

```
def basic_multivector_operations_3D():
    Print_Function()
    g3d = Ga('e*x|y|z')
    (ex,ey,ez) = g3d.mv()
    A = g3d.mv('A','mv')
    A.Fmt(1,'A')
    A.Fmt(2,'A')
    A.Fmt(3,'A')
    A.even().Fmt(1,'%A_{+}')
    A.odd().Fmt(1,'%A_{-}')
    X = g3d.mv('X','vector')
    Y = g3d.mv('Y','vector')
    print 'g_{ij} = ',g3d.g
    X.Fmt(1,'X')
    Y.Fmt(1,'Y')
    (X*Y).Fmt(2,'X*Y')
    (X^Y).Fmt(2,'X^Y')
    (X|Y).Fmt(2,'X|Y')
    return
```

Code Output:

$$A = A + A^x e_x + A^y e_y + A^z e_z + A^{xy} e_x \wedge e_y + A^{xz} e_x \wedge e_z + A^{yz} e_y \wedge e_z + A^{xyz} e_x \wedge e_y \wedge e_z$$

$$\begin{aligned} A = & A \\ & + A^x e_x + A^y e_y + A^z e_z \\ & + A^{xy} e_x \wedge e_y + A^{xz} e_x \wedge e_z + A^{yz} e_y \wedge e_z \\ & + A^{xyz} e_x \wedge e_y \wedge e_z \end{aligned}$$

$$\begin{aligned} A = & A \\ & + A^x e_x \\ & + A^y e_y \\ & + A^z e_z \\ & + A^{xy} e_x \wedge e_y \\ & + A^{xz} e_x \wedge e_z \\ & + A^{yz} e_y \wedge e_z \\ & + A^{xyz} e_x \wedge e_y \wedge e_z \end{aligned}$$

$$A_+ = A + A^{xy} e_x \wedge e_y + A^{xz} e_x \wedge e_z + A^{yz} e_y \wedge e_z$$

$$A_- = A^x e_x + A^y e_y + A^z e_z + A^{xyz} e_x \wedge e_y \wedge e_z$$

$$g_{ij} = \begin{bmatrix} (e_x \cdot e_x) & (e_x \cdot e_y) & (e_x \cdot e_z) \\ (e_x \cdot e_y) & (e_y \cdot e_y) & (e_y \cdot e_z) \\ (e_x \cdot e_z) & (e_y \cdot e_z) & (e_z \cdot e_z) \end{bmatrix}$$

$$X = X^x e_x + X^y e_y + X^z e_z$$

$$Y = Y^x e_x + Y^y e_y + Y^z e_z$$

$$\begin{aligned} XY = & (X^x Y^x (e_x \cdot e_x) + X^x Y^y (e_x \cdot e_y) + X^x Y^z (e_x \cdot e_z) + X^y Y^x (e_x \cdot e_y) + X^y Y^y (e_y \cdot e_y) + X^y Y^z (e_y \cdot e_z) + X^z Y^x (e_x \cdot e_z) + X^z Y^y (e_y \cdot e_z) + X^z Y^z (e_z \cdot e_z)) \\ & + (X^x Y^y - X^y Y^x) e_x \wedge e_y + (X^x Y^z - X^z Y^x) e_x \wedge e_z + (X^y Y^z - X^z Y^y) e_y \wedge e_z \end{aligned}$$

$$X \wedge Y = (X^x Y^y - X^y Y^x) e_x \wedge e_y + (X^x Y^z - X^z Y^x) e_x \wedge e_z + (X^y Y^z - X^z Y^y) e_y \wedge e_z$$

$$X \cdot Y = X^x Y^x (e_x \cdot e_x) + X^x Y^y (e_x \cdot e_y) + X^x Y^z (e_x \cdot e_z) + X^y Y^x (e_x \cdot e_y) + X^y Y^y (e_y \cdot e_y) + X^y Y^z (e_y \cdot e_z) + X^z Y^x (e_x \cdot e_z) + X^z Y^y (e_y \cdot e_z) + X^z Y^z (e_z \cdot e_z)$$

```
def basic_multivector_operations_2D():
    Print_Function()
    g2d = Ga('e*x|y')
    (ex,ey) = g2d.mv()
    print 'g-{ ij } =',g2d.g
    X = g2d.mv('X','vector')
    A = g2d.mv('A','spinor')
    X.Fmt(1,'X')
    A.Fmt(1,'A')
    (X|A).Fmt(2,'X|A')
    (X<A).Fmt(2,'X<A')
    (A>X).Fmt(2,'A>X')
    return
```

Code Output:

