Chapter 7

Multilinear Functions (Tensors)

A multivector multilinear function[1] is a multivector function $T(A_1, \ldots, A_r)$ that is linear in each of its arguments[2] The tensor could be non-linearly dependent on a set of additional arguments such as the position coordinates x^i in the case of a tensor field defined on a manifold. If x denotes the coordinate tuple for a manifold we denote the dependence of T on x by $T(A_1, \ldots, A_r; x)$.

T is a *tensor* of degree r if each variable $A_j \in \mathcal{V}_n$ ($\mathcal{V}_n$ is an n-dimensional vector space). More generally if each $A_j \in \mathcal{G}(\mathcal{V}_n)$ (the geometric algebra of $\mathcal{V}_n$), we call T an *extensor* of degree-r on $\mathcal{G}(\mathcal{V}_n)$.

If the values of $T(a_1, \ldots, a_r)$ ($a_j \in \mathcal{V}_n \; \forall \; 1 \le j \le r$) are s-vectors (pure grade s multivectors in $\mathcal{G}(\mathcal{V}_n)$) we say that T has grade s and rank $r+s$. A tensor of grade zero is called a *multilinear form*.

In the normal definition of tensors as multilinear functions the tensor is defined as a multilinear mapping

$$T : \bigtimes_{i=1}^{r} \mathcal{V}_n \to \Re,$$

so that the standard tensor definition is an example of a grade zero degree/rank r tensor in our definition.

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[1]We are following the treatment of Tensors in section 3–10 of [4].

[2]We assume that the arguments are elements of a vector space or more generally a geometric algebra so that the concept of linearity is meaningful.

7.1 Algebraic Operations

The properties of tensors are ($\alpha \in \Re$, $a_j, b \in \mathcal{V}_n$, T and S are tensors of rank r, and $\circ$ is any multivector multiplicative operation)

$$T(a_1, \ldots, \alpha a_j, \ldots, a_r) = \alpha T(a_1, \ldots, a_j, \ldots, a_r), \tag{7.1}$$

$$T(a_1, \ldots, a_j + b, \ldots, a_r) = T(a_1, \ldots, a_j, \ldots, a_r) + T(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_r), \tag{7.2}$$

$$(T \pm S)(a_1, \ldots, a_r) \equiv T(a_1, \ldots, a_r) \pm S(a_1, \ldots, a_r). \tag{7.3}$$

Now let T be of rank r and S of rank s then the product of the two tensors is

$$(T \circ S)(a_1, \ldots, a_{r+s}) \equiv T(a_1, \ldots, a_r) \circ S(a_{r+1}, \ldots, a_{r+s}), \tag{7.4}$$

where "$\circ$" is any multivector multiplicative operation.

7.2 Covariant, Contravariant, and Mixed Representations

The arguments (vectors) of the multilinear fuction can be represented in terms of the basis vectors or the reciprocal basis vectors[3]

$$a_j = a^{i_j} \boldsymbol{e}_{i_j}, \tag{7.5}$$

$$= a_{i_j} \boldsymbol{e}^{i_j}. \tag{7.6}$$

Equation (7.5) gives a_j in terms of the basis vectors and eq (7.6) in terms of the reciprocal basis vectors. The index j refers to the argument slot and the indices i_j the components of the vector in terms of the basis. The Einstein summation convention is used throughout. The covariant representation of the tensor is defined by

$$T_{i_1 \ldots i_r} \equiv T(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) \tag{7.7}$$

$$\begin{aligned} T(a_1, \ldots, a_r) &= T\left(a^{i_1} \boldsymbol{e}_{i_1}, \ldots, a^{i_r} \boldsymbol{e}_{i_r}\right) \\ &= T(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r})\, a^{i_1} \ldots a^{i_r} \\ &= T_{i_1 \ldots i_r}\, a^{i_1} \ldots a^{i_r}. \end{aligned} \tag{7.8}$$

[3]When the a_j vectors are expanded in terms of a basis we need a notatation that lets one determine which vector argument, j, the scalar components are associated with. Thus when we expand the vector in terms of the basis we write $a_j = a^{i_j} \boldsymbol{e}_{i_j}$ with the Einstein summation convention applied over the i_j indices. In the expansion the j in the a^{i_j} determines which argument in the tensor function the a^{i_j} coefficients are associated with. Thus it is always the subscript of the component super or subscript that determines the argument the coefficient is associated with.

