Chapter 9

Lie Groups as Spin Groups[1]

9.1 Introduction

A Lie group, G, is a group that is also a differentiable manifold. So that if $g \in G$ and $x \in \Re^N$ then there is a differentiable mapping $\phi : \Re^N \to G$ so that for each $g \in G$ we can define a tangent space $\mathcal{T}_g$.

A example (the one we are most concerned with) of a Lie group is the group of $n \times n$ non-singular matrices. The coordinates of the manifold are simply the elements of the matrix so that the dimension of the group manifold is n^2 and the matrix is obviously a continuous differentiable function of its coordinates.

A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear operator $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket* which satifies ($x, y, z \in \mathfrak{g}$ and $\alpha, \beta \in \Re$):

$$[ax + by, z] = a\,[x, z] + b\,[y, z]\,, \tag{9.1}$$

$$[x, x] = 0, \tag{9.2}$$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0. \tag{9.3}$$

Equations (9.1) and (9.2) imply $[x, y] = -\,[y, x]$, while eq (9.3) is the Jacobi identity.

The purpose of the following analysis is to show that the Lie algebra of the general linear group of dimension n over the real numbers (the group of $n \times n$ invertible matrices), $GL\,(n, \Re)$, can be

[1]This chapter follows [5]

represented by a rotation group in an appropriate vector space, $\mathcal{M}$ (let (p,q) be the signature of the new vector space). Furthermore, since the rotation group, $SO(p,q)$ can be represented by the spin group, $Spin(p,q)$, in the same vector space. Then by use of geometric algebra we can construct any rotor, $R \in Spin(p,q)$, as $R = e^{\frac{B}{2}}$ where B is a bi-vector in the geometric algebra of $\mathcal{M}$ and the bi-vectors form a Lie algebra under the commutator product.[2]

The main trick in doing this is to construct the appropriate vector space, $\mathcal{M}$, with subspaces isomorphic to $\Re^n$ so that a rotation in $\mathcal{M}$ is equivalent to a general linear transformation in a subspace of $\mathcal{M}$ isomorphic to $\Re^n$. We might suspect that $\mathcal{M}$ cannot be a Euclidean space since a general linear transformation can cause the input vector to grow as well as shrink.

9.2 Simple Examples

9.2.1 $SO(2)$ - Special Orthogonal Group of Order 2

The group is represented by all 2×2 real matrices[3] $\underline{R}$ where[4] $\underline{R}\underline{R}^T = \underline{I}$ and $\det(\underline{R}) = 1$. The group product is matrix multiplication and it is a group since if $\underline{R}_1\underline{R}_1^T = \underline{I}$ and $\underline{R}_2\underline{R}_2^T = \underline{I}$ then

$$\left(\underline{R}_1\underline{R}_2\right)\left(\underline{R}_1\underline{R}_2\right)^T = \underline{R}_1\underline{R}_2\underline{R}_2^T\underline{R}_1^T = \underline{R}_1\underline{I}\underline{R}_1^T = \underline{R}_1\underline{R}_1^T = \underline{I} \tag{9.4}$$

$$\det\left(\underline{R}_1\underline{R}_2\right) = \det\left(\underline{R}_1\right)\det\left(\underline{R}_2\right) = 1. \tag{9.5}$$

$SO(2)$ is also a Lie group since all $\underline{R} \in SO(2)$ can be represented as a continuous function of the coordinate θ:

$$\underline{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}. \tag{9.6}$$

In this case the spin representation of $SO(2)$ is trivial, namely[5] $R(\theta) = e^{\frac{I}{2}\theta} = \cos\left(\frac{\theta}{2}\right) + I\sin\left(\frac{\theta}{2}\right)$, $R(\theta)^\dagger = e^{-\frac{I}{2}\theta} = \cos\left(\frac{\theta}{2}\right) - I\sin\left(\frac{\theta}{2}\right)$ where I is the pseudo-scalar for $\mathcal{G}(\Re^2)$. Then we have $\underline{R}(\theta)\, a = R(\theta)\, a R(\theta)^\dagger$.

[2] Since we could be dealing with vector spaces of arbitrary signature (p,q) we use the following nomenclature:

$SO(p,q)$	Special orthogonal group in vector space with signature (p,q)
$SO(n)$	Special orthogonal group in vector space with signature $(n,0)$
$Spin(p,q)$	Spin group in vector space with signature (p,q)
$Spin(n)$	Spin group in vector space with signature $(n,0)$
$GL(p,q,\Re)$	General linear group in real vector space with signature (p,q)
$GL(n,\Re)$	General linear group in real vector space with signature $(n,0)$

[3] We denote linear transformations with an underbar.

[4] In this case $\underline{I}$ is the identity matrix and not the pseudo scalar.

[5] For multivectors such as the rotor R there is no underbar. We have $\underline{R}(a) = \underline{R}a = RaR^\dagger$ where a is a vector.

9.2.2 $GL(2,\Re)$ - General Real Linear Group of Order 2

The group is represented by all 2×2 real matrices $\underline{A}$ where $\det(\underline{A}) \neq 0$. Again the the group product is matrix multiplication and it is a group because if $\underline{A}$ and $\underline{B}$ are 2×2 real matrices then $\det(\underline{AB}) = \det(\underline{A})\det(\underline{B})$ and if $\det(\underline{A}) \neq 0$ and $\det(\underline{B}) \neq 0$ then $\det(\underline{AB}) \neq 0$. Any member of the group is represented by the matrix ($a = (a^1, a^2, a^3, a^4) \in \Re^4$):

$$\underline{A}(a) = \begin{bmatrix} a^1 & a^2 \\ a^3 & a^4 \end{bmatrix}. \tag{9.7}$$

Thus any element $\underline{A} \in GL(2,\Re)$ is a continuous function of $a = (a^1, a^2, a^3, a^4)$. Thus $GL(2,\Re)$ is a four dimensional Lie group while $SO(2)$ is a one dimensional Lie group.

Another difference is that $SO(2)$ is compact while $GL(2,\Re)$ is not. $SO(2)$ is compact since for any convergent sequence (θ_i),

$$\lim_{i\to\infty} \underline{R}(\theta_i) \in SO(2).$$

$GL(2,\Re)$ is not compact since there is at least one convergent sequence (a_i) such that

$$\lim_{i\to\infty} \det(\underline{A}(a_i)) = 0.$$

After we develop the required theory we will calculate the spin representation of $GL(2,\Re)$.

9.3 The Grassmann Algebra

Let $\mathcal{V}^n$ be an n-dimensional real vector space with basis $\{\boldsymbol{w}_i\} : 1 \leq i \leq n$ and $\wedge$ is the outer (wedge) product for the geometric algebra define on $\mathcal{V}^n$. As before the geometric object

$$v_1 \wedge v_2 \wedge \ldots \wedge v_k \tag{9.8}$$

is a k-blade in the Grassmann or the geometric algebra where the v_i's are k independent vectors in $\mathcal{V}_n$ and a linear combination of k-blades is called a k-vector. The space of all k-vectors is just all the k-grade multivectors in the geometric algebra $\mathcal{G}(\mathcal{V}_n)$. Denote the space of all k-vectors in $\mathcal{V}^n$ by $\Lambda^k_n = \Lambda^k(\mathcal{V}^n)$ with

$$\dim\left(\Lambda^k_n\right) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}. \tag{9.9}$$

Letting $\Lambda_n^1 = \mathcal{V}^n$ and $\Lambda_n^0 = \Re$ the entire Grassmann algebra is a 2^n-dimensional space.[6]

$$\Lambda_n = \sum_{k=0}^{n} \Lambda_n^k. \tag{9.10}$$

The Grassmann algebra of a vector space $\mathcal{V}^n$ is denoted by $\Lambda(\mathcal{V}^n)$.