$$g_{ij} = \begin{bmatrix} (e_x \cdot e_x) & (e_x \cdot e_y) \\ (e_x \cdot e_y) & (e_y \cdot e_y) \end{bmatrix}$$
$$X = X^x \boldsymbol{e_x} + X^y \boldsymbol{e_y}$$
$$A = A + A^{xy} \boldsymbol{e_x} \wedge \boldsymbol{e_y}$$
$$X \cdot A = -A^{xy} (X^x (e_x \cdot e_y) + X^y (e_y \cdot e_y)) \boldsymbol{e_x} + A^{xy} (X^x (e_x \cdot e_x) + X^y (e_x \cdot e_y)) \boldsymbol{e_y}$$
$$X|A = (AX^x - A^{xy}X^x (e_x \cdot e_y) - A^{xy}X^y (e_y \cdot e_y)) \boldsymbol{e_x} + (AX^y + A^{xy}X^x (e_x \cdot e_x) + A^{xy}X^y (e_x \cdot e_y)) \boldsymbol{e_y}$$
$$A|X = (AX^x + A^{xy}X^x (e_x \cdot e_y) + A^{xy}X^y (e_y \cdot e_y)) \boldsymbol{e_x} + (AX^y - A^{xy}X^x (e_x \cdot e_x) - A^{xy}X^y (e_x \cdot e_y)) \boldsymbol{e_y}$$

```
def basic_multivector_operations_2D_orthogonal():
    Print_Function()
    o2d = Ga('e*x|y',g=[1,1])
    (ex,ey) = o2d.mv()
    print 'g-{ ii } =',o2d.g
    X = o2d.mv('X','vector')
    A = o2d.mv('A','spinor')
    X.Fmt(1,'X')
    A.Fmt(1,'A')
    (X*A).Fmt(2,'X*A')
    (X|A).Fmt(2,'X|A')
    (X<A).Fmt(2,'X<A')
    (X>A).Fmt(2,'X>A')
    (A*X).Fmt(2,'A*X')
    (A|X).Fmt(2,'A|X')
    (A<X).Fmt(2,'A<X')
    (A>X).Fmt(2,'A>X')
    return
```

Code Output:

$$g_{ii} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$X = X^x \boldsymbol{e_x} + X^y \boldsymbol{e_y}$$
$$A = A + A^{xy} \boldsymbol{e_x} \wedge \boldsymbol{e_y}$$
$$XA = (AX^x - A^{xy}X^y) \boldsymbol{e_x} + (AX^y + A^{xy}X^x) \boldsymbol{e_y}$$
$$X \cdot A = -A^{xy}X^y \boldsymbol{e_x} + A^{xy}X^x \boldsymbol{e_y}$$
$$X|A = (AX^x - A^{xy}X^y) \boldsymbol{e_x} + (AX^y + A^{xy}X^x) \boldsymbol{e_y}$$
$$X|A = AX^x \boldsymbol{e_x} + AX^y \boldsymbol{e_y}$$
$$AX = (AX^x + A^{xy}X^y) \boldsymbol{e_x} + (AX^y - A^{xy}X^x) \boldsymbol{e_y}$$

$$A \cdot X = A^{xy}X^ye_{\boldsymbol{x}} - A^{xy}X^xe_{\boldsymbol{y}}$$

$$A\rfloor X = AX^xe_{\boldsymbol{x}} + AX^ye_{\boldsymbol{y}}$$

$$A\llcorner X = (AX^x + A^{xy}X^y)\,e_{\boldsymbol{x}} + (AX^y - A^{xy}X^x)\,e_{\boldsymbol{y}}$$

```
def check_generalized_BAC_CAB_formulas():
    Print_Function()
    g4d = Ga('a b c d')
    (a,b,c,d) = g4d.mv()
    print 'g_{ij} =',g4d.g
    print '\\bm{a|(b*c)} =',a|(b*c)
    print '\\bm{a|(b^c)} =',a|(b^c)
    print '\\bm{a|(b^c^d)} =',a|(b^c^d)
    print '\\bm{a|(b^c)+c|(a^b)+b|(c^a)} =',(a|(b^c))+(c|(a^b))+(b|(c^a))
    print '\\bm{a*(b^c)-b*(a^c)+c*(a^b)} =',a*(b^c)-b*(a^c)+c*(a^b)
    print '\\bm{a*(b^c^d)-b*(a^c^d)+c*(a^b^d)-d*(a^b^c)} =',a*(b^c^d)-b*(a^c^d)+c*(a^b^d)-d*(a^b^c)
    print '\\bm{(a^b)|(c^d)} =',(a^b)|(c^d)
    print '\\bm{((a^b)|c)|d} =',((a^b)|c)|d
    print '\\bm{(a^b)\\times (c^d)} =',Com(a^b,c^d)
    return
```

Code Output:

$$g_{ij} = \begin{bmatrix} (a \cdot a) & (a \cdot b) & (a \cdot c) & (a \cdot d) \\ (a \cdot b) & (b \cdot b) & (b \cdot c) & (b \cdot d) \\ (a \cdot c) & (b \cdot c) & (c \cdot c) & (c \cdot d) \\ (a \cdot d) & (b \cdot d) & (c \cdot d) & (d \cdot d) \end{bmatrix}$$

$$\boldsymbol{a} \cdot (\boldsymbol{bc}) = - (a \cdot c)\,\boldsymbol{b} + (a \cdot b)\,\boldsymbol{c}$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c}) = - (a \cdot c)\,\boldsymbol{b} + (a \cdot b)\,\boldsymbol{c}$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c} \wedge \boldsymbol{d}) = (a \cdot d)\,\boldsymbol{b} \wedge \boldsymbol{c} - (a \cdot c)\,\boldsymbol{b} \wedge \boldsymbol{d} + (a \cdot b)\,\boldsymbol{c} \wedge \boldsymbol{d}$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c}) + \boldsymbol{c} \cdot (\boldsymbol{a} \wedge \boldsymbol{b}) + \boldsymbol{b} \cdot (\boldsymbol{c} \wedge \boldsymbol{a}) = 0$$