Likewise for the contravariant representation

$$\begin{aligned}
T^{i_1 \dots i_r} \equiv & T\left(\boldsymbol{e}^{i_1}, \dots, \boldsymbol{e}^{i_r}\right) && (7.9) \\
T\left(a_1, \dots, a_r\right) = & T\left(a_{i_1}\boldsymbol{e}^{i_1}, \dots, a_{i_r}\boldsymbol{e}^{i_r}\right) \\
= & T\left(\boldsymbol{e}^{i_1}, \dots, \boldsymbol{e}^{i_r}\right) a_{i_1} \dots a_{i_r} \\
= & T^{i_1 \dots i_r} a_{i_1} \dots a_{i_r}. && (7.10)
\end{aligned}$$

One could also have a mixed representation

$$\begin{aligned}
T_{i_1 \dots i_s}{}^{i_{s+1} \dots i_r} \equiv & T\left(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_s}, \boldsymbol{e}^{i_{s+1}} \dots \boldsymbol{e}^{i_r}\right) && (7.11) \\
T\left(a_1, \dots, a_r\right) = & T\left(a^{i_1}\boldsymbol{e}_{i_1}, \dots, a^{i_s}\boldsymbol{e}_{i_s}, a_{i_{s+1}}\boldsymbol{e}^{i_s}, \dots, a_{i_r}\boldsymbol{e}^{i_r}\right) \\
= & T\left(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_s}, \boldsymbol{e}^{i_{s+1}}, \dots, \boldsymbol{e}^{i_r}\right) a^{i_1} \dots a^{i_s} a_{i_{s+1}} \dots a_{i_r} \\
= & T_{i_1 \dots i_s}{}^{i_{s+1} \dots i_r} a^{i_1} \dots a^{i_s} a_{i_{s+1}}, \dots a_{i_r}. && (7.12)
\end{aligned}$$

In the representation of T one could have any combination of covariant (lower) and contravariant (upper) indices.

To convert a covariant index to a contravariant index simply consider

$$\begin{aligned}
T\left(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}^{i_j}, \dots, \boldsymbol{e}_{i_r}\right) = & T\left(\boldsymbol{e}_{i_1}, \dots, g^{i_j k_j}\boldsymbol{e}_{k_j}, \dots, \boldsymbol{e}_{i_r}\right) \\
= & g^{i_j k_j} T\left(\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{k_j}, \dots, \boldsymbol{e}_{i_r}\right) \\
T_{i_1 \dots}{}^{i_j}{}_{\dots i_r} = & g^{i_j k_j} T_{i_1 \dots i_j \dots i_r}. && (7.13)
\end{aligned}$$

Similarly one could raise a lower index with $g_{i_j k_j}$.

7.3 Contraction

The contraction of a tensor between the j^{th} and k^{th} variables (slots) is[4]

$$T\left(a_i,\ldots,a_{j-1},\nabla_{a_k},a_{j+1},\ldots,a_r\right)=\nabla_{a_j}\cdot\left(\nabla_{a_k}T\left(a_1,\ldots,a_r\right)\right). \tag{7.14}$$

This operation reduces the rank of the tensor by two. This definition gives the standard results for *metric contraction* which is proved as follows for a rank r grade zero tensor (the circumflex "$\breve{\ }$" indicates that a term is to be deleted from the product).

$$\begin{aligned}
T\left(a_1,\ldots,a_r\right) =& a^{i_1}\ldots a^{i_r}T_{i_1\ldots i_r} && (7.15)\\
\nabla_{a_j}T =& \boldsymbol{e}^{l_j}a^{i_1}\ldots\left(\partial_{a^{l_j}}a^{i_j}\right)\ldots a_{i_r}T_{i_1\ldots i_r}\\
=& \boldsymbol{e}^{l_j}\delta^{i_j}_{l_j}a^{i_1}\ldots\breve{a}^{i_j}\ldots a^{i_r}T_{i_1\ldots i_r} && (7.16)\\
\nabla_{a_m}\cdot\left(\nabla_{a_j}T\right) =& \boldsymbol{e}^{k_m}\cdot\boldsymbol{e}^{l_j}\delta^{i_j}_{l_j}a^{i_1}\ldots\breve{a}^{i_j}\ldots\left(\partial_{a^{k_m}}a^{i_m}\right)\ldots a^{i_r}T_{i_1\ldots i_r}\\
=& g^{k_m l_j}\delta^{i_j}_{l_j}\delta^{i_m}_{k_m}a^{i_1}\ldots\breve{a}^{i_j}\ldots\breve{a}^{i_m}\ldots a^{i_r}T_{i_1\ldots i_r}\\
=& g^{i_m i_j}a^{i_1}\ldots\breve{a}^{i_j}\ldots\breve{a}^{i_m}\ldots a^{i_r}T_{i_1\ldots i_j\ldots i_m\ldots i_r}\\
=& g^{i_j i_m}a^{i_1}\ldots\breve{a}^{i_j}\ldots\breve{a}^{i_m}\ldots a^{i_r}T_{i_1\ldots i_j\ldots i_m\ldots i_r}\\
=& \left(g^{i_j i_m}T_{i_1\ldots i_j\ldots i_m\ldots i_r}\right)a^{i_1}\ldots\breve{a}^{i_j}\ldots\breve{a}^{i_m}\ldots a^{i_r} && (7.17)
\end{aligned}$$

Equation (7.17) is the correct formula for the metric contraction of a tensor.

