Practice Set 4 Solutions

Problem 1 The transmit covariances for the various cases are given as follows.

a) When the transmitter has no knowledge of the channel, use $\mathbf{\Sigma} = \frac{P}{2} I$

b) With no phase information, it is impossible to correctly aim a beam towards the receiver. Thus it is best to transmit equal power to all spatial directions. Mathematically, the power transmitted to angle θ is given by $\mathrm{E}\{|\mathbf{a}^H(\theta)\mathbf{s}|^2\}$, and should be independent of θ. Here, $\mathbf{a}^H(\theta)$ is the array steering vector and is of the form $[1, e^{j2\pi\Delta\sin(\theta)/\lambda}]^T$ for a two-element array.

Let

$$\mathbf{R_{ss}} = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

be the transmit covariance matrix. Then,

$$\begin{aligned} \mathrm{E}\{|\mathbf{a}^H(\theta)\mathbf{s}|^2\} &= \mathbf{a}(\theta)^H \mathbf{R_{ss}} \mathbf{a}(\theta) \\ &= a + c + 2\,\mathrm{Re}(b e^{j2\pi\Delta\sin(\theta)/\lambda}) \end{aligned}$$

If follows that $b = 0$ and thus $\mathbf{R_{ss}}$ is a diagonal matrix. Plugging into the capacity formula, we obtain

$$\begin{aligned} C &= \log\left(1 + \frac{E_s}{N_0 M_T}[h_1, h_2]\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} h_1^* \\ h_2^* \end{bmatrix}\right) \\ &= \log\left(1 + \frac{E_s}{N_0 M_T}\left(|h_1|^2 a + |h_2|^2 b\right)\right), \end{aligned}$$

which we should maximize subject to $\mathrm{trace}(\mathbf{R_{ss}}) = a + c \leq M_T$. Clearly, C is maximized by

$$\mathbf{R_{ss}} = \begin{cases} \begin{bmatrix} M_T & 0 \\ 0 & 0 \end{bmatrix} & \text{if } |h_1| \geq |h_2|, \\ \begin{bmatrix} 0 & 0 \\ 0 & M_T \end{bmatrix} & \text{if } |h_1| < |h_2|. \end{cases}$$

The transmit covariance matrix $\mathbf{\Sigma}$ is then given as

$$\mathbf{\Sigma} = \begin{cases} \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} & \text{if } |h_1| \geq |h_2|, \\ \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} & \text{if } |h_1| < |h_2|. \end{cases}$$

c) When the transmitter has full knowledge of the channel, use MRC at the transmitter. This can also be shown using the SVD.

$$\mathbf{\Sigma} = \frac{P}{2(|h_1|^2 + |h_2|^2)}\begin{bmatrix} |h_1|^2 & h_1^* h_2 \\ h_2^* h_1 & |h_2|^2 \end{bmatrix}$$

Problem 2

a) Fig. 1 plots the capacity versus SNR for different receive correlation values. Here, we use equation (4.39) in the book with

$$\mathbf{R}_t = \mathbf{I} \text{ and } \mathbf{R}_r = \begin{bmatrix} 1 & \rho \\ \rho^* & 1 \end{bmatrix}.$$

The capacity for the case with no correlation ($\rho = 0$) and the case with $\rho = 0.2$ have almost identical capacities. Use equation (4.40) in the book to see why the two capacities are so similar (the high SNR difference is only about 0.06 bps/Hz). Only higher values for correlation seem to affect the ergodic capacity. The Matlab script follows:

```
clear all
close all
clc
M = 1000;
rho = [0 .2 .8];
SNR = [0:2:16];

s = {'b-','r-.','k:'};
figure;
xlabel('SNR [dB]');
ylabel('Capacity');
title('Capacity vs SNR with and without correlation')
grid on
hold on;

for l = 1:length(rho),
    l
    R_t = eye(2);
    R_r = [1 rho(l);rho(l) 1]
    for snr_idx = 1:length(SNR),
        snr = 10^(SNR(snr_idx)/10);
        for m = 1:M,
            Hw = (randn(2,2)+1i*randn(2,2))/sqrt(2);
            H = R_r^(.5)*Hw*R_t^(.5);
            C(m,snr_idx)=log2((det(eye(2)+snr*H*H'/2)));
        end
        Capacity(snr_idx,l) = mean(C(:,snr_idx));
    end
    linetype = s(l);
    plot(SNR,Capacity(:,l),deal(linetype{:}));
    pause
end

legend('\rho=0','\rho=0.2','\rho=0.8');
```

