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投资组合优化

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摘　要

本文主要介绍并讨论了现今常用的几种资产组合分配方式，具体包括等权重组合、反向波动率组合、风险贡献平价组合、最小方差组合、最大分化度组合以及最小条件风险在值组合。根据他们性质上的差异与关联，我将它们分成了两种主要类型，一种是分散投资导向的投资类型，另一种是优化风险导向的投资类型。此外，本文还提出通过添加正规化项，直接设置权重上限以及其他参数调整的方式来使得一些方法兼具分散投资和降低风险这两种特点，从而达到进一步优化投资组合的目的。

根据本文提出的方法，我将它们运用在申万 28 个一级行业指数的投资组合问题中，不同投资组合方式的权重分布正对应了他们的类型，并且我们引入了多个指标来衡量投资组合的表现，总体来说，最小方差组合在我们的问题中表现较好；不过面对不同的可选投资集，相同投资组合表现的变化很大。

关键词: 等权重投资组合; 反向波动率组合；风险贡献平价组合；最小方差组合；最大分化度组合；最小条件风险在值组合；正规化限制；

ABSTRACT

This paper mainly introduces and discusses several commonly used assets allocation strategies, including Equally-weighted Portfolio(EW), Inverse Volatility Portfolio(InVol), Risk Contribution Parity Portfolio(RCP), Minimum Variance Portfolio(MV), Maximum Diversification Ratio Portfolio(MD) and Minimum Conditional Value-at-Risk Portfolio(Min CVaR). Based on their differences and similarities, I divide them into two types, the first type is diversify-oriented portfolio, the second is risk-oriented portfolio. Moreover, this paper puts forward that by adding regularization term, setting weight upper bound or adjusting other optional parameters, we can diversify the investment and lower the risk at the same time, in this way we can step further in portfolio optimization.

I apply these strategies into the investable set consisting of 28 Shenwan first-level industry indices, the weight distribution of different strategies corresponds to the type it belongs to. In addition, I include several indicators to measure the performance of these portfolios, in general, MV performs relatively better in our problem. However, faced with different investable sets, the performance of the same strategy differs greatly.

Key words: Equally-weighted Portfolio; Inverse Volatility Portfolio; Risk Contribution Parity Portfolio; Minimum Variance Portfolio; Maximum Diversification Ratio Portfolio; Minimum Conditional Value-at-Risk Portfolio; Regularization

Contents

1 Introduction

Portfolio optimization, the process of efficiently allocating wealth different among asset classes and securities, has a longstanding history both in academy and in industry. Over 60 years ago, Markowitz [1, 2] formalized the problem in a mean-variance framework where one assumes that the rational investors hope to maximize the expected return and to keep the volatility lower than a prespecified level. While brilliant, it suffers several serious drawbacks in implementation. Firstly, though we hope to lower the risk by enlarging our investment sets, the optimal portfolio, on the contrary, is excessively concentrated in a limited subset. Secondly, the mean-variance solution is overly sensitive to the input parameters—expected return and covariance matrix—which need robust and accurate estimates, small changes in these parameters, especially in expected returns can lead to significant variations in the composition of the result, which will obviously, raise the cost in turnover.

Therefore, portfolio construction techniques based on risk, without expected returns, have become increasingly popular recently. Two well-known examples are the Minimum Variance(MV) and the Equally-Weighted(EW). The first one is a special portfolio on the mean-variance efficient frontier which minimize the ex ante variance with no regard to the expected returns. This portfolio, though is easy to compute and the solution is unique, suffers the drawback of concentration too. The second technique is a natural and naive way to resolve this issue by allocating the same weight to all assets in the investable set. Equally-weighted portfolios are widely used in practice and they have been shown to be efficient [3, 4], the drawback is that it doesn't take any information about the investable set into consideration, which can only partly diversify the risk if individual risk is significantly different among classes.

Another technique which slightly adds more information in EW is Inverse Volatility(InVol), as its name tells, it includes the information of volatility of each asset and simply put less weight on volatile ones. To some extent, it can minimize risk by putting less weight on volatile assets, but the information of correlation between assets is missed, so in general, it isn't a convincing technique theoretically.

There are still other heuristic approaches. [5] introduces Maximum Diversification(MD) portfolio to maximize the diversification ratio defined in his paper. the objective function is motivated by maximizing the portfolio Sharpe Ratio, that's to say, MV is the tangent portfolio in the efficient frontier is expected asset returns are assumed to be proportional to asset risk.

Another technique is to equalize the risk contribution in each asset in the investable set, which is called Risk Contribution Parity(RCP). The concept of Risk Parity has evolved over time, initially, risk parity ignored correlations even when it was applied to merely two assets. [6] formalized a complete definition which considers correlation where weights are adjusted so that each asset has the same contribution to the portfolio risk, and [7, 8] offered more properties of RCP portfolio.

In addition to using standard deviation to measure risk, Value-at-risk, or VaR is also a popular measure of risk which has been widely in industry. Although it suffers from being unstable and difficult to work with numerically when losses are not ”normally” distributed—which in fact is often the case, an alternative measure that quantify the losses in the tail is conditional value-at-risk, or CVaR. It maintains consistency with VaR , for portfolio assumed with ”elliptical” distribution, working with CVaR or VaR are equivalent [9]. What's more, although VaR is difficult to optimize when it's calculated from scenarios, CVaR is a coherent risk measure which is easier to optimize. A complete description of the approach can be found in [10], and we call the corresponding portfolio Minimum CVaR(Min CVaR) portfolio.

This paper devotes to analyzing properties of portfolio optimization strategies talked above in a intuitive way, then we apply them to construct portfolio in Shenwan First-Level Industry Indice so that we can check whether they preserve these properties. Based on their weight distribution, we divide these techniques into two types. What's more, we add a regularization term to some objective functions and set constraints to the weight to drive the strategy to lower the risk, and at the same time, to diversify the investment, which is the contribution of this paper.

2 Minimum Variance Portfolio

MV portfolio has been increasingly popular to cater to the increasing needs in risk management due to the financial crisis. Moreover, a lots of empirical evidence show that high-market-beta stocks aren't rewarded with correspondingly high return[11], therefore, people tend to actively find the portfolio with lower volatility .

Originally, the optimization problem is to minimize portfolio variance:

$$\begin{aligned} &\text{Min: } \sigma_p^2 = \boldsymbol{\omega'\Sigma\omega} \\ &s.t. \quad \boldsymbol{\omega' l} = 1 \end{aligned} \tag{2.1}$$

where $\boldsymbol{\Sigma}$ is the N-by-N covariance matrix of assets in the investable set, $\boldsymbol{\omega}$ is the N-by-1 vector including the weight of each asset, $\boldsymbol{l}$ is the N-by-1 vector of ones. The solution to this optimization problem is:

$$\boldsymbol{\omega}_{MV} = \frac{\boldsymbol{\Sigma^{-1} l}}{\boldsymbol{l'\Sigma^{-1} l}} \tag{2.2}$$

where $\boldsymbol{\Sigma}^{-1}$ is the inverse matrix[1].

Using CAPM model [7, 12], the returns on the ith security are assumed to follow $r_i = \alpha_i + \beta_i r_M + \varepsilon_i$, where ε_i is a zero-mean, $\sigma^2_{\varepsilon,i}$-variance random variable that is uncorrelated with the market return and other factors. Then we can get the N-by-N covariance matrix associated with the CAPM model is:

$$\boldsymbol{\Sigma} = \boldsymbol{\beta\beta'}\sigma_M^2 + \text{Diag}(\boldsymbol{\sigma}_\varepsilon^2) \tag{2.3}$$

where $\boldsymbol{\beta}$ is the N-by-1 vector of β_i, σ_M is the risk factor variance(the market variance in CAPM) and $Diag(\sigma_\varepsilon^2)$ is a diagonal matrix with $\sigma_{\varepsilon,i}$ in the diagonal line and 0 in other areas. Combining 2.2, 2.3 and algebraic computation, we can get a simple expression for the individual weights:

$$\omega_i = \frac{\sigma_{MV}^2}{\sigma_{\varepsilon,i}^2}(1 - \frac{\beta_i}{\beta_{LS}}) \tag{2.4}$$

where β_i is the ex ante market beta for asset i, β_{LS}[2] is the long short threshold beta.

β_{LS} is called long short beta because the value determines positive and negative weights in an unconstrained portfolio optimization. From 2.4, securities with $\beta_i < \beta_{LS}$ have positive weights and that securities with $\beta_i > \beta_{LS}$ have negative weights.

[1] $\boldsymbol{\Sigma}^{-1} = \text{Diag}(1/\boldsymbol{\sigma}^2) - \frac{(\boldsymbol{\beta}/\boldsymbol{\sigma}^2)(\boldsymbol{\beta}/\boldsymbol{\sigma}^2)'}{\frac{1}{\sigma_M^2} + (\boldsymbol{\beta}/\boldsymbol{\sigma}^2)'\boldsymbol{\beta}}$

[2] The implicit solution for β_{LS} is $\beta_{LS} = \frac{1/\sigma_M^2 + \sum \beta_i^2/\sigma_{\varepsilon,i}^2}{\sum \beta_i^2/\sigma_{\varepsilon,i}^2}$.

