Topological Data Analysis

12 November 2019

2 Simplicial homology

2.1 Chains and homology

Let K be an abstract simplicial complex with vertex set $V = \{v_i\}_{i \in I}$. Assume given a total order on the set I. For simplicity of notation, faces of K will be denoted by $(i_0 \cdots i_n)$ with $i_0 < \cdots < i_n$ instead of $\{v_{i_0}, \ldots, v_{i_n}\}$.

For every $n \geq 0$, we denote by $C_n(K)$ the free abelian group over the set of all n-faces of K, and assume that $C_{-1}(K) = 0$. The elements of $C_n(K)$ are called *n-chains* in K. Thus an n-chain in K is a formal sum

$$\lambda_1(i_0^1 \cdots i_n^1) + \cdots + \lambda_m(i_0^m \cdots i_n^m) \tag{2.1}$$

where $\lambda_k \in \mathbb{Z}$ for all k and each $(i_0^k \cdots i_n^k)$ is an n-face of K.

More generally, for any commutative ring R with 1, we denote by $C_n(K;R)$ the free R-module over the set of n-faces of K. Its elements are called *n-chains* in K with *coefficients* in R. Hence $C_n(K) = C_n(K;\mathbb{Z})$ and $C_n(K;R) = R \otimes_{\mathbb{Z}} C_n(K)$ for every R. Elements of $C_n(K;R)$ are sums as in (2.1), but with $\lambda_k \in R$ for all k.

The *boundary operator*

$$\partial_n \colon C_n(K;R) \longrightarrow C_{n-1}(K;R)$$

is the R-module homomorphism defined on basis elements as

$$\boxed{\partial_n(i_0 \cdots i_n) = \textstyle\sum_{k=0}^{n} (-1)^k \, (i_0 \cdots \widehat{i_k} \cdots i_n),}$$

where the notation $\widehat{i_k}$ means that i_k is omitted.

The fundamental property of the boundary operator is that

$$\partial_n \circ \partial_{n+1} = 0 \qquad \text{for all } n.$$

Consequently, if we denote by $B_n(K;R)$ the image of ∂_{n+1} and by $Z_n(K;R)$ the kernel of ∂_n, then

$$B_n(K;R) \subseteq Z_n(K;R) \qquad \text{for all } n.$$

The elements in $Z_n(K;R)$ are called *n-cycles* and the elements in $B_n(K;R)$ are called *n-boundaries*. Hence every n-boundary is an n-cycle, but not conversely. The quotient

$$\boxed{H_n(K;R) = Z_n(K;R)/B_n(K;R)}$$

is called the nth *homology* of K with coefficients in R. It is an abelian group if $R = \mathbb{Z}$ and, more generally, an R-module if R is any ring. If R is a field, then R-modules are vector spaces over R. The fields of main interest will be $R = \mathbb{R}$ (the reals), $R = \mathbb{Q}$ (the rationals) or $R = \mathbb{F}_p$ for some prime p (the field of p elements).

2.2 Induced homomorphisms

Let K and L be abstract simplicial complexes with vertex sets V_K and V_L. Suppose given an injective function $f\colon V_K \to V_L$ that extends to a function $f\colon K \to L$; that is, $\{f(v_{i_0}), \ldots, f(v_{i_n})\}$ is a face of L whenever $\{v_{i_0}, \ldots, v_{i_n}\}$ is a face of K. Then the resulting function $f\colon K \to L$ is called a *simplicial map*.

Every simplicial map $f\colon K \to L$ induces an R-module homomorphism

$$f_n\colon C_n(K;R) \longrightarrow C_n(L;R)$$

for each n, defined by extending f linearly:

$$\begin{aligned} f_n\Big(\lambda_1\,\{v_{i_0^1}, \ldots, v_{i_n^1}\} + \cdots + \lambda_m\,\{v_{i_0^m}, \ldots, v_{i_n^m}\}\Big) \\ = \lambda_1\,\{f(v_{i_0^1}), \ldots, f(v_{i_n^1})\} + \cdots + \lambda_m\,\{f(v_{i_0^m}), \ldots, f(v_{i_n^m})\}. \end{aligned}$$

These homomorphisms satisfy

$$f_{n-1} \circ \partial_n = \partial_n \circ f_n$$

for all n and therefore induce well-defined R-module homomorphisms

$$f_*\colon H_n(K;R) \longrightarrow H_n(L;R),$$

namely $f_*([z]) = [f_n(z)]$ for each n-cycle $z \in Z_n(K;R)$.