$$\boldsymbol{a}(\boldsymbol{b} \wedge \boldsymbol{c}) - \boldsymbol{b}(\boldsymbol{a} \wedge \boldsymbol{c}) + \boldsymbol{c}(\boldsymbol{a} \wedge \boldsymbol{b}) = 3\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}$$

$$\boldsymbol{a}(\boldsymbol{b} \wedge \boldsymbol{c} \wedge \boldsymbol{d}) - \boldsymbol{b}(\boldsymbol{a} \wedge \boldsymbol{c} \wedge \boldsymbol{d}) + \boldsymbol{c}(\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{d}) - \boldsymbol{d}(\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}) = 4\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} \wedge \boldsymbol{d}$$

$$(\boldsymbol{a} \wedge \boldsymbol{b}) \cdot (\boldsymbol{c} \wedge \boldsymbol{d}) = - (a \cdot c)\,(b \cdot d) + (a \cdot d)\,(b \cdot c)$$

$$((\boldsymbol{a} \wedge \boldsymbol{b}) \cdot \boldsymbol{c}) \cdot \boldsymbol{d} = - (a \cdot c)\,(b \cdot d) + (a \cdot d)\,(b \cdot c)$$

$$(\boldsymbol{a} \wedge \boldsymbol{b}) \times (\boldsymbol{c} \wedge \boldsymbol{d}) = - (b \cdot d)\,\boldsymbol{a} \wedge \boldsymbol{c} + (b \cdot c)\,\boldsymbol{a} \wedge \boldsymbol{d} + (a \cdot d)\,\boldsymbol{b} \wedge \boldsymbol{c} - (a \cdot c)\,\boldsymbol{b} \wedge \boldsymbol{d}$$

```
def rounding_numerical_components():
    Print_Function()
    o3d = Ga('e_x e_y e_z',g=[1,1,1])
    (ex,ey,ez) = o3d.mv()
    X = 1.2*ex+2.34*ey+0.555*ez
    Y = 0.333*ex+4*ey+5.3*ez
    print 'X =',X
    print 'Nga(X,2) =',Nga(X,2)
    print 'X*Y =',X*Y
    print 'Nga(X*Y,2) =',Nga(X*Y,2)
    return
```

Code Output:

$$X = 1 \cdot 2e_{\boldsymbol{x}} + 2 \cdot 34e_{\boldsymbol{y}} + 0 \cdot 555e_{\boldsymbol{z}}$$

$$Nga(X,2) = 1 \cdot 2e_{\boldsymbol{x}} + 2 \cdot 3e_{\boldsymbol{y}} + 0 \cdot 55e_{\boldsymbol{z}}$$

$$XY = 12 \cdot 7011 + 4 \cdot 02078e_{\boldsymbol{x}} \wedge e_{\boldsymbol{y}} + 6 \cdot 175185e_{\boldsymbol{x}} \wedge e_{\boldsymbol{z}} + 10 \cdot 182e_{\boldsymbol{y}} \wedge e_{\boldsymbol{z}}$$

$$Nga(XY,2) = 13 \cdot 0 + 4 \cdot 0e_{\boldsymbol{x}} \wedge e_{\boldsymbol{y}} + 6 \cdot 2e_{\boldsymbol{x}} \wedge e_{\boldsymbol{z}} + 10 \cdot 0e_{\boldsymbol{y}} \wedge e_{\boldsymbol{z}}$$

```
def derivatives_in_rectangular_coordinates():
    Print_Function()
    X = (x,y,z) = symbols('x y z')
    o3d = Ga('e_x e_y e_z ',g=[1,1,1],coords=X)
    (ex,ey,ez) = o3d.mv()
    grad = o3d.grad
    f = o3d.mv('f','scalar',f=True)
    A = o3d.mv('A','vector',f=True)
    B = o3d.mv('B','bivector',f=True)
    C = o3d.mv('C','mv')
    print 'f =',f
    print 'A =',A
    print 'B =',B
    print 'C =',C
    print 'grad*f =',grad*f
    print 'grad|A =',grad|A
    print 'grad*A =',grad*A
    print '-I*(grad^A) =',-o3d.i*(grad^A)
    print 'grad*B =',grad*B
    print 'grad^B =',grad^B
    print 'grad|B =',grad|B
    return
```

Code Output:

