# Manipulations of the Metric

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Let us consider three different ways of realizing a manifold which I call "implicit", "derived", and "explicit".

The "implicit" manifold is defined by a set of coordinates $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_n\}$, a set of basis vectors $\{\boldsymbol{e}_{\theta_1}, \ldots, \boldsymbol{e}_{\theta_n}\}$, and a metric tensor $g_{ij}(\boldsymbol{\theta})$ where the metric tensor is related to the basis vectors by

$$g_{ij} = \boldsymbol{e}_{\theta_i} \cdot \boldsymbol{e}_{\theta_j}, \tag{1}$$

and the $\boldsymbol{e}_{\theta_i}$ are implicitly a function of the coordinates. That is to say for an "implicit" manifold the metric tensor defines the dot products of the basis vectors and not vice versa. This way of defining a manifold is usually encountered in general relativity.

The "derived" manifold is given by a single vector field, $\boldsymbol{X}(\boldsymbol{\theta})$, that is a function of the coordinates that defined in some embedding vector space. In this case the basis vectors, $\{\boldsymbol{e}_{\theta_1}, \ldots, \boldsymbol{e}_{\theta_n}\}$ and the metric tensor, $g_{ij}(\boldsymbol{\theta})$, are derived from $\boldsymbol{X}(\boldsymbol{\theta})$ via

$$\boldsymbol{e}_{\theta_i} = \frac{\partial \boldsymbol{X}}{\partial \theta_i} \tag{2}$$

$$g_{ij} = \boldsymbol{e}_{\theta_i} \cdot \boldsymbol{e}_{\theta_j}, \tag{3}$$

where the dot product is that of the embedding vector space. In this case the basis vectors and the metric tensor are derived from $\boldsymbol{X}(\boldsymbol{\theta})$. An example of this would be a two dimensional manifold embedded in a three dimensional space such as a sphere defined by

$$\boldsymbol{X}(\theta, \phi) = \cos(\theta)\,\boldsymbol{e}_z + \sin(\theta)\left(\cos(\phi)\,\boldsymbol{e}_x + \sin(\phi)\,\boldsymbol{e}_y\right).$$

The "explicit" manifold is one in which the basis vectors are given vector fields, $\boldsymbol{e}_{\theta_i}(\boldsymbol{\theta})$, in an embedding vector space and the metric tensor is calculated as in the case of the "derived" manifold. An example of this would be the spherical coordinate basis vectors

$$\begin{aligned}
\boldsymbol{e}_r &= \cos(\theta)\,\boldsymbol{e}_z + \sin(\theta)\left(\cos(\phi)\,\boldsymbol{e}_x + \sin(\phi)\,\boldsymbol{e}_y\right), \\
\boldsymbol{e}_\theta &= \sin(\theta)\,\boldsymbol{e}_z - \cos(\theta)\left(\cos(\phi)\,\boldsymbol{e}_x + \sin(\phi)\,\boldsymbol{e}_y\right), \\
\boldsymbol{e}_\phi &= -\sin(\phi)\,\boldsymbol{e}_x + \cos(\phi)\,\boldsymbol{e}_y.
\end{aligned}$$

In this case the basis vectors can be normalized vector fields and the metric tensor is calculated as in the "derived" case.

In both the "derived" and "explicit" cases the derivatives of the basis vectors can be calculated by direct differentiation of the $\boldsymbol{e}_{\theta_i}(\boldsymbol{\theta})$'s. However, in the case of the "implicit" manifold we must proceed as follows.

The derivatives of the basis vectors are calculated from the Christ-Awful symbols, $\Gamma^k_{ij}(\boldsymbol{\theta})$, by (Einstein summation convention used)

$$\frac{\partial \boldsymbol{e}_{\theta_i}}{\partial \theta_j} = \Gamma^k_{ij}(\boldsymbol{\theta})\, \boldsymbol{e}_{\theta_k}. \tag{4}$$

Where the $\Gamma^k_{ij}(\boldsymbol{\theta})$ are derived from the $g_{ij}(\boldsymbol{\theta})$.

Now let us derive all the relevant manifold quatities in terms of normalized basis vectors.