9.4 The Dual Space to $\mathcal{V}_n$

The dual space $\mathcal{V}^{n*}$ to $\mathcal{V}^n$ is defined as follows. Let $\{\boldsymbol{w}_i\}$ be a basis for $\mathcal{V}^n$ and define the basis, $\{\boldsymbol{w}_i^*\}$ for $\mathcal{V}^{n*}$ by

$$\boldsymbol{w}_i^* \cdot \boldsymbol{w}_j = \frac{1}{2}\delta_{ij}. \tag{9.11}$$

Again the dual space $\mathcal{V}^{n*}$ has its own Grassmann algebra $\Lambda^*(\mathcal{V}^n)$ given by

$$\Lambda^*(\mathcal{V}^n) = \Lambda_n^* = \sum_{k=0}^{n} \Lambda_n^{k*}. \tag{9.12}$$

$\Lambda^*(\mathcal{V}^n)$ can be represented as a geometric algebra by imposing the null metric condition $(\boldsymbol{w}_i^*)^2 = 0$ as in the case of $\Lambda(\mathcal{V}^n)$.

9.5 The Mother Algebra

From the base vector space and it's dual space one can construct a $2n$ dimensional vector space from the direct sum of the two vector spaces as defined in Appendix F

$$\Re^{n,n} \equiv \mathcal{V}^n \oplus \mathcal{V}^{n*} \tag{9.13}$$

[6]Consider the geometric algebra $\mathcal{G}(\mathcal{V}^n)$ of $\mathcal{V}^n$ with a null basis $\{\boldsymbol{w}_i\} : \boldsymbol{w}_i^2 = 0, 1 \le i \le n$. Then

$$\begin{aligned}
(\boldsymbol{w}_i + \boldsymbol{w}_j)^2 &= \boldsymbol{w}_i^2 + \boldsymbol{w}_i\boldsymbol{w}_j + \boldsymbol{w}_j\boldsymbol{w}_i + \boldsymbol{w}_j^2 \\
0 &= \boldsymbol{w}_i\boldsymbol{w}_j + \boldsymbol{w}_j\boldsymbol{w}_i \\
\boldsymbol{w}_i\boldsymbol{w}_j &= -\boldsymbol{w}_j\boldsymbol{w}_i \\
0 &= 2\boldsymbol{w}_i \cdot \boldsymbol{w}_j.
\end{aligned}$$

If the basis set is null the metric tensor is null and $\boldsymbol{w}_i\boldsymbol{w}_j = \boldsymbol{w}_i \wedge \boldsymbol{w}_j$ even if $i = j$ and the geometric algebra is a Grassmann algebra.

with basis $\{\boldsymbol{w}_i, \boldsymbol{w}_j^*\}$. An orthogonal basis for $\Re^{n,n}$ can be constructed as follows (using equation 9.11):

$$
\begin{aligned}
\boldsymbol{e}_i &= \boldsymbol{w}_i + \boldsymbol{w}_i^* && (9.14)\\
\bar{\boldsymbol{e}}_i &= \boldsymbol{w}_i - \boldsymbol{w}_i^* && (9.15)\\
\boldsymbol{e}_i \cdot \boldsymbol{e}_j &= (\boldsymbol{w}_i + \boldsymbol{w}_i^*) \cdot (\boldsymbol{w}_j + \boldsymbol{w}_j^*) \\
&= \boldsymbol{w}_i \cdot \boldsymbol{w}_j + \boldsymbol{w}_i \cdot \boldsymbol{w}_j^* + \boldsymbol{w}_i^* \cdot \boldsymbol{w}_j + \boldsymbol{w}_i^* \cdot \boldsymbol{w}_j^* \\
&= \delta_{ij} && (9.16)\\
\bar{\boldsymbol{e}}_i \cdot \bar{\boldsymbol{e}}_j &= (\boldsymbol{w}_i - \boldsymbol{w}_i^*) \cdot (\boldsymbol{w}_j \boldsymbol{w}_j^*) \\
&= \boldsymbol{w}_i \cdot \boldsymbol{w}_j \boldsymbol{w}_i \cdot \boldsymbol{w}_j^* - \boldsymbol{w}_i^* \cdot \boldsymbol{w}_j + \boldsymbol{w}_i^* \cdot \boldsymbol{w}_j^* \\
&= -\delta_{ij} && (9.17)\\
\boldsymbol{e}_i \cdot \bar{\boldsymbol{e}}_j &= (\boldsymbol{w}_i + \boldsymbol{w}_i^*) \cdot (\boldsymbol{w}_j - \boldsymbol{w}_j^*) \\
&= \boldsymbol{w}_i \cdot \boldsymbol{w}_j - \boldsymbol{w}_i \cdot \boldsymbol{w}_j^* + \boldsymbol{w}_i^* \cdot \boldsymbol{w}_j - \boldsymbol{w}_i^* \cdot \boldsymbol{w}_j^* \\
&= 0. && (9.18)
\end{aligned}
$$

Thus $\Re^{n,n}$ can also be represented by the direct sum of an n-dimensional Euclidian vector space, E^n, and a n-dimensional anti-Euclidian vector space (square of all basis vectors is -1) $\bar{E}^n$

$$\Re^{n,n} = E^n \oplus \bar{E}^n. \tag{9.19}$$

In this case E^n and $\bar{E}^n$ are orthogonal[7] since $\boldsymbol{e}_i \cdot \bar{\boldsymbol{e}}_j = 0$. The geometric algebra of $\Re^{n,n}$ is defined and denoted by

$$\Re_{n,n} \equiv \mathcal{G}(\Re^{n,n}) \tag{9.20}$$

has dimension 2^{2n} with k-vector subspaces $\Re_{n,n}^k = \mathcal{G}^k(\Re^{n,n})$ and is called the *mother algebra*.[8]

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[7] Every vector in E^n is orthogonal to every vector in $\bar{E}^n$.

[8] As an example of the equivalence of $E^n \oplus \bar{E}^n$ and $\mathcal{V}^n \oplus \mathcal{V}^{n*}$ consider the metric tensors of each representation of $\Re^{2,2}$. The metric tensor of $E^2 \oplus \bar{E}^2$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

with eigenvalues $[1, 1, -1, -1]$. Thus the signature of the metric is $(2, 2)$. The metric tensor of $\mathcal{V}^2 \oplus \mathcal{V}^{2*}$ is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} /2$$

From the basis $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_i\}$ we can construct $(p+q)$-blades

$$E_{p,q} = E_p \wedge \bar{E}_q^\dagger = E_p \bar{E}_q^\dagger, \tag{9.21}$$

where[9]

$$E_p = \boldsymbol{e}_{i_1}\boldsymbol{e}_{i_2}\ldots\boldsymbol{e}_{i_p} = E_{p,0} \;\; 1 \le i_1 < i_2 < \cdots < i_p \le n \tag{9.22}$$
$$\bar{E}_q = \bar{\boldsymbol{e}}_{j_1}\bar{\boldsymbol{e}}_{j_2}\ldots\bar{\boldsymbol{e}}_{j_q} = \bar{E}_{0,q} \;\; 1 \le j_1 < j_2 < \cdots < j_q \le n. \tag{9.23}$$

Each blade determines a projection of $\underline{E}_{p,q}$ of $\Re^{n,n}$ into a $(p+q)$-dimensional subspace $\Re^{p,q}$ defined by (see section 5.1.2 and equation 1.42)[10]

$$\underline{E}_{p,q} a \equiv (a \cdot E_{p,q})\, E_{p,q}^{-1} = \frac{1}{2}\left(a - (-1)^{p+q} E_{p,q} a E_{p,q}^{-1}\right). \tag{9.24}$$

A vector, a, is in $\Re^{p,q}$ if and only if

$$a \wedge E_{p,q} = 0 = aE_{p,q} + (-1)^{p+q} E_{p,q} a. \tag{9.25}$$

For $p+q=n$, the blade $E_{p,q}$ determines a split of $\Re^{n,n}$ into orthogonal subspaces with *complementary signature*[11], as expressed by

$$\Re^{n,n} = \Re^{p,q} \oplus \bar{\Re}^{p,q}. \tag{9.26}$$

For the case of $q=0$ equation 9.24 can be written as

$$\underline{E}_n a = \frac{1}{2}\left(a + a^*\right), \tag{9.27}$$

where a^* is defined by

$$a^* \equiv (-1)^{n+1} E_n a E_n^{-1}. \tag{9.28}$$

It follows immediately that $\boldsymbol{e}_i^* = \boldsymbol{e}_i$ and $(\bar{\boldsymbol{e}}_i)^* = -\bar{\boldsymbol{e}}_i$.[12] The split of $\Re^{n,n}$ given by equation 9.13 cannot be constructed as the split in equation 9.19 since the vectors expanded in the $\left\{\boldsymbol{w}_i, \boldsymbol{w}_j^*\right\}$

with eigenvalues $[1/2, 1/2, -1/2, -1/2]$. Thus the signature of the metric is $(2,2)$. As expected both representations have the same signature.