The long only case is similar:

$$\omega_i = \begin{cases} \frac{\sigma_{LMV}}{\sigma_{\varepsilon,i}}(1 - \frac{\beta_i}{\beta_L}), & \text{if } \beta_i < \beta_L \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Where σ_{LMV} is ex ante variance of the long only minimum-variance portfolio,β_L[3] is the long-only threshold beta.

As we can see in 2.5, higher idiosyncratic volatility $\sigma_{\epsilon,i}$ and higher market beta(still less than β_L) would drive the security weight to zero. And the long-only threshold beta usually falls within the lowest beta quintile of the investable set, so it's common to see that more than 80% securities in the MV portfolio have zero weights which we will see in our application as well.

Therefore, the optimal security weight derived in this minimum-variance framework is sufficient to abandon a large proportion of disqualified securities in the investable set. What's more, the mathematics indicates that many parameter estimates in the covariance matrix are unused, at least, not important to calculate as accurately as others. For example, if there are only 20% securities end up with positive weights in the long only MV portfolio, then there are only 4%(20% squared) values in covariance matrix directly affect the optimal weights. Thus, a more accurate and robust covariance estimation on a subset of investable set may lead to a better result.

[3] $\beta_L = \frac{1/\sigma_M^2 + \sum_{\beta_i < \beta_L} \frac{\beta_i^2}{\sigma_{\varepsilon,i}^2}}{\sum_{\beta_i < \beta_L} \frac{\beta_i}{\sigma_{\varepsilon,i}^2}}$

3 Equally Weighted and Inverse Volatility Weighted Portfolio

3.1 The "$1/n$" Portfolio

Due to the fact that MV portfolio adopt a strategy which is not as diversified as previously thought and that MV needs much computation on matrix of large dimensions, there is evidence[13] that many participants use simple heuristic diversification rules in allocating their contributions among available assets. One popular rule, which indeed need no calculation at all, is the "$1/n$" rule. Under this rule, investors equally divides his wealth to all assets available. This strategy has raised many doubts since it is not an optimal portfolio, at least, there is no evidence to show that it lies on the efficient frontier or other efficient set. However, as we will see below, the $1/n$ portfolio is consistent with the Markowitz efficient portfolios given more assumptiion, and even with all of the available historical information available to investment professionals, the performance of the $1/n$ strategy is not unreasonable[3, 4].

We can see that if all assets have the same correlation coefficients as well as means and variances, or to be more specific, all assets are homogeneous, then the mean-variance framework becomes:

$$\begin{aligned} \max \quad & 2\boldsymbol{\omega}'\boldsymbol{R_0} - \boldsymbol{\omega}'\boldsymbol{\Sigma_0}\boldsymbol{\omega} \\ s.t. \quad & \boldsymbol{\omega}'\boldsymbol{l} = 1 \end{aligned} \tag{3.1}$$

where $\boldsymbol{R_0} = \overline{R}\boldsymbol{l}$ and $\boldsymbol{\Sigma_0} = \overline{\sigma}^2\boldsymbol{I}$ with $\boldsymbol{I}$ denotes the N-by-N identity matrix. Then the optimal weights are given as:

$$\boldsymbol{\omega} = \boldsymbol{\omega_{mv}} + \tau \triangle \boldsymbol{\omega}_{risk} \tag{3.2}$$

where $\boldsymbol{\omega_{mv}}$ represents the minimum-variance portfolio in which case equals to $1/n\boldsymbol{l}$, and $\triangle\boldsymbol{\omega}_{risk}$ represents the adjustment that optimally trades off risk versus reward in which case equals to $\mathbf{0}$.

It's reasonable to think that, under the circumstance where all assets available are homogeneous, investors have no reason to put more weight on a specific asset as they can't even distinguish them in the sense of return and volatility. However, this ideal assumption about the market is not accurate in many empirical results like [14] where for a sample of 500 U.S. stocks, the correlation range from

-0.37 to 0.92 with the average correlation is 0.28. In most cases, the EW portfolio lies below the so-called efficient frontier and hence is suboptimal theoretically[1].

3.2 Inverse Volatility Portfolio

As talked above, we want to minimize the variance of the portfolio and to diversify our investments at the same time, while MV focuses on a minor subset of investable assets and EW is suboptimal in most cases. Therefore we hope to take more conservative strategy to diversify our holdings, here comes the Inverse Volatility portfolio. Mathematically, it can be expressed as follows:

$$
\begin{aligned}
&\omega_i = \frac{1/\sigma_i}{\sum_{j=1}^{n} 1/\sigma_j} \quad \text{for } j = 1, 2, \ldots, n \\
\text{s.t. } &\sum_{i=1}^{n} \omega_i = 1 \\
&\omega_i \geq 0 \quad \text{for } i = 1, 2, \ldots, n
\end{aligned}
\tag{3.3}
$$

Similar to EW, since σ_i is strictly positive, the weight of each assets in investable set is positive through which we can diversify the risk. In addition, it takes individual risk into account, thus the asset with higher risk exposure, that is σ_i is large, is down weighted to lower the risk of the portfolio, so it is also similar to MV to some extent.

Like EW, the main drawback of InVol is that it doesn't include the information of the correlation between assets. An asset may be unnecessarily penalized simply because it's relatively more volatile, while it may be an ideal instrument for investors to hedge systematic risk should correlation be considered.

[1]The reason I use "theoretically" here is that armed with simple and practical implementation and efficient performance [3], it's still popular in industry even with no information about the dependency.

4 Maximum Diversification Portfolio

Diversification ratio was first introduced by [5] that maximize the ratio of weighted average assets volatility to portfolio volatility. Compared with MV portfolio which equalizes the marginal contribution of each asset to portfolio risk, it equalizes each asset's marginal contribution, given a small change in the asset's weight.

Mathematically, diversification ratio can be defined as below:

$$\text{Max: } D(\boldsymbol{\omega}) = \frac{\boldsymbol{\omega'\sigma}}{\sqrt{\boldsymbol{\omega'\Sigma\omega}}} \tag{4.1}$$

where $\boldsymbol{\sigma}$ is the N-by-1 vector containing the volatility of each asset, and $D(\boldsymbol{\omega})$ is called diversification ratio.

If the assumption that the expected excess returns of assets are proportional to their risk holds, then $E[R(\boldsymbol{\omega})] = k\boldsymbol{\omega'\sigma}$, so maximizing $D(P)$ is equivalent to maximizing $\frac{E[R(\boldsymbol{\omega})]}{\sqrt{\boldsymbol{\omega'\Sigma\omega}}}$, which is the Sharpe ratio of the portfolio. Therefore, if the correlation between excess return and volatility is linear, MD portfolio is also the tangent portfolio on the efficient frontier.

Moreover, we can transpose the problem to a synthetic universe to gain a better understanding about $D(\boldsymbol{\omega})$. Suppose we synthesize assets $Y_1, Y_2, \dots, Y_n$ by

$$Y_i = \frac{X_i}{\sigma_i} + (1 - \frac{1}{\sigma_i})r_f$$

where X_i is the asset in the real world, r_f is the risk-free rate. Therefore, the volatility $\sigma_{S_i}$of Y_i is equal to 1 according to the definition.

Naturally, if we want to maximize $D(\boldsymbol{\omega})$ in the synthetic world, the numerator $\boldsymbol{\omega'\sigma}_s = \sum_{i=1}^{n} \omega_i = 1$, then what we need to do is to minimize the denominator in 4.1. Furthermore, $\boldsymbol{\Sigma}_s = \boldsymbol{\sigma}_s'\boldsymbol{\rho\sigma}_s = \boldsymbol{\rho}$, so at last, maximizing the diversification rario is equivalent to minimizing $\boldsymbol{\omega'\rho\omega}$:

$$\text{Max: } \frac{\boldsymbol{\omega'\sigma}}{\sqrt{\boldsymbol{\omega'\Sigma\omega}}} \Longleftrightarrow \text{Min: } \boldsymbol{\omega'\rho\omega} \tag{4.2}$$

where $\boldsymbol{\rho}$ is the correlation matrix of assets in the real world.

Solving the optimization problem, we get:

$$\boldsymbol{\omega}_{MD} = (\frac{\sigma_{MD}^2}{\sigma_A})\boldsymbol{\Sigma}^{-1}\boldsymbol{\sigma} \tag{4.3}$$

where $\sigma_A = \sum_{i=1}^{n} \omega_i\sigma_i$.