$$f = f$$
$$A = A^x e_x + A^y e_y + A^z e_z$$
$$B = B^{xy} e_x \wedge e_y + B^{xz} e_x \wedge e_z + B^{yz} e_y \wedge e_z$$
$$C = C + C^x e_x + C^y e_y + C^z e_z + C^{xy} e_x \wedge e_y + C^{xz} e_x \wedge e_z + C^{yz} e_y \wedge e_z + C^{xyz} e_x \wedge e_y \wedge e_z$$
$$\nabla f = \partial_x f e_x + \partial_y f e_y + \partial_z f e_z$$
$$\nabla \cdot A = \partial_x A^x + \partial_y A^y + \partial_z A^z$$
$$\nabla A = (\partial_x A^x + \partial_y A^y + \partial_z A^z) + (-\partial_y A^x + \partial_x A^y) e_x \wedge e_y + (-\partial_z A^x + \partial_x A^z) e_x \wedge e_z + (-\partial_z A^y + \partial_y A^z) e_y \wedge e_z$$
$$-I(\nabla \wedge A) = (-\partial_z A^y + \partial_y A^z) e_x + (\partial_z A^x - \partial_x A^z) e_y + (-\partial_y A^x + \partial_x A^y) e_z$$
$$\nabla B = (-\partial_y B^{xy} - \partial_z B^{xz}) e_x + (\partial_x B^{xy} - \partial_z B^{yz}) e_y + (\partial_x B^{xz} + \partial_y B^{yz}) e_z + (\partial_z B^{xy} - \partial_y B^{xz} + \partial_x B^{yz}) e_x \wedge e_y \wedge e_z$$
$$\nabla \wedge B = (\partial_z B^{xy} - \partial_y B^{xz} + \partial_x B^{yz}) e_x \wedge e_y \wedge e_z$$
$$\nabla \cdot B = (-\partial_y B^{xy} - \partial_z B^{xz}) e_x + (\partial_x B^{xy} - \partial_z B^{yz}) e_y + (\partial_x B^{xz} + \partial_y B^{yz}) e_z$$

```
def derivatives_in_spherical_coordinates():
    Print_Function()
    X = (r,th,phi) = symbols('r theta phi')
    s3d = Ga('e_r e_theta e_phi ',g=[1,r**2,r**2*sin(th)**2],coords=X,norm=True)
    (er,eth,ephi) = s3d.mv()
    grad = s3d.grad
    f = s3d.mv('f','scalar',f=True)
    A = s3d.mv('A','vector',f=True)
    B = s3d.mv('B','bivector',f=True)
    print 'f =',f
    print 'A =',A
    print 'B =',B
    print 'grad*f =',grad*f
    print 'grad|A =',grad|A
    print '-I*(grad^A) =',(-s3d.i*(grad^A)).simplify()
    print 'grad^B =',grad^B
```

Code Output:

$$f = f$$
$$A = A^r \mathbf{e}_r + A^\theta \mathbf{e}_\theta + A^\phi \mathbf{e}_\phi$$
$$B = B^{r\theta} \mathbf{e}_r \wedge \mathbf{e}_\theta + B^{r\phi} \mathbf{e}_r \wedge \mathbf{e}_\phi + B^{\phi\phi} \mathbf{e}_\theta \wedge \mathbf{e}_\phi$$
$$\nabla f = \partial_r f \mathbf{e}_r + \frac{1}{r} \partial_\theta f \mathbf{e}_\theta + \frac{\partial_\phi f}{r \sin(\theta)} \mathbf{e}_\phi$$
$$\nabla \cdot A = \frac{1}{r} \left(r \partial_r A^r + 2 A^r + \frac{A^\theta}{\tan(\theta)} + \partial_\theta A^\theta + \frac{\partial_\phi A^\phi}{\sin(\theta)} \right)$$
$$-I(\nabla \wedge A) = \frac{1}{r} \left(\frac{A^\phi}{\tan(\theta)} + \partial_\theta A^\phi - \frac{\partial_\phi A^\theta}{\sin(\theta)} \right) \mathbf{e}_r + \frac{1}{r} \left(-r \partial_r A^\phi - A^\phi + \frac{\partial_\phi A^r}{\sin(\theta)} \right) \mathbf{e}_\theta + \frac{1}{r} \left(r \partial_r A^\theta + A^\theta - \partial_\theta A^r \right) \mathbf{e}_\phi$$
$$\nabla \wedge B = \frac{1}{r} \left(r \partial_r B^{\phi\phi} - \frac{B^{r\phi}}{\tan(\theta)} + 2 B^{\phi\phi} - \partial_\theta B^{r\phi} + \frac{\partial_\phi B^{r\theta}}{\sin(\theta)} \right) \mathbf{e}_r \wedge \mathbf{e}_\theta \wedge \mathbf{e}_\phi$$

```
def noneuclidian_distance_calculation():
    Print_Function()
    from sympy import solve,sqrt
    g = '0 # #,# 0 #,# # 1'
    nel = Ga('X Y e',g=g)
    (X,Y,e) = nel.mv()
    print 'g-{ij} =',nel.g
    print '%(X\\W Y)^{2} =',(X^Y)*(X^Y)
    L = X^Y^e
    B = L*e # DℳL 10.152
    Bsq = (B*B).scalar()
    print '#ℳL = X\\W Y\\W e \\text{ is a non-euclidian line}'
    print 'B = L*e =',B
    BeBr =B*e*B.rev()
    print '%BeB^{\\dagger} =',BeBr
    print '%B^{2} =',B*B
    print '%L^{2} =',L*L # DℳL 10.153
    (s,c,Binv,M,S,C,alpha) = symbols('s c (1/B) M S C alpha')
    XdotY = nel.g[0,1]
    Xdote = nel.g[0,2]
    Ydote = nel.g[1,2]
    Bhat = Binv*B # DℳL 10.154
    R = c+s*Bhat # Rotor R = exp(alpha*Bhat/2)
    print '#ℳs = \\f{\\sinh}{\\alpha/2} \\text{ and } c = \\f{\\cosh}{\\alpha/2}'
    print '%e^{\\alpha B/{2\\abs{B}}} =',R
    Z = R*X*R.rev() # DℳL 10.155
    Z.obj = expand(Z.obj)
    Z.obj = Z.obj.collect([Binv,s,c,XdotY])
    Z.Fmt(3,'%R XR^{\\dagger}')
    W = Z|Y # Extract scalar part of multivector
    # From this point forward all calculations are with sympy scalars
    #print '# Objective is to determine value of C = cosh(alpha) such that W = 0'
    W = W.scalar()
    print '%W = Z\\cdot Y =',W
    W = expand(W)
    W = simplify(W)
    W = W.collect([s*Binv])
    M = 1/Bsq
    W = W.subs(Binv**2,M)
    W = simplify(W)
    Bmag = sqrt(XdotY**2-2*XdotY*Xdote*Ydote)
    W = W.collect([Binv*c*s,XdotY])
    #Double angle substitutions
```

```
W = W.subs(2*XdotY**2-4*XdotY*Xdote*Ydote,2/(Binv**2))
W = W.subs(2*c*s,S)
W = W.subs(c**2,(C+1)/2)
W = W.subs(s**2,(C-1)/2)
W = simplify(W)
W = W.subs(1/Binv,Bmag)
W = expand(W)
print '%S = \\f{\\sinh}{\\alpha} \\text{ and } C = \\f{\\cosh}{\\alpha}',
print 'W=',W
Wd = collect(W,[C,S],exact=True,evaluate=False)
Wd_1 = Wd[one]
Wd_C = Wd[C]
Wd_S = Wd[S]
print '%\\text{Scalar Coefficient} =',Wd_1
print '%\\text{Cosh Coefficient} =',Wd_C
print '%\\text{Sinh Coefficient} =',Wd_S
print '%\\abs{B} =',Bmag
Wd_1 = Wd_1.subs(Bmag,1/Binv)
Wd_C = Wd_C.subs(Bmag,1/Binv)
Wd_S = Wd_S.subs(Bmag,1/Binv)
lhs = Wd_1+Wd_C*C
rhs = -Wd_S*S
lhs = lhs**2
rhs = rhs**2
W = expand(lhs-rhs)
W = expand(W.subs(1/Binv**2,Bmag**2))
W = expand(W.subs(S**2,C**2-1))
W = W.collect([C,C**2],evaluate=False)
a = simplify(W[C**2])
b = simplify(W[C])
c = simplify(W[one])
print '%\\text{Require } aC^{2}+bC+c = 0',
print 'a =',a
print 'b =',b
print 'c =',c
x = Symbol('x')
C = solve(a*x**2+b*x+c,x)[0]
print '%b^{2}-4ac =',simplify(b**2-4*a*c)
print '%\\f{\\cosh}{\\alpha} = C = -b/(2a) =',expand(simplify(expand(C)))
return
```