[9]Since the $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_i\}$ form an orthogonal set and specifying that no factors are repeated we can use the geometric product in equations 9.22 and 9.23 instead of the outer (wedge) product.

[10]The underbar notation as in $\underline{E}_{p,q}(a)$ allows one to distinguish linear operators from elements in the algebra such as $E_{p,q}$.

[11]The signature of $\Re^{p,q}$ is (p,q) as opposed to $\bar{\Re}^{p,q}$ with signature (q,p).

[12]This is obvious by

$$\boldsymbol{e}_i^* = (-1)^{n+1} E_n \boldsymbol{e}_i E_n^{-1} = (-1)^{n+1}(-1)^{n-1}\boldsymbol{e}_i E_n E_n^{-1} = (-1)^{2n}\boldsymbol{e}_i = \boldsymbol{e}_i,$$
$$(\bar{\boldsymbol{e}}_i)^* = (-1)^{n+1} E_n (\bar{\boldsymbol{e}}_i) E_n^{-1} = (-1)^{n+1}(-1)^{n}(\bar{\boldsymbol{e}}_i) E_n E_n^{-1} = (-1)^{2n+1}(\bar{\boldsymbol{e}}_i) = -(\bar{\boldsymbol{e}}_i).$$

basis cannot be normalized since they are null vectors. Instead consider a bivector[13] K in $\Re^{n,n}$

$$K = \sum_{i=0}^{n} K_i, \tag{9.29}$$

where the K_i are distinct commuting blades, $K_i \times K_j = 0$ (commutator product), normalized to $K_i^2 = 1$. The bivector K defines the automorphism $\underline{K} : \Re^{n,n} \to \Re^{n,n}$

$$\bar{a} = \underline{K}a \equiv a \times K = a \cdot K. \tag{9.30}$$

This maps every vector a into the vector $\bar{a}$ which is called the *complement* of a with respect to K.

Each K_i is a bivector blade that defines a two dimensional Minkowski (since $K_i^2 = 1$) subspace of $\Re^{n,n}$. Since the blades commute, $K_i \times K_j = 0$, they define disjoint subspaces of $\Re^{n,n}$ and since there are n of them they span $\Re^{n,n}$. Since $K_i^2 = 1$ there exists an orthonormal Minkowski basis $\boldsymbol{e}_i$ and $\bar{\boldsymbol{e}}_i$ such that

$$\boldsymbol{e}_i \cdot \bar{\boldsymbol{e}}_j = 0,\ i \neq j, \tag{9.31}$$

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_i = -\bar{\boldsymbol{e}}_i \cdot \bar{\boldsymbol{e}}_i = 1. \tag{9.32}$$

Then if $a \in \Re^{n,n}$ and using equations 9.31, 9.32, and from appendix B equations B.2 and B.5 we have

$$a \cdot K_i = a \cdot (\boldsymbol{e}_i \bar{\boldsymbol{e}}_i) = -\left(a \cdot \bar{\boldsymbol{e}}_i\right) \boldsymbol{e}_i + \left(a \cdot \boldsymbol{e}_i\right) \bar{\boldsymbol{e}}_i, \tag{9.33}$$

$$\begin{aligned}(a \cdot K_i) \cdot K_i &= (a \cdot (\boldsymbol{e}_i \bar{\boldsymbol{e}}_i)) \cdot (\boldsymbol{e}_i \bar{\boldsymbol{e}}_i) \\ &= \left(a \cdot \boldsymbol{e}_i\right) \boldsymbol{e}_i - \left(a \cdot \bar{\boldsymbol{e}}_i\right) \bar{\boldsymbol{e}}_i \\ &= \left(a \cdot \boldsymbol{e}^i\right) \boldsymbol{e}_i + \left(a \cdot \bar{\boldsymbol{e}}^i\right) \bar{\boldsymbol{e}}_i = a,\end{aligned} \tag{9.34}$$

$$\begin{aligned}(a \cdot K_i) \cdot K_j &= \left(\left(a \cdot \bar{\boldsymbol{e}}_i\right)\left(\boldsymbol{e}_i \cdot \bar{\boldsymbol{e}}_j\right) - \left(a \cdot \boldsymbol{e}_i\right)\left(\bar{\boldsymbol{e}}_i \cdot \bar{\boldsymbol{e}}_j\right)\right) \boldsymbol{e}_j \\ &\quad + \left(\left(a \cdot \boldsymbol{e}_i\right)\left(\bar{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j\right) - \left(a \cdot \bar{\boldsymbol{e}}_i\right)\left(\boldsymbol{e}_i \cdot \boldsymbol{e}_j\right)\right) \bar{\boldsymbol{e}}_j = 0,\ \ \forall\ i \neq j\end{aligned} \tag{9.35}$$

Then

$$\underline{K}a = \sum_{i=1}^{n} a \cdot K_i = \sum_{i=1}^{n} \left(\left(a \cdot \boldsymbol{e}_i\right) \bar{\boldsymbol{e}}_i - \left(a \cdot \bar{\boldsymbol{e}}_i\right) \boldsymbol{e}_i\right) \tag{9.36}$$

[13] In general the basis blades for bivectors in $\Re_{n,n}$ do not commute since the dimension of the bivector subspace, $\Re^2_{n,n}$ is

$$\binom{2n}{2} = \frac{(2n)!}{2!\,(2n-2)!} = n\,(2n-1) = 2n^2 - 1.$$

If one has more than n bivector blades the planes defined by at least two of the blades will intersect.

and[14]

$$
\begin{aligned}
\underline{K}^2 a &= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a\cdot K_i\right)\cdot K_j \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} (a\cdot K_i)\cdot K_j \\
&= \sum_{i=1}^{n} (a\cdot K_i)\cdot K_i \\
&= \sum_{i=1}^{n} \left((a\cdot \boldsymbol{e}_i)\,\boldsymbol{e}_i - (a\cdot \bar{\boldsymbol{e}}_i)\,\bar{\boldsymbol{e}}_i\right) \\
&= \sum_{i=1}^{n} \left(\left(a\cdot \boldsymbol{e}^i\right)\boldsymbol{e}_i + \left(a\cdot \bar{\boldsymbol{e}}^i\right)\bar{\boldsymbol{e}}_i\right) \\
&= a. \qquad (9.37)
\end{aligned}
$$

Also

$$
\begin{aligned}
a\cdot\bar{a} &= \sum_{i=1}^{n} a\cdot(a\cdot K_i) \\
&= \sum_{i=1}^{n} a\cdot\left((a\cdot \boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i - (a\cdot \bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i\right) \\
&= \sum_{i=1}^{n} \left((a\cdot \boldsymbol{e}_i)\,(a\cdot \bar{\boldsymbol{e}}_i) - (a\cdot \bar{\boldsymbol{e}}_i)\,(a\cdot \boldsymbol{e}_i)\right) \\
&= 0, \qquad (9.38)\\
a^2 &= \sum_{i=1}^{n} \left((a\cdot \boldsymbol{e}_i)^2 - (a\cdot \bar{\boldsymbol{e}}_i)^2\right), \qquad (9.39)\\
\bar{a}^2 &= \sum_{i=1}^{n} \left((a\cdot \bar{\boldsymbol{e}}_i)^2 - (a\cdot \boldsymbol{e}_i)^2\right), \qquad (9.40)\\
a^2+\bar{a}^2 &= 0. \qquad (9.41)
\end{aligned}
$$

