Practice Set 3 Solutions

Problem 1

$$\mathbf{G_{ZF}} = \sqrt{M_T}(\mathbf{H^H H})^{-1}\mathbf{H^H} = \begin{bmatrix} -0.8787 - 1.3185j & 0.0132 + 1.1870j \\ -0.6193 - 0.0484i & 1.5225 + 0.3681i \end{bmatrix}$$

To verify this set $\mathbf{X} = [1 + j \quad -1 + j]^T$, then $\mathbf{Y} = \sqrt{(}1/M_T)\mathbf{HX}$

$$\widehat{\mathbf{X}} = \mathbf{G_{ZF}Y} = \begin{bmatrix} 1+j \\ -1+j \end{bmatrix}$$

Problem 2 *Diversity Gain*

a) We can express the received signal model as

$$\mathbf{y} = \sqrt{E_s}\mathbf{h}s + \mathbf{n}$$

With MRC,

$$z = \sum_{i=1}^{M} h_i^* y_i = \mathbf{h}^H\mathbf{y} = \sqrt{E_s}\mathbf{h}^H\mathbf{h}s + \mathbf{h}^H\mathbf{n}$$

Signal Power

$$\mathcal{E}\left\{\left[\sqrt{E_s}\mathbf{h}^H\mathbf{h}s\right]\left[\sqrt{E_s}\mathbf{h}^H\mathbf{h}s\right]^H\right\} = E_s\left(\|\mathbf{h}\|^2\right)^2\mathcal{E}\{ss^*\} = E_s\left(\|\mathbf{h}\|^2\right)^2$$

Noise Power

$$\mathcal{E}\left\{\left[\mathbf{h}^H\mathbf{n}\right]\left[\mathbf{h}^H\mathbf{n}\right]^H\right\} = \mathbf{h}^H\mathcal{E}\{\mathbf{n}\mathbf{n}^H\}\mathbf{h} = \mathbf{h}^H(N_0\mathbf{I})\mathbf{h} = N_0\|h\|^2$$

SNR

$$\eta = \frac{E_s(\|\mathbf{h}\|^2)^2}{N_0\|h\|^2} = \|\mathbf{h}\|^2\rho$$

b) The processed signal is given by

$$z = \mathbf{h}^H\mathbf{y} = \sqrt{E_s}\|\mathbf{h}\|^2 s + \mathbf{h}^H\mathbf{n}$$

Dividing it by $\sqrt{E_s}\|\mathbf{h}\|^2$ does not change the decision at the receiver.

$$\tilde{z} = s + \frac{\mathbf{h}^H\mathbf{n}}{\sqrt{\frac{E_s}{M}}\|\mathbf{h}\|^2}$$

$$\Rightarrow \mathbf{n} \rightarrow \mathcal{N}(0, \frac{N_0}{E_s\|h\|^2}) \rightarrow \mathcal{N}(0, \frac{1}{\eta})$$

Using the Nearest Neighbor Union Bound (NNUB), the probability of symbol error P_e is given as

$$P_e \leq \overline{N}_e Q\left[\frac{d_{min}}{2\sigma_n}\right]$$

where d_{min} is the minimum distance between any two points in the constellation and $\sigma_n^2 = \frac{\mathcal{N}_0}{2} = \frac{1}{2\eta}$ is the variance per dimension of the AWGN. $Q(\cdot)$ is the Q-function defined as

$$Q(x) = \frac{1}{2\pi}\int_x^{\infty} e^{\frac{-u^2}{2}} du$$

$\overline{N}_e$ is the average number of nearest neighbors for the signal constellation. The probability of symbol error becomes

$$\begin{aligned} P_e &\leq \overline{N}_e Q\left[\frac{d_{min}\sqrt{2\eta}}{2}\right] \\ &= \overline{N}_e Q\left[\sqrt{\frac{d_{min}^2 \eta}{2}}\right] \\ &= \overline{N}_e Q\left[\sqrt{\frac{d_{min}^2 \|h\|^2 \rho}{2}}\right] \end{aligned}$$

c) Using the Chernoff bound, i.e. $Q(x) \leq e^{-\frac{x^2}{2}}$

$$\begin{aligned} P_e &\leq \overline{N}_e e^{-\frac{\eta d_{min}^2}{4}} \\ &= \overline{N}_e e^{-\frac{\|h\|^2 \rho d_{min}^2}{4}} \end{aligned}$$