Code Output:

$$g_{ij} = \begin{bmatrix} 0 & (X \cdot Y) & (X \cdot e) \\ (X \cdot Y) & 0 & (Y \cdot e) \\ (X \cdot e) & (Y \cdot e) & 1 \end{bmatrix}$$

$$(X \wedge Y)^2 = (X \cdot Y)^2$$

$L = X \wedge Y \wedge e$ is a non-euclidian line

$$B = Le = \boldsymbol{X} \wedge \boldsymbol{Y} - (Y \cdot e) \boldsymbol{X} \wedge e + (X \cdot e) \boldsymbol{Y} \wedge e$$

$$BeB^\dagger = (X \cdot Y) (- (X \cdot Y) + 2 (X \cdot e) (Y \cdot e)) e$$

$$B^2 = (X \cdot Y) ((X \cdot Y) - 2 (X \cdot e) (Y \cdot e))$$

$$L^2 = (X \cdot Y) ((X \cdot Y) - 2 (X \cdot e) (Y \cdot e))$$

$$s = \sinh(\alpha/2) \text{ and } c = \cosh(\alpha/2)$$

$$e^{\alpha B/2|B|} = c + (1/B)s\boldsymbol{X} \wedge \boldsymbol{Y} - (1/B)(Y \cdot e)s\boldsymbol{X} \wedge e + (1/B)(X \cdot e)s\boldsymbol{Y} \wedge e$$

$$\begin{aligned}
RXR^\dagger = & \left((1/B)^2 (X \cdot Y)^2 s^2 - 2(1/B)^2 (X \cdot Y) (X \cdot e) (Y \cdot e) s^2 + 2(1/B) (X \cdot Y) cs - 2(1/B) (X \cdot e) (Y \cdot e) cs + c^2 \right) \mathbf{X} \\
& + 2(1/B) (X \cdot e)^2 cs \mathbf{Y} \\
& + 2(1/B) (X \cdot Y) (X \cdot e) s (-1/B) (X \cdot Y) s + 2(1/B) (X \cdot e) (Y \cdot e) s - c) \mathbf{e}
\end{aligned}$$

$$W = Z \cdot Y = (1/B)^2 (X \cdot Y)^3 s^2 - 4(1/B)^2 (X \cdot Y)^2 (X \cdot e) (Y \cdot e) s^2 + 4(1/B)^2 (X \cdot Y) (X \cdot e)^2 (Y \cdot e)^2 s^2 + 2(1/B) (X \cdot Y)^2 cs - 4(1/B) (X \cdot Y) (X \cdot e) (Y \cdot e) cs + (X \cdot Y) c^2$$

$$S = \sinh(\alpha) \text{ and } C = \cosh(\alpha)$$

$$W = (1/B)C(X \cdot Y) \sqrt{(X \cdot Y)^2 - 2(X \cdot Y)(X \cdot e)(Y \cdot e)} - (1/B)C(X \cdot e)(Y \cdot e) \sqrt{(X \cdot Y)^2 - 2(X \cdot Y)(X \cdot e)(Y \cdot e)} + (1/B)(X \cdot e)(Y \cdot e) \sqrt{(X \cdot Y)^2 - 2(X \cdot Y)(X \cdot e)(Y \cdot e)} + S \sqrt{(X \cdot Y)^2 - 2(X \cdot Y)(X \cdot e)(Y \cdot e)}$$