Thus defining

$$a_{\pm} \equiv a \pm \bar{a} = a \pm \underline{K}a = a \pm a\cdot K, \qquad (9.42)$$

[14]Remember that since K_i defines a Minkowski subspace we have for the reciprocal basis $\boldsymbol{e}^i = \boldsymbol{e}_i$ and $\bar{\boldsymbol{e}}^i = -\bar{\boldsymbol{e}}_i$.

we have

$$
\begin{aligned}
(a_{\pm})^2 &= 0, && (9.43)\\
a_+ \cdot a_- &= a^2 - \bar{a}^2 = 2\sum_{i=1}^{n} (a\cdot \boldsymbol{e}_i)^2 . && (9.44)
\end{aligned}
$$

Thus the sets $\{a_+\}$ and $\{a_-\}$ of all such vectors are in dual n-dimensional vector spaces ($a_+ \in \mathcal{V}^n$ and $a_- \in \mathcal{V}^{n*}$), so K determines the desired null space decomposition of the form in equation 9.13 without referring to a vector basis.[15]

The K for a given $\Re^{n,n}$ is constructed from the basis given in equations 9.14 and 9.15. Then we have from equation 9.36

$$
\begin{aligned}
\underline{K}\boldsymbol{e}_i &= \bar{\boldsymbol{e}}_i, && (9.45)\\
\underline{K}\bar{\boldsymbol{e}}_i &= \boldsymbol{e}_i. && (9.46)
\end{aligned}
$$

Also from equation 9.36

$$
\begin{aligned}
\underline{K}\boldsymbol{w}_i &= (\boldsymbol{w}_i\cdot\boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i - (\boldsymbol{w}_i\cdot\bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i,\\
&= \frac{1}{2}\left(((\boldsymbol{e}_i+\bar{\boldsymbol{e}}_i)\cdot\boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i - ((\boldsymbol{e}_i+\bar{\boldsymbol{e}}_i)\cdot\bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i\right),\\
&= \frac{1}{2}\left(\bar{\boldsymbol{e}}_i+\boldsymbol{e}_i\right),\\
&= \boldsymbol{w}_i, && (9.47)\\
\underline{K}\boldsymbol{w}_i^* &= (\boldsymbol{w}_i^*\cdot\boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i - (\boldsymbol{w}_i^*\cdot\bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i,\\
&= \frac{1}{2}\left(((\boldsymbol{e}_i-\bar{\boldsymbol{e}}_i)\cdot\boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i - ((\boldsymbol{e}_i-\bar{\boldsymbol{e}}_i)\cdot\bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i\right),\\
&= -\frac{1}{2}\left(\boldsymbol{e}_i-\bar{\boldsymbol{e}}_i\right),\\
&= -\boldsymbol{w}_i^*. && (9.48)
\end{aligned}
$$

The basis $\{\boldsymbol{w}_i, \boldsymbol{w}_i^*\}$ is called a *Witt basis* in the theory of quadratic forms.

[15]The properties $K_i \times K_j = 0$ and $K_i^2 = 1$ allows us to construct K from the basis $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_i\}$, but do not specify any particular basis.

9.6 The General Linear Group as a Spin Group

We will now use the results in section 1.15 and the extension of a linear vector function to blades (equation 1.77)[16] and the geometric algebra definition of the determinant of a linear transformation (equation 1.79) [17].

We are concerned here with linear transformations on $\Re^{n,n}$ and its subspaces, especially orthogonal transformations. An orthogonal transformation $\underline{R}$ is defined by the property

$$(\underline{R}a)\cdot(\underline{R}b) = a\cdot b \tag{9.49}$$

$\underline{R}$ is called a rotation if $\det(\underline{R}) = 1$, that is, if

$$\underline{R}E_{n,n} = E_{n,n}, \tag{9.50}$$

where $E_{n,n} = E_n\bar{E}_n^\dagger$ is the pseudoscalar for $\Re_{n,n}$ (equation 9.21). These rotations form a group called the *special orthogonal group* SO(n).

From section 1.10 we know that every rotation can be expressed by the canonical form

$$\underline{R}a = RaR^\dagger, \tag{9.51}$$

where R is an even multivector (*rotor*) satisfying

$$RR^\dagger = 1. \tag{9.52}$$

The rotors form a multiplicative group called the *spin group* or *spin representation* of SO(n), and it is denoted by Spin(n). Spin(n) is said to be a double covering of SO(n), since equation 9.51 shows that both $\pm R$ correspond to the same $\underline{R}$.

From equation 9.52 it follows that $R^{-1} = R^\dagger$ and that the inverse of the rotation is

$$\underline{R}^\dagger a = R^\dagger aR. \tag{9.53}$$

This implies that from the definition of the adjoint (using **RR5** in appendix C)

$$a\cdot(\underline{R}b) = \left\langle aRbR^\dagger\right\rangle = \left\langle bR^\dagger aR\right\rangle = b\cdot\left(\underline{R}^\dagger a\right). \tag{9.54}$$

The adjoint of a rotation is equal to its inverse.

---

[16] $\underline{f}(a\wedge b\wedge\ldots) = \underline{f}(a)\wedge\underline{f}(a)\wedge\ldots$

[17] $\underline{f}(I) = \det\left(\underline{f}\right) I$ where I is the pseudoscalar.

"It can be shown that every rotor can be expressed in exponential form

$$R = \pm e^{\frac{1}{2}B}, \text{ with } R^{\dagger} = \pm e^{-\frac{1}{2}B}, \tag{9.55}$$

where B is a bivector (section 1.10.2) called the generator of R or $\underline{R}$, and the minus sign can usually be eliminated by a change in the definition of B. Thus every bivector determines a unique rotation. The bivector generators of a spin or rotation group form a Lie algebra under the commutator product. **This reduces the description of Lie groups to Lie algebras.** The Lie algebra of SO(n) and Spin(n) is designated by $\mathfrak{so}(n)$. It consists of the entire bivector space $\Re^2_{n,n}$. Our task will be to prove that and develop a systematic way to find them.

Lie groups are classified according to their invariants. For the *classical groups* the invariants are nondegenerate bilinear (quadratic) forms. Geometric algebra supplies us with a simpler alternative of invariants, namely, the multivectors which determine the bilinear forms. As emphasized in reference [4], every bilinear form can be written as $a \cdot \left(\underline{Q}(b)\right)$ where $\underline{Q}$ is a linear operator, and the form is nondegenerate if $\underline{Q}$ is nonsingular (i.e., det $\left(\underline{Q}\right) \neq 0$)." [18]

To prove that $e^{\frac{B}{2}}$ is a rotation if B is a general bivector consider the function[19]

$$a(\lambda) = e^{\frac{\lambda B}{2}} a e^{-\frac{\lambda B}{2}} \tag{9.56}$$

where a is a vector. Then differentiate $a(\lambda)$

$$\begin{aligned} \frac{da}{d\lambda} &= \frac{B}{2} e^{\frac{\lambda B}{2}} a e^{-\frac{\lambda B}{2}} - e^{\frac{\lambda B}{2}} a e^{-\frac{\lambda B}{2}} \frac{B}{2} \\ &= \frac{B}{2} a(\lambda) - a(\lambda) \frac{B}{2} \\ &= B \cdot a(\lambda) \end{aligned} \tag{9.57}$$

$$\frac{d^2 a}{d\lambda^2} = B \cdot \left(\frac{da}{d\lambda}\right) = B \cdot (B \cdot a(\lambda)) \tag{9.58}$$

$$\left.\frac{d^r a}{d\lambda^r}\right|_{\lambda=0} = \underbrace{B \cdot (B \cdot (\ldots (B \cdot a)) \ldots)}_{r \text{ copies of } B}. \tag{9.59}$$

so that the Taylor expansion of $a(1)$ is

$$e^{\frac{B}{2}} a e^{-\frac{B}{2}} = a + B \cdot a + \frac{1}{2!} B \cdot (B \cdot a) + \frac{1}{3!} B \cdot (B \cdot (B \cdot a)) + \cdots. \tag{9.60}$$

[18] From reference [5].