Replace the inverse covariance matrix, similar to 2.5, we can get the weight expression of MD portfolio:

$$\omega_{MD,i} = \begin{cases} \frac{\sigma_{MD}^2}{\sigma_{\varepsilon,i}^2}\frac{\sigma_i}{\sigma_A}(1-\frac{\rho_i}{\rho_L}), & \text{if } \rho_i < \rho_L \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

where ρ_L is the long only threshold correlation:

$$\rho_L = \frac{\sigma_{MD}^2}{\sigma_A \beta_{MD} \sigma_M} = \frac{1+\sum_{\rho_i<\rho_L}\frac{\rho_i^2}{1-\rho_i^2}}{\sum_{\rho_i<\rho_L}\frac{\rho_i^2}{1-\rho_i^2}} \tag{4.5}$$

According to 4.4, assets are only included in the long-only MD portfolio if their correlation to the market is lower than the threshold correlation. High idiocratic risk in the denominator of the first term lowers the asset weight but can't drive itself out of the MD portfolio. And the asset's risk σ_i tend to offset the negative effect brought by $\sigma_{\varepsilon,i}$, so MV portfolio weights are approximately proportional to the inverse of standard deviation of the asset return, as opposed to the inverse of variance of assets in MV portfolio. The outcome is that weights in the MD portfolio tend to be less concentrated than MV portfolio since standard deviation is less likely to have extreme value than variance. However, even though it slightly relieve the concentration effect, MD still suffers the same drawback of MV

5 Risk Contribution Parity Portfolio

As its name tells, RCP strategy seeks to equalized risk contribution from each component of the portfolio. Suppose a portfolio $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$ of n risky assets, then the marginal volatility contribution of asset i is defined as:

$$c_i(\omega) = \frac{\partial \sigma(\omega)}{\partial \omega_i} = \frac{\partial \sqrt{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}}{\partial \omega_i} = \frac{(\Sigma\boldsymbol{\omega})_i}{\sqrt{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}} \tag{5.1}$$

where $(\Sigma\boldsymbol{\omega})_i$ is the ith row of the N-by-1 vector, so the marginal volatility contribution of asset i gives the change in volatility of the portfolio caused by a small change in the weight of one component. If we denote $\sigma_i(\boldsymbol{\omega}) = \omega_i \times c_i(\boldsymbol{\omega})$ the total risk contribution of the ith asset, then we can decompose the risk of the portfolio as:

$$\sigma(\boldsymbol{\omega}) = \sum_{i=1}^{n} \omega_i c_i(\boldsymbol{\omega}) = \boldsymbol{\omega}' \frac{\Sigma\boldsymbol{\omega}}{\sqrt{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}} = \sqrt{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}$$

Intuitively, we can find the corresponding weights by implementing the following optimization problem:

$$\begin{aligned} \text{Min: } & \sum_{i=1}^{n} (\sigma(\boldsymbol{\omega})/n - \omega_i c_i(\boldsymbol{\omega}))^2 \\ s.t. & \sum_{i=1}^{n} \omega_i = 1 \end{aligned} \tag{5.2}$$

Let us begin to analyze the property of RCP with a special case. Suppose the volatility of all assets are equal denoted as σ, but the correlation between them is different, through algebraic computation, we get implicit expression of weight:

$$\omega_i = \frac{(\sum_{k=1}^{n} \omega_k \rho_{ik})^{-1}}{\sum_{j=1}^{n} (\sum_{k=1}^{n} \omega_k \rho_{jk})^{-1}} \tag{5.3}$$

As we can see, the expression is very similar 3.3. And to be more exact, if we let the correlation of each pais be equal, then 5.3 will regress to the naive inverse volatility case. In other words, ECP is a improved version of InVol strategy by taking the dependence information into account.

In a more general case where both volatilities and correlations differ, we can define the covariance of asset i with the portfolio, $\sigma_{i,P} = cov(\boldsymbol{R}_i, \sum_{j=1}^{n} \omega_j \boldsymbol{R}_j)$, we have $\sigma_i(\boldsymbol{\omega}) = \omega_i \sigma_{i,P}/\sigma(\boldsymbol{\omega})$. And let $\beta_i = \sigma_{i,P}/\sigma(\boldsymbol{\omega})^2$, $\sigma_i(\boldsymbol{\omega}) = \omega_i \beta_i \sigma(\boldsymbol{\omega})$, then RCP portfolio being defined by $\sigma_i(\boldsymbol{\omega}) = \sigma_j(\boldsymbol{\omega}) = \frac{\sigma(\boldsymbol{\omega})}{n}$ for all i, j, it follows that:

$$\omega_i = \frac{\beta_i^{-1}}{\sum_{j=1}^{n} \beta_j^{-1}} = \frac{\beta_i^{-1}}{n} \tag{5.4}$$

which means the weight of asset i is inversely proportional to its beta. Therefore, the assets with higher volatility or higher correlation with other assets will be down weighted. Recall that this solution is endogenous since ω_i is a function of β_i which, in turn, depends on $\boldsymbol{\omega}$.

Generally, we can sum up the intuition behind MV, EW, RCP portfolios, they are as follows:

$$\begin{aligned} &\omega_i = \omega_j && \text{(EW)} \\ &\partial_{\omega_i}\sigma(\boldsymbol{\omega}) = \partial_{\omega_j}\sigma(\boldsymbol{\omega}) && \text{(MV)} \\ &\omega_i \times \partial_{\omega_i}\sigma(\boldsymbol{\omega}) = \omega_j \times \partial_{\omega_j}\sigma(\boldsymbol{\omega}) && \text{(RCP)} \end{aligned}$$

Thus, RCP may be viewed as a combination of MV and EW portfolio. To be clear, we consider a modified version[15] of the optimization problem 5.2:

$$\begin{aligned} &\mathrm{Min}_{\omega}\sqrt{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}} \\ &s.t. \begin{cases} \sum_{i=1}^{n} \ln \omega_i \geq c \\ \boldsymbol{l}'\boldsymbol{\omega} = 1 \\ \boldsymbol{\omega} \geq 0 \end{cases} \end{aligned} \tag{5.5}$$

where c is a constant which can be interpreted as the minimum level of diversification among assets in order to get the RCP portfolio. If $c = -\infty$, then we get the MV portfolio; and if $c = -n \ln n$, we can get EW portfolio which is the largest case for $\sum_{i=1}^{n} \ln \omega_i$. So it shows RCP is an intermediate portfolio between MV and EW ones, they all can be given by the formulation 5.5 with different constraints c. And naturally, we shall see that the ex ante volatilities of them are ordered in the following way:

$$\sigma_{MV} \leq \sigma_{RCP} \leq \sigma_{EW}$$

6 Minimum CVaR Portfolio

6.1 Introduction and Definition

Usually, the deviation of the return from its mean is a symmetric way in measuring risk, while several serious financial crisis drive more interests towards quantile-based measures, such as value-at-risk(VaR).

VaR, though has lots of great properties and has been widely used in industry, if studied in terms of coherent risk measures [16], lacks subadditivity, and therefore convexity, in the general loss distribution(it may be subadditivity for some special classes). Lacking subadditivity means the inconsistency with the popular principle of diversification that diversification reduces risk, and non-convexity means that the optimization process can be non-convergent through common numerical iteration in which convexity is an important prerequisite. In addition, another serious shortcoming of VaR is that it provides no information about the extent of losses that might be suffered beyond the threshold. In other words, we may have adequate(usually 90%,95%,99%) confidence that the loss wouldn't exceed the corresponding VaR, while we don't know how much we may lose if the loss exceed VaR in extreme cases. Indeed, it tends to have a bias toward optimism instead of the conservatism that ought to prevail in risk management.

Conditional value at risk, CVaR, is also called Mean Excess Loss, Expected Shortfall or tail VaR. As its name tells, it does quantify the losses that might be encountered in the tail. Different with VaR, it's coherent, and it maintains consistency with VaR by yielding the same results in cases where VaR computations are tractable. More important, with the existence of convexity, it can be expressed by a minimization formula which can be incorporated into problems of optimization.