The random channel tap is given by $h_i = a_i + jb_i$ where a_i and b_i are $\mathcal{N}(0, 0.5)$. $|h_i|^2$ is a chi-squared variable with two degrees of freedom and pdf given by

Chi-squared distribution with n degrees of freedom $X_i \sim \mathcal{N}(0, \sigma^2)$ $Y = X_1^2 + \cdots + X_n^2$ has density f_n with $f_Y^{(n)}(y) = \frac{1}{2^{n/2}\sigma^n \Gamma(\frac{n}{2})} y^{\frac{n}{2}-1} e^{-\frac{y}{2\sigma^2}}, \quad y \geq 0,\, n \geq 0$ $f_Y^{(2)}(y) = \frac{1}{2\sigma^2} e^{-\frac{y}{2\sigma^2}}$

$$z = |h_i|^2$$
$$f_Z(z) = e^{-z}$$

The average probability of error is found to be

$$\begin{aligned} P_e &= \overline{N}_e e^{-\sum_{i=1}^{M}|h_i|^2\gamma} \\ &= \overline{N}_e \prod_{i=1}^{M} e^{-|h_i|^2\gamma} \\ \overline{P}_e &= \int_0^{\infty} P_e f(P_e) \end{aligned}$$

where $\gamma = \frac{\rho d_{min}^2}{4}$. As the elements of $\mathbf{h}$ are independent, the joint pdf is obtained by the product of the individidual pdfs. Therefore,

$$\begin{aligned} \overline{P}_e &\leq \overline{N}_e \prod_{i=1}^{M} \int_0^{\infty} e^{-|h_i|^2\gamma} e^{-|h_i|^2} d|h_i|^2 \\ &= \overline{N}_e \prod_{i=1}^{M} \frac{1}{1+\gamma} = \overline{N}_e \prod_{i=1}^{M} \frac{1}{1+\frac{\rho d_{min}^2}{4}} \end{aligned}$$

d) At high SNRs, $\gamma >> 1$ and we can approximate $\overline{P}_e$ as

$$\overline{P}_e \approx \overline{N}_e \prod_{i=1}^{M} \gamma^{-M}$$

where M is the diversity order of MRC.

Problem 3 *Dominant eigenmode transmission.*

a) For single-mode transmission, the received signal is given by

$$\mathbf{y} = \sqrt{\frac{E_s}{M_T}} \mathbf{Hws} + \mathbf{n},$$

with power constraint $\|\mathbf{w}\|_F^2 \leq M_T$. At the receiver,

$$z = \mathbf{g}^H \mathbf{y}$$

and thus the effective SNR is given as

$$\eta = \frac{\left|\mathbf{g}^H \mathbf{Hw}\right|^2}{M_T \|\mathbf{g}\|^2} \rho.$$

Recall the singular value decomposition

$$\mathbf{H} = \mathbf{U\Sigma V}^H.$$

If $\mathbf{w}/\sqrt{M_T} = \mathbf{v}_1$ and $\mathbf{g} = \mathbf{u}_1$, i.e., we use the input and output singular vectors corresponding to maximum singular value of $\mathbf{H}$, the effective I/O relation for the channel becomes

$$z = \sqrt{E_s} \sigma_{max} s + n,$$

with signal power $E_s\sigma^2_{max}$ and noise power N_0. Therefore the instantaneous SNR becomes

$$\eta = \frac{E_s\sigma^2_{max}}{N_o} = \sigma^2_{max}\rho = \lambda_{max}\rho$$

where λ_{max} is the maximum eigenvalue of $\mathbf{HH}^H$.

The expected array gain of a transmission technique is calculated as the ratio of expected value of the SNR with the technique to the SNR without the technique,

$$\alpha = \frac{\mathcal{E}\{\eta\}}{\rho} = \mathcal{E}\{\lambda_{max}\}$$

b) To derive a range for the $\overline{P}_e$, first recall that the Chernoff bound is tight for high SNR, and we can thus write

$$P_e = \overline{N}_e Q\left(\sqrt{\frac{d^2_{\min}\eta}{2}}\right) \approx \overline{N}_e \exp\left(-\frac{d^2_{\min}\eta}{4}\right),$$

where $\eta = \rho\lambda_{\max}$ as shown in part (a). We are interested in $\overline{P}_e = \mathcal{E}(P_e)$, i.e., the average error rate over the channel randomness, which is hard to compute exactly due to the dependence on $\lambda_{\max}$.

