Practice Set 2 Solutions

Problem 1 Use the following equations to solve the problem.

$$C_{SIMO} = \log(1 + \frac{E_s}{N_o}\sum_{j=1}^{M_R}\|h_j\|^2)$$

$$C_{MISO} = \log(1 + \frac{E_s}{M_T N_o}\sum_{j=1}^{M_T}\|h_j\|^2)$$

Solutions are $C_{SIMO} = 7.6726$ and $C_{MISO} = 6.6797$

Problem 2 Use the following equations to solve the problem.

$$C_{MIMO} = \log_2 \det\left(\mathbf{I}_{M_R} + \frac{E_s}{M_T N_o}\mathbf{H}\mathbf{H}^H\right)$$

Solution is $C_{MIMO} = 14.8927$

Problem 3

```
clc
clear all
close all
figure
SNR_dB=10;  SNR_linear=10.^(SNR_dB/10.);
N_iter=50000; sq2=sqrt(0.5); grps = ['m-'; 'b-'];
for Icase=1:2
   if Icase==1,  nT=2; nR=2;  % 2x2
    else   nT=4; nR=4;       % 4x4
   end
   n=min(nT,nR);  I = eye(n);
   for iter=1:N_iter
      H = sq2*(randn(nR,nT)+j*randn(nR,nT));
      C(iter) = log2(real(det(I+SNR_linear/nT*H'*H)));
   end
   [PDF,Rate] = hist(C,50);
   PDF = PDF/N_iter;
   CDF(Icase,:)=cumsum(PDF);
   plot(Rate,CDF(Icase,:),grps(Icase,:),'LineWidth',1.2); hold on
end
xlabel('Rate[bps/Hz]'); ylabel('CDF')
axis([1 18 0 1]); grid on; set(gca,'fontsize',10);
legend('{\it M_T}={\it M_R}=2','{\it M_T}={\it M_R}=4');
```

See Figure 1 for output.