If we have a mixed representation of a tensor, $T_{i_1\ldots}{}^{i_j}{}_{\ldots i_k\ldots i_r}$, and wish to contract between an upper and lower index (i_j and i_k) first lower the upper index and then use eq (7.17) to contract the result. Remember lowering the index does *not* change the tensor, only the *representation* of the tensor, while contraction results in a *new* tensor. First lower index

$$T_{i_1\ldots}{}^{i_j}{}_{\ldots i_k\ldots i_r}\xRightarrow{\text{Lower Index}}g_{i_j k_j}T_{i_1\ldots}{}^{k_j}{}_{\ldots i_k\ldots i_r} \tag{7.18}$$

[4]The notation of the l.h.s. of eq (7.14) is new and is defined by $\nabla_{a_k}=\boldsymbol{e}^{l_k}\partial_{a^{l_k}}$ and (the assumption of the notation is that the $\partial_{a^{l_k}}$ can be factored out of the argument like a simple scalar)

$$\begin{aligned}
T\left(a_i,\ldots,a_{j-1},\nabla_{a_k},a_{j+1},\ldots,a_r\right)\equiv& T\left(a_i,\ldots,a_{j-1},\boldsymbol{e}^{l_k}\partial_{a^{l_k}},a_{j+1},\ldots,a^{i_k}\boldsymbol{e}_{i_k},\ldots,a_r\right)\\
=& T\left(a_i,\ldots,a_{j-1},\boldsymbol{e}_{j_k}g^{j_k l_k}\partial_{a^{l_k}},a_{j+1},\ldots,a^{i_k}\boldsymbol{e}_{i_k},\ldots,a_r\right)\\
=& g^{j_k l_k}\partial_{a^{l_k}}a^{i_k}T\left(a_i,\ldots,a_{j-1},\boldsymbol{e}_{j_k},a_{j+1},\ldots,\boldsymbol{e}_{i_k},\ldots,a_r\right)\\
=& g^{j_k l_k}\delta^{i_k}_{l_k}T\left(a_i,\ldots,a_{j-1},\boldsymbol{e}_{j_k},a_{j+1},\ldots,\boldsymbol{e}_{i_k},\ldots,a_r\right)\\
=& g^{j_k i_k}T\left(a_i,\ldots,a_{j-1},\boldsymbol{e}_{j_k},a_{j+1},\ldots,\boldsymbol{e}_{i_k},\ldots,a_r\right)\\
=& g^{j_k i_k}T_{i_1\ldots i_{j-1}j_k i_{j+1}\ldots i_k\ldots i_r}a^{i_1}\ldots\breve{a}^{i_j}\ldots\breve{a}^{i_k}\ldots a^{i_r}.
\end{aligned}$$

Now contract between i_j and i_k and use the properties of the metric tensor.

$$\begin{aligned} g_{i_j k_j} T_{i_1 \ldots}{}^{k_j}{}_{\ldots i_k \ldots i_r} \xRightarrow{\text{Contract}} & g^{i_j i_k} g_{i_j k_j} T_{i_1 \ldots}{}^{k_j}{}_{\ldots i_k \ldots i_r} \\ = & \delta^{i_k}_{k_j} T_{i_1 \ldots}{}^{k_j}{}_{\ldots i_k \ldots i_r} . \end{aligned} \tag{7.19}$$

Equation (7.19) is the standard formula for contraction between upper and lower indices of a mixed tensor.

7.4 Differentiation

If $T(a_1, \ldots, a_r; x)$ is a tensor field (a function of the position vector, x, for a vector space or the coordinate tuple, x, for a manifold) the tensor directional derivative is defined as

$$\mathcal{D}T(a_1, \ldots, a_r; x) \equiv (a_{r+1} \cdot \nabla) T(a_1, \ldots, a_r; x), \tag{7.20}$$

assuming the a^{i_j} coefficients are not a function of the coordinates.