$$\text{Scalar Coefficient} = (1/B)(X \cdot e)(Y \cdot e) \sqrt{(X \cdot Y)^2 - 2(X \cdot Y)(X \cdot e)(Y \cdot e)}$$

$$\text{Cosh Coefficient} = (1/B)(X \cdot Y) \sqrt{(X \cdot Y)^2 - 2(X \cdot Y)(X \cdot e)(Y \cdot e)} - (1/B)(X \cdot e)(Y \cdot e) \sqrt{(X \cdot Y)^2 - 2(X \cdot Y)(X \cdot e)(Y \cdot e)}$$

$$\text{Sinh Coefficient} = \sqrt{(X \cdot Y)^2 - 2(X \cdot Y)(X \cdot e)(Y \cdot e)}$$

$$|B| = \sqrt{(X \cdot Y)^2 - 2(X \cdot Y)(X \cdot e)(Y \cdot e)}$$

$$\text{Require } aC^2 + bC + c = 0$$

$$a = (X \cdot e)^2 (Y \cdot e)^2$$

$$b = 2(X \cdot e)(Y \cdot e)((X \cdot Y) - (X \cdot e)(Y \cdot e))$$

$$c = (X \cdot Y)^2 - 2(X \cdot Y)(X \cdot e)(Y \cdot e) + (X \cdot e)^2 (Y \cdot e)^2$$

$$b^2 - 4ac = 0$$

$$\cosh(\alpha) = C = -b/(2a) = -\frac{(X \cdot Y)}{(X \cdot e)(Y \cdot e)} + 1$$

```

def conformal_representations_of_circles_lines_spheres_and_planes():
    Print_Function()
    global n,nbar
    g = '1 0 0 0 0,0 1 0 0 0,0 0 1 0 0,0 0 0 0 2,0 0 0 2 0'
    c3d = Ga('e_1 e_2 e_3 n \bar{n}',g=g)
    (e1,e2,e3,n,nbar) = c3d.mv()
    print 'g_{ij} =',c3d.g
    e = n+nbar
    #conformal representation of points
    A = make_vector(e1, ga=c3d) # point a = (1,0,0) A = F(a)
    B = make_vector(e2, ga=c3d) # point b = (0,1,0) B = F(b)
    C = make_vector(-e1, ga=c3d) # point c = (-1,0,0) C = F(c)
    D = make_vector(e3, ga=c3d) # point d = (0,0,1) D = F(d)
    X = make_vector('x',3, ga=c3d)
    print 'F(a) =',A
    print 'F(b) =',B
    print 'F(c) =',C
    print 'F(d) =',D
    print 'F(x) =',X
    print '#a = e1, b = e2, c = -e1, and d = e3'
    print '#A = F(a) = 1/2*(a*a*n+2*a-nbar), etc.'
    print '#Circle through a, b, and c'
    print 'Circle: A^B^C^X = 0 =',(A^B^C^X)
    print '#Line through a and b'
    print 'Line : A^B^n^X = 0 =',(A^B^n^X)
    print '#Sphere through a, b, c, and d'
    print 'Sphere: A^B^C^D^X = 0 =',(((A^B)^C)^D)^X
    print '#Plane through a, b, and d'
    print 'Plane : A^B^n^D^X = 0 =',(A^B^n^D^X)
    L = (A^B^e)^X
    L.Fmt(3,'Hyperbolic\\;\\; Circle: (A^B^e)^X = 0')
    return

```

Code Output:

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$F(a) = \boldsymbol{e_1} + \frac{1}{2}\boldsymbol{n} - \frac{1}{2}\bar{\boldsymbol{n}}$$

$$F(b) = \boldsymbol{e_2} + \frac{1}{2}\boldsymbol{n} - \frac{1}{2}\bar{\boldsymbol{n}}$$

$$F(c) = -1\boldsymbol{e_1} + \frac{1}{2}\boldsymbol{n} - \frac{1}{2}\bar{\boldsymbol{n}}$$

$$F(d) = \boldsymbol{e_3} + \frac{1}{2}\boldsymbol{n} - \frac{1}{2}\bar{\boldsymbol{n}}$$

$$F(x) = x_1\boldsymbol{e_1} + x_2\boldsymbol{e_2} + x_3\boldsymbol{e_3} + \left(\frac{1}{2}(x_1)^2 + \frac{1}{2}(x_2)^2 + \frac{1}{2}(x_3)^2\right)\boldsymbol{n} - \frac{1}{2}\bar{\boldsymbol{n}}$$

a = e1, b = e2, c = -e1, and d = e3 A = F(a) = 1/2*(a*a*n+2*a-nbar), etc. Circle through a, b, and c

$$Circle : A \wedge B \wedge C \wedge X = 0 = -x_3\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_3} \wedge \boldsymbol{n} + x_3\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_3} \wedge \bar{\boldsymbol{n}} + \left(\frac{1}{2}(x_1)^2 + \frac{1}{2}(x_2)^2 + \frac{1}{2}(x_3)^2 - \frac{1}{2}\right)\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{n} \wedge \bar{\boldsymbol{n}}$$