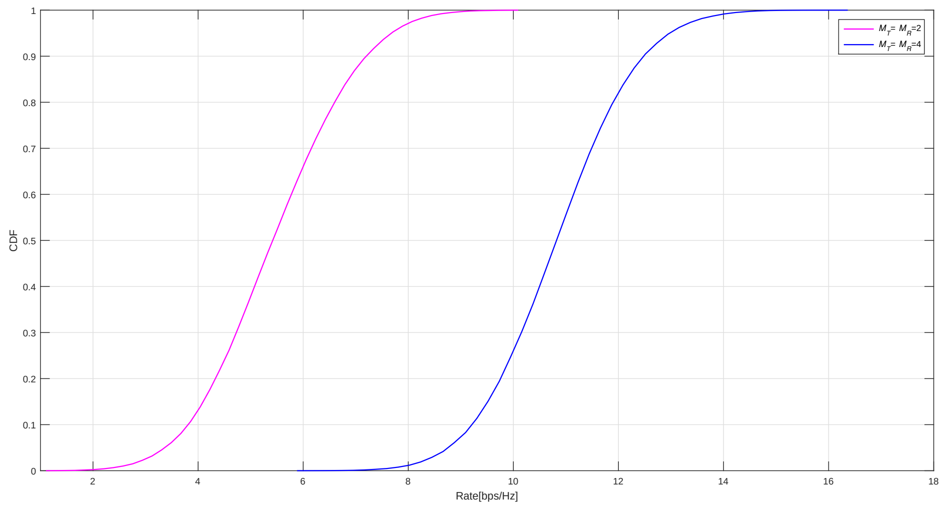

Figure 1: CDF

Problem 4

```
clc
clear all,  close all
SNR_dB=[0:5:20];  SNR_linear=10.^(SNR_dB/10.);
N_iter=1000;
for Icase=1:5
   if Icase==1,  nT=4; nR=4;      % 4x4
    elseif Icase==2, nT=2;  nR=2; % 2x2
    elseif Icase==3, nT=1;  nR=1; % 1x1
    elseif Icase==4, nT=1;  nR=2; % 1x2
    else   nT=2;  nR=1;          % 2x1
   end
   n=min(nT,nR);  I = eye(n);
   C(Icase,:) = zeros(1,length(SNR_dB));
   for iter=1:N_iter
      H = sqrt(0.5)*(randn(nR,nT)+j*randn(nR,nT));
      if nR>=nT,  HH = H'*H;  else  HH = H*H';  end
      for i=1:length(SNR_dB) %random channel generation
         C(Icase,i) = C(Icase,i)+log2(real(det(I+SNR_linear(i)/nT*HH)));
      end
   end
end
C = C/N_iter;
figure, plot(SNR_dB,C(1,:),'g-o', SNR_dB,C(2,:),'k-<', SNR_dB,C(3,:),'b-s','LineWidth',
hold on, plot(SNR_dB,C(4,:),'m->', SNR_dB,C(5,:),'r-^','LineWidth',1.2);
xlabel('SNR[dB]'); ylabel('bps/Hz');
s1='{\it M_T}=1,{\it M_R}=1'; s2='{\it M_T}=1,{\it M_R}=2';
```

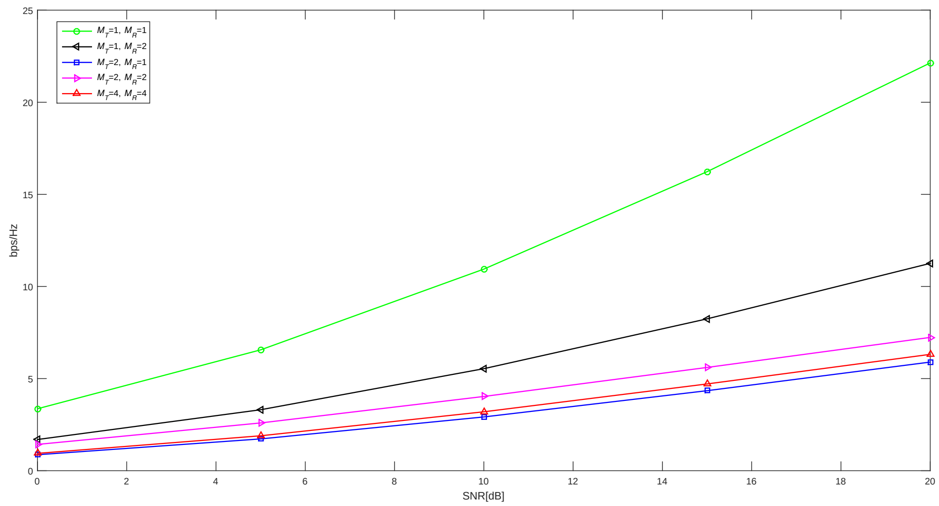

Figure 2: Ergodic Capacity Vs SNR

```
s3='{\it M_T}=2,{\it M_R}=1'; s4='{\it M_T}=2,{\it M_R}=2'; s5='{\it M_T}=4,{\it M_R}=4
legend(s1,s2,s3,s4,s5)
```

See Figure 2 for output.

Problem 5

```
clc
clear all
close all
SNR_dB=[0:5:20];  SNR_linear=10.^(SNR_dB/10);
N_iter=1000;  N_SNR=length(SNR_dB);
nT=2;  nR=2;  n=min(nT,nR);  I = eye(n);  sq2=sqrt(0.5);
Rr=[1        0.7 ;
   0.7     1    ];
Rt=[1        0.5 ;
   0.5     1    ];
C_22_iid=zeros(1,N_SNR);  C_22_corr=zeros(1,N_SNR);
for iter=1:N_iter
   H_iid = sq2*(randn(nR,nT)+j*randn(nR,nT));
   H_corr = Rr^(1/2)*H_iid*Rt^(1/2);
   tmp1 = H_iid'*H_iid/nT;  tmp2 = H_corr'*H_corr/nT;
   for i=1:N_SNR
      C_22_iid(i) = C_22_iid(i) + log2(det(I+SNR_linear(i)*tmp1));
      C_22_corr(i) = C_22_corr(i) + log2(det(I+SNR_linear(i)*tmp2));
   end
end
```