This gives for a grade zero rank r tensor

$$\begin{aligned} (a_{r+1} \cdot \nabla) T(a_1, \ldots, a_r) = & a^{i_{r+1}} \partial_{x^{i_{r+1}}} a^{i_1} \ldots a^{i_r} T_{i_1 \ldots i_r}, \\ = & a^{i_1} \ldots a^{i_r} a^{i_{r+1}} \partial_{x^{i_{r+1}}} T_{i_1 \ldots i_r}. \end{aligned} \tag{7.21}$$

7.5 From Vector/Multivector to Tensor

A rank one tensor corresponds to a vector since it satisfies all the axioms for a vector space, but a vector in not necessarily a tensor since not all vectors are multilinear (actually in the case of vectors a linear function) functions. However, there is a simple isomorphism between vectors and rank one tensors defined by the mapping $v(a) : \mathcal{V} \to \Re$ such that if $v, a \in \mathcal{V}$

$$v(a) \equiv v \cdot a. \tag{7.22}$$

So that if $v = v^i \boldsymbol{e}_i = v_i \boldsymbol{e}^i$ the covariant and contravariant representations of v are (using $\boldsymbol{e}^i \cdot \boldsymbol{e}_j = \delta^i_j$)

$$v(a) = v_i a^i = v^i a_i. \tag{7.23}$$

The equivalent mapping from a pure r-grade multivector A to a rank-r tensor $A(a_1, \ldots, a_r)$ is

$$A(a_1, \ldots, a_r) = A \cdot (a_1 \wedge \ldots \wedge a_r). \tag{7.24}$$

Note that since the sum of two tensor of different ranks is not defined we cannot represent a spinor with tensors. Additionally, even if we allowed for the summation of tensors of different ranks we would also have to redefine the tensor product to have the properties of the geometric wedge product. Likewise, multivectors can only represent completely antisymmetric tensors of rank less than or equal to the dimension of the base vector space.

7.6 Parallel Transport Definition and Example

The defintion of parallel transport is that if a and b are tangent vectors in the tangent spaced of the manifold then

$$(a\cdot\nabla_x)\,b = 0 \tag{7.25}$$

if b is parallel transported in the direction of a (infinitesimal parallel transport). Since $b = b^i\boldsymbol{e}_i$ and the derivatives of $\boldsymbol{e}_i$ are functions of the x^i's then the b^i's are also functions of the x^i's so that in order for eq (7.25) to be satisfied we have

$$\begin{aligned}
(a\cdot\nabla_x)\,b =& a^i\partial_{x^i}\left(b^j\boldsymbol{e}_j\right)\\
=& a^i\left(\left(\partial_{x^i}b^j\right)\boldsymbol{e}_j + b^j\partial_{x^i}\boldsymbol{e}_j\right)\\
=& a^i\left(\left(\partial_{x^i}b^j\right)\boldsymbol{e}_j + b^j\Gamma^k_{ij}\boldsymbol{e}_k\right)\\
=& a^i\left(\left(\partial_{x^i}b^j\right)\boldsymbol{e}_j + b^k\Gamma^j_{ik}\boldsymbol{e}_j\right)\\
=& a^i\left(\left(\partial_{x^i}b^j\right) + b^k\Gamma^j_{ik}\right)\boldsymbol{e}_j = 0.
\end{aligned} \tag{7.26}$$

Thus for b to be parallel transported (infinitesimal parallel transport in any direction a) we must have

$$\partial_{x^i}b^j = -b^k\Gamma^j_{ik}. \tag{7.27}$$

The geometric meaning of parallel transport is that for an infinitesimal rotation and dilation of the basis vectors (cause by infinitesimal changes in the x^i's) the direction and magnitude of the vector b does not change to first order.

If we apply eq (7.27) along a parametric curve defined by $x^j(s)$ we have

$$\begin{aligned}
\frac{db^j}{ds} =& \frac{dx^i}{ds}\frac{\partial b^j}{\partial x^i}\\
=& -b^k\frac{dx^i}{ds}\Gamma^j_{ik},
\end{aligned} \tag{7.28}$$

and if we define the initial conditions $b^j(0)\,\boldsymbol{e}_j$. Then eq (7.28) is a system of first order linear differential equations with intial conditions and the solution, $b^j(s)\,\boldsymbol{e}_j$, is the parallel transport of the vector $b^j(0)\,\boldsymbol{e}_j$.

An equivalent formulation for the parallel transport equation is to let $\gamma(s)$ be a parametric curve in the manifold defined by the tuple $\gamma(s) = (x^1(s), \ldots, x^n(s))$. Then the tangent to $\gamma(s)$ is given by