Let's start by define VaR first, let $f(\boldsymbol{x}, \boldsymbol{y})$ be the loss function associated with decision vector $\boldsymbol{x}$, to be chosen from a subset $\boldsymbol{X}$ in $\mathbb{R}^n$, and random vector $\boldsymbol{y}$ in $\mathbb{R}^m$. The $\boldsymbol{x}$ usually represents a constructed portfolio, with $\boldsymbol{X}$ as the investable set. The vector $\boldsymbol{y}$ stands for the uncertainties that can affect the loss. Thus, for each given $\boldsymbol{x}$, $f(\boldsymbol{x}, \boldsymbol{y})$ is a random variable having a distribution induced by $\boldsymbol{y}$ whose density function denoted by $p(\boldsymbol{y})$. Then the probability of $f(\boldsymbol{x}, \boldsymbol{y})$ not exceeding

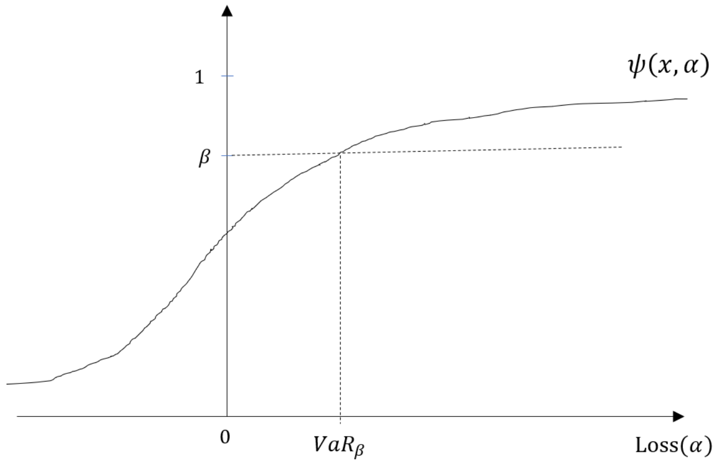


Figure 6.1: VaR Definition

a threshold loss α is given by:

$$\psi(\boldsymbol{x},\alpha) = \int_{f(\boldsymbol{x},\boldsymbol{y})<\alpha} p(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \tag{6.1}$$

For fixed $\boldsymbol{x}$, $\psi(\boldsymbol{x},\alpha)$ is the cumulative distribution for the loss.

Definition 6.1.1 ($\beta-$VaR, β-CVaR) The $VaR_\beta(\boldsymbol{x})$ and $CVaR_\beta(\boldsymbol{x})$ values for the loss random variable associated with $\boldsymbol{x}$ and any confidence level β is defined as:

$$VaR_\beta(\boldsymbol{x}) = \min\{\alpha \in \mathbb{R} : \psi(\boldsymbol{x},\alpha) \geq \beta\} \tag{6.2}$$

$$CVaR_\beta(\boldsymbol{x}) = (1-\beta)^{-1} \int_{f(\boldsymbol{x},\boldsymbol{y})>VaR_\beta(\boldsymbol{x})} f(\boldsymbol{x},\boldsymbol{y})p(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \tag{6.3}$$

As we can see from figure 6.1, if $\psi(\boldsymbol{x},\alpha)$ is continuous and nondecreasing with respect to α, $VaR_\beta(\boldsymbol{x})$ is the left endpoint of the interval consisting of the values α such that $\psi(\boldsymbol{x}, VaR_\beta(\boldsymbol{x})) = \beta$. Thus, the probability that $f(\boldsymbol{x},\boldsymbol{y}) \geq \alpha_\beta(\boldsymbol{x})$ is equal to $1-\beta$, so CVaR defined by 6.3 is the conditional expectation of the loss associated with loss no less than $CVaR_\beta(\boldsymbol{x})$.

What's more, we can expand the calculus in 6.3 in the global event space so that we can optimize it easier in the future.

$$CVaR_\beta(\boldsymbol{x}) = \alpha + (1-\beta)^{-1} \int_{y\in\mathbb{R}} [f(\boldsymbol{x},\boldsymbol{y}) - \alpha]^+ p(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \tag{6.4}$$

where $[t]^+ = t$ when $t \geq 0$ and $[t]^+ = 0$ when $t \leq 0$.

6.2 Practical CVaR Optimization

We consider the case where vector $\boldsymbol{x}$ represents a portfolio of assets with x_i being the weight of the ith asset. And $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)$ is the return vector with density distribution $p(\boldsymbol{y})$. Therefore, the loss function is given by

$$f(\boldsymbol{x}, \boldsymbol{y}) = -[x_1 y_1 + x_2 y_2 + \cdots + x_n y_n] = -\boldsymbol{x'y}$$

Since we only observe finite returns from the $\boldsymbol{y}$, a sample set $\boldsymbol{y_1}, \boldsymbol{y_2}, \ldots, \boldsymbol{y_q}$ yields the approximate function

$$CVaR_\beta(\boldsymbol{x}, \alpha) = \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} [-\boldsymbol{x'y} - \alpha]^+ \tag{6.5}$$

The minimization of $CVaR_\beta(\boldsymbol{x}, \alpha)$ over $\boldsymbol{X} \times \mathbb{R}$ can be reduced to convex programming by including auxiliary real variables u_k for $k = 1, 2, \ldots, q$, so it's equivalent to minimizing the linear expression:

$$\begin{aligned} &\text{Min: } CVaR_\beta(\boldsymbol{x}, \alpha) = \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} u_k \\ &s.t. \quad u_k \geq 0 \\ &\qquad \boldsymbol{x'y} + \alpha + u_k \geq 0, \quad \text{for } k = 1, 2, \ldots, q \end{aligned} \tag{6.6}$$

It's worthwhile to note that minimizing $CVaR_\beta(\boldsymbol{x}, \alpha)$ over $\boldsymbol{X} \times \mathbb{R}$ also determines the VaR_β of the portfolio $\boldsymbol{x}$ that minimizes $CVaR_\beta$, which means VaR_β is a by-product when solving problem 6.6. Moreover, it's a linear programming problem, which means we can solve it easily using simplex method, while most optimization problems require quadratic programming, where we used SLSQP(Sequential Least Squares Quadratic Programming) algorithm.

7 Extension on Diversification

We must be aware that optimizer tend to allocate more risk to factors whose volatility has been underestimated, this is especially true for long-short portfolios built from large universe. In this case, instead of trying to be diversified, some optimization problems tend to be overly concentrated on some assets with no consideration to other assets. This phenomenon certainly contradicts our original goal to diversify the risk, thus we hope to add constraints to these strategies to drive them to select more assets.

Here, we have 2 options. On the one hand, we set upper bound to the weight, e.g. if we set the upper bound $u = 0.1$, then we should at least select 10 assets in order to satisfy the constraint. In this way, we can restrict the influence of some assets, which otherwise, occupy a large proportion in the portfolio.

On the other hand, we modified the objective function by adding a regularization term to force them to select all assets evenly. For example, the MV problem becomes:

$$\begin{aligned} &\text{Min: } \boldsymbol{\omega'\Sigma\omega} + \lambda(\boldsymbol{\omega}' - \overline{\boldsymbol{\omega}'})^2\boldsymbol{l} \\ &s.t. \quad \boldsymbol{\omega' l} = 1 \end{aligned} \tag{7.1}$$

where $\overline{\boldsymbol{\omega}}$ is a N-by-1 vector with each element equals to $1/n$, $\lambda \geq 0$ is a parameter which adjusts the influence of regularization. Intuitively, we can imagine that if $\lambda \to \infty$, then the regularization term is so big that the first term of 7.1 is inconspicuous which drive the problem to select all assets evenly; if $\lambda \to 0$, then the result of 7.1 tend to be the same with that of 2.2. In this way, by adjusting λ, we can balance the tradeoff between minimizing the risk and diversifying the investments.

Similarly, we can also add a regularization term to 4.2:

$$\begin{aligned} &\text{Min: } \boldsymbol{\omega'\rho\omega} + \lambda(\boldsymbol{\omega}' - \overline{\boldsymbol{\omega}'})^2\boldsymbol{l} \\ &s.t. \quad \boldsymbol{\omega' l} = 1 \end{aligned} \tag{7.2}$$

one more thing we need to do is to rescale the weight of modified MD portfolio according to the volatility of each asset:

$$\boldsymbol{\xi} = \boldsymbol{D}^{-1/2}\boldsymbol{\omega}$$

where $\boldsymbol{D}$ is a diagnal matrix with $\sigma_{i,i}^2$ as its (i,i)th element and zero on all off-diagnal elements, and $\boldsymbol{\xi}$ is rescaled so that the sum of final weights equal to 1.

$$w_i = \frac{\xi_i}{\sum_{j=1}^{n} \xi_j}$$

In general, we sum up all available parameters for technique. For some techniques, both parameters can be used together, but we wouldn't focus too much on it.

Portfolio	**Optional Parameter**
Equal Weight	N/A
Inverse Volatility	Covariance Estimator
Minimum Variance	λ, u, Covariance Estimator
Maximum Diversification	λ, u, Covariance Estimator
Risk Parity	u, Covariance Estimator
Minimum CVaR	u,β

Table 7.1: Optional Parameters

Note that, fundamentally, we can add regularization to RCP and Min CVaR too. But we have to bear in mind that, we really don't need to add regularization to diversified portfolio like RCP, which will spoil the intuition of it. And the dimension of CVaR and variance is different, so even if we include λ in Min CVaR, it isn't comparable to MV and MD.

It's worthwhile to think that, if the assumption that future returns is positively correlated with risk holds, since volatility and CVaR are risk measures, the technique in this chapter, indeed, is a tradeoff between risk and return. We hope to minimize the risk in the future, and at the same time, yield as much benefits as possible, and λ is the risky compensation measuring your marginal tolerance of risk by gaining one more unit of return. Thus, to make the model more complicated, we can also set different λ for the same portfolio in different cases, e.g., in a bull market, λ can be larger so that we are willing to take risks to gain more yields, while in a bear market, λ may be smaller so that we take a more conservative strategy.

