List of exercises 3: One dimensional optimization

Exercise 1. List the maximal faces of the Čech complex $C_\varepsilon(X)$ and the Vietoris–Rips complex $R_\varepsilon(X)$, depending on ε, if X is the set of vertices of a regular hexagon of radius 1.

Proof. Let us first compute the distances between each point

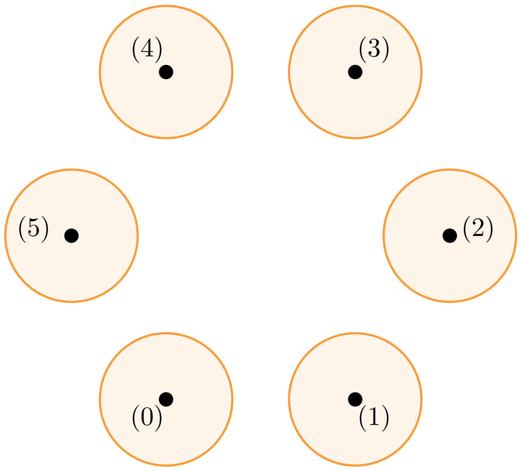
	(0)	(1)	(2)	(3)	(4)	(5)
(0)	0					
(1)	1	0				
(2)	$\sqrt{3}$	1	0			
(3)	2	$\sqrt{3}$	1	0		
(4)	$\sqrt{3}$	2	$\sqrt{3}$	1	0	
(5)	1	$\sqrt{3}$	2	$\sqrt{3}$	1	0

So, we will have the following cases:

- $\boldsymbol{\varepsilon < 1}$

  In this case, $C_\varepsilon(X) = R_\varepsilon(X)$, since all points are at distance 1 and we only hace the 0-faces. The complex has maximal faces

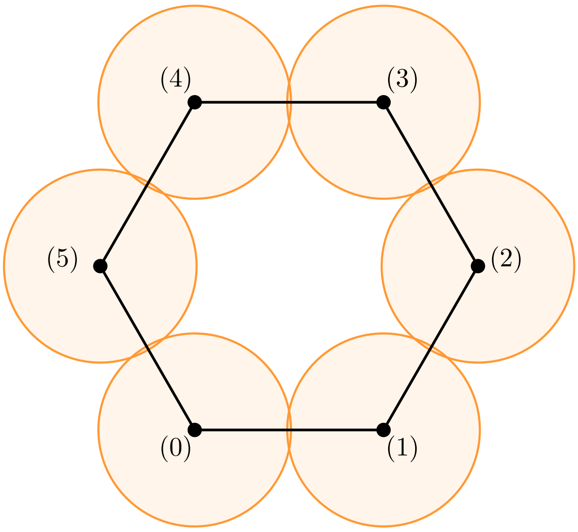
$$(0),\ (1),\ (2),\ (3),\ (4),\ (5)$$



- $\boldsymbol{1 \leq \varepsilon < \sqrt{3}}$

  In this case all points are at distance 1 to their adjacent and at distance at least $\sqrt{3}$ to any other point which is not adjacent. Hence, $C_\varepsilon(X) = R_\varepsilon(X)$, and the complex has maximal faces