$$\frac{d\gamma}{ds} \equiv \frac{dx^i}{ds}\boldsymbol{e}_i \tag{7.29}$$

and if $v(x)$ is a vector field on the manifold then

$$\begin{aligned}
\left(\frac{d\gamma}{ds}\cdot\nabla_x\right)v =& \frac{dx^i}{ds}\frac{\partial}{\partial x^i}\left(v^j\boldsymbol{e}_j\right)\\
=& \frac{dx^i}{ds}\left(\frac{\partial v^j}{\partial x^i}\boldsymbol{e}_j + v^j\frac{\partial \boldsymbol{e}_j}{\partial x^i}\right)\\
=& \frac{dx^i}{ds}\left(\frac{\partial v^j}{\partial x^i}\boldsymbol{e}_j + v^j\Gamma^k_{ij}\boldsymbol{e}_k\right)\\
=& \frac{dx^i}{ds}\frac{\partial v^j}{\partial x^i}\boldsymbol{e}_j + \frac{dx^i}{ds}v^k\Gamma^j_{ik}\boldsymbol{e}_j\\
=& \left(\frac{dv^j}{ds} + \frac{dx^i}{ds}v^k\Gamma^j_{ik}\right)\boldsymbol{e}_j\\
=& 0.
\end{aligned} \tag{7.30}$$

Thus eq (7.30) is equivalent to eq (7.28) and parallel transport of a vector field along a curve is equivalent to the directional derivative of the vector field in the direction of the tangent to the curve being zero.

As a specific example of parallel transport consider a spherical manifold with a series of concentric circular curves and paralle transport a vector along each curve. Note that the circular curves are defined by

$$\begin{aligned}
u(s) =& u_0 + a\cos\left(\frac{s}{2\pi a}\right)\\
v(s) =& v_0 + a\sin\left(\frac{s}{2\pi a}\right)
\end{aligned}$$

where u and v are the manifold coordinates. The spherical manifold is defined by

$$\begin{aligned}
x =& \cos(u)\,cos(v)\\
y =& \cos(u)\,sin(v)\\
z =& \sin(u).
\end{aligned}$$

Note that due to the dependence of the metric on the coordinates circles do not necessarily appear to be circular in the plots depending on the values of u_0 and v_0 (see fig 7.2). For symmetrical circles we have fig 7.1 and for asymmetrical circles we have fig 7.2. Note that the appearance of the transported (black) vectors is an optical illusion due to the projection. If the sphere were rotated we would see that the transported vectors are in the tangent space of the manifold.

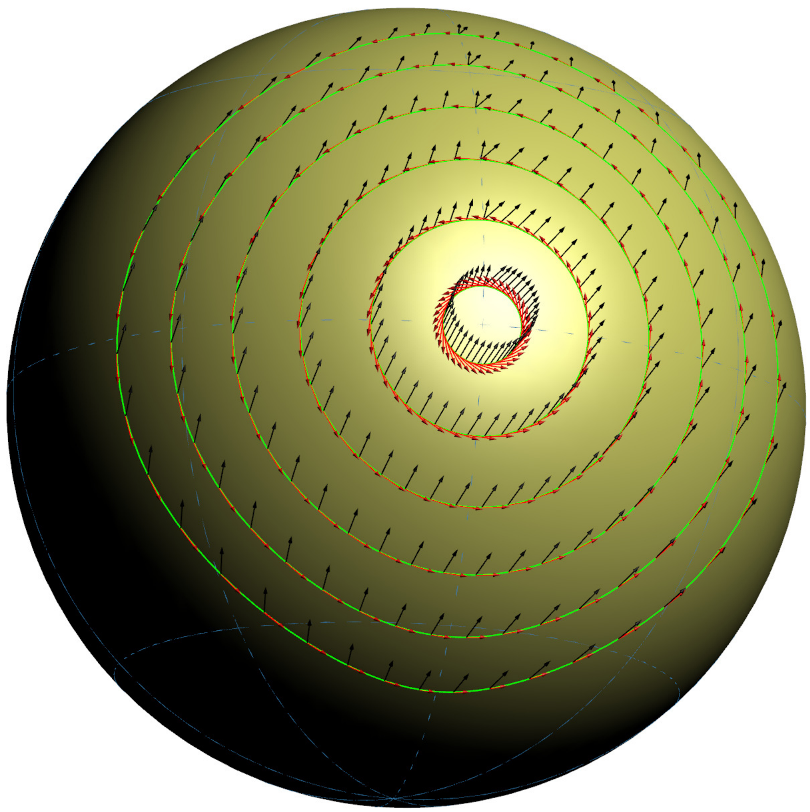

Figure 7.1: Parallel transport for $u_0 = 0$ and $v_0 = 0$. Red vectors are tangents to circular curves and black vectors are the vectors being transported.