Line through a and b

$$Line : A \wedge B \wedge n \wedge X = 0 = -x_3\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_3} \wedge \boldsymbol{n} + \left(\frac{x_1}{2} + \frac{x_2}{2} - \frac{1}{2}\right)\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{n} \wedge \bar{\boldsymbol{n}} + \frac{x_3}{2}\boldsymbol{e_1} \wedge \boldsymbol{e_3} \wedge \boldsymbol{n} \wedge \bar{\boldsymbol{n}} - \frac{x_3}{2}\boldsymbol{e_2} \wedge \boldsymbol{e_3} \wedge \boldsymbol{n} \wedge \bar{\boldsymbol{n}}$$

Sphere through a, b, c, and d

$$Sphere : A \wedge B \wedge C \wedge D \wedge X = 0 = \left(-\frac{1}{2}(x_1)^2 - \frac{1}{2}(x_2)^2 - \frac{1}{2}(x_3)^2 + \frac{1}{2}\right)\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_3} \wedge \boldsymbol{n} \wedge \bar{\boldsymbol{n}}$$

Plane through a, b, and d

$$Plane : A \wedge B \wedge n \wedge D \wedge X = 0 = \left(-\frac{x_1}{2} - \frac{x_2}{2} - \frac{x_3}{2} + \frac{1}{2}\right)\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_3} \wedge \boldsymbol{n} \wedge \bar{\boldsymbol{n}}$$

$$\begin{aligned} Hyperbolic \ Circle : (A \wedge B \wedge e) \wedge X = 0 = & -x_3\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_3} \wedge \boldsymbol{n} \\ & -x_3\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{e_3} \wedge \bar{\boldsymbol{n}} \\ & + \left(-\frac{1}{2}(x_1)^2 + x_1 - \frac{1}{2}(x_2)^2 + x_2 - \frac{1}{2}(x_3)^2 - \frac{1}{2}\right)\boldsymbol{e_1} \wedge \boldsymbol{e_2} \wedge \boldsymbol{n} \wedge \bar{\boldsymbol{n}} \\ & + x_3\boldsymbol{e_1} \wedge \boldsymbol{e_3} \wedge \boldsymbol{n} \wedge \bar{\boldsymbol{n}} \\ & - x_3\boldsymbol{e_2} \wedge \boldsymbol{e_3} \wedge \boldsymbol{n} \wedge \bar{\boldsymbol{n}} \end{aligned}$$

```
def properties_of_geometric_objects():
    Print_Function()
    global n, nbar
    g = '# # # 0 0, '+ \
        '# # # 0 0, '+ \
        '# # # 0 0, '+ \
        '0 0 0 0 2, '+ \
        '0 0 0 2 0'
    c3d = Ga('p1 p2 p3 n \\\bar{n}',g=g)
    (p1,p2,p3,n,nbar) = c3d.mv()
    print 'g-{'ij} =',c3d.g
    P1 = F(p1)
    P2 = F(p2)
    P3 = F(p3)
```



```
print '\\text{Extracting direction of line from }L = P1\\W P2\\W n'
L = P1^P2^n
delta = (L|n)|nbar
print '(L|n)|\\bar{n} = ', delta
print '\\text{Extracting plane of circle from }C = P1\\W P2\\W P3'
C = P1^P2^P3
delta = ((C^n)|n)|nbar
print '((C^n)|n)|\\bar{n}= ', delta
print '(p2-p1)^(p3-p1)= ', (p2-p1)^(p3-p1)
return
```

Code Output:

$$g_{ij} = \begin{bmatrix} (p_1 \cdot p_1) & (p_1 \cdot p_2) & (p_1 \cdot p_3) & 0 & 0 \\ (p_1 \cdot p_2) & (p_2 \cdot p_2) & (p_2 \cdot p_3) & 0 & 0 \\ (p_1 \cdot p_3) & (p_2 \cdot p_3) & (p_3 \cdot p_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

Extracting direction of line from $L = P1 \wedge P2 \wedge n$

$$(L \cdot n) \cdot \bar{n} = 2\boldsymbol{p_1} - 2\boldsymbol{p_2}$$

Extracting plane of circle from $C = P1 \wedge P2 \wedge P3$

$$((C \wedge n) \cdot n) \cdot \bar{n} = 2\boldsymbol{p_1} \wedge \boldsymbol{p_2} - 2\boldsymbol{p_1} \wedge \boldsymbol{p_3} + 2\boldsymbol{p_2} \wedge \boldsymbol{p_3}$$

$$(p2 - p1) \wedge (p3 - p1) = \boldsymbol{p_1} \wedge \boldsymbol{p_2} - 1\boldsymbol{p_1} \wedge \boldsymbol{p_3} + \boldsymbol{p_2} \wedge \boldsymbol{p_3}$$

```
def extracting_vectors_from_conformal_2_blade():
    Print_Function()
    print r'B = P1\W P2'
    g = '0 -1 #,'+ \
        '-1 0 #,'+ \
        '# # #'
    c2b = Ga('P1 P2 a',g=g)
    (P1,P2,a) = c2b.mv()
    print 'g_{ij} = ', c2b.g
    B = P1^P2
    Bsq = B*B
    print '%B^{2} = ', Bsq
    ap = a-(a^B)*B
    print "a' = a-(a^B)*B =", ap
    Ap = ap+ap*B
    Am = ap-ap*B
    print "A+ = a'+a'*B =", Ap
    print "A- = a'-a'*B =", Am
    print '%(A+)^{2} = ', Ap*Ap
    print '%(A-)^{2} = ', Am*Am
    aB = a|B
    print 'a|B = ', aB
    return
```