b) Fig. 2 plots the capacity vs SNR for various K-factors. As K-factor increases the capacity at high SNR values decreases. I used the following script:

```
clear all
```

```
close all
clc
M = 1000;
K = [0 1 10];
SNR = [0:2:16];
H0 = [1 1; 1 1];
s = {'b-','r-.','k:'};
figure;
xlabel('SNR [dB]');
ylabel('Capacity');
title('Capacity vs SNR with and without K-factor')
grid on
hold on;

for l = 1:length(K),
    l
    k = K(l);
    for snr_idx = 1:length(SNR),
        snr = 10^(SNR(snr_idx)/10);
        for m = 1:M,
            Hw = (randn(2,2)+1i*randn(2,2))/sqrt(2);
            H = sqrt(k/(k+1))*H0+sqrt(1/(k+1))*Hw;
            C(m,snr_idx)=log2((det(eye(2)+snr*H*H'/2)));
        end
        Capacity(snr_idx,l) = mean(C(:,snr_idx));
    end
    linetype = s(l);
    plot(SNR,Capacity(:,l),deal(linetype{:}));
    pause
end

legend('K=0','K=1','K=10');
```

Problem 3 *Alamouti Code and Channel Estimation Errors*

a) With Alamouti coding, the received signals can be expressed as

$$\begin{bmatrix} y_1 \\ y_2^* \end{bmatrix} = \sqrt{\frac{E_s}{2}} \begin{bmatrix} h_1+\epsilon_1 & h_2+\epsilon_2 \\ h_2^*+\epsilon_2^* & -h_1^*-\epsilon_1^* \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}$$

$$\mathbf{y} = \sqrt{\frac{E_s}{2}}\mathbf{Hs}+\mathbf{n}$$

With a channel estimate $\widehat{\mathbf{H}}$, Alamouti decoding becomes

$$\widehat{\mathbf{y}} = \sqrt{\frac{E_s}{2}}\widehat{\mathbf{H}}^H\mathbf{H}+\widehat{\mathbf{H}}^H\mathbf{n},$$

where the channel estimate is

$$\widehat{\mathbf{H}} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}.$$

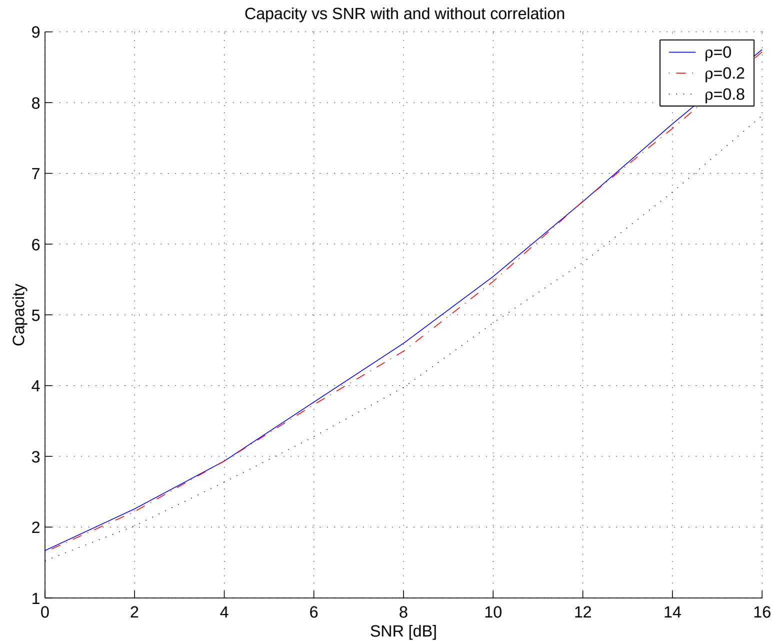

Figure 1: Capacity vs SNR with and without correlation.

