of $(V, \pi)$ is represented by a point $(b_i, y_\infty)$ in the persistence diagram, where $y_\infty = a_m$ is the largest spectral point of $(V, \pi)$. Accordingly, for the validity of (6.1) we need to assume that $V_t = \{0\}$ if $t > a_m$ (although $V_{a_m}$ can be nonzero).

- If the left-hand set in (6.1) is empty, then its supremum is taken to be 0. This convention has an interpretation that is normally found in the bibliography about persistence diagrams: The diagonal $b = d$ is included in every persistence diagram, and its points are counted with infinite multiplicity. Consequently, the inequality $\beta_{t,t} \geq k$ is assumed to hold for all $t \in \mathbb{R}$ and all $k$, and this implies that $x = 0$ belongs to the set $\{x \geq 0 \mid \beta_{t-x,\,t+x} \geq k\}$, which is therefore never empty. This convention ensures that $\lambda_k(t) \geq 0$ for all $t \in \mathbb{R}$ and all $k$.

- The function kmax takes the value 0 if there is no $k$th largest element in the right-hand set in (6.1).

It follows from these conventions that $\lambda_k(t) = 0$ for $t \leq a_0$ and all $k$, and $\lambda_k(t) = 0$ as well for $t \geq a_m$ and all $k$. This is due to the fact that $V_{a_0 - \varepsilon} = \{0\}$ for all $\varepsilon > 0$, and similarly $V_{a_m + \varepsilon} = \{0\}$ for all $\varepsilon > 0$. In other words, the support of each function $\lambda_k$ is contained in the closed interval $[a_0, a_m]$.

Now, if $V = \mathbb{F}[b, d)$, then

$$\lambda_1(t) = \sup\{x \geq 0 \mid \beta_{t-x,\,t+x} \geq 1\} = \min\{t - b, d - t\} = \Lambda_{(b,d)}(t)$$

for $a_0 < t < a_m$, while $\lambda_k(t) = 0$ for $k \geq 2$ and all $t$.

If $V = \mathbb{F}[b,d)^n$ for some $n \geq 2$, then, for $a_0 < t < a_m$ and $1 \leq k \leq n$,

$$\lambda_k(t) = \sup\{x \geq 0 \mid \beta_{t-x,\,t+x} \geq k\} = \min\{t-b, d-t\} = \Lambda_{(b,d)}(t)$$

and the kmax function returns $\Lambda_{(b,d)}(t)$ for $1 \leq k \leq n$ as well, since the value $\Lambda_{(b,d)}(t)$ is counted with multiplicity $n$.

If $V = \mathbb{F}[b_1,d_1) \oplus \mathbb{F}[b_2,d_2)$ with $b_1 \leq b_2 < d_1 \leq d_2$, then $\lambda_1(t) \neq 0$ only for $b_1 < t < d_2$, and in this range of values we have

$$\lambda_1(t) = \sup\{\min\{t-b_1, d_1-t\},\, \min\{t-b_2, d_2-t\}\} = \max\{\Lambda_{(b_1,d_1)}(t),\, \Lambda_{(b_2,d_2)}(t)\},$$

while

$$\lambda_2(t) = \min\{t-b_2, d_1-t\} = 2\max\{\Lambda_{(b_1,d_1)}(t),\, \Lambda_{(b_2,d_2)}(t)\},$$

and $\lambda_k(t) = 0$ for $k \geq 3$ and all $t \in \mathbb{R}$.

In the general case, assume that $V = \mathbb{F}[b,d) \oplus V'$ and suppose, inductively, that (6.1) holds for $V'$. Then (6.1) also holds for $V$ by a combination of the arguments given in the two previous steps. Specifically, for $k = 1$, the module $\mathbb{F}[b,d)$ attains the supremum in the left-hand term of (6.1) only if $\min\{t-b, d-t\}$ exceeds the supremum in $V'$, in which case $\mathbb{F}[b,d)$ also attains the maximum in the right-hand term. If $\mathbb{F}[b,d)$ does not attain the supremum for $k = 1$, pick the smallest value of $k$ for which it does, and in this case (and not for smaller values of $k$) it also attains the maximum in the right-hand term, by the induction hypothesis.