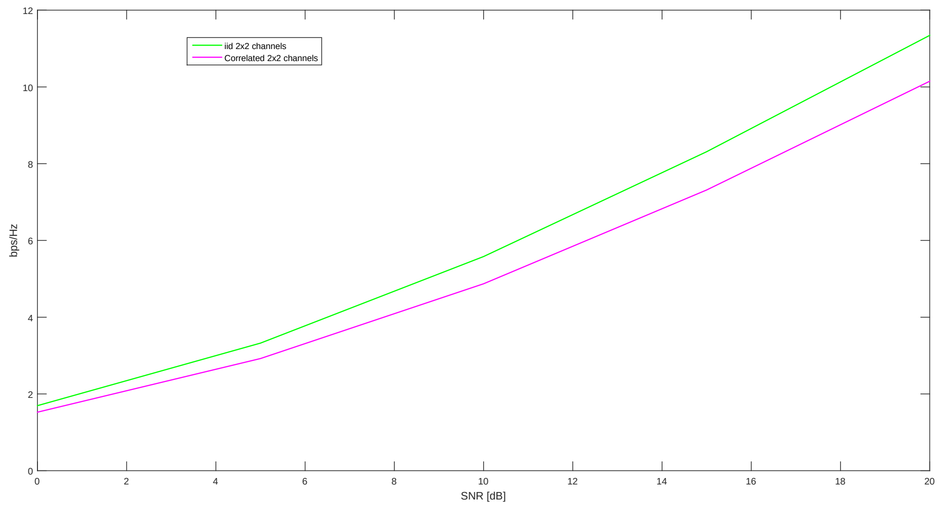

Figure 3: Impact of correlated and iid channels

```
C_22_iid = real(C_22_iid)/N_iter;  C_22_corr = real(C_22_corr)/N_iter;
plot(SNR_dB,C_22_iid,'-g', SNR_dB,C_22_corr,'-m','LineWidth',1.2);
xlabel('SNR [dB]'); ylabel('bps/Hz');
legend('iid 2x2 channels','Correlated 2x2 channels');
```

See Figure 3 for output.

Problem 6

```
clc,clear all,close all
SNR_dB=10;
SNR_linear=10.^(SNR_dB/10);
K_dB=[-10:2:20];
N_iter=50000;  N_K=length(K_dB);
nT=2;  nR=2;  n=min(nT,nR);  I = eye(n);  sq2=sqrt(0.5);
H1_bar=ones(nR,nT);
H2_bar=H1_bar;
H2_bar(1,2,:)=-1;
C_22_1=zeros(1,N_K);  C_22_2=zeros(1,N_K);
for i=1:N_K
 K=10^(K_dB(i)/10);
 for iter=1:N_iter
   H_w=sq2*(randn(nR,nT)+j*randn(nR,nT));
   H_1 = sqrt(K/(K+1)).*H1_bar + sqrt(1/(K+1)).*H_w;
   H_2 = sqrt(K/(K+1)).*H2_bar + sqrt(1/(K+1)).*H_w;
   tmp1 = H_1'*H_1/nT;
   tmp2 = H_2'*H_2/nT;
   C_22_1(i) = C_22_1(i)+log2(real(det(I+SNR_linear*tmp1)));
```