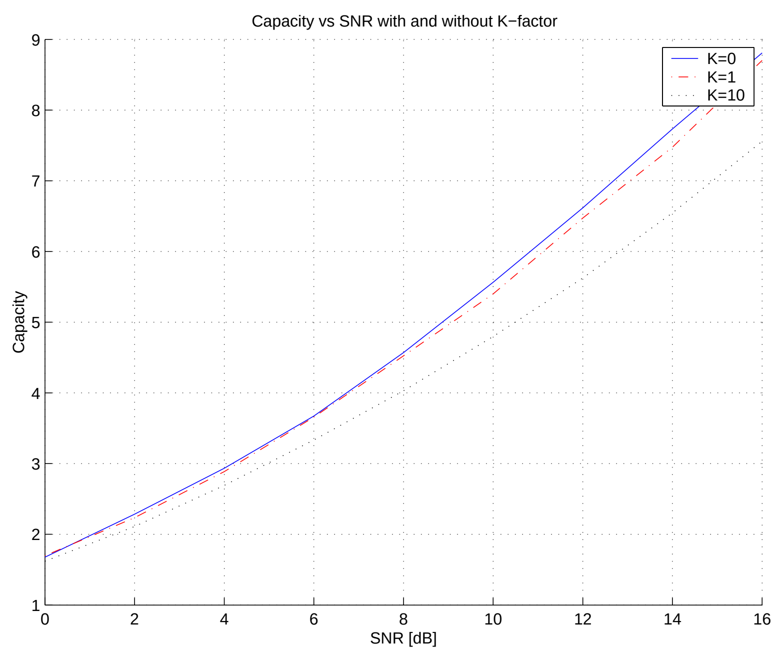

Figure 2: Capacity vs SNR with and without K-factor.

On decoding we have

$$\widehat{\mathbf{H}}\mathbf{H} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}^H \begin{bmatrix} h_1+\epsilon_1 & h_2+\epsilon_2 \\ h_2^*+\epsilon_2^* & -h_1^*-\epsilon_1^* \end{bmatrix}$$
$$= \begin{bmatrix} |h_1|^2 + h_1^*\epsilon_1 + |h_2|^2 + h_2\epsilon_2^* & h_1^*\epsilon_2 - h_2\epsilon_1^* \\ h_2^*\epsilon_1 - h_1\epsilon_2^* & |h_1|^2 + h_1\epsilon_1^* + |h_2|^2 + h_2^*\epsilon_2 \end{bmatrix}$$
$$= \begin{bmatrix} |h_1|^2+|h_2|^2 & 0 \\ 0 & |h_1|^2+|h_2|^2 \end{bmatrix} + \begin{bmatrix} h_1^*\epsilon_1 + h_2\epsilon_2^* & h_1^*\epsilon_2 - h_2\epsilon_1^* \\ h_2^*\epsilon_1 - h_1\epsilon_2^* & h_1\epsilon_1^* + h_2^*\epsilon_2 \end{bmatrix}$$

Therefore our decoded signal $\widehat{\mathbf{y}}$ becomes

$$\widehat{\mathbf{y}} = \sqrt{\frac{E_s}{2}} \begin{bmatrix} |h_1|^2+|h_2|^2 & 0 \\ 0 & |h_1|^2+|h_2|^2 \end{bmatrix}\mathbf{s} + \sqrt{\frac{E_s}{2}} \begin{bmatrix} h_1^*\epsilon_1 + h_2\epsilon_2^* & h_1^*\epsilon_2 - h_2\epsilon_1^* \\ h_2^*\epsilon_1 - h_1\epsilon_2^* & h_1\epsilon_1^* + h_2^*\epsilon_2 \end{bmatrix}\mathbf{s} + \widehat{\mathbf{H}}^H\mathbf{n}$$
$$= \tilde{\mathbf{y}}_{signal} + \tilde{\mathbf{y}}_{mismatch} + \tilde{\mathbf{n}}_{noise}$$

b) The main diagonal terms represent the noise due to channel estimation errors from the symbol itself. The off-diagonal terms represent inter-symbol interference generated by the channel estimation errors.

c) We assume that ϵ_i and s_i are independent.

$$\mathbf{y}_{signal,1} = \sqrt{\frac{E_s}{2}}(|h_1|^2+|h_2|^2)s_1 \Rightarrow (|h_1|^2+|h_2|^2)^2\frac{E_s}{2}$$
$$\mathbf{y}_{mismatch,1} = \sqrt{\frac{E_s}{2}}(h_1^*\epsilon_1 + h_2\epsilon_2^*)s_1 + \sqrt{\frac{E_s}{2}}(h_1^*\epsilon_2 - h_2\epsilon_1^*)s_2 \Rightarrow 2(|h_1|^2\sigma_e^2 + |h_2|^2\sigma_e^2)\frac{E_s}{2}$$
$$\mathbf{y}_{noise} = h_1 n_1 + h_2 n_2$$
$$= \left(|h_1|^2+|h_2|^2\right)\sigma_n^2 = (|h_1|^2+|h_2|^2)\sigma_n^2$$