In particular, this kind of tricks will be more effective for assets having great potential growth. Moreover, the $\overline{\boldsymbol{\omega}}$ can be replaced by other return-based factor. For example, if we have information $\boldsymbol{\theta}$ so that assets with higher θ are more likely to yield higher return, then $\overline{\boldsymbol{\omega}} = \frac{\boldsymbol{\theta}}{\sum_{i=1}^{n} \theta_i}$ is more suitable. In this way, we take return into consideration indirectly, which, tend to be like mean-variance theory.

8 Portfolio Back-test

8.1 Data Description

Without losing generality, we choose *Shenwan First Level Indice* as our investable set, that's to say, we take them as investable assets like stocks, buy and hold them to yield return. The reason we choose them is that, there are in total 28 investable industries[1] which is small enough to optimize the portfolio efficiently, and the correlation between them may not be as strong as that of stocks since they are less sensitive to market risk.

The data contains 2955 daily close log return ranging from Jan. 1st 2006 to Mar. 1st 2018. If we use Shanghai Composite Index as the market return, we shall see that the correlation coefficients between the market return and each industry are almost around 0.8 ranging from 0.747 to 0.906 with the public industry being the highest and the military defense industry being the lowest, so they all have strong correlation with the market. While if we check the correlation coefficient between themselves, the difference is larger with "other industry" and "textile and garment industry" being the highest 0.943, and "bank industry" and "computer industry" being the lowest 0.428.

In figure 8.1, we plot cumulative returns of several typical industries, as we can see, though the trend of different industries is generally similar(e.g. the crash in 2008 and 2015), their behaviors in normal years is different(e.g. at the end of data, bank industry is still rising while others tend to decrease), so it's reasonable for us to believe that, the performance of portfolio would be better using previous strategies.

8.2 Portfolio Construction

We take following steps to construct the portfolio and to do analysis:

1. We used 240 days of data points as time window, which is almost the trading days of one year,

[1] It includes agriculture, extractive industry, chemical industry, iron industry, electronic industry, nonferrous metals industry, household electrical appliance industry, food and beverage industry, textile and garment industry, light industry, pharmaceutical industry, public industry, traffic industry, estate, commerce, entertainment, building materials, architectural decoration industry, electrical equipment, military defense industry, computer, media, communication, bank, non-bank financial industry, vehicle industry, mechanical equipment industry, other industry.

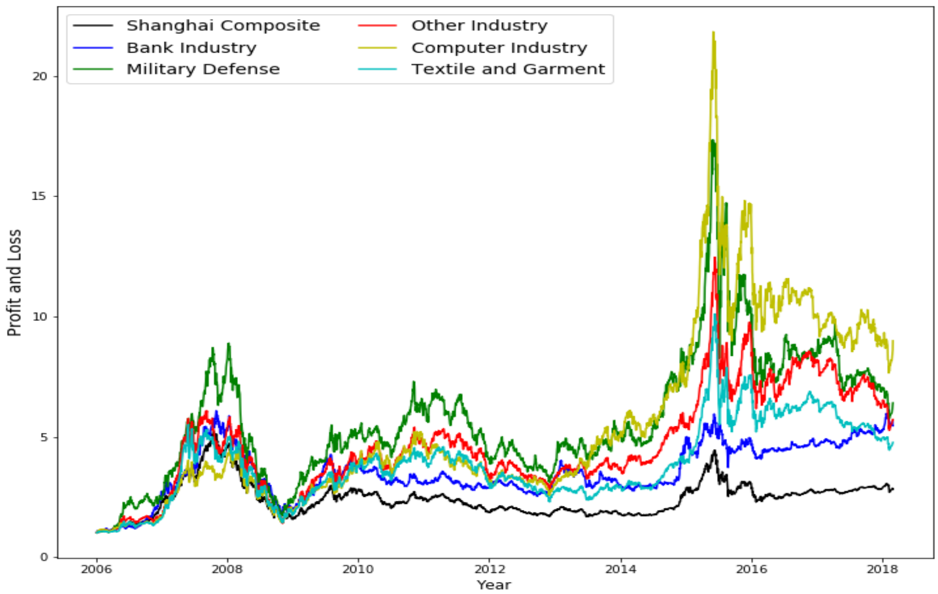


Figure 8.1: Cumulate Returns of Several Industries

to calculate covariance matrix of these industries, thus we start our investment on Dec. 28th 2006.

2. We will mainly use $u = 1$, $\lambda = 0$, $\beta = 0.95$ to do analysis, and we will analyze the influence of other parameters too.
3. We will use sample empirical covariance matrix as well as other robust covariance estimators like "Minimum Covariance Determinant"[17, 18] and "Shrunk Covariance"[19] which we won't talk about their theory and property, we just use it to see if there are any difference in calculating the weight.
4. After getting the covariance, we used quadratic programming method(SLSQP) to calculate the weight in MV, MD, RCP portfolios, and we used linear programming method(Simplex Method) to calculate the weight in MCVaR portfolio.
5. We don't take trading cost into consideration, thus we buy and hold the portfolio for 5 trading days, then we proceed to roll our investments in the next 5 trading days, etc.
6. Finally, we can get the weight in each holding period and back-test the portfolio.
7. We compare different portfolios in various aspects to correspond to their properties.

8.3 Comparison and Analysis

8.3.1 Weight distribution

Figure 8.2 shows the weight of different portfolios when we use $u = 1$, $\lambda = 0$, $\beta = 0.95$ and empirical covariance matrix. Intuitively, we can summarize several results from the figure.

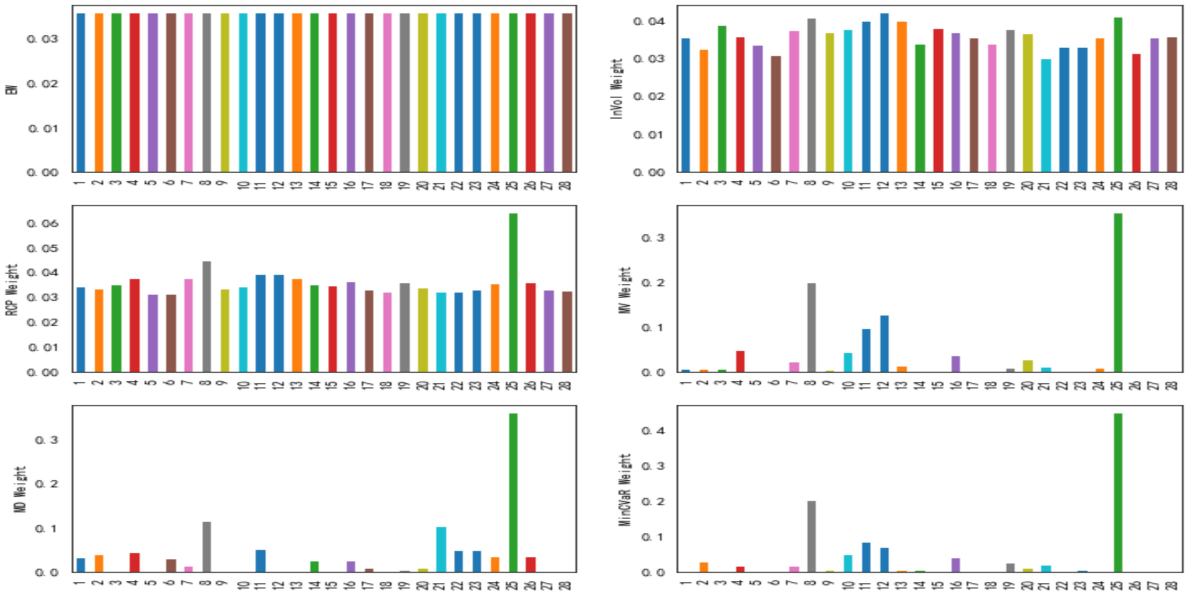
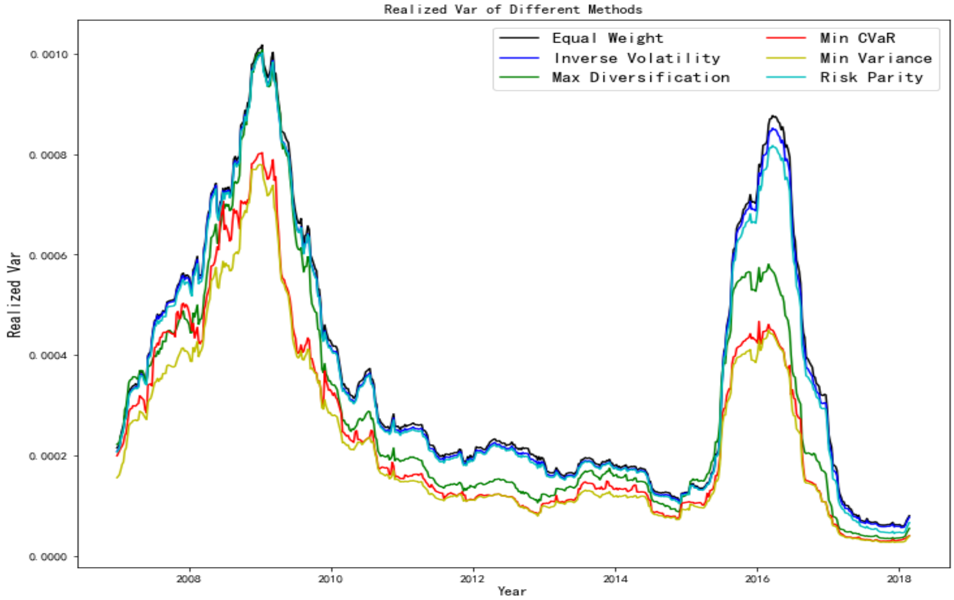


