Short exercises Lecture 3

Exercise 1. Draw a barcode for the persistent homology with $\mathbb{Z}$ coefficients for the Vietoris-Rips filtration of the following point cloud in $\mathbb{R}^2$:

$$X = \{(0,0),(1,2),(2,1),(4,3),(4,-3)\}$$

Proof. Let us start by defining $(0) := (0,0)$, $(1) := (1,2)$, $(2) := (2,1)$, $(3) := (4,3)$, $(4) := (4,-3)$. We have the following distance table:

	(0)	(1)	(2)	(3)	(4)
(0)	0				
(1)	$\sqrt{5}$	0			
(2)	$\sqrt{5}$	$\sqrt{2}$	0		
(3)	5	$\sqrt{10}$	$2\sqrt{2}$	0	
(4)	5	$\sqrt{34}$	$2\sqrt{5}$	6	0

Let us now compute the Vietoris-Rips filtration.

- $\varepsilon < \mathbf{2}$:

  In this case, the distance is smaller than the distance at which any point is from each other. Therefore, $R_\varepsilon(X)$ has maximal faces:

  $$(0),\ (1),\ (2),\ (3),\ (4).$$

  Then, since we have 5 connected components, $H_0(R_\varepsilon(X)) = \mathbb{Z}^5$, and all the other dimensions are 0.

- $\sqrt{\mathbf{2}} \le \varepsilon < \sqrt{\mathbf{5}}$:

  Now the only two points at distance less than $\sqrt{5}$ are (1) and (2). Therefore, we add an edge there and $R_\varepsilon(X)$ has maximal faces:

  $$(0),\ (12),\ (3),\ (4).$$

  Since we have 4 connected components, $H_0(R_\varepsilon(X)) = \mathbb{Z}^4$. The other dimensions are zero because there is no "n-hole" for $n \ge 1$.

- $\sqrt{\mathbf{5}} \le \varepsilon < \mathbf{2}\sqrt{\mathbf{2}}$:

  For these values of ε, the points (0), (1) and (2) are at reach one from each other. Since the Vietoris-Rips complex is a flag complex, we will have the 2-face (012). Hence, $R_\varepsilon(X)$ has maximal faces:

  $$(012),\ (3),\ (4).$$

  We have 3 connected components, hence $H_0(R_\varepsilon(X)) = \mathbb{Z}^3$. Also, $H_1(R_\varepsilon(X)) = 0$, because the 1-faces come from a 2-face (hence, there is a boundary). The higher dimension homologies are zero.

- $\mathbf{2}\sqrt{\mathbf{2}} \le \varepsilon < \sqrt{\mathbf{10}}$:

  In this case, we add (23). Therefore, $R_\varepsilon(X)$ has maximal faces:

  $$(012),\ (23),\ (4).$$

  We have 2 connected components ((012), (23) and (4)), hence, $H_0(R_\varepsilon(X)) = \mathbb{Z}^2$. For the higher dimensions, this is a combination of the previous one and the one before, so they are 0.

- $\sqrt{\mathbf{10}} \le \varepsilon < \mathbf{2}\sqrt{\mathbf{5}}$:

  For ε in this range, we add the 1-face (13). Then we have (12), (13), (23), which is the boundary of (123). Since Vietoris-Rips is a flag complex, we have this 2-face. Hence, the maximal faces of $R_\varepsilon(X)$ are:

  $$(012),\ (123),\ (4).$$

  We still have 2 connected components, hence, $H_0(R_\varepsilon(X)) = \mathbb{Z}^2$, and since we have two 2-faces, the higher dimensions are 0.

- $\mathbf{2\sqrt{5}} \le \varepsilon < \mathbf{5}$:

  Now we add the edge (34) and therefore we only have a connected component. The Vietoris-Rips complex $R_\varepsilon(X)$ has maximal faces:

  $$(012),\ (123),\ (34).$$

  Then, $H_0(R_\varepsilon(X)) = \mathbb{Z}$ and all the higher dimensions are zero.

- $\mathbf{5} \le \varepsilon < \sqrt{\mathbf{34}}$:

  For this case, we add the faces (013) and (023), which in combination of the ones we had before, (012) and (123), implies that we have the 3-face (0123). We also add (04) and hence the 2-face (024). The maximal faces are:

  $$(0123),\ (024).$$

  The homology is the same, $H_0(R_\varepsilon(X)) = \mathbb{Z}$ and all the higher dimensions are 0.

- $\sqrt{\mathbf{34}} \le \varepsilon < \mathbf{6}$:

  Now we add the 2-faces (014) and (124), which implies that we have the 3-face (0124). The maximal faces of the Vietoris-Rips complex $R_\varepsilon(X)$ are:

  $$(0123),\ (0124).$$

  The homology remains the same, $H_0(R_\varepsilon(X)) = \mathbb{Z}$ and all the higher ones are zero.