The induced homomorphisms satisfy the *functoriality* relations:

$$(g \circ f)_* = g_* \circ f_*, \qquad\qquad \mathrm{id}_* = \mathrm{id}.$$

An important special case is the inclusion $i\colon K \subset L$ of a subcomplex. Let us emphasize that, in spite of the fact that i is obviously injective, the induced homomorphisms $i_*\colon H_n(K;R) \to H_n(L;R)$ need not be monomorphisms.

2.3 Some properties of homology

- If an abstract simplicial complex K is finite, then $C_n(K;R)$ is a finitely generated R-module for every n. If R is a principal ideal domain, then $Z_n(K;R)$ is also finitely generated and consequently $H_n(K;R)$ is finitely generated for every n. In the special case $R = \mathbb{Z}$, this means that

  $$H_n(K;\mathbb{Z}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}/p_1^{\alpha_1} \oplus \cdots \oplus \mathbb{Z}/p_r^{\alpha_r}$$

  for some primes $p_1, \ldots, p_r$ and positive exponents $\alpha_1, \ldots \alpha_r$. The number of $\mathbb{Z}$ summands is called the *rank* of $H_n(K;\mathbb{Z})$. It is equal to the dimension of $H_n(K;\mathbb{Q})$ as a $\mathbb{Q}$-vector space.

- If $K = A \cup B$ where A and B are subcomplexes with $A \cap B = \emptyset$, then $H_n(K;R) \cong H_n(A;R) \oplus H_n(B;R)$ for all n and all R.

- The 0th homology group $H_0(K;\mathbb{Z})$ is necessarily torsion-free and counts the number of connected components of the geometric realization $|K|$. More precisely, $H_0(K;\mathbb{Z}) \cong \mathbb{Z}^N$ if and only if $|K|$ has N connected components.

- If $|K|$ is a finite connected graph, then $H_1(K;\mathbb{Z}) \cong \mathbb{Z}^N$ if and only if there are precisely N independent cycles in $|K|$, that is, if the complement of a maximal tree in $|K|$ has precisely N edges. As a special case, if $|K|$ is a connected graph then $H_1(K;\mathbb{Z}) = 0$ if and only if $|K|$ is a tree.

- If K has dimension n, then $H_n(K;\mathbb{Z})$ is necessarily a free abelian group, since $B_n(K;\mathbb{Z}) = 0$ and $Z_n(K;\mathbb{Z}) \subseteq C_n(K;\mathbb{Z})$.

- If K is the abstract simplicial complex determined by a geometric n-simplex with $n \geq 1$, then $H_i(K;\mathbb{Z}) = 0$ for $i \geq 1$, since every cycle of any positive dimension is a boundary.

- If K is determined by a geometric $(n+1)$-simplex with $n \geq 1$, and S is the n-skeleton of K (hence $|S|$ is homeomorphic to an n-sphere), then $H_i(S;\mathbb{Z}) = 0$ for $1 \leq i \leq n-1$ while $H_n(S;\mathbb{Z}) \cong \mathbb{Z}$. Thus H_n detects "n-chambers".

Short exercise

(1) Find the homology groups with coefficients in $\mathbb{Z}$ of the abstract simplicial complex whose maximal faces are

(01) (02) (03) (12) (13) (234) (345) (346) (356) (456).

Longer exercise

(1) Let K be the abstract simplicial complex whose maximal faces are

(124) (125) (135) (136) (146) (234) (236) (256) (345) (456).

(a) Compute the homology groups $H_i(K;\mathbb{Z})$ using

GUDHI (http://gudhi.gforge.inria.fr),

Dyonisus (https://mrzv.org/software/dionysus2/),

or any other software of your preference.

(b) Find a generator of $H_1(K;\mathbb{Z})$.

(c) Draw a picture showing that the geometric realization of K is homeomorphic to the real projective plane.