[19] We let $a = a(0)$.

Since every term in eq (9.60) is a vector then $e^{\frac{B}{2}}ae^{-\frac{B}{2}}$ is a vector. Finally let a and b be vectors then (using RR5 in appendix C)

$$\begin{aligned}\left(e^{\frac{B}{2}}ae^{-\frac{B}{2}}\right)\cdot\left(e^{\frac{B}{2}}be^{-\frac{B}{2}}\right) &= \left\langle e^{\frac{B}{2}}ae^{-\frac{B}{2}}e^{\frac{B}{2}}be^{-\frac{B}{2}}\right\rangle \\ &= \left\langle e^{\frac{B}{2}}abe^{-\frac{B}{2}}\right\rangle \\ &= \left\langle e^{-\frac{B}{2}}e^{\frac{B}{2}}ab\right\rangle \\ &= \langle ab\rangle = a\cdot b. \end{aligned} \tag{9.61}$$

Thus $e^{\frac{B}{2}}$ is a rotor since $e^{\frac{B}{2}}ae^{-\frac{B}{2}}$ is a vector and the transformation generated by $e^{\frac{B}{2}}$ preserves the inner product of two vectors.

For an n-dimensional vector space the number of linearly independent bivectors is $\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n^2-n}{2}$, which is also the number of generators for the spin group $Spin\,(n)$ which are $e^{\frac{\theta_{ij}\boldsymbol{e}_i\wedge\boldsymbol{e}_j}{2}} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\boldsymbol{e}_i\wedge\boldsymbol{e}_j\ \ \forall\ \ 1\le i<j\le n$ where θ_{ij} is the rotation in the plane defined by $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$. Thus there is a one to one correspondence between the bivectors, B, and the rotations in an n-dimensional vector space.

$\underline{Q}$ is invariant under a rotation $\underline{R}$ if

$$(\underline{R}a)\cdot\left(\underline{Q}\,\underline{R}b\right) = a\cdot\left(\underline{Q}b\right). \tag{9.62}$$

Using equation 9.54 transforms equation 9.62 to

$$\begin{aligned}(\underline{R}a)\cdot\left(\underline{Q}\,\underline{R}b\right) &= a\cdot\left(\underline{Q}b\right), \\ \left(\underline{Q}\,\underline{R}b\right)\cdot(\underline{R}a) &= a\cdot\left(\underline{Q}b\right), \\ a\cdot\left(\underline{R}^\dagger\left(\underline{Q}\,\underline{R}b\right)\right) &= a\cdot\left(\underline{Q}b\right), \\ a\cdot\left(\underline{R}^\dagger\underline{Q}\,\underline{R}b\right) &= a\cdot\left(\underline{Q}b\right), \end{aligned} \tag{9.63}$$

or since the a and b in equation 9.63 are arbitrary

$$\underline{R}^\dagger\underline{Q}\,\underline{R} = \underline{Q} = \underline{R}\,\underline{Q}\,\underline{R}^\dagger, \tag{9.64}$$

$$\underline{R}\,\underline{R}^\dagger\underline{Q}\,\underline{R} = \underline{R}\,\underline{Q}, \tag{9.65}$$

$$\underline{Q}\,\underline{R} = \underline{R}\,\underline{Q}. \tag{9.66}$$

Thus the invariance group of consists of those rotations which commute with $\underline{Q}$.

This is obviously a group since if

$$\underline{R}_1\underline{Q} = \underline{Q}\underline{R}_1, \tag{9.67}$$
$$\underline{R}_2\underline{Q} = \underline{Q}\underline{R}_2, \tag{9.68}$$

then

$$\underline{R}_1\underline{R}_2\underline{Q} = \underline{R}_1\underline{Q}\underline{R}_2, \tag{9.69}$$
$$\underline{R}_1\underline{R}_2\underline{Q} = \underline{Q}\underline{R}_1\underline{R}_2. \tag{9.70}$$

As a specific case consider the quadratic form where $\underline{Q}b = b^*$. Then

$$\begin{aligned}
\underline{Q}b &= (-1)^{n+1}\, E_n b E_n^{-1}, && (9.71)\\
\underline{Q}\underline{R}b &= (-1)^{n+1}\, E_n R b R^\dagger E_n^{-1}, \\
\underline{R}\underline{Q}b &= (-1)^{n+1}\, R E_n b E_n^{-1} R^\dagger, \\
E_n R b R^\dagger E_n^{-1} &= R E_n b E_n^{-1} R^\dagger, \\
E_n R &= R E_n, \\
E_n &= R E_n R^\dagger. && (9.72)
\end{aligned}$$

Thus, the invariance of the bilinear form $a \cdot b^*$ is equivalent to the invariance of the n-blade E_n.

We can determine a representation for R by noting the generators of any rotation in $\Re^{n,n}$ can be written in the form

$$\begin{aligned}
R &= e^{\frac{\theta}{2}\boldsymbol{uv}}, \\
&= \left\{ \begin{array}{c} \cos(\theta/2) \\ \cosh(\theta/2) \end{array} \right\} + \left\{ \begin{array}{c} \sin(\theta/2) \\ \sinh(\theta/2) \end{array} \right\} \boldsymbol{uv}, \\
&= \left\{ \begin{array}{c} \cos(\theta/2) \\ \cosh(\theta/2) \end{array} \right\} + \left\{ \begin{array}{c} \sin(\theta/2) \\ \sinh(\theta/2) \end{array} \right\} \sum_{i<j} \left(u^i \boldsymbol{e}_i + \bar{u}^i \bar{\boldsymbol{e}}_i\right)\left(v^j \boldsymbol{e}_j + \bar{v}^j \bar{\boldsymbol{e}}_j\right), && (9.73)
\end{aligned}$$

where $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ and $\left|(\boldsymbol{uv})^2\right| = 1$. Equation (9.73) is what makes corresponding the Lie algebras with the bivector commuatator algebra possible and also allows one to calculate the generators of the corresponding Lie group. The generators of the most general rotation in $\Re^{n,n}$ are $\boldsymbol{e}_i\boldsymbol{e}_j$, $\bar{\boldsymbol{e}}_i\boldsymbol{e}_j$, and $\bar{\boldsymbol{e}}_i\bar{\boldsymbol{e}}_j$. The question is which of these generators commute with E_n. The answer is

$$\boldsymbol{e}_i\boldsymbol{e}_j E_n = E_n \boldsymbol{e}_i\boldsymbol{e}_j \tag{9.74}$$
$$\boldsymbol{e}_i\bar{\boldsymbol{e}}_j E_n = -E_n \boldsymbol{e}_i\bar{\boldsymbol{e}}_j \tag{9.75}$$
$$\bar{\boldsymbol{e}}_i\bar{\boldsymbol{e}}_j E_n = E_n \bar{\boldsymbol{e}}_i\bar{\boldsymbol{e}}_j. \tag{9.76}$$

Equation 9.76 is obvious since $\bar{\boldsymbol{e}}_i$ and $\bar{\boldsymbol{e}}_j$ give the same number of sign flips, n, in passing through E_n totalling $2n$ an even number. Likewise, in equation 9.74 as $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ give the same number of sign flips, $n-1$, in passing through E_n totalling $2(n-1)$ an even number. In equation 9.75 $\boldsymbol{e}_i$ produces $n-1$ sign flips in traversing E_n and $\bar{\boldsymbol{e}}_j$ produces n sign flips for the same traverse so that the total number of sign flips is $2n-1$ and odd number.