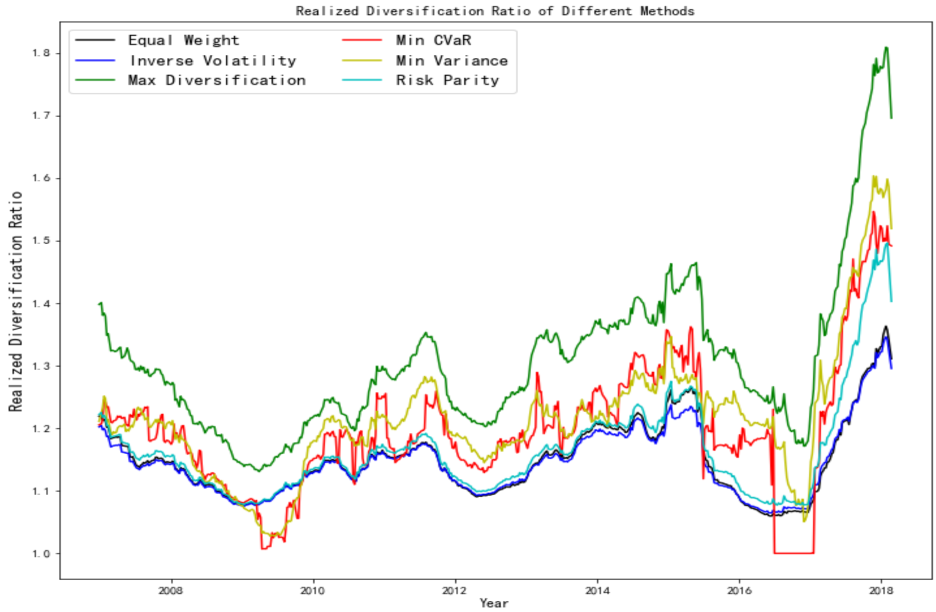
Figure 8.2: Average Weight of Different Portfolios

- Generally speaking, the first three portfolios are similar and the last three are similar. EW, RCP, InVol always try to select all investable assets which seems to put more attention in diversifying the investment, while MV, MD, MinCVaR merely select a few available assets in each period which focus on minimizing the risk. Therefore, we call the first 3 portfolios as diversify-oriented portfolio(type I), the last 3 portfolios as risk-oriented portfolio(type II)
- Industry 25(Bank) have superior status since it has the highest weight in each portfolio regardless the type, thus, naturally, the bank industry is less volatile than other industries and less correlated with the market, as the blue line in figure 8.1 shows.
- From the definition and the weight of RCP portfolio, in addition to bank industry, the weight of almost all industries are around 0.03, therefore, in a way, it indirectly tells that other industries are similar in terms of volatility and correlation. Therefore, it may be a signal indicating that assets in this investable set are indistinguishable which stimulate us to include more heterogeneous assets into the set.

8.3.2 Risk Measures

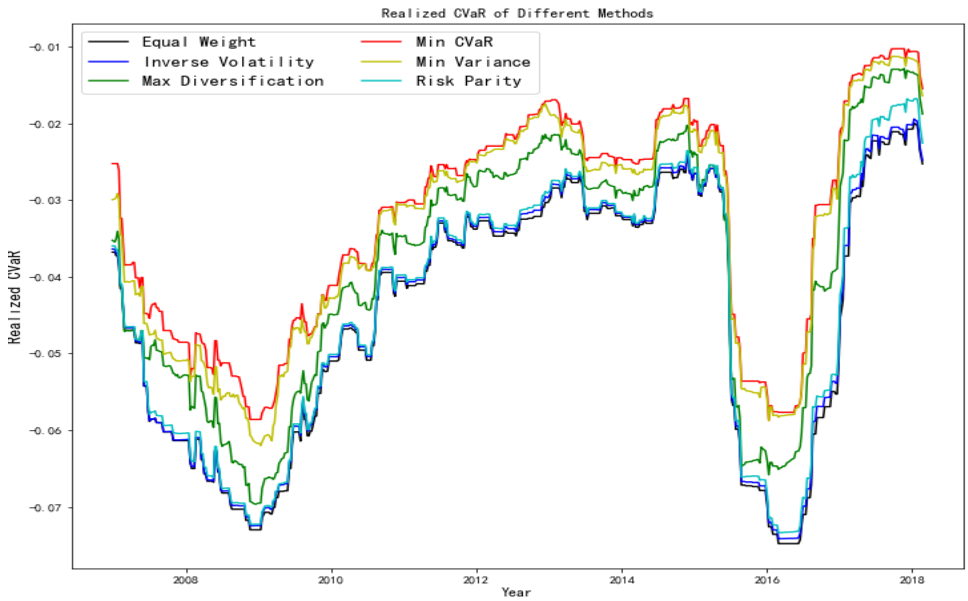
By including risk plots of these portfolios, we can also separate them into 2 types.





- In the "Realized Var" figrue, MV portfolio is certainly the portfolio with the lowest variance, in addition, the variance of type II portfolio is generally lower than type I, which is accord with the previous conclusion that type II portfolio focus on minimizing the risk.
- In the "Realized Diversification Ratio" figure, although sometimes the DR of MV and MinCVaR portfolio is lower than type I portfolio, the DR of type II portfolio is higher than type I in most time.
- In the "Realized CVaR" figure, CVaR of type II portfolio is higher than that of type I.

Therefore, we conclude that, though diversifying the investment in all investable set may de-



crease the risk is the common sense, it's likely to increase the risk, at least in our ex ante risk measures. On the contrary, the portfolio seemingly undiversified may have a lower risk.

8.3.3 Back-test Performance

We summarize the performance of these portfolios using several important indicators.

Portfolio	Yearly Return	Yearly Volatility	Sharp Ratio	Max Drawdown	Mean Drawdown	Turnover
EW	10.2%	29.39%	0.347	-71.4%	-34.0%	0.00%
InVol	10.61%	29.18%	0.364	-71.0%	-32.9%	0.28%
RCP	10.61%	28.92%	0.367	-71.4%	-34.0%	0.36%
MV	12.36%	24.94%	0.496	-69.8%	-32.9%	3.86%
MD	9.69%	27.35%	0.354	-73.2%	-41.8%	3.54%
MinCVaR	7.89%	25.89%	0.305	-78.1%	-53.7%	6.68%

Table 8.1: Performance Summary

In Table 8.1 "Sharp Ratio" $= \frac{\text{Yearly Return}}{\text{Yearly Volatility}}$ measures the maximum return you can get by taking one more unit of risk; "Mean Drawdown" measures the recovering efficiency, as we can imagine, the portfolio is more resistant to the crash if "Mean Drawdown" is higher; "Turnover" measures the average proportion of weight need to change in each period.

