Differential Geometry versus Geometric Algebra.

In differential geometry I refer to " Lectures on Differential Geometry " by Wulf Rossmann, which you can find on Internet.

Preliminary observation.

One should resist to the temptation to utilize notations or algebraic formulations specific to one language when working in the other system. For example $e_i\,dx^i = dx$ has a clear signification in GA but none in DG, because in the latter you juxtapose a vector e_i , which indeed should be written $\partial_i = \partial/\partial x^i$, with a covector dx^i . Such a suggested algebraic operation is not defined in DG. That looks even worse with $d^2X = \partial_m \wedge \partial_n\, dx^m dx^n$ .... ! Remember, all these beasts are vectors, and in fact differential operators.

Definition of differential one- and two-forms.

In DG the $\partial/\partial x^i$ are the natural basis vectors in the tangent plane TM_p to the manifold M at the point p . Then $x^i(p)$ is the ith coordinate function $M \longrightarrow R$ and dx^i is a basis vector in the dual space of the tangent space and thus a one-form. We have by definition :

(1) $\quad dx^i(\partial_j) = \delta^i_j \qquad$ and $\qquad dx^i(v) = dx^i(v^j\partial_j) = v^j\delta^i_j = v^i$

A general one-form is expressed as :

(2) $\quad \alpha(v) = \alpha_i\, dx^i(v) = \alpha_i\, v^i \qquad$ where v is a tangent vector at p

Usually one writes :

(3) $\quad \alpha = \alpha_i\, dx^i$

omitting to mention the vector with which contraction is done to execute the function $M \rightarrow R$.

(Question : Should we consider that (3) is onely a useful abbreviation of (2) ?)

What about two-forms ? A simple two-form is written :

(4) $\quad \alpha(u,v) = dx^r \wedge dx^s(u,v) = dx^r(u)dx^s\,(v) - dx^r(v)dx^s\,(u) = u^r v^s - u^s v^r$

A general two-form is defined by :

(5) $\quad \alpha(u,v) = 1/2\,\alpha_{rs}\, dx^r \wedge dx^s(u,v) = 1/2\,\alpha_{rs}(u^r v^s - u^s v^r)$

with $\alpha_{sr} = -\alpha_{rs}$, and usually written :

(6) $\quad \alpha = 1/2\,\alpha_{rs}\, dx^r \wedge dx^s$

Relationship with GA.

The relation (3) is obviously expressed in GA by :

(7) $\quad \alpha(v) = a.v \qquad$ where $\quad a = a_i\, e^i \qquad v = v^i\, e_i \qquad e^i\, e_j = \delta^i_j$

The equivalence with relation (5) is given by :

(8) $\quad \alpha(u,v) = A.\tilde{X} \quad$ with $\quad A = 1/2\,\alpha_{\mathrm{rs}}\, e^r \wedge e^s \qquad X = u \wedge v = u^i\, v^j\, e_i \wedge e_j$

(9) $\quad \alpha(u,v) = 1/2\,\alpha_{rs}\,(e^r \wedge e^s).(v \wedge u) = 1/2\,\alpha_{rs}(u^r v^s - u^s v^r)$

Of course the e^i and e_j vectors correspond respectively to dx^i and ∂/∂^j but here they are true vectors and not abstract differential operators.