Thus the reqired rotation generators for R are

$$\boldsymbol{e}_{ij} = \boldsymbol{e}_i\boldsymbol{e}_j, \text{ for } i < j = 1, \ldots, n, \tag{9.77}$$
$$\bar{\boldsymbol{e}}_{ij} = \bar{\boldsymbol{e}}_i\bar{\boldsymbol{e}}_j, \text{ for } i < j = 1, \ldots, n. \tag{9.78}$$

Also note that since $\boldsymbol{e}_{ij} \times \bar{\boldsymbol{e}}_{kl} = 0$, the barred and unbarred generators commute. Any generator in the algebra can be written in the form

$$B = \boldsymbol{\alpha} : \boldsymbol{e} + \boldsymbol{\beta} : \bar{\boldsymbol{e}}, \tag{9.79}$$

where

$$\boldsymbol{\alpha} : \boldsymbol{e} \equiv \sum_{i<j} \alpha^{ij}\boldsymbol{e}_{ij}, \tag{9.80}$$

and the α^{ij} are scalar coefficients. So that $\boldsymbol{\alpha} : \boldsymbol{e}$ is the generator for any rotation in $\Re^n$ and $\boldsymbol{\beta} : \bar{\boldsymbol{e}}$ is the generator for any rotation in $\bar{\Re}^n$. The corresponding group rotor is (we can factor the exponent since $\boldsymbol{e}$ and $\bar{\boldsymbol{e}}$ generators commute)

$$R = e^{\frac{1}{2}(\boldsymbol{\alpha}:\boldsymbol{e}+\boldsymbol{\beta}:\bar{\boldsymbol{e}})} = e^{\frac{1}{2}\boldsymbol{\alpha}:\boldsymbol{e}}e^{\frac{1}{2}\boldsymbol{\beta}:\bar{\boldsymbol{e}}}. \tag{9.81}$$

This is the spin representation of the product group $\mathrm{SO}(n) \otimes \mathrm{SO}(n)$.

In most cases the generators of the invariance group are not as obvious and in the case of the $a \cdot b^*$ form. Thus we need some general methods for such determinations. Consider a skew-symmetric bilinear form $\underline{Q}$[20]

$$a \cdot \left(\underline{Q}b\right) = -b \cdot \left(\underline{Q}a\right). \tag{9.82}$$

The form $\underline{Q}$ can be written

$$a \cdot \left(\underline{Q}b\right) = a \cdot (b \cdot Q) = (a \wedge b) \cdot Q, \tag{9.83}$$

[20]This selection is done with malice aforethought. To generate the memembers of $GL(n, \Re)$ we do not need the most general linear transformation on $\Re^{n,n}$ since the dimension of that group is $4n^2$ and not the n^2 of $GL(n, \Re)$.

where Q is a bivector.[21] We say that the bivector Q in *involutory* if $\underline{Q}$ is nonsingular and

$$\underline{Q}^2 = \pm\underline{1}. \tag{9.84}$$

Note that the operator equation 9.84 only applies to vectors.

Note that

$$\begin{aligned}
(\underline{R}a \wedge \underline{R}b) \cdot Q &= \underline{R}\,(a \wedge b) \cdot Q \\
&= \left(R\,(a \wedge b)\,R^\dagger\right) \cdot Q \\
&= \left\langle R\,(a \wedge b)\,R^\dagger Q\right\rangle \\
&= \left\langle (a \wedge b)\,R^\dagger Q R\right\rangle \\
&= (a \wedge b) \cdot \left(\underline{R}^\dagger Q\right).
\end{aligned} \tag{9.85}$$

Then equation 9.83 gives for a stability condition

$$\begin{aligned}
(\underline{R}a) \cdot \left(\underline{Q}\underline{R}b\right) &= a \cdot \left(\underline{Q}b\right), \\
((\underline{R}a) \cdot (\underline{R}b)) \cdot Q &= (a \wedge b) \cdot Q, \\
((\underline{R}a) \wedge (\underline{R}b)) \cdot Q &= \\
\underline{R}\,(a \wedge b) \cdot Q &= \\
\left\langle R\,(a \wedge b)\,R^\dagger Q\right\rangle &= \\
\left\langle (a \wedge b)\,R^\dagger Q R\right\rangle &= \\
(a \wedge b) \cdot \left(R^\dagger Q R\right) &= (a \wedge b) \cdot Q, \\
R^\dagger Q R &= Q. \\
QR &= RQ
\end{aligned} \tag{9.86}$$

From equation 9.86 we have that generators of the stability group $G\,(Q)$ for Q must commute with Q. To learn more about this requirement, we study the commutator of Q with an arbitrary

[21] This is obvious from the properties of the bilinear form that if $a \cdot \left(\underline{Q}b\right) = (a \wedge b) \cdot Q$ then $a \cdot \left(\underline{Q}b\right)$ is linear in a and b and is skew-symmetric

$$\begin{aligned}
b \cdot \left(\underline{Q}a\right) &= (b \wedge a) \cdot Q \\
&= -\,(a \wedge b) \cdot Q \\
&= -a \cdot \left(\underline{Q}b\right)
\end{aligned}$$

Finally the maximum number of free parameters (coefficients) for the bivector, Q, in an n-dimensional space is $\binom{n}{2} = \frac{n(n-1)}{2}$. This is also the number of independent coefficients in the $n \times n$ antisymmetric matrix that represents the skew-symmetric bilinear form.

bivector blade $a \wedge b$. Since $a \wedge b = a \times b$ and $a \times Q = a \cdot Q$ the Jacobi identity gives

$$\begin{aligned}
(a \wedge b) \times Q &= (a \times b) \times Q, \\
&= (a \times Q) \times b + a \times (b \times Q), \\
&= (a \times Q) \wedge b + a \wedge (b \times Q), \\
&= (a \cdot Q) \wedge b + a \wedge (b \cdot Q), \\
&= \left(\underline{Q}a\right) \wedge b + a \wedge \left(\underline{Q}b\right), \qquad (9.87)
\end{aligned}$$

and then

$$((a \wedge b) \times Q) \times Q = \left(\left(\underline{Q}a\right) \wedge b + a \wedge \left(\underline{Q}b\right)\right) \times Q. \qquad (9.88)$$

Now using the Jacobi identity and the extension of linear functions to blades we have

$$\begin{aligned}
\left(\left(\underline{Q}a\right) \wedge b\right) \times Q &= \left(\left(\underline{Q}a\right) \times b\right) \times Q, \\
&= \left(\underline{Q}a\right) \times (b \times Q) - b \times \left(\left(\underline{Q}a\right) \times Q\right), \\
&= \left(\underline{Q}a\right) \wedge (b \cdot Q) - b \wedge \left(\left(\underline{Q}a\right) \cdot Q\right), \\
&= \left(\underline{Q}a\right) \wedge \left(\underline{Q}b\right) - b \wedge \left(\underline{Q}^2 a\right), \\
&= \left(\underline{Q}^2 a\right) \wedge b + \underline{Q}\,(a \wedge b). \qquad (9.89)
\end{aligned}$$

Similarly

$$\left(a \wedge \left(\underline{Q}b\right)\right) \times Q = \underline{Q}\,(a \wedge b) + a \wedge \left(\underline{Q}^2 b\right), \qquad (9.90)$$

so that, since Q is involutary $\left(\underline{Q}^2 = \pm\underline{1}\right)$

$$\begin{aligned}
((a \wedge b) \times Q) \times Q &= \left(\underline{Q}^2 a\right) \wedge b + 2\underline{Q}\,(a \wedge b) + a \wedge \left(\underline{Q}^2 b\right), \\
&= 2\left(\underline{Q}\,(a \wedge b) \pm a \wedge b\right). \qquad (9.91)
\end{aligned}$$

By linearity and superposition since equation 9.91 holds for any blade $a \wedge b$ it also holds for any bivector (superposition of 2-blades) B so that

$$(B \times Q) \times Q = 2\left(\underline{Q}B \pm B\right). \qquad (9.92)$$

If B commutes with Q then $B \times Q = 0$ and

$$\begin{aligned}
\underline{Q}B \pm B &= 0, \qquad (9.93) \\
\underline{Q}B &= \mp B. \qquad (9.94)
\end{aligned}$$

Thus the generators of $G(Q)$ are the eigenbivectors of $\underline{Q}$ with eigenvalues ∓ 1.