To circumvent this, recall that

$$\|\mathbf{H}\|_F^2 = \text{Trace}(\mathbf{HH}^H) = \sum_{i=1}^{r}\lambda_i,$$

where r is the rank of $\mathbf{HH}^H$. From $\lambda_i \geq 0$ and $\lambda_{\max} \geq \lambda_i$ for all i, we have

$$\frac{\sum_{i=1}^r \lambda_i}{r} \leq \lambda_{max} \leq \sum_{i=1}^{r}\lambda_i$$

and thus

$$\frac{\|\mathbf{H}\|_F^2}{r} \leq \lambda_{max} \leq \|\mathbf{H}\|_F^2.$$

We can use these bounds instead of $\lambda_{\max}$ in the computation of $\overline{P}_e$. First, conclude

$$\overline{P}_e \;\; \geq \;\; \overline{N}_e \; \mathcal{E}\left\{\exp\left(-\frac{d^2_{\min}\rho}{4}\|\mathbf{H}\|_F^2\right)\right\}.$$

Using the moment-generating function given in (3.44) in the textbook for the special case of $\mathbf{H} = \mathbf{H}_w$, write

$$\overline{P}_e \;\; \geq \;\; \overline{N}_e \;\left(\frac{1}{1+\frac{\rho d^2_{\min}}{4}}\right)^{M_R M_T}.$$

On the other hand, using the same technique,

$$\begin{aligned}\overline{P}_e &\leq \overline{N}_e\, \mathcal{E}\left\{\exp\left(-\frac{d_{\min}^2\rho}{4r}\|\mathbf{H}\|_F^2\right)\right\} \\ &= \overline{N}_e \left(\frac{1}{1+\frac{\rho d_{\min}^2}{4r}}\right)^{M_R M_T}.\end{aligned}$$

For large SNR ρ, the two bounds can be simplified as

$$\overline{N}_e \left(\frac{\rho d_{\min}^2}{4}\right)^{-M_R M_T} \leq \overline{P}_e \leq \overline{N}_e \left(\frac{\rho d_{\min}^2}{4r}\right)^{-M_R M_T}.$$

Finally, note that for $\mathbf{H} = \mathbf{H}_w$, we have $r = \min(M_T, M_R)$ with probability 1.

c) From the exponent of $\overline{P}_e$ in the high SNR regime, it can be seen that dominant eigenmode transmission extracts full diversity, namely $M_R M_T$.

Problem 4 *Comparison of MIMO diversity schemes*

$$\begin{aligned}\overline{P}_e &= \overline{N}_e Q\left(\sqrt{\frac{d_{min}^2\eta}{2}}\right) \\ \eta &= \rho\,\|h_{\text{eff}}\|^2 \\ Q(x) &= \frac{1}{2}\,\text{erfc}\left(\frac{x}{\sqrt{2}}\right)\end{aligned}$$