$$(01),\ (12),\ (23),\ (34),\ (45),\ (05)$$



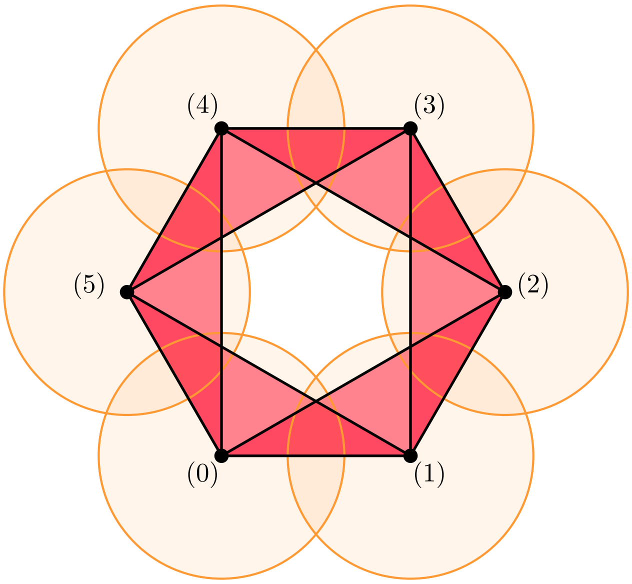
- **$\sqrt{3} \leq \varepsilon < 2$**

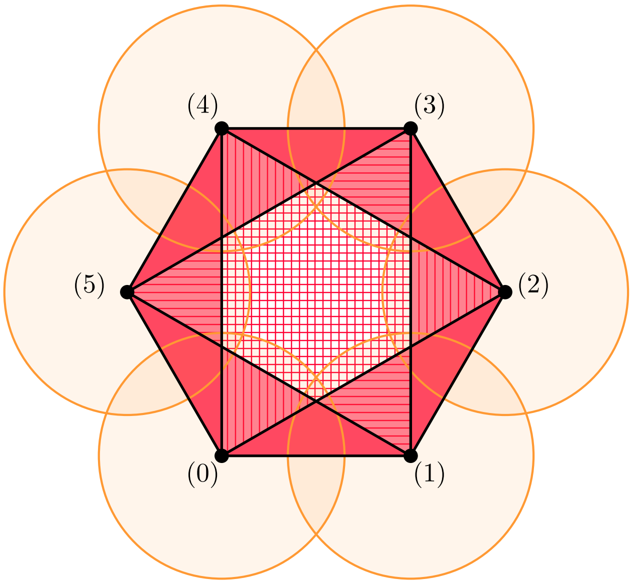
  In this case, $C_\varepsilon(X) \subsetneq R_\varepsilon(X)$. We have the following maximal faces for each complex:

  $$C_\varepsilon(X): \quad (015),\ (012),\ (123),\ (234),\ (345),\ (045)$$

  $$R_\varepsilon(X): \quad (015),\ (012),\ (123),\ (234),\ (345),\ (045),\ (024),\ (135)$$

  Why this discrepancy? Well, notice that even if $d((0),(2))$, $d((2),(4))$ and $d((0),(4))$ are $\sqrt{3}$ there intersection between $\overline{B}_{\varepsilon/2}((0)), \overline{B}_{\varepsilon/2}((2))$ and $\overline{B}_{\varepsilon/2}((4))$ is empty. Therefore, this triangle is in Vietoris-Rips but not in Čech. The same happens with (135).



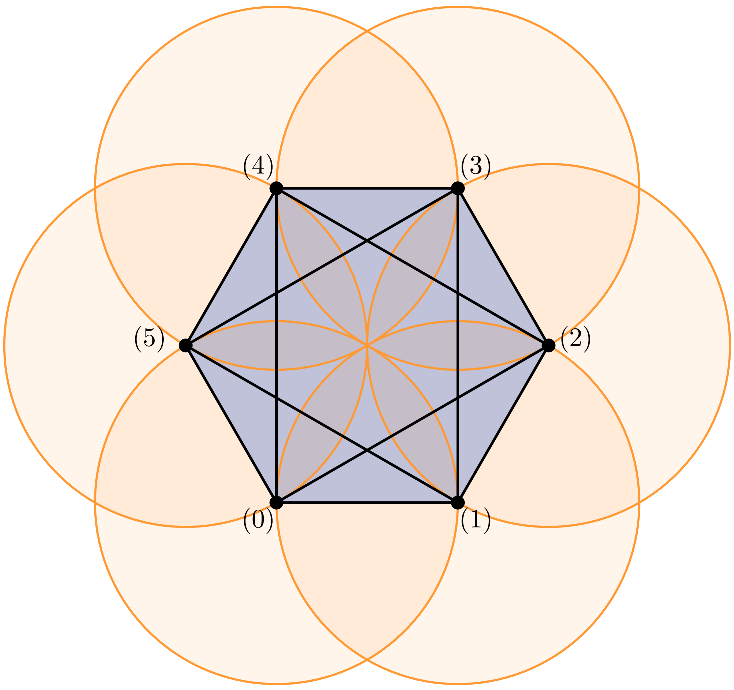


  On the left we can see $C_\varepsilon(X)$ and on the right $R_\varepsilon(X)$.