Now define the bivectors $E^{\pm}(a,b)$ and $F(a,b)$ by

$$\begin{aligned} E^{\pm}(a,b) &\equiv a\wedge b \pm \left(\underline{Q}a\right)\left(\underline{Q}b\right), && (9.95)\\ F(a,b) &\equiv \left(\underline{Q}a\right)\wedge b - a\wedge\left(\underline{Q}b\right). && (9.96)\end{aligned}$$

Then using equation 9.87 we get

$$\begin{aligned} E^{\pm}(a,b)\times Q &= \left(\underline{Q}a\right)\wedge b + a\wedge\left(\underline{Q}b\right) \mp \left(\underline{Q}^2a\right)\wedge\left(\underline{Q}b\right) \mp \left(\underline{Q}a\right)\wedge\left(\underline{Q}^2b\right),\\ &= \left(\underline{Q}a\right)\wedge b + a\wedge\left(\underline{Q}b\right) - a\wedge\left(\underline{Q}b\right) - \left(\underline{Q}a\right)\wedge b,\\ &= 0, && (9.97)\\ F(a,b)\times Q &= \left(\underline{Q}a\right)\wedge\left(\underline{Q}b\right) + a\wedge\left(\underline{Q}^2b\right) - \left(\underline{Q}^2a\right)\wedge b - \left(\underline{Q}a\right)\wedge\left(\underline{Q}b\right),\\ &= \pm a\wedge b \mp a\wedge b,\\ &= 0. && (9.98)\end{aligned}$$

Thus $E^{\pm}(a,b)$ and $F(a,b)$ are the generators of the stability qroup for Q. A basis for the Lie algebra is obtained by inserting basis vectors for a and b. The commutation relations for the generators $E^{\pm}(a,b)$ and $F(a,b)$ can be found from equations 9.95, 9.96, and B.13. Evaluation of the commutation relations is simplified by using the eigenvectors of $\underline{Q}$ for a basis, so it is best to defer the task until $\underline{Q}$ is completely specified.

As a concrete example let $Q = K$ where K is from equation 9.29 and use equations 9.45 and 9.46 to get[22]

$$\begin{aligned} E_{ij} = E^{+}(\boldsymbol{e}_i,\boldsymbol{e}_j) &= \boldsymbol{e}_i\wedge\boldsymbol{e}_j - \left(\underline{K}\boldsymbol{e}_i\right)\wedge\left(\underline{K}\boldsymbol{e}_i\right),\\ &= \boldsymbol{e}_i\boldsymbol{e}_j - \bar{\boldsymbol{e}}_i\bar{\boldsymbol{e}}_j\ \ (i<j), && (9.99)\\ F_{ij} = F(\boldsymbol{e}_i,\boldsymbol{e}_j) &= \boldsymbol{e}_i\wedge\left(\underline{K}\boldsymbol{e}_j\right) - \left(\underline{K}\boldsymbol{e}_i\right)\wedge\boldsymbol{e}_j,\\ &= \boldsymbol{e}_i\bar{\boldsymbol{e}}_j - \bar{\boldsymbol{e}}_i\boldsymbol{e}_j\ \ (i<j), && (9.100)\\ K_i = \frac{1}{2}F_{ii} &= \boldsymbol{e}_i\bar{\boldsymbol{e}}_i. && (9.101)\end{aligned}$$

At this point we should note that if a bivector B is a linear combination of E_{ij}, F_{ij}, and F_i, B will commute with Q. Then $R = e^{\frac{B}{2}}$ also commutes with Q since $e^{\frac{B}{2}}$ only can contain powers of B. Also E_{ij}, F_{ij}, and F_i are called the generators of the Lie algebra associated with the bilinear form defined by $\underline{K}$ and the number of generators are $\frac{n^2-n}{2}+\frac{n^2-n}{2}+n = n^2$.

[22] Since $\underline{K}^2 = 1$ we only need consider $E^{+}(\boldsymbol{e}_i,\boldsymbol{e}_j)$.

The structure equations for the Lie algebra (non-zero commutators of the Lie algebra generators) of the bilinear form $\underline{K}$ are

$$E_{ij} \times F_{ij} = 2\left(K_i - K_j\right) \tag{9.102}$$
$$E_{ij} \times K_i = -F_{ij} \tag{9.103}$$
$$F_{ij} \times K_i = -E_{ij} \tag{9.104}$$
$$E_{ij} \times E_{il} = -E_{jl} \tag{9.105}$$
$$F_{ij} \times F_{il} = E_{jl} \tag{9.106}$$
$$F_{ij} \times E_{il} = F_{jl}. \tag{9.107}$$

The structure equations close the algebra with respect to the commutator product (see appendix G for how to calculate the structure equations).

Thus (using the notation of eq (9.80)[23]) the rotors for the stablity group of $\underline{K}$ can be written

$$e^{\frac{B}{2}} = e^{\frac{1}{2}(\boldsymbol{\alpha}:\boldsymbol{E}+\boldsymbol{\beta}:\boldsymbol{F}+\boldsymbol{\gamma}:\boldsymbol{K})}, \tag{9.108}$$

where the n^2 coefficients are α_{ij}, β_{ij}, and γ_i.

The stability group of K can be identified with the *general linear group* $\mathrm{GL}(n,\Re)$. First we must show that $\underline{K}$ does not mix the subspaces $\mathcal{V}^n$ and $\mathcal{V}^{n*}$ of $\Re^{n,n}$. Using equations 9.47 and 9.48 we have (where W_n and W_n^* are the pseudoscalars for $\mathcal{V}^n$ and $\mathcal{V}^{n*}$).

$$W_n = \boldsymbol{w}_1 \ldots \boldsymbol{w}_n, \tag{9.109}$$
$$W_n^* = \boldsymbol{w}_1^* \ldots \boldsymbol{w}_n^* \tag{9.110}$$
$$\underline{K}\left(W_n\right) = W_n, \tag{9.111}$$
$$\underline{K}\left(W_n^*\right) = (-1)^n W_n^*. \tag{9.112}$$

Thus when restricted to $\mathcal{V}^n$ or $\mathcal{V}^{n*}$, $\underline{K}$ is non-singular since the $\det\left(\underline{K}\right) \neq 0$.

Since each group element leaves $\mathcal{V}^n$ invariant

$$\underline{R}\underline{K}\left(W_n\right) = \underline{R}W_n = RW_nR^\dagger = W_n, \tag{9.113}$$

we can write

$$\underline{R}\boldsymbol{w}_j = \sum_{k=1}^{n} \boldsymbol{w}_k \rho_{kj}. \tag{9.114}$$

[23]The $\boldsymbol{\alpha}:\boldsymbol{E}$ notation simply says that indices of the scalar coefficients, α_{ij}, are balanced by the indices of the bivectors, E_{ij}. So that $\boldsymbol{\alpha}:\boldsymbol{E} = \sum_{i<j} \alpha_{ij} E_{ij}$ or $\boldsymbol{\gamma}:\boldsymbol{K} = \sum_i \gamma_i K_i$.

