Short exercises Lecture 10

Exercise 1. Consider the functions $f, g\colon [-1,1] \longrightarrow \mathbb{R}$ given by

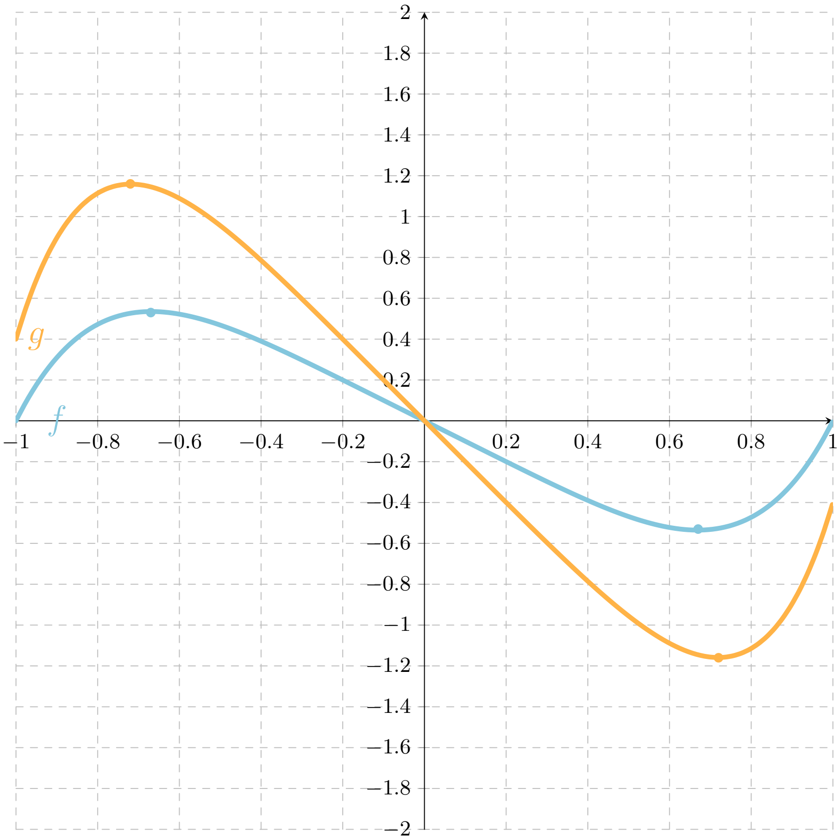
$$f(x) = x^5 - x, \qquad g(x) = \frac{1}{5}(x^9 + 7x^5 - 10x).$$

(a) Find the persistence modules $V(f)$ and $V(g)$ and the spectrum of each.

(b) Compute the interleaving distance $d_{int}(V(f), V(g))$.

(c) Prove that $d_{int}(V(f), V(g)) \leq \|f - g\|_\infty$ on $[-1,1]$.

Solution. (a) Let us first start by plotting f and g:



Now we want to compute $V_t(h)$. In order to do so, let us first compute $L_t(h)$, for all t.

- We first compute $L_t(f)$. We have the following points to look for:

$$t \in \left\{ f(-1) = 0, f\left(\frac{1}{\sqrt[4]{5}}\right), f\left(-\frac{1}{\sqrt[4]{5}}\right), f(1) = 0 \right\}.$$

  Then,

  – For $t \in \left(-\infty, f\left(-\frac{1}{\sqrt[4]{5}}\right)\right)$, $L_t(f) = \varnothing$.

  – For $t \in \left[f\left(-\frac{1}{\sqrt[4]{5}}\right), 0\right)$, $L_t(f)$ is 1 interval.

  – For $t \in \left[0, f\left(\frac{1}{\sqrt[4]{5}}\right)\right)$, $L_t(f)$ is 2 intervals.

  – For $t \in \left[f\left(\frac{1}{\sqrt[4]{5}}\right), \infty\right)$, $L_t(f) = [-1,1]$.