a) For a SISO link,

$$\|h_{\text{eff}}\|^2 = |h|^2$$

b) For 2×2 Alamouti,

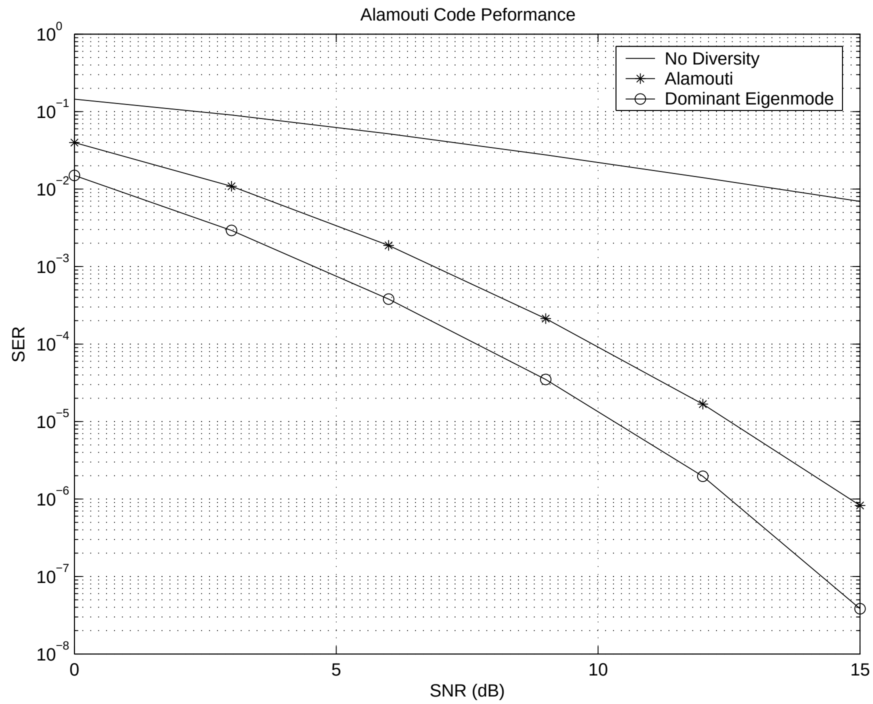
$$\|h_{\text{eff}}\|^2 = \frac{\|\mathbf{H}\|_F^2}{2}$$

where $\|\mathbf{H}\|_F^2$ is the squared Frobenius norm of the channel (cf. (5.40) in the textbook).

c) For dominant eigenmode transmission,

$$\|h_{\text{eff}}\|^2 = \lambda_{max}$$

where λ_{max} is the maximum eigenvalue of $\mathbf{H}\mathbf{H}^H$.




```
% EE492m Homework 3 no. 3
% Kome Oteri
clear all
close all
SnrdB = 0:3:15;
Ne = 1;
dmin = 2;
ResultsSISO = zeros(1,length(SnrdB));
ResultsAC = zeros(1,length(SnrdB));
ResultsED = zeros(1,length(SnrdB));

max_iter = 10000
for iter = 1:max_iter,
    h = sqrt(1/2)*(randn(1) + j*randn(1));
    for count = 1:length(SnrdB),
        rho = 10.^(SnrdB(count)/10);
        eta = rho * abs(h).^2;
        x = sqrt((eta *dmin^2)/2);
        ResultsSISO(count) = ResultsSISO(count)+ 0.5* Ne* erfc(x/sqrt(2)) ;
    end
end
semilogy(SnrdB,ResultsSISO/max_iter);

for iter = 1:max_iter,
    H = sqrt(1/2)*(randn(2,2) + j*randn(2,2));
    for count = 1:length(SnrdB),
        rho = 10.^(SnrdB(count)/10);
        etaAC = rho * (norm(H,'fro').^2)/2;
        xAC = sqrt((etaAC *dmin.^2)/2);
        ResultsAC(count) = ResultsAC(count)+ 0.5* Ne* erfc(xAC/sqrt(2)) ;
        a = svd(H);
```

```
        etaED = rho *a(1).^2;
        xED = sqrt((etaED *dmin.^2)/2);
        ResultsED(count) = ResultsED(count)+ 0.5*  Ne* erfc(xED/sqrt(2)) ;
    end
end
hold on
semilogy(SnrdB,ResultsAC/max_iter,'-*');
semilogy(SnrdB,ResultsED/max_iter,'-o');
% grid on
legend('No Diversity','Alamouti','Dominant Eigenmode',0)
xlabel ('SNR (dB)')
ylabel ('SER')
title('Alamouti Code Peformance')
```

