Short exercises Lecture 2

Exercise 1. Find the homology groups with coefficients in $\mathbb{Z}$ of the abstract simplicial complex whose maximal faces are

$$(01)\ (02)\ (03)\ (12)\ (13)\ (234)\ (345)\ (346)\ (356)\ (456).$$

Proof. Let X be the abstract simplicial complex whose maximal faces are the ones stated above. We write the chain complex for X:

$$0 \xrightarrow[D_3]{\partial_3} C_2(X) \cong \mathbb{Z}^5 \xrightarrow[D_2]{\partial_2} C_1(X) \cong \mathbb{Z}^{13} \xrightarrow[D_1]{\partial_1} C_0(X) \cong \mathbb{Z}^7 \xrightarrow[D_0]{\partial_0} 0$$

Where, given the basis of $C_i(X)$ in lexicographic order,

$$D_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The geometric realisation of this simplex is the following:

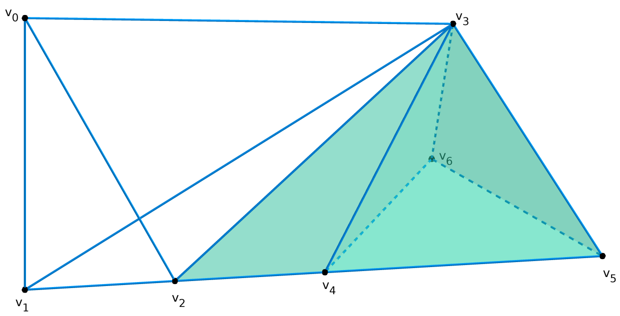

Figure 1: The geometric realisation of X

Now, seeing the geometric realisation, we can make a guess of what the homology is going to be.

- $\mathbf{H_0(X)}$. Clearly, it is a connected figure, so we know that $H_0(X) \cong \mathbb{Z}$.

- $\mathbf{H_1(X)}$. All the edges coming from a face don't count towards the homology. Therefore, we only have to compute the homology of the subsimplex $\widetilde{X}$ with maximal faces $(01), (02), (03), (12), (13), (23)$, which is the 1-skeleton of the 3-simplex. Therefore (this has been computed in class using the maximal tree), $H_1(X) \cong H_1(\widetilde{X}) \cong \mathbb{Z}^3$.

- $\mathbf{H_2(X)}$. There is a "2-hole" (a cavity enclosed by 2-faces). Therefore, $H_2(X) \cong \mathbb{Z}$.

We can check our hypothesis computing the Betti numbers:

$$1 = \beta_0 = F_0 - \operatorname{rank}(D_0) - \operatorname{rank}(D_1) = 7 - 0 - \operatorname{rank}(D_1)$$

Hence, $\operatorname{rank}(D_1) = 6$.

$$\beta_1 = F_1 - \operatorname{rank}(D_1) - \operatorname{rank}(D_2) = 13 - 6 - \operatorname{rank}(D_2) = 7 - \operatorname{rank}(D_2)$$

$$\beta_2 = F_2 - \operatorname{rank}(D_2) - \operatorname{rank}(D_3) = 5 - \operatorname{rank}(D_2) - 0 = 5 - \operatorname{rank}(D_2)$$

Now, we may compute $\operatorname{rank}(D_2)$ using the matrix of D_2. If one does that, we find out $\operatorname{rank}(D_2) = 4$, and hence

$$\beta_1 = 7 - 4 = 3, \qquad \beta_2 = 5 - 4 = 1$$

Another way to compute the homology (without using the matrices) is using a software, such as Sage.

```
S = SimplicialComplex(maximal_faces=[[0,1],[0,2],[0,3],[1,2],[1,3],[2,3,4],
        [3,4,5],[3,4,6],[3,5,6],[4,5,6]], is_mutable=False)
S.homology(reduced=False)
```

This code outputs

```
{0: Z, 1: Z x Z x Z, 2: Z}
```

□