Where using eq (9.11) we get

$$\rho_{ij} = 2\boldsymbol{w}_i^* \cdot (\underline{R}\boldsymbol{w}_j) = 2\left\langle \boldsymbol{w}_i^* R \boldsymbol{w}_j R^\dagger \right\rangle. \tag{9.115}$$

The rotations can be described on $\mathcal{V}^n$ without reference to $\Re^n$. For any member $\underline{R}$ of $GL(n,\Re)$ we have

$$\underline{R} W_n = W_n \det\nolimits_{\mathcal{V}^n}(\underline{R}), \tag{9.116}$$

where $\det_{\mathcal{V}^n}$ is the determinant of the linear transformation restricted to the $\mathcal{V}^n$ subspace of $\Re^{n,n}$ and not on the entire vector space. First note that

$$W_n^* \cdot W_n^\dagger = (w_1^* \dots w_n^*) \cdot (w_n \dots w_1) = \langle w_1^* \dots w_n^* w_n \dots w_1 \rangle = 2^{-n}. \tag{9.117}$$

Therefore,

$$\underline{R} W_n^\dagger = W_n^\dagger \det\nolimits_{\mathcal{V}^n}\left(\underline{R}^{-1}\right) \tag{9.118}$$

9.7 Endomorphisms of $\Re^n$

For a vector space an *endomorphism* is a linear mapping of the vector space onto itself. If the *endomorphism* has an inverse it is an *automorphism*. By studying the *endomorphisms* of $\mathfrak{R}_n$, the geometric algebra of $\Re^n$ we can also show in an alternative way that the mother algebra, $\mathfrak{R}_{n,n}$, is the appropriate arena for the study of linear transformations and Lie groups. First note that ($\simeq$ is the symbol for "is isomorphic to")

$$\mathcal{E}nd(\mathfrak{R}_n) \simeq \Re^{2^{2n}}, \tag{9.119}$$

since $End(\mathfrak{R}_n)$ is isomorphic to the algebra of all $2^n \times 2^n$ matrices.

For an arbitrary multivector $A \in \mathfrak{R}_n$, left and right multiplications by orthonormal basis vectors $\boldsymbol{e}_i$ determine endomorphisms of $\mathfrak{R}_n$ defined by

$$\underline{\boldsymbol{e}}_i : A \to \underline{\boldsymbol{e}}_i(A) \equiv \boldsymbol{e}_i A, \tag{9.120}$$
$$\underline{\bar{\boldsymbol{e}}}_i : A \to \underline{\bar{\boldsymbol{e}}}_i(A) \equiv \bar{A}\boldsymbol{e}_i \tag{9.121}$$

and the involution operator $\left(\bar{\bar{A}} = A\right)$ is defined by[24]

$$\overline{(AB)} = \bar{A}\bar{B} \tag{9.122}$$
$$\bar{\boldsymbol{e}}_i = -\boldsymbol{e}_i. \tag{9.123}$$

Thus for any multivector A we have

$$\underline{\boldsymbol{e}}_i\underline{\boldsymbol{e}}_j\,(A) = \underline{\boldsymbol{e}}_i\,(\boldsymbol{e}_jA) = \boldsymbol{e}_i\boldsymbol{e}_jA, \tag{9.124}$$
$$\underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_j\,(A) = \underline{\boldsymbol{e}}_i\left(\bar{A}\boldsymbol{e}_j\right) = \boldsymbol{e}_i\bar{A}\boldsymbol{e}_j, \tag{9.125}$$
$$\underline{\bar{\boldsymbol{e}}}_j\underline{\boldsymbol{e}}_i\,(A) = \underline{\bar{\boldsymbol{e}}}_j\,(\boldsymbol{e}_iA) = \overline{\boldsymbol{e}_iA}\boldsymbol{e}_j = -\boldsymbol{e}_i\bar{A}\boldsymbol{e}_j, \tag{9.126}$$
$$\underline{\bar{\boldsymbol{e}}}_i\underline{\bar{\boldsymbol{e}}}_j\,(A) = \underline{\bar{\boldsymbol{e}}}_i\left(\bar{A}\boldsymbol{e}_j\right) = \overline{\bar{A}\boldsymbol{e}_j}\boldsymbol{e}_i = -A\boldsymbol{e}_j\boldsymbol{e}_i. \tag{9.127}$$

Thus

$$\begin{aligned}
\left(\underline{\boldsymbol{e}}_i\underline{\boldsymbol{e}}_j + \underline{\boldsymbol{e}}_j\underline{\boldsymbol{e}}_i\right)(A) &= \left(\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i\right)A = 2\delta_{ij}A,\\
\underline{\boldsymbol{e}}_i\underline{\boldsymbol{e}}_j + \underline{\boldsymbol{e}}_j\underline{\boldsymbol{e}}_i &= 2\delta_{ij}, && (9.128)\\
\left(\underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_j + \underline{\bar{\boldsymbol{e}}}_j\underline{\boldsymbol{e}}_i\right)(A) &= \boldsymbol{e}_iA\boldsymbol{e}_j - \boldsymbol{e}_iA\boldsymbol{e}_j = 0,\\
\underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_j + \underline{\bar{\boldsymbol{e}}}_j\underline{\boldsymbol{e}}_i &= 0, && (9.129)\\
\left(\underline{\bar{\boldsymbol{e}}}_i\underline{\bar{\boldsymbol{e}}}_j + \underline{\bar{\boldsymbol{e}}}_j\underline{\bar{\boldsymbol{e}}}_i\right)(A) &= -A\left(\boldsymbol{e}_j\boldsymbol{e}_i + \boldsymbol{e}_i\boldsymbol{e}_j\right)A = -2\delta_{ij}A,\\
\underline{\boldsymbol{e}}_i\underline{\boldsymbol{e}}_j + \underline{\boldsymbol{e}}_j\underline{\boldsymbol{e}}_i &= -2\delta_{ij}. && (9.130)
\end{aligned}$$

Thus equations (9.128), (9.129), and (9.130) are isomorphic to equations (9.16), (9.18), and (9.17) which define the orthogonal basis $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_j\}$ of $\Re^{n,n}$. This establishes the isomorphism[25]

$$\Re_{n,n} \simeq \mathcal{E}nd\left(\Re_n\right). \tag{9.131}$$

The differences between $\Re_{n,n}$ and $\mathcal{E}nd\left(\Re_n\right)$ are that $\Re_{n,n}$ is the geometric algebra of a vector space with signature (n,n), basis $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_j\}$, and dimension 2^{2n}, while $\mathcal{E}nd\left(\Re_n\right)$ are the linear mappings of the geometric algebra $\Re_n$ on to itself with linear basis functions $\left\{\underline{\boldsymbol{e}}_i, \underline{\bar{\boldsymbol{e}}}_j\right\}$ that are isomorphic to $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_j\}$ and have the same anti-commutation relations (dot products) as $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_j\}$.

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[24]Since any grade r basis blade is of the form $\boldsymbol{e}_{j_1}\ldots\boldsymbol{e}_{j_r}$ with the j_k's in normal order then

$$\overline{\boldsymbol{e}_{j_1}\ldots\boldsymbol{e}_{j_r}} = \bar{\boldsymbol{e}}_{j_1}\ldots\bar{\boldsymbol{e}}_{j_r} = (-1)^r\,\boldsymbol{e}_{j_1}\ldots\boldsymbol{e}_{j_r}$$

and the involute of any multivector, A, can be determined from equations (9.122) and (9.123). Likewise $\overline{\overline{\boldsymbol{e}_{j_1}\ldots\boldsymbol{e}_{j_r}}} = \boldsymbol{e}_{j_1}\ldots\boldsymbol{e}_{j_r}$.

[25] $\dim\left(\Re_{n,n}\right) = 2^{2n}$.

Note that in defining $\underline{\bar{\boldsymbol{e}}}_i(A) = \bar{A}\boldsymbol{e}_i$, the involution, $\bar{A}$, is required to get the proper anti-commutation (dot product) relations that insure $\underline{\boldsymbol{e}}_i$ and $\underline{\bar{\boldsymbol{e}}}_j$ are orthogonal and that the signature of $\mathcal{E}nd(\Re_n)$ is (n, n).

Additionally, the composite operators

$$\underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_i : A \rightarrow \underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_i(A) = \boldsymbol{e}_i\bar{A}\boldsymbol{e}_i \tag{9.132}$$

$$\begin{aligned} \underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_i(AB) &= \boldsymbol{e}_i\bar{A}\bar{B}\boldsymbol{e}_i \\ &= \boldsymbol{e}_i\bar{A}\boldsymbol{e}_i\boldsymbol{e}_i\bar{B}\boldsymbol{e}_i \\ &= \underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_i(A)\,\underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_i(B) \end{aligned} \tag{9.133}$$

preserve the geometric product and generate $\mathcal{A}ut(\Re_n)$, a subgroup of $\mathcal{E}nd(\Re_n)$.