We define a norm for the basis vectors by

$$|\boldsymbol{e}_{\theta_i}| \equiv \sqrt{|g_{ii}|} \tag{5}$$

and

$$\boldsymbol{e}_{\theta_i} = |\boldsymbol{e}_{\theta_i}|\, \hat{\boldsymbol{e}}_{\theta_i} \tag{6}$$

so that $\hat{\boldsymbol{e}}_{\theta_i} \cdot \hat{\boldsymbol{e}}_{\theta_i} = \pm 1$.

Now consider what happens if we use a normalized basis $\{\hat{\boldsymbol{e}}_{\theta_1}, \ldots, \hat{\boldsymbol{e}}_{\theta_n}\}$.

$$\begin{aligned}
\frac{\partial \boldsymbol{e}_{\theta_i}}{\partial \theta_j} &= \frac{\partial}{\partial \theta_j}\left(|\boldsymbol{e}_{\theta_i}|\, \hat{\boldsymbol{e}}_{\theta_i}\right) \\
&= \frac{\partial |\boldsymbol{e}_{\theta_i}|}{\partial \theta_j} \hat{\boldsymbol{e}}_{\theta_i} + |\boldsymbol{e}_{\theta_i}| \frac{\partial \hat{\boldsymbol{e}}_{\theta_i}}{\partial \theta_j} \\
\frac{\partial \hat{\boldsymbol{e}}_{\theta_i}}{\partial \theta_j} &= \frac{1}{|\boldsymbol{e}_{\theta_i}|}\left(\frac{\partial \boldsymbol{e}_{\theta_i}}{\partial \theta_j} - \frac{\partial |\boldsymbol{e}_{\theta_i}|}{\partial \theta_j} \hat{\boldsymbol{e}}_{\theta_i}\right),
\end{aligned} \tag{7}$$

but

$$\begin{aligned}
\frac{\partial |\boldsymbol{e}_{\theta_i}|^2}{\partial \theta_j} &= 2\, |\boldsymbol{e}_{\theta_i}| \frac{\partial |\boldsymbol{e}_{\theta_i}|}{\partial \theta_j} \\
\frac{\partial |\boldsymbol{e}_{\theta_i}|}{\partial \theta_j} &= \frac{1}{2\, |\boldsymbol{e}_{\theta_i}|} \frac{\partial |\boldsymbol{e}_{\theta_i}|^2}{\partial \theta_j} \\
&= \frac{|\boldsymbol{e}_{\theta_i}|}{2\, |g_{ii}|} \frac{\partial |g_{ii}|}{\partial \theta_j}.
\end{aligned} \tag{8}$$

So that

$$\begin{aligned}
\frac{\partial \hat{\boldsymbol{e}}_{\theta_i}}{\partial \theta_j} &= \frac{1}{|\boldsymbol{e}_{\theta_i}|} \frac{\partial \boldsymbol{e}_{\theta_i}}{\partial \theta_j} - \frac{1}{2\, |g_{ii}|} \frac{\partial |g_{ii}|}{\partial \theta_j} \hat{\boldsymbol{e}}_{\theta_i} \\
&= \frac{|\boldsymbol{e}_{\theta_k}|}{|\boldsymbol{e}_{\theta_i}|} \Gamma^k_{ij} \hat{\boldsymbol{e}}_{\theta_k} - \frac{1}{2\, |g_{ii}|} \frac{\partial |g_{ii}|}{\partial \theta_j} \hat{\boldsymbol{e}}_{\theta_i}.
\end{aligned} \tag{9}$$

Note that the derivatives of the normalized basis vectors are not even in the same direction as the derivatives of the original basis vectors.

My thesis is if the $\boldsymbol{e}_{\theta_i}$'s are normalized you cannot calculate the $\dfrac{\partial \boldsymbol{e}_{\theta_i}}{\partial \theta_j}$'s unless you explicitly know the $\boldsymbol{e}_{\theta_i}(\boldsymbol{\theta})$ as vector fields in an embedding vector space. This can be done for "derived" and "explicit" manifolds, but not for "implicit" manifolds. For "implicit" manifolds you can derive a set of normalized basis vectors and their derivatives in terms of the normalized basis vectors from a metric for unnormalized basis vectors as shown above.

An exception to my thesis could be a case of non-orthogonal basis vectors for then the off diagonal elements of the metric tensor could encode the necessary information or again maybe not?