If $\gamma(s) = (u(s), v(s))$ defines the transport curve then

$$\frac{d\gamma}{ds} = \frac{du}{ds}\boldsymbol{e}_u + \frac{dv}{ds}\boldsymbol{e}_v \tag{7.31}$$

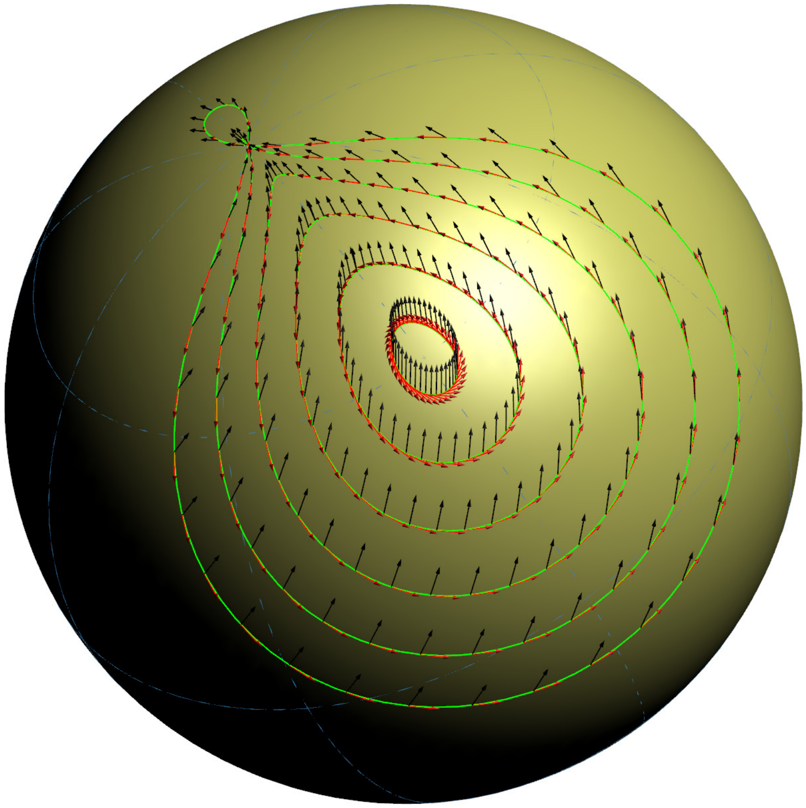

Figure 7.2: Parallel transport for $u_0 = \pi/4$ and $v_0 = \pi/4$. Red vectors are tangents to circular curves and black vectors are the vectors being transported.

and the transport equations are

$$\begin{aligned}
\left(\frac{d\gamma}{ds}\cdot\nabla\right) f &= \left(\frac{du}{ds}\frac{\partial f^u}{\partial u} + \frac{dv}{ds}\frac{\partial f^u}{\partial v} - \sin{(u)}\cos{(u)}\frac{dv}{ds}f^v\right)\boldsymbol{e}_u + \\
&\quad \left(\frac{du}{ds}\frac{\partial f^v}{\partial u} + \frac{dv}{ds}\frac{\partial f^v}{\partial v} + \frac{\cos{(u)}}{\sin{(u)}}\left(\frac{du}{ds}f^v + \frac{dv}{ds}f^u\right)\right)\boldsymbol{e}_u \\
&= \left(\frac{df^u}{ds} - \sin{(u)}\cos{(u)}\frac{dv}{ds}f^v\right)\boldsymbol{e}_u + \\
&\quad \left(\frac{df^v}{ds} + \frac{\cos{(u)}}{\sin{(u)}}\left(\frac{du}{ds}f^v + \frac{dv}{ds}f^u\right)\right)\boldsymbol{e}_u = 0 \qquad (7.32)
\end{aligned}$$

$$\frac{df^u}{ds} = \sin{(u)}\cos{(u)}\frac{dv}{ds}f^v \qquad (7.33)$$

$$\frac{df^v}{ds} = -\frac{\cos{(u)}}{\sin{(u)}}\left(\frac{du}{ds}f^v + \frac{dv}{ds}f^u\right) \qquad (7.34)$$

If the tensor component representation is contra-variant (superscripts instead of subscripts) we must use the covariant component representation of the vector arguements of the tensor, $a = a_i \boldsymbol{e}^i$. Then the definition of parallel transport gives

$$\begin{aligned}(a \cdot \nabla_x) b =& a^i \partial_{x^i} \left(b_j \boldsymbol{e}^j\right) \\ =& a^i \left(\left(\partial_{x^i} b_j\right) \boldsymbol{e}^j + b_j \partial_{x^i} \boldsymbol{e}^j\right), \end{aligned} \tag{7.35}$$

and we need

$$\left(\partial_{x^i} b_j\right) \boldsymbol{e}^j + b_j \partial_{x^i} \boldsymbol{e}^j = 0. \tag{7.36}$$