The effective SNR becomes

$$SNR_{AC} = \frac{(|h_1|^2+|h_2|^2)^2 E_s/2}{(|h_1|^2+|h_2|^2)(\sigma_n^2 + 2\sigma_e^2 E_s/2)} = \frac{\frac{1}{2}\rho\|\mathbf{h}\|^2}{1+\rho\sigma_e^2}$$

d) AC is sensitive to channel estimation errors.

```
clear all
close all
symbols = [ -1 1];
eps_var = 0:0.1:1;
errors_AC = zeros(size(eps_var));

SNR = 10.^(15/10);
noise_var = 1/SNR;
max_iter = 10000
for iter = 1:max_iter,
```

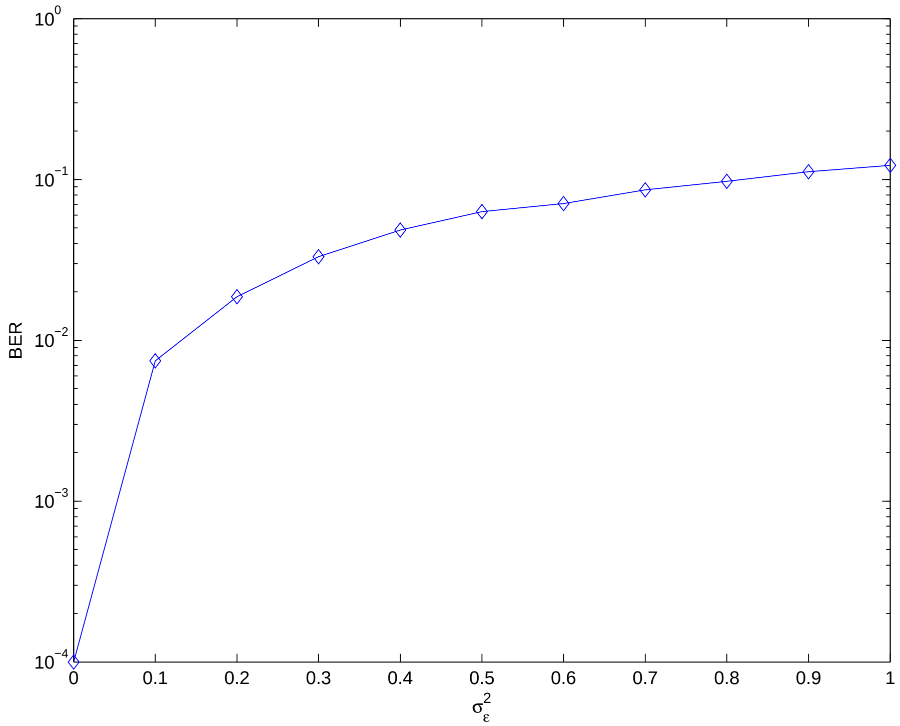

Figure 3: AC performance with estimation errors.

```
    [iter]
    h =  sqrt(0.5)* (randn(2,1)+ j* randn(2,1));
    H = [h(1) h(2) ; h(2)' -h(1)'];
    s_index = round(rand(2,1));
    s = symbols(s_index+1).';
    n = sqrt(noise_var/2)*(randn(2,1)+j*randn(2,1));
    y = H*s + n;

    for count = 1:length(eps_var),
        index = eps_var(count);
        eps = sqrt(index/2)*(randn(2,1)+j*randn(2,1));
        H_hat = [h(1)+ eps(1)   (h(2)+eps(2))
         (h(2) + eps(2))' -(h(1) + eps(1))'];
        z = H_hat'*y;
        decoded_AC = sign(real(z));
        decoded_AC(find(decoded_AC == -1)) = 0;
        s_index;
        errors = sum(abs(decoded_AC - s_index));
        errors_AC(count) = errors_AC(count) + errors;
    end
end
BER_AC = errors_AC/(max_iter*2)
```

```
semilogy(eps_var, BER_AC, '-d')
xlabel(texlabel('sigma_epsilon^2'))
ylabel('BER')
```

Problem 4 *Spatial Multiplexing*
Recall that we are using zero-forcing receivers. Let $\rho = E_s/\sigma^2$.

a) (Horizontal encoding with successive cancellation) Decoding the first stream with treating the second stream as noise yields the equivalent SINR

$$\eta_1 = \frac{\rho}{M_T} \cdot \frac{1}{[(\mathbf{H}^H\mathbf{H})^{-1}]_{1,1}} \approx 0.1285\rho$$