## Longer exercises

(1) Install and try the JavaPlex library for MATLAB:

http://appliedtopology.github.io/javaplex/

(2) Draw a barcode for $H_0$, $H_1$ and $H_2$, and landscape functions for $H_1$ of the Vietoris–Rips persistence module of the following point cloud in $\mathbb{R}^3$:

| | | | |
|---|---|---|---|
| $(1.05, 2.00, 2.16)$ | $(2.96, 1.84, 2.09)$ | $(2.49, 2.77, 1.87)$ | $(1.99, 2.42, 2.82)$ |
| $(1.59, 1.42, 2.66)$ | $(2.27, 1.59, 1.16)$ | $(1.69, 1.24, 2.60)$ | $(2.22, 2.76, 2.53)$ |
| $(2.50, 2.74, 2.07)$ | $(2.55, 1.59, 1.30)$ | $(1.80, 1.15, 1.47)$ | $(1.31, 2.40, 1.44)$ |
| $(1.92, 1.06, 1.64)$ | $(2.29, 1.97, 1.01)$ | $(2.25, 2.94, 2.29)$ | $(2.38, 2.57, 2.71)$ |
| $(1.19, 1.83, 1.55)$ | $(1.53, 2.80, 2.24)$ | $(1.52, 2.82, 1.81)$ | $(1.66, 2.58, 2.66)$ |
| $(2.05, 1.76, 2.98)$ | $(1.41, 1.96, 1.29)$ | $(1.70, 2.51, 2.81)$ | $(1.79, 1.74, 2.88)$ |
| $(2.89, 1.84, 2.16)$ | $(2.16, 1.63, 1.08)$ | $(1.62, 1.27, 2.63)$ | $(2.91, 2.27, 2.29)$ |
| $(1.05, 2.05, 1.59)$ | $(2.60, 1.91, 2.82)$ | $(1.21, 1.68, 2.64)$ | $(1.67, 3.00, 1.80)$ |
| $(1.76, 1.09, 2.21)$ | $(0.99, 1.66, 2.29)$ | $(2.57, 1.95, 1.17)$ | $(1.31, 1.45, 2.63)$ |
| $(0.99, 1.90, 1.84)$ | $(2.88, 2.63, 2.05)$ | $(2.13, 1.96, 1.08)$ | $(2.62, 2.76, 2.47)$ |
| $(2.35, 2.64, 2.69)$ | $(2.52, 2.94, 2.19)$ | $(1.68, 1.25, 2.55)$ | $(1.03, 2.11, 1.74)$ |
| $(2.49, 2.15, 2.89)$ | $(1.37, 2.46, 1.35)$ | $(2.21, 1.05, 2.40)$ | $(1.20, 1.73, 2.36)$ |
| $(1.67, 1.15, 1.95)$ | $(2.45, 2.27, 1.14)$ | | |

(3) Compute the Betti numbers with coefficients in $\mathbb{Z}/2$ and with coefficients in $\mathbb{Z}$ of the simplicial complexes $T$ and $R$ with the following maximal faces.

(a) Maximal faces of $T$:

$$\begin{array}{ccccccc} (123) & (145) & (156) & (345) & (167) & (467) & (247) \\ (124) & (236) & (256) & (346) & (257) & (357) & (137) \end{array}$$

(b) Maximal faces of $R$ (here $a = 10$ and $b = 11$):

$$\begin{array}{cccccccc} (1237) & (123b) & (1269) & (126b) & (1279) & (135a) & (135b) & (137a) \\ (1479) & (147a) & (1489) & (148a) & (1568) & (156b) & (158a) & (1689) \\ (2348) & (234b) & (2378) & (246a) & (246b) & (248a) & (2578) & (2579) \\ (258a) & (259a) & (269a) & (3459) & (345b) & (3489) & (359a) & (3678) \\ (367a) & (3689) & (369a) & (4567) & (456b) & (4579) & (467a) & (5678) \end{array}$$

Can you find out what are their geometric realizations $|T|$ and $|R|$?