- $\varepsilon \ge \mathbf{6}$:

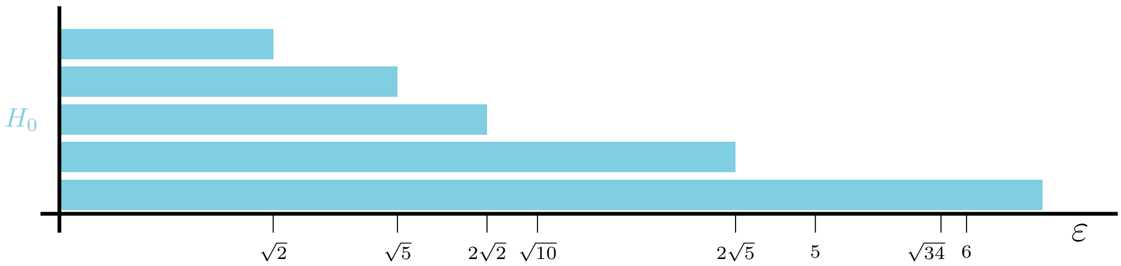
  Finally, all the points are within reach of each other, so $R_\varepsilon(X) = \Delta^4$, with a unique maximal face

  $$(01234).$$

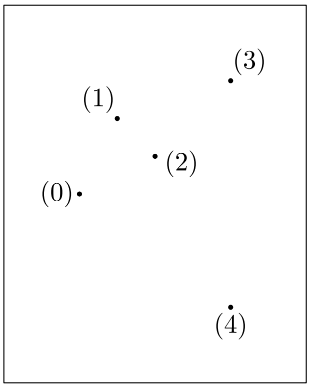
  Since its homotopy equivalent to a point, the homology is isomorphic to that of a point, hence $H_0(R_\varepsilon(X)) = \mathbb{Z}$ and all higher dimensions are zero.

In the next page, we have the geometric realisations for each interval that we are going to discuss now. You will notice that only 0-faces and 1-faces have been drawn. This is on purpose, since the Vietoris-Rips complex is a flag complex, so whenever there is a boundary for a higher face, that higher face must be there. Therefore, we know the face must be there, but its easier to see if it is not drawn.

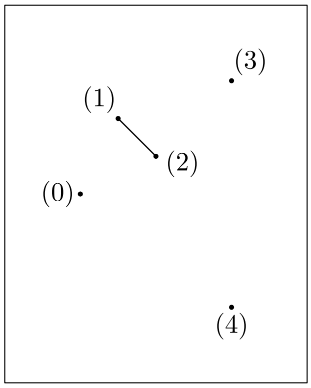
Barcode



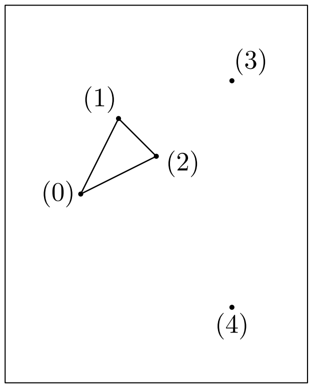
$\square$



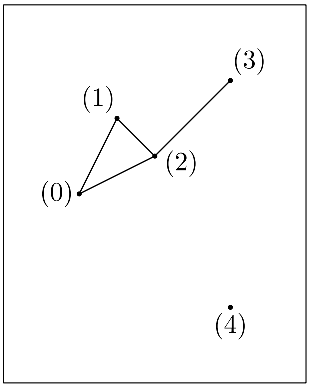
$\varepsilon < \sqrt{2}$



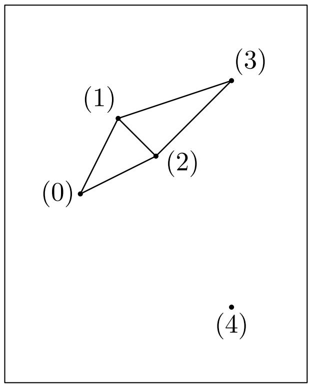
$\sqrt{2} \leq \varepsilon < \sqrt{5}$



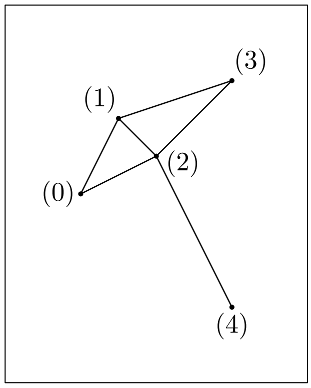
$\sqrt{5} \leq \varepsilon < 2\sqrt{2}$



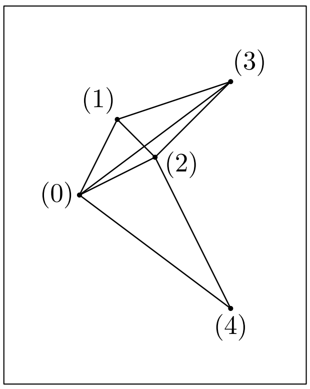
$2\sqrt{2} \leq \varepsilon < \sqrt{10}$



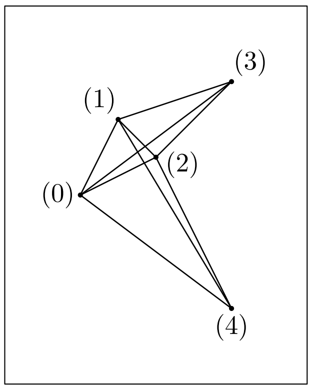
$\sqrt{10} \leq \varepsilon < 2\sqrt{5}$



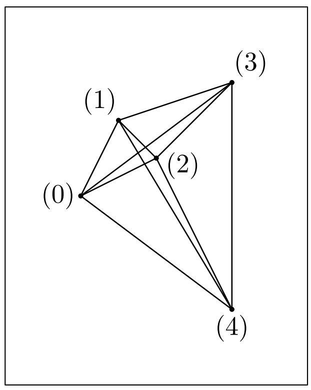
$2\sqrt{5} \leq \varepsilon < 5$



$5 \leq \varepsilon < \sqrt{34}$



$\sqrt{34} \leq \varepsilon < 6$



$\varepsilon \geq 6$