Thus the mutual information for this stream is

$$I_1^{\text{HE}} \approx \log_2(1 + 0.1285\rho)$$

After cancelation of the first stream, the second stream experiences only thermal noise. With $\mathbf{h}_2$ being the second column of $\mathbf{H}$, we have

$$\eta_2 = \frac{\rho}{M_T}\|\mathbf{h}_2\|^2 \approx 1.125\rho$$

Thus the mutual information for this stream is

$$I_2^{\text{HE}} \approx \log_2(1 + 1.125\rho)$$

b) (Diagonal encoding) Recall the encoding structure

$$\begin{bmatrix} S_{11} & S_{21} & 0 \\ 0 & S_{12} & S_{22} \end{bmatrix}$$

When S_{11} is decoded, only thermal noise will play a role. Thus

$$\eta_{11} = \frac{\rho}{M_T}\|\mathbf{h}_1\|^2 \approx 0.5312\rho$$

where $\mathbf{h}_1$ is the first column of $\mathbf{H}$. For S_{12}, the component S_{21} will act as interference, leading to

$$\eta_{12} = \frac{\rho}{M_T} \cdot \frac{1}{[(\mathbf{H}^H\mathbf{H})^{-1}]_{2,2}} \approx 0.2721\rho$$

Next, for S_{21}, perfect cancelation applies and only thermal noise remains,

$$\eta_{21} = \frac{\rho}{M_T}\|\mathbf{h}_1\|^2 \approx 0.5312\rho$$

Finally, S_{22} does not experience interference by construction. Thus

$$\eta_{22} = \frac{\rho}{M_T}\|\mathbf{h}_2\|^2 \approx 1.125\rho$$

where again, $\mathbf{h}_2$ is the second column of $\mathbf{H}$. The average mutual information for the two data streams are thus

$$\begin{aligned} I_1^{\text{DE}} &\approx 1/2\ \log_2(1 + 0.5312\rho) + 1/2\ \log_2(1 + 0.2721\rho) \\ I_2^{\text{DE}} &\approx 1/2\ \log_2(1 + 0.5312\rho) + 1/2\ \log_2(1 + 1.125\rho). \end{aligned}$$

Problem 5 *(Channel correlations)* Consider a 2×2 MIMO channel

(a) The transmit and receive correlation matrices can be expressed as follows:

$$\mathbf{R}_T = \begin{bmatrix} 1 & \alpha \\ \alpha^* & 1 \end{bmatrix}, \text{ and } \mathbf{R}_R = \begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix}$$

Then the Kronecker product can be expanded as:

$$\mathbf{R}_T^T \otimes \mathbf{R}_R = \begin{bmatrix} 1 & \beta & \alpha^* & \alpha^*\beta \\ \beta^* & 1 & \alpha^*\beta^* & \alpha^* \\ \alpha & \alpha\beta & 1 & \beta \\ \alpha\beta^* & \alpha & \beta^* & 1 \end{bmatrix}$$

Consider a general 4×4 correlation matrix $\mathbf{R} = [r_{ij}]$. In order for the relation in question to hold one must have:

$$\begin{aligned} r_{14} &= r_{24}r_{12} \Rightarrow E[h_{11}h_{22}^*] = E[h_{21}h_{22}^*]E[h_{11}h_{21}^*] && (1) \\ r_{23} &= r_{13}r_{21} \Rightarrow E[h_{21}h_{12}^*] = E[h_{11}h_{12}^*]E[h_{21}h_{11}^*] && (2) \end{aligned}$$

and

$$\begin{aligned} r_{13} &= r_{24} \Rightarrow E[h_{11}h_{12}^*] = E[h_{21}h_{22}^*] && (3) \\ r_{12} &= r_{34} \Rightarrow E[h_{11}h_{21}^*] = E[h_{12}h_{22}^*] && (4) \end{aligned}$$

(b) The first set of relation implies that if there is both transmit and receive correlation, the correlation between channel elements without any common transmitter and receiver will be non-zero. The second set of relations means that in order to compute the transmit correlation, it doesn't matter which receive antenna is chosen as the reference. Additionally, it doesn't matter which transmit antenna is chosen as the reference for computing the receive correlation. This model assumes that the antennas in each set of transmit antennas interact with the same scatterers. The same is true for the receive antennas.