Although we can separate these portfolios into 2 types according to the weight distribution, we can't easily get a clear boundary from the indicators in table 8.1

- If we simply focus on the volatility, we are glad to see that all portfolios have a lower volatility compared with EW(except for itself), that's to say, these strategies are effective to minimize

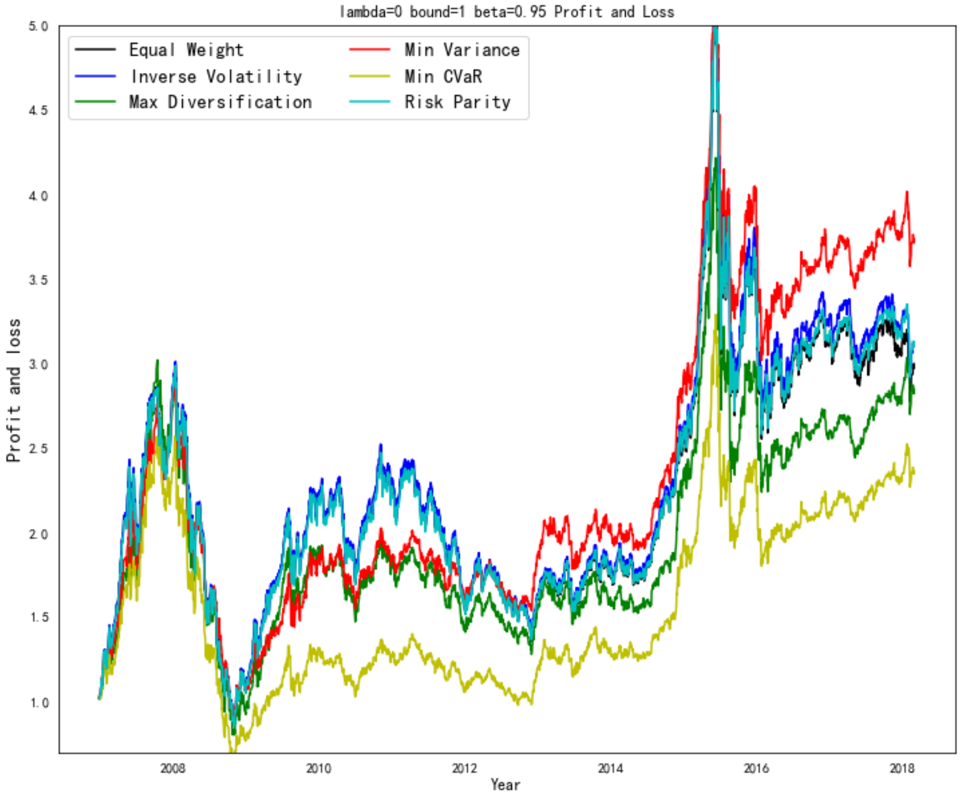


Figure 8.3: Daily Profit and Loss

risk more or less.

- Among these strategies, MV portfolio seems to be the best in terms of all indicators except "Turnover", while MD, MinCVaR are worse than type I portfolio in most parts, the return and drawdown of MD and MinCVaR is lower than type I portfolio, while MV is higher.
- The most clear boundary may lie in the "Turnover", as we can imagine, the diversify-oriented portfolio has a lower turnover rate since it always select all available assets, while the components of risk-oriented portfolio is much changeable since it merely selects a few assets in each period.

8.3.4 Extension on Parameters

u=0.08

We firstly change $u = 0.08$, therefore, we force the strategy to select at least 13 industries into portfolio[2]. Through table 8.2 and figure 8.4, the weight distribution of MV and MD become less volatile, so the turnover decrease, and the performance of them tend to be like EW.

[2]Note that after the adjustment of volatility, the weight of MD portfolio may exceed u

Portfolio	Yearly Return	Yearly Volatility	Sharp Ratio	Max Drawdown	Mean Drawdown	Turnover
RCP	10.54%	28.93%	0.364	-71.4%	-34.0%	0.32%
MV	10.92%	27.21%	0.401	-69.1%	-34.82%	2.36%
MD	9.2%	28.39%	0.324	-73.1%	-42.7%	2.07%

Table 8.2: Portfolio Performance When $u = 0.08$

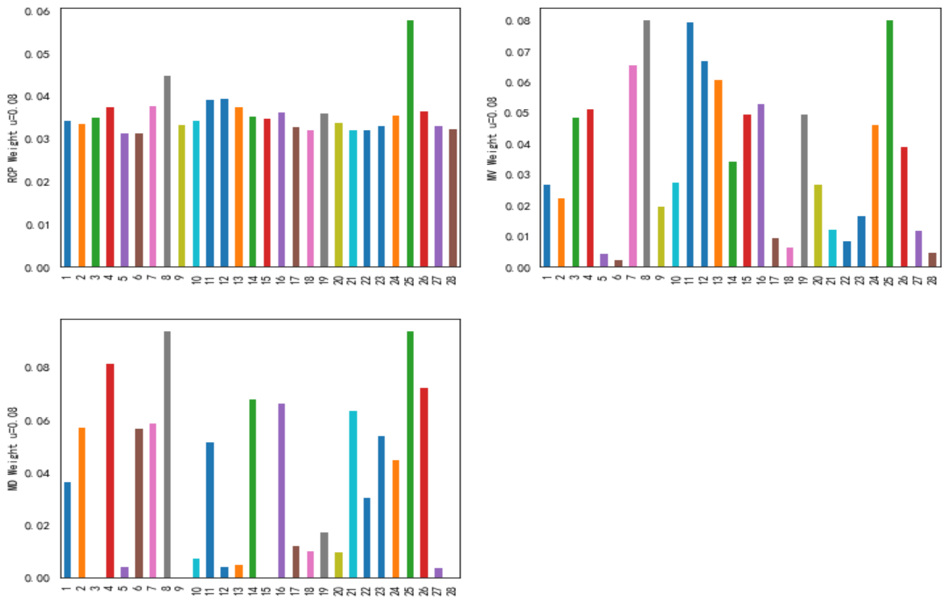


Figure 8.4: Weight Distribution When $u = 0.08$

$\lambda = 0.5$

We then set u back to zero, and change $\lambda = 0.5$. Still the regularization term drive the MV portfolio more like EW like the influence of u. On the contrary, MD portfolio becomes much worse than the benchmark, it's likely that the adjustment of each asset's volatility contaminate the objective function too much which indicates us to select a smaller λ , after all, the influence of the same parameter to different objective function is different.

Portfolio	Yearly Return	Yearly Volatility	Sharp Ratio	Max Drawdown	Mean Drawdown	Turnover
MV	11.54%	24.91%	0.463	-69.2%	-33.75%	3.56%
MD	8.89%	27.5%	0.323	-72.8%	-38.4%	2.44%

Table 8.3: Portfolio Performance When $\lambda = 0.5$

MinDet

We set all parameters to benchmark, then we change the covariance estimator to Minimum Determinant Matrix. As we can see, the difference between MinDet and benchmark isn't large, the performance of MV becomes slightly better, while at the same time, the weight in each period becomes more volatile which result in a large increase in the turnover.

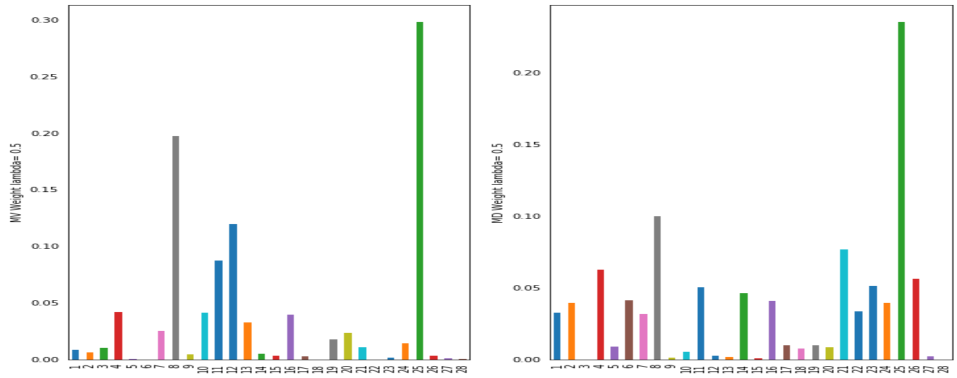


Figure 8.5: Weight Distribution When $\lambda = 0.5$

Portfolio	Yearly Return	Yearly Volatility	Sharp Ratio	Max Drawdown	Mean Drawdown	Turnover
InVol	10.58%	29.18%	0.363	-71.23%	-33.63%	0.77%
RCP	10.67%	28.97%	0.368	-71.28%	-33.59%	0.97%
MV	13.19%	25.44%	0.518	-66.8%	-32.92%	10.88%
MD	9.28%	27.68%	0.335	-74.5%	-38.2%	9.69%