Problem 5

a) The plots for ergodic and outage capacity are shown in Fig. 1 and 2. The pair of curves at the bottom of each plot is for the case of a 2×2 system. The other pairs going up correspond to increasing number of antennas. At low SNR values the channel capacity improves dramatically with channel knowledge. Channel knowledge becomes less and less important as SNR increases. The script follows:

```
clear all;
close all;
clc;
M = 1000;
Nt = [1:2:5];

SNR = [0:2:16]; %dB
figure(1);
title('Plot of capacity for an MxM system vs SNR')
xlabel('SNR [dB]');
ylabel('Capacity');
grid on
hold on;
figure(2);
title('Plot of 10% outage capacity for an MxM system vs SNR')
xlabel('SNR [dB]');
ylabel('Outage Capacity');
grid on
hold on;
for n = 1:length(Nt),
    N = Nt(n);
    for m = 1:M,
        H = (randn(N,N)+1i*randn(N,N))/sqrt(2);
        for snr_idx = 1:length(SNR),
            rho = 10^(SNR(snr_idx)/10);

            % Find the capacity for unknown channel at Tx:
            CU(m,snr_idx) = log2(real(det(eye(N)+rho*H*H'/N)));

            % Find the capacity for known channel at Tx:
            [gamma,eigs] = pwr_modes(H,rho);
            CK(m,snr_idx) = sum(log2(real(1+eigs.*gamma*rho/N)));
```

```
        end
    end
    C_unknown(:,n) = mean(CU)';
    C_known(:,n) = mean(CK)';

    for snr_idx = 1:length(SNR),

        [cdf_u,co_u] = hist(CU(:,snr_idx),100);
        cdf_u = cumsum(cdf_u);
        idx_ten_percent = find(abs(cdf_u-100)==min(abs(cdf_u-100)));
        C_unknown_outage(snr_idx,n) = co_u(idx_ten_percent(1));

        [cdf_k,co_k] = hist(CK(:,snr_idx),100);
        cdf_k = cumsum(cdf_k);
        idx_ten_percent = find(abs(cdf_k-100)==min(abs(cdf_k-100)));
        C_known_outage(snr_idx,n) = co_k(idx_ten_percent(1));
    end

    figure(1);
    plot(SNR,C_unknown(:,n));
    plot(SNR,C_known(:,n),'r:');

    figure(2);
    plot(SNR,C_unknown_outage(:,n));
    plot(SNR,C_known_outage(:,n),'r:');

    if n == 1
        figure(1)
        legend('Channel Unknown','Channel known');
        figure(2)
        legend('Channel Unknown','Channel known');
    end

    pause
end

hold off
figure
plot(Nt,C_unknown(6,:))
hold on
plot(Nt,C_known(6,:),'r:')
title('Plot of capacity vs number of antennas - channel known and unknown')
xlabel('Number of antennas')
ylabel('Capacity');
legend('Channel Unknown','Channel known');
grid on


function [g,l] = pwr_modes(H,rho)

N_tilde = size(H,1);
l_tilde = real(eig(H*H'));

l = l_tilde(find(l_tilde~=0));
N = length(l);
```

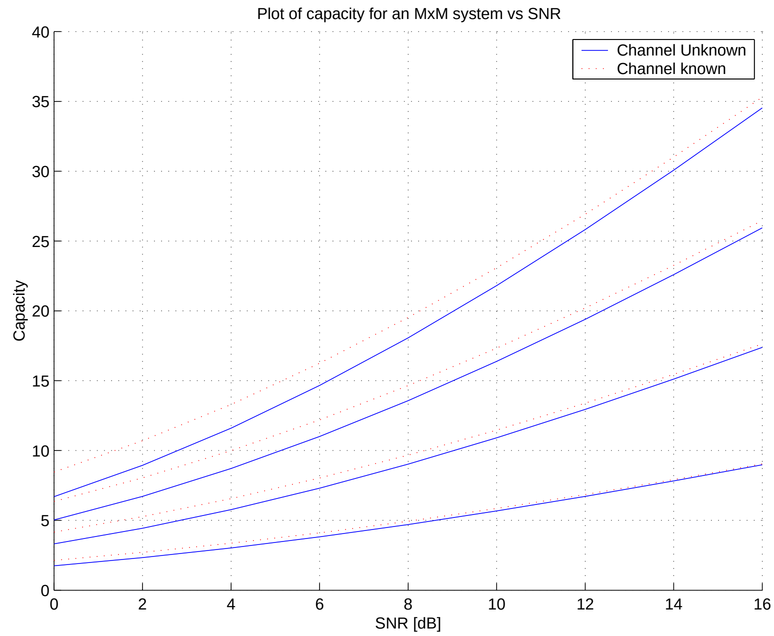


Figure 1: Ergodic capacity vs SNR. Channel known (dotted line) and channel unknown(solid line)

```
mu = (N_tilde+sum(N_tilde./(rho*l)))/N;
g = mu-N_tilde./(rho*l);

while (length(find(g <= 0)) ~= 0)
    l = l(find(g > 0));
    N = length(l);
    mu = (N_tilde+sum(N_tilde./(rho*l)))/N;
    g = mu-N_tilde./(rho*l);
end
```