- Now compute $L_t(g)$. We have the following points to look for:

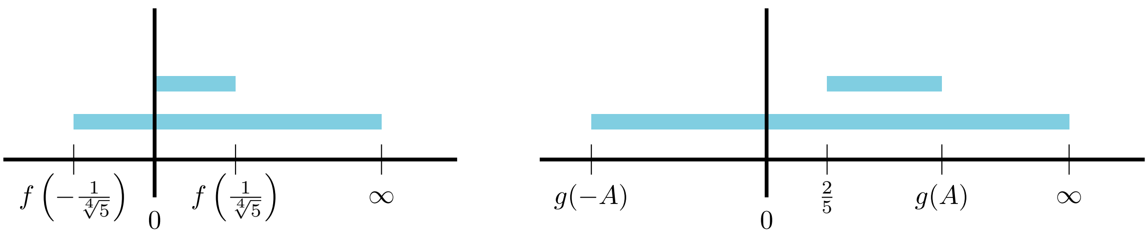
$$t \in \left\{ g(1) = -\frac{2}{5}, g\left(\frac{\sqrt[4]{0.5(-35+\sqrt{1585})}}{\sqrt{3}}\right), f\left(-\frac{\sqrt[4]{0.5(-35+\sqrt{1585})}}{\sqrt{3}}\right), g(-1) = \frac{2}{5} \right\}.$$

Then, We are going to let $A = \frac{\sqrt[4]{0.5(-35+\sqrt{1585})}}{\sqrt{3}}$ for convenience.

  - For $t \in (-\infty, g(-A))$, $L_t(g) = \varnothing$.
  - For $t \in \left[g(-A), \frac{2}{5}\right)$, $L_t(g)$ is 1 interval.
  - For $t \in \left[\frac{2}{5}, g(A)\right)$, $L_t(g)$ is 2 intervals.
  - For $t \in [g(A), \infty)$, $L_t(g) = [-1, 1]$.

We have not forgotten about the interval $\left(-\frac{2}{5}, \frac{2}{5}\right)$. There is just no change in the H_0 of $L_t(g)$.

The barcodes are the following:



(a) Barcode for $V_t(f)$ (b) Barcode for $V_t(g)$

Finally,

$$\mathrm{Spec}(V(f)) = \left\{ f\left(-\frac{1}{\sqrt[4]{5}}\right) \approx -0.534992,\ f(\pm 1) = 0,\ f\left(\frac{1}{\sqrt[4]{5}}\right) \approx 0.534992 \right\}$$

$$V(f) = \mathbb{R}[-0.534992, \infty) \oplus \mathbb{R}[0, 0.534992)$$

$$\mathrm{Spec}(V(g)) = \left\{ g(-A) \approx -1.15872,\ g(-1) = \frac{2}{5},\ g(A) \approx 1.15872 \right\}$$

$$V(g) = \mathbb{R}[-1.15872, \infty) \oplus \mathbb{R}[0.4, 1.15872)$$

(b) Now to compute the interleaving distance we recall that:

$$d_{int}(V, V') = \inf\{\delta > 0 \mid (V, \pi) \text{ and } (V', \pi') \text{ are } \delta\text{-interleaved}\}$$

Then, if $a < b$ and $c < d$,

$$d_{int}(\mathbb{F}[a,b), \mathbb{F}[c,d)) = \min\{\max\{0.5(b-a), 0.5(d-c)\}, \max\{|a-c|, |b-d|\}\}$$

And also,

$$d_{int}(\mathbb{F}[a,\infty), \mathbb{F}[c,\infty)) = |a-c|.$$

Hence, to compute the interleaving distance between $V(f)$ and $V(g)$ we must compute:
Since $0 < 0.534992$ and $0.4 < 1.15872$, then

$$d_{int}(\mathbb{F}[0, 0.534992), \mathbb{F}[0.4, 1.15872)) = 0.37936$$

Also,

$$d_{int}(\mathbb{F}[-0.534992, \infty), \mathbb{F}[-1.15872, \infty)) = 0.623728$$

Hence,

$$d_{int}(V(f), V(g)) = \max\{0.37936, 0.623728\} = 0.623728.$$

(c) We need to compute the maximum of the function

$$h(x) = |f(x) - g(x)| = \left| \frac{1}{5} x(-5 + 2x^4 + x^8) \right|$$

in $[-1, 1]$. We can split the function in two intervals: between $[-1, 0]$ and $[0, 1]$. We know that h is not differentiable at 0 and $h(0) = 0$. So, let us find the maximum in both intervals. The critical points of h are:

$$x^* = \pm \frac{\sqrt[4]{-5 + \sqrt{70}}}{\sqrt{3}}$$

And in both values we have a maximum. The value of h is:

$$h(x^*) = -\frac{8\left(\sqrt{70} - 50\right) \sqrt[4]{\sqrt{70} - 5}}{405\sqrt{3}} \approx 0.643153$$

□