- **$\varepsilon \geq 2$**

  In this case, since all points are at distance at most 2, $C_\varepsilon(X) = R_\varepsilon(X) = \Delta^5$. That is, the only maximal face is

  $$(012345)$$



□

Exercise 2. List the maximal faces of the Vietoris–Rips complex for every ε for the point cloud $X = \{(0,0,0),(1,1,1),(0,2,0),(0,0,-1)\}$ in $\mathbb{R}^3$.

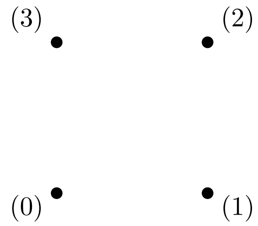
Proof. First, let $(0) := (0,0,0)$, $(1) := (1,1,1)$, $(2) := (0,2,0)$, $(3) := (0,0,-1)$. Now let us compute the distances between each point:

	(0)	(1)	(2)	(3)
(0)	0			
(1)	$\sqrt{3}$	0		
(2)	2	$\sqrt{3}$	0	
(3)	1	$\sqrt{6}$	$\sqrt{5}$	0

- $\varepsilon < \mathbf{1}$

  Since the minimum distance between two points is 1, we have only the 4 points. The maximal faces of $R_\varepsilon(X)$ are:

$$(0),\ (1),\ (2),\ (3).$$

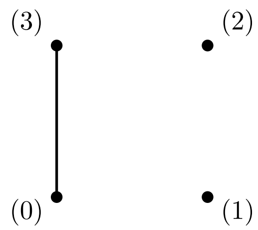


  Remark. Notice that this configuration of points is not the original cloud point. This is just a representation of the Vietoris-Rips complex (although it should be in 3 dimensions, but that would just make it less clear).

- $\mathbf{1} \leq \varepsilon < \sqrt{\mathbf{3}}$

  Only the vertices (0) and (3) are at distance 1, hence, we only have a 1-face. The maximal faces of $R_\varepsilon(X)$ are:

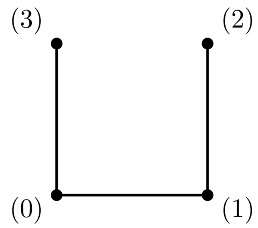
$$(03),\ (1),\ (2).$$



- $\sqrt{\mathbf{3}} \leq \varepsilon < \mathbf{2}$

  We add (01) and (12) to the 1-faces, since (0) and (1) are at distance $\sqrt{3}$ and so are (1) and (2). The maximal faces of $R_\varepsilon(X)$ are:

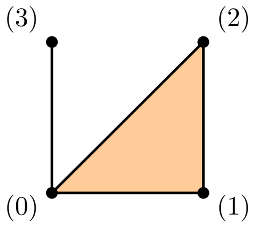
$$(01),\ (03),\ (12)$$



- $\mathbf{2} \leq \varepsilon < \sqrt{\mathbf{5}}$

  Now we add the 2-face (012), since the distance between the three vertices is less or equal than 2 (notice that this adds the 1-face (02) immediately, since we are just counting maximal faces). The maximal faces of $R_\varepsilon(X)$ are:

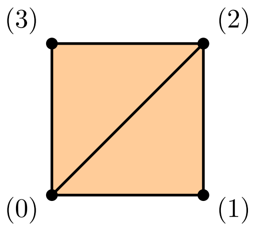
$$(012),\ (03).$$



- $\sqrt{\mathbf{5}} \leq \varepsilon < \sqrt{\mathbf{6}}$

  We now add the 2-face (023) (which immediately adds the edge (23)). The maximal faces of $R_\varepsilon(X)$ are

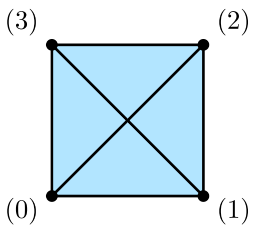
$$(012),\ (023).$$



- $\varepsilon \geq \sqrt{\mathbf{6}}$

  All points are at distance less or equal than $\sqrt{6}$, hence, we $R_\varepsilon(X) = \Delta^3$ has only one maximal face

$$(0123)$$



$\square$