Short exercises Lecture 9

Exercise 1. What is the Gromov-Hausdorff distance between a point and the vertices of a standars n-simplex for $n \geq 1$?

Solution. We are going to compute the Gramov-Hausdorff distance using the following identity:

$$d_{GH}(X, Y) = \frac{1}{2} \inf\{\text{dis}(C) \mid C \subseteq X \times Y\}$$

where C is a correspondence between X and Y and $\text{dis}(C)$ is the distortion of the correspondence C,

$$\text{dis}(C) = \max\{|d_X(x, x') - d_Y(y, y')| \mid (x, y), (x', y') \in C\}$$

Now, let X be the standard n-simplex Δ^n (i.e. X is the convex hull of the standard basis in $\mathbb{R}^{n+1}$) and let $Y = \{y_0\}$ be a point. Then, we only have one correspondence C, namely

$$C = \{(x, y_0) \mid x \in X\}$$

Then,

$$\text{dis}(C) = \max\{|d_X(x, x') - d_Y(y_0, y_0)| \mid (x, y_0), (x', y_0) \in C\} = \max_{x, x' \in X}\{d_X(x, x')\} = \text{diam}(X),$$

where $\text{diam}(X)$ is the diameter of X. Hence,

$$d_{GH}(X, Y) = \frac{\text{diam}(X)}{2}$$

Now, the diameter of Δ^n is the distance between any of the vertices, i.e., is the distance between any two vectors of the canonical basis. Particularly,

$$\text{diam}(\Delta^{\text{n}}) = \sqrt{2}$$

This is an easy exercise: Suppose that $\text{diam}(X) = d(x, y)$, for some x, y. We are going to show that $d(x, y) \leq d(x, e_i)$ for some element of the standard basis of $\mathbb{R}^{n+1}$, e_i. We know that we can write $y = \sum_{i=1}^{n+1} \lambda_i e_i$ for some $0 \leq \lambda_i \leq 1$ with $\sum_{i=1}^{n+1} \lambda_i = 1$. Then,

$$d(x, y) = d\left(\sum_{i=1}^{n+1} \lambda_i x - \sum_{i=1}^{n+1} \lambda_i e_i\right) \leq \sum_{i=1}^{n+1} \lambda_i d(x, e_i) \leq \sum_{i=1}^{n+1} \lambda_i \max_{1 \leq i \leq n+1}\{d(x, e_i)\} = d(x, e_i)$$

□