b) Fig 3 shows the ergodic capacity as a function of the number of antennas, M, at 10 dB SNR. Here, channel knowledge becomes more important for increasing M. This is because there are more modes in the channel if there are more antennas. Thus the gain with waterfilling increases.

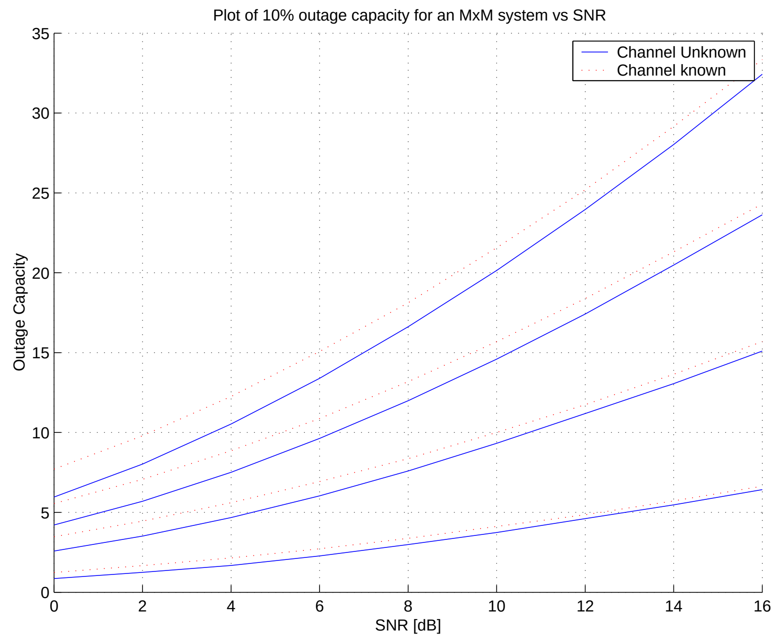


Figure 2: 10% outage capacity vs SNR. Channel known (dotted line) and channel unknown(solid line)

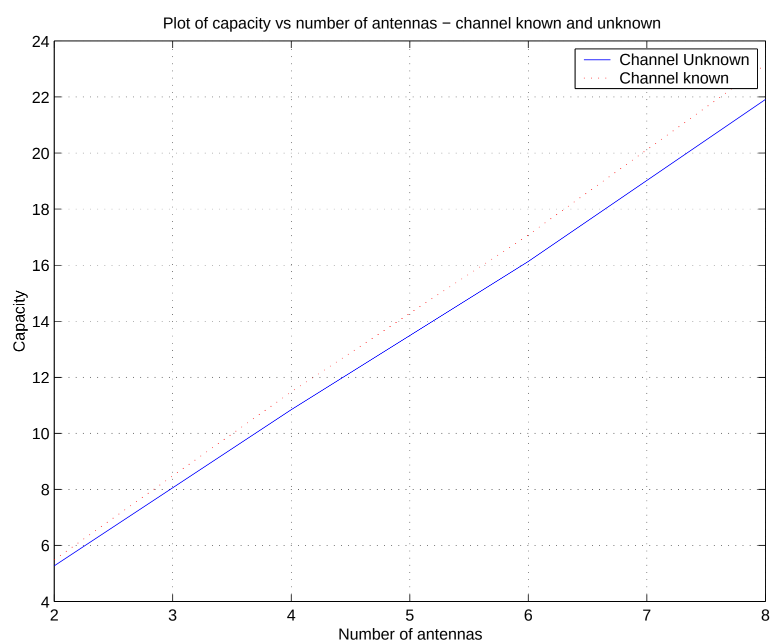


Figure 3: Ergodic capacity vs number of antennas. Channel known (dotted line) and channel unknown(solid line)