Table 8.4: Portfolio Performance When Covariance Estimator Change

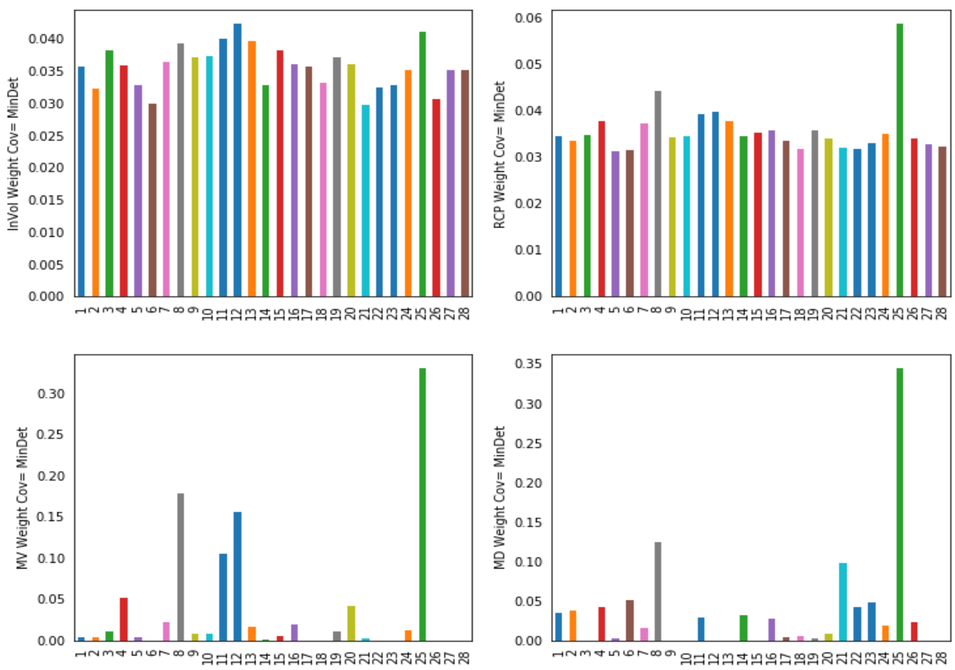


Figure 8.6: Weight Distribution When Covariance matrix=MinDet

$\beta = 0.8, 0.9$

Simply consider the MinCVaR portfolio, we change β to different confidence level to check its performance. Interestingly, we find that though the weight distrtribution of different β is almost the same, the performance of them are completely different. By decreasing the confidence level β, yearly return increases and the drawdown decreases, I think it may lie in the fact that we directly use sample returns to do CVaR optimization without any further robust methods[20] to prevent the variance and bias, so if we use robust methods and more samples to calculate

CVaR or even merely slightly lower the confidence level, we may get better result.

β	Yearly Return	Yearly Volatility	Sharp Ratio	Max Drawdown	Mean Drawdown	Turnover
0.95	7.89%	25.89%	0.305	-74.45%	-41.47%	6.68%
0.9	10.66%	25.36%	0.420	-71.39%	-34.22%	5.67%
0.8	12.29%	25.13%	0.489	-68.5%	-28.31%	5.97%

Table 8.5: Portfolio Performance When β Change

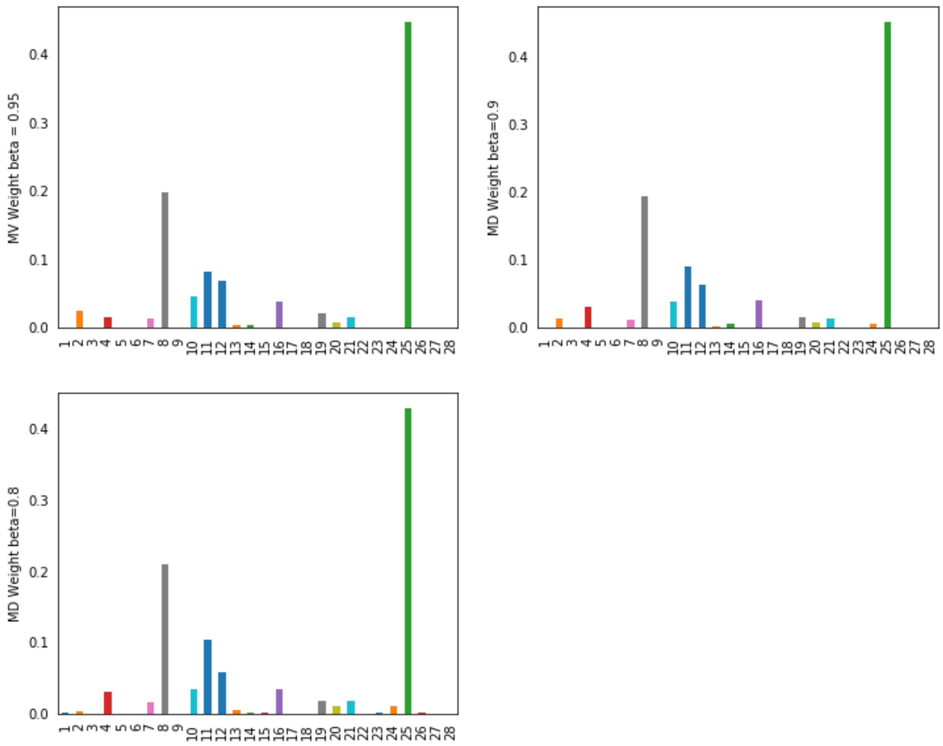


Figure 8.7: Weight Distribution When β Change

9 Conclusion

In this paper, we provide definition of MV, EW, InVol, RCP, MD, MinCVaR asset allocations and do some analysis about them. Generally, we divide them into 2 types, EW, InVol, RCP are diversify-oriented(type I) asset allocation strategies, while MV, MD, MinCVaR are risk-oriented(type II) asset allocation strategies. As their names tell, type I portfolios focus on diversifying the investment in the investable set, type II portfolios focus on minimizing the risk defined in their strategies respectively by selecting a few stable assets.

Although these strategies have many similarities and differences, they can all, to some extent, lower the risk of the portfolio compared with EW which is the most commonly used strategy. And their performance may differ greatly facing different investable sets. In our set consisting of 28 industry indices, taking all aspects into consideration, MV performs relatively good.

One bright feature of this paper is that, instead of directly taking expected return into account to form the classical mean-variance theory, we choose to use "1/n" weight or other return-based factor as regularization term to drive the optimization process towards our anticipation. Moreover, we indicate some optional parameters of these strategies and give their performance by slight changing the parameters. And We do see that the performance of these strategies develop towards our anticipation. Different strategies have different optimal parameter combinations, we merely show the effect of these parameters, and we can adjust the parameter combinations faced with different assets allocation problems.

Reference

[1] Harry Markowitz. Portfolio selection. *Journal of Finance*, 7(1):77–91, 1952.

[2] Harry Markowitz. The optimization of a quadratic function subject to linear constraints. *Naval Research Logistics*, 3(1-2):111–133, 1956.

[3] DeMiguel, Victor, Garlappi, Lorenzo, and Uppal. Optimal versus naive diversification: How inefficient is the 1/n portfolio strategy? *Review of Financial Studies*, 22(5):1915–1953, 2009.

[4] Heath Windcliff and Phelim P. Boyle. The 1/ n pension investment puzzle. *North American Actuarial Journal*, 8(3):32–45, 2004.

[5] Yves Choueifaty and Yves Coignard. Toward maximum diversification. *Journal of Portfolio Management*, 35(1):40–51, 2008.

[6] Edward E Qian. On the financial interpretation of risk contribution: Risk budgets do add up. *Social Science Electronic Publishing*, 4(4):3, 2006.

[7] Roger Clarke, Harindra De Silva, and Steven Thorley. Risk parity, maximum diversification, and minimum variance: An analytic perspective. *Journal of Portfolio Management*, 39(3):39, 2013.

[8] Jérôme Teiletche, Thierry Roncalli, and Sébastien Maillard. The properties of equally-weighted risk contributions portfolios. pages 60–70, 2010.

[9] R. T Rockfellar. Optimization of conditional value-at risk. *Journal of Risk*, 2(1):1071–1074, 2000.

[10] S. Uryasev. Conditional value-at-risk: optimization algorithms and applications. In *Computational Intelligence for Financial Engineering*, pages 49–57, 2002.

[11] Roger G Clarke, Harindra de Silva, and Steven Thorley. Minimum-variance portfolios in the u.s. equity market. *The Journal of Portfolio Management*, 33(1):10–24, 2006.

[12] Roger Clarke, Harindra De Silva, and Steven Thorley. Minimum-variance portfolio composition. *Journal of Portfolio Management*, 37(2):31, 2011.

[13] Shlomo Benartzi and Richard H Thaler. Naive diversification strategies in defined contribution saving plans. *American economic review*, 91(1):79–98, 2001.

[14] Louis KC Chan, Jason Karceski, and Josef Lakonishok. On portfolio optimization: Forecasting

covariances and choosing the risk model. *The Review of Financial Studies*, 12(5):937–974, 1999.

[15] Sébastien Maillard, Thierry Roncalli, and Jerome Teiletche. On the properties of equally-weighted risk contributions portfolios. *Social Science Electronic Publishing*, 36(4):60–70, 2008.

[16] Freddy Delbaen. Coherent risk measures. *Blätter Der Dgvfm*, 24(4):733–739, 2000.

[17] Mia Hubert and Michiel Debruyne. Minimum covariance determinant. *Wiley Interdisciplinary Reviews Computational Statistics*, 2(1):36–43, 2010.

[18] Peterj. Rousseeuw and KatrienVanDriessen. A fast algorithm for the minimum covariance determinant estimator. *Technometrics*, 41(3):212–223, 1999.

[19] Olivier Ledoit and Michael Wolf. A well-conditioned estimator for large-dimensional covariance matrices. *Journal of Multivariate Analysis*, 88(2):365–411, 2004.

[20] Anna Grazia Quaranta and Alberto Zaffaroni. Robust optimization of conditional value at risk and portfolio selection. *Journal of Banking & Finance*, 32(10):2046–2056, 2008.