Topological Data Analysis

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10 Stability for smooth functions

10.1 Sublevel sets

For a continuous function $f\colon [a,b] \to \mathbb{R}$ denote, for each $t \in \mathbb{R}$,

$$L_t(f) = \{x \in [a,b] \,:\, f(x) \le t\},$$

and observe that $L_s(f) \subseteq L_t(f)$ if $s \le t$, and $L_t(f) = \emptyset$ if $t < t_0$ and f is bounded below by t_0. The sets $L_t(f)$ are called *sublevel sets* of f.

Assuming that the function f is differentiable, a *critical point* of f is a point x_0 with $a < x_0 < b$ and $f'(x_0) = 0$. The *critical values* of f are the values of f at critical points, together with $f(a)$ and $f(b)$. If f is differentiable and has finitely many critical points, then we obtain a persistence module over $\mathbb{R}$ by defining

$$V_t(f) = H_0(L_t(f); \mathbb{R}),$$

and letting $\pi_{s,t}\colon V_s(f) \to V_t(f)$ be induced by the inclusion $L_s(f) \subseteq L_t(f)$ if $s \le t$. The spectrum of $(V(f), \pi)$ is contained in the set of critical values of f.

Note that $V(f)[\delta] = V(f - \delta)$ for every $\delta \in \mathbb{R}$. From this fact if follows that

$$d_{\mathrm{int}}(V(f), V(g)) \le \|f - g\|_\infty$$

where d_{int} is the interleaving distance and $\|f-g\|_\infty = \sup\{|f(x)-g(x)| \,:\, x \in [a,b]\}$.

10.2 Morse persistence modules

Let M be a closed smooth manifold (that is, a compact smooth manifold without boundary). For a smooth function $f\colon M \to \mathbb{R}$, a *critical point* of f is a point $p \in M$ such that $(\partial f/\partial x_i)(p) = 0$ for all i, where $(x_1, \ldots, x_n)$ are coordinates on some neighbourhood of p.

Similarly as above, if f is a smooth function on M with finitely many critical points, then we obtain a persistence module, called a *Morse persistence module*, by defining

$$L_t(f) = \{x \in M \mid f(x) \le t\}$$

and

$$V_t(f) = H_*(L_t(f); \mathbb{R}),$$

with $\pi_{s,t}\colon V_s(f) \to V_t(f)$ induced by the inclusion $L_s(f) \subseteq L_t(f)$ if $s \le t$.

Then the following inequality holds:

$$d_{\mathrm{int}}(V(f), V(g)) \le \|f - g\|_\infty$$

where $\|f - g\|_\infty = \sup\{|f(x) - g(x)| \,:\, x \in M\}$.

Short exercise

(1) Consider the functions $f, g\colon [-1,1] \to \mathbb{R}$ given by

$$f(x) = x^5 - x, \qquad g(x) = \frac{1}{5}(x^9 + 7x^5 - 10x).$$

(a) Find the persistence modules $V(f)$ and $V(g)$ and the spectrum of each.

(b) Compute the interleaving distance $d_{\mathrm{int}}(V(f), V(g))$.

(c) Prove that $d_{\mathrm{int}}(V(f), V(g)) \leq \|f - g\|_\infty$ on $[-1,1]$.

Longer exercises

(1) Prove that $d_{\mathrm{int}}(V(f), V(g)) \leq \|f - g\|_\infty$ for any two differentiable functions $f, g\colon [a,b] \to \mathbb{R}$, where $a < b$, assuming that f and g have finitely many critical points.

(2) Find information about the PHAT software library (*Persistent Homology Algorithm Toolbox*) at https://bitbucket.org/phat-code/phat/src/master/. For a descriptive reference, see the article [U. Bauer, M. Kerber, J. Reininghaus, H. Wagner, PHAT – Persistent Homology Algorithms Toolbox, *J. Symb. Comput.* 78 (2017), 76–90].