To satisfy equation (7.36) consider the following

$$\begin{aligned}\partial_{x^i} \left(\boldsymbol{e}^j \cdot \boldsymbol{e}_k\right) =& 0 \\ \left(\partial_{x^i} \boldsymbol{e}^j\right) \cdot \boldsymbol{e}_k + \boldsymbol{e}^j \cdot \left(\partial_{x^i} \boldsymbol{e}_k\right) =& 0 \\ \left(\partial_{x^i} \boldsymbol{e}^j\right) \cdot \boldsymbol{e}_k + \boldsymbol{e}^j \cdot \boldsymbol{e}_l \Gamma^l_{ik} =& 0 \\ \left(\partial_{x^i} \boldsymbol{e}^j\right) \cdot \boldsymbol{e}_k + \delta^j_l \Gamma^l_{ik} =& 0 \\ \left(\partial_{x^i} \boldsymbol{e}^j\right) \cdot \boldsymbol{e}_k + \Gamma^j_{ik} =& 0 \\ \left(\partial_{x^i} \boldsymbol{e}^j\right) \cdot \boldsymbol{e}_k =& -\Gamma^j_{ik} \end{aligned} \tag{7.37}$$

Now dot eq (7.36) into $\boldsymbol{e}_k$ giving[5]

$$\begin{aligned}\left(\partial_{x^i} b_j\right) \boldsymbol{e}^j \cdot \boldsymbol{e}_k + b_j \left(\partial_{x^i} \boldsymbol{e}^j\right) \cdot \boldsymbol{e}_k =& 0 \\ \left(\partial_{x^i} b_j\right) \delta^k_j - b_j \Gamma^j_{ik} =& 0 \\ \left(\partial_{x^i} b_k\right) = b_j \Gamma^j_{ik}. \end{aligned} \tag{7.39}$$

7.7 Covariant Derivative of Tensors

The covariant derivative of a tensor field $T(a_1, \ldots, a_r; x)$ (x is the coordinate tuple of which T can be a non-linear function) in the direction a_{r+1} is (remember $a_j = a^{k_j} \boldsymbol{e}_{k_j}$ and the $\boldsymbol{e}_{k_j}$ can be functions of x) the directional derivative of $T(a_1, \ldots, a_r; x)$ where all the a_i vector arguments of T are parallel transported.

[5] These equations also show that

$$\partial_{x^i} \boldsymbol{e}^j = -\Gamma^j_{ik} \boldsymbol{e}^k. \tag{7.38}$$

Thus if we have a mixed representation of a tensor

$$T\left(a_1,\dots,a_r;x\right)=T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}\left(x\right)a^{i_1}\dots a^{i_s}a_{i_{s+1}}\dots a_{i_r},\tag{7.40}$$

the covariant derivative of the tensor is

$$\begin{aligned}
\left(a_{r+1}\cdot D\right)T\left(a_1,\dots,a_r;x\right)=&\frac{\partial T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}}{\partial x^{r+1}}a^{i_1}\dots a^{i_s}a_{i_{s+1}}\dots a_{i_r}a^{i_{r+1}}\\
&+\sum_{p=1}^{s}\frac{\partial a^{i_p}}{\partial x^{i_{r+1}}}T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}a^{i_1}\dots\breve{a}^{i_p}\dots a^{i_s}a_{i_{s+1}}\dots a_{i_r}a^{i_{r+1}}\\
&+\sum_{q=s+1}^{r}\frac{\partial a_{i_p}}{\partial x^{i_{r+1}}}T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}a^{i_1}\dots a^{i_s}a_{i_{s+1}}\dots\breve{a}_{i_q}\dots a_{i_r}a^{i_{r+1}}\\
=&\frac{\partial T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}}{\partial x^{r+1}}a^{i_1}\dots a^{i_s}a_{i_{s+1}}\dots a^{r}_{i_r}a^{i_{r+1}}\\
&-\sum_{p=1}^{s}\Gamma^{i_p}_{i_{r+1}l_p}T_{i_1\dots i_p\dots i_s}{}^{i_{s+1}\dots i_r}a^{i_1}\dots a^{l_p}\dots a^{i_s}a_{i_{s+1}}\dots a_{i_r}a^{i_{r+1}}\\
&+\sum_{q=s+1}^{r}\Gamma^{l_q}_{i_{r+1}i_q}T_{i_1\dots i_s}{}^{i_{s+1}\dots i_q\dots i_r}a^{i_1}\dots a^{i_s}a_{i_{s+1}}\dots a_{l_q}\dots a_{i_r}a^{i_{r+1}}.
\end{aligned}\tag{7.41}$$

From eq (7.41) we obtain the components of the covariant derivative to be

$$\frac{\partial T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}}{\partial x^{r+1}}-\sum_{p=1}^{s}\Gamma^{i_p}_{i_{r+1}l_p}T_{i_1\dots i_p\dots i_s}{}^{i_{s+1}\dots i_r}+\sum_{q=s+1}^{r}\Gamma^{l_q}_{i_{r+1}i_q}T_{i_1\dots i_s}{}^{i_{s+1}\dots i_q\dots i_r}.\tag{7.42}$$