Code Output:

$$B = P1 \wedge P2$$

$$g_{ij} = \begin{bmatrix} 0 & -1 & (P_1 \cdot a) \\ -1 & 0 & (P_2 \cdot a) \\ (P_1 \cdot a) & (P_2 \cdot a) & (a \cdot a) \end{bmatrix}$$

$$B^2 = 1$$

$$a' = a - (a \wedge B)B = -(P_2 \cdot a) \boldsymbol{P_1} - (P_1 \cdot a) \boldsymbol{P_2}$$

$$A+ = a' + a' B = -2 (P_2 \cdot a) \boldsymbol{P_1}$$

$$A- = a' - a' B = -2 (P_1 \cdot a) \boldsymbol{P_2}$$

$$(A+)^2 = 0$$

$$(A-)^2 = 0$$

$$a \cdot B = - (P_2 \cdot a) \boldsymbol{P_1} + (P_1 \cdot a) \boldsymbol{P_2}$$

```
def reciprocal_frame_test():
    Print_Function()
    g = '1 # #,'+ \
        '# 1 #,'+ \
        '# # 1 '
    ng3d = Ga('e1 e2 e3 ',g=g)
    (e1,e2,e3) = ng3d.mv()
    print 'g-{ij} =',ng3d.g
    E = e1^e2^e3
    Esq = (E*E).scalar()
    print 'E =',E
    print '%E^{2} =' ,Esq
    Esq_inv = 1/Esq
    E1 = (e2^e3)*E
    E2 = (-1)*(e1^e3)*E
    E3 = (e1^e2)*E
    print 'E1 = (e2^e3)*E =',E1
    print 'E2 =-(e1^e3)*E =',E2
    print 'E3 = (e1^e2)*E =',E3
    w = (E1|e2)
    w = w.expand()
    print 'E1|e2 =',w
    w = (E1|e3)
    w = w.expand()
    print 'E1|e3 =',w
    w = (E2|e1)
    w = w.expand()
    print 'E2|e1 =',w
    w = (E2|e3)
    w = w.expand()
    print 'E2|e3 =',w
    w = (E3|e1)
    w = w.expand()
    print 'E3|e1 =',w
    w = (E3|e2)
    w = w.expand()
    print 'E3|e2 =',w
    w = (E1|e1)
    w = (w.expand()).scalar()
    Esq = expand(Esq)
    print '%(E1\\cdot e1)/E^{2} =' ,simplify(w/Esq)
    w = (E2|e2)
    w = (w.expand()).scalar()
    print '%(E2\\cdot e2)/E^{2} =' ,simplify(w/Esq)
    w = (E3|e3)
    w = (w.expand()).scalar()
    print '%(E3\\cdot e3)/E^{2} =' ,simplify(w/Esq)
    return
```

Code Output:

$$g_{ij} = \begin{bmatrix} 1 & (e_1 \cdot e_2) & (e_1 \cdot e_3) \\ (e_1 \cdot e_2) & 1 & (e_2 \cdot e_3) \\ (e_1 \cdot e_3) & (e_2 \cdot e_3) & 1 \end{bmatrix}$$

$$E = \mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}$$

$$E^2 = (e_1 \cdot e_2)^2 - 2 (e_1 \cdot e_2) (e_1 \cdot e_3) (e_2 \cdot e_3) + (e_1 \cdot e_3)^2 + (e_2 \cdot e_3)^2 - 1$$

$$E1 = (e2 \wedge e3)E = \left((e_2 \cdot e_3)^2 - 1 \right) \mathbf{e_1} + ((e_1 \cdot e_2) - (e_1 \cdot e_3) (e_2 \cdot e_3)) \mathbf{e_2} + (- (e_1 \cdot e_2) (e_2 \cdot e_3) + (e_1 \cdot e_3)) \mathbf{e_3}$$

$$E2 = -(e1 \wedge e3)E = ((e_1 \cdot e_2) - (e_1 \cdot e_3) (e_2 \cdot e_3)) \mathbf{e_1} + \left((e_1 \cdot e_3)^2 - 1 \right) \mathbf{e_2} + (- (e_1 \cdot e_2) (e_1 \cdot e_3) + (e_2 \cdot e_3)) \mathbf{e_3}$$

$$E3 = (e1 \wedge e2)E = (- (e_1 \cdot e_2) (e_2 \cdot e_3) + (e_1 \cdot e_3)) \mathbf{e_1} + (- (e_1 \cdot e_2) (e_1 \cdot e_3) + (e_2 \cdot e_3)) \mathbf{e_2} + \left((e_1 \cdot e_2)^2 - 1 \right) \mathbf{e_3}$$

$$E1 \cdot e2 = 0$$

$$E1 \cdot e3 = 0$$

$$E2 \cdot e1 = 0$$

$$E2 \cdot e3 = 0$$

$$E3 \cdot e1 = 0$$

$$E3 \cdot e2 = 0$$

$$(E1 \cdot e1)/E^2 = 1$$

$$(E2 \cdot e2)/E^2 = 1$$

$$(E3 \cdot e3)/E^2 = 1$$