To extend the covariant derivative to tensors with multivector values in the tangent space (geometric algebra of the tangent space) we start with the coordinate free definition of the covariant derivative of a conventional tensor using the following notation. Let $T\left(a_1,\dots,a_r;x\right)$ be a conventional tensor then the directional covariant derivative is

$$\left(b\cdot D\right)T=a^{i_1}\dots a^{i_r}\left(b\cdot\nabla\right)T\left(e_{i_1},\dots,e_{i_r};x\right)-\sum_{j=1}^{r}T\left(a_1,\dots,\left(b\cdot\nabla\right)a_j,\dots,a_r;x\right).\tag{7.43}$$

The first term on the r.h.s. of eq (7.44) is the directional derivative of T if we assume that the component coefficients of each of the a_j does not change if the coordinate tuple changes. The remaining terms in eq (7.44) insure that for the totality of eq (7.44) the directional derivative $\left(b\cdot\nabla\right)T$ is the same as that when all the a_j vectors are parallel transported. If in eq (7.44) we let $b \cdot \nabla$ be the directional derivative for a multivector field we have generalized the defin-intion of covariant derivative to include the cases where $T(a_1, \ldots, a_r; x)$ is a multivector and not only a scalar. Basically in eq (7.44) the terms $T(e_{i_1}, \ldots, e_{i_r}; x)$ are multivector fields and $(b \cdot \nabla) T(e_{i_1}, \ldots, e_{i_r}; x)$ is the direction derivative of each of the multivector fields that make up the component representation of the multivector tensor. The remaining terms in eq (7.44) take into account that for parallel transport of the a_i's the coefficients a^{i_j} are implicit functions of the coordinates x^k. If we define the symbol ∇_x to only refer to taking the geometric derivative with respect to an explicit dependence on the x coordinate tuple we can recast eq (7.44) into

$$(b \cdot D) T = (b \cdot \nabla_x) T(a_1, \ldots, a_r; x) - \sum_{j=1}^{r} T(a_1, \ldots, (b \cdot \nabla) a_j, \ldots, a_r; x). \tag{7.44}$$

7.8 Coefficient Transformation Under Change of Variable

In the previous sections on tensors a transformation of coordinate tuples $\bar{x}(x) = (\bar{x}^i(x), \ldots, \bar{x}^n(x))$, where $x = (x^1, \ldots, x^n)$, is not mentioned since the definition of a tensor as a multilinear function is invariant to the representation of the vectors (coordinate system). From our tensor definitions the effect of a coordinate transformation on the tensor components is simply calculated.

If $R(x) = R(\bar{x})$ is the defining vector function for a vector manifold (R is in the embedding space of the manifold) then[6]

$$\boldsymbol{e}_i = \frac{\partial R}{\partial x^i} = \frac{\partial \bar{x}^j}{\partial x^i} \bar{\boldsymbol{e}}_j \tag{7.45}$$

$$\bar{\boldsymbol{e}}_i = \frac{\partial R}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i} \boldsymbol{e}_j. \tag{7.46}$$

[6]For an abstract manifold the equation $\bar{\boldsymbol{e}}_i = \dfrac{\partial x^j}{\partial \bar{x}^i} \boldsymbol{e}_j$ can be used as an defining relationship.

Thus we have

$$T\left(\boldsymbol{e}_{i_1},\ldots,\boldsymbol{e}_{i_1}\right) = T_{i_1\ldots i_r} \tag{7.47}$$

$$T\left(\bar{\boldsymbol{e}}_{j_1},\ldots,\bar{\boldsymbol{e}}_{j_1}\right) = \bar{T}_{j_1\ldots j_r} \tag{7.48}$$

$$T\left(\boldsymbol{e}_{i_1},\ldots,\boldsymbol{e}_{i_1}\right) = T\left(\frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}}\bar{\boldsymbol{e}}_{j_1},\ldots,\frac{\partial \bar{x}^{j_r}}{\partial x^{i_r}}\bar{\boldsymbol{e}}_{j_1}\right) \tag{7.49}$$

$$= \frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}}\cdots\frac{\partial \bar{x}^{j_r}}{\partial x^{i_r}} T\left(\bar{\boldsymbol{e}}_{j_1},\ldots,\bar{\boldsymbol{e}}_{j_1}\right) \tag{7.50}$$

$$T_{i_1\ldots i_r} = \frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}}\cdots\frac{\partial \bar{x}^{j_r}}{\partial x^{i_r}} \bar{T}_{j_1\ldots jr}. \tag{7.51}$$

Equation (7.51) is the standard formula for the transformation of tensor components.