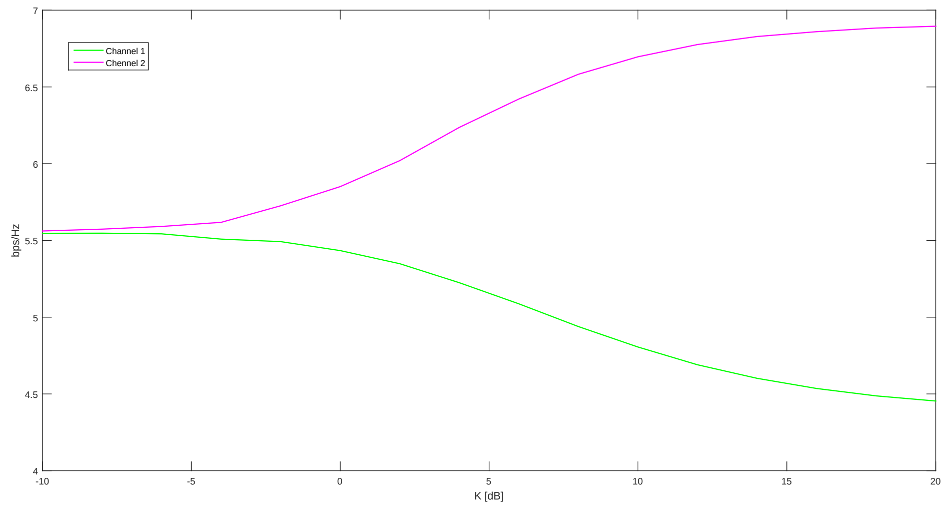

Figure 4: Ergodic capacity vs K-factor for a MIMO channel with H_1 and H_2 LOS components. The channel geometry has a significant impact on capacity at a high K-factor.

```
   C_22_2(i) = C_22_2(i)+log2(real(det(I+SNR_linear*tmp2)));
  end
end
C_22_1 = C_22_1/N_iter;  C_22_2 = C_22_2/N_iter;
plot(K_dB,C_22_1,'-g', K_dB,C_22_2,'-m','LineWidth',1.2);
xlabel('K [dB]'); ylabel('bps/Hz');
legend('Channel 1','Chennel 2');
```

See Figure 4 for output.

Problem 7 Find XX^H

$$\mathbf{XX}^H = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

From the result we can conclude that $\mathbf{X}$ is a orthogonal matrix.

Problem 8 Let $x_1 = 1 + j$, $x_2 = -1 + j$ and $\mathbf{h} = [h_1 \quad h_2]$ then

$$\mathbf{X} = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix} = \begin{bmatrix} 1+j & 1+j \\ -1+j & 1-j \end{bmatrix}$$

Received signal vector is $\mathbf{Y} = [y_1 \quad y_2^*]$, Where

$$y_1 = h_1 x_1 + h_2 x_2$$

$$y_2 = -h_1 x_2^* + h_2 x_1^*$$

Channel matrix is

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} = \begin{bmatrix} 0.2362 - 1.0244j & -0.3048 + 0.2587j \\ -0.3048 - 0.2587j & -0.3048 - 0.2587j \end{bmatrix}$$

We know $|\mathbf{h}|^2 = 1.2650$

$\tilde{\mathbf{X}}$ can be estimated as follows,

$$\widehat{\mathbf{X}} = \mathbf{H^H Y}$$

$$\widehat{\mathbf{X}} = \mathbf{H^H Y} = \begin{bmatrix} 1+j \\ -1+j \end{bmatrix}$$

Problem 9 *(MISO array gain)*

a) Applying the power constraint, we have $\mathcal{E}(\mathbf{x}^H\mathbf{x}) = 2a^2 < P$, so that the optimal gain is $a = \sqrt{P/2}$. The received signal for the MISO system is then $y = 2as + n = \sqrt{2P}s + n$ with signal to noise ratio $\mathrm{SNR_{MISO}} = 2P/N_0$.

The reference signal to noise ratio in a SISO system with transmit power P, channel gain $H = 1$ and noise power N_0 is $\mathrm{SNR_{SISO}} = P/N_0$, and hence the MISO transmit array gain is $\mathrm{SNR_{MISO}}/\mathrm{SNR_{SISO}} = 2 = 3\,\mathrm{dB}$.

b) In this case, the received signal is $y = -2as+n = -\sqrt{2P}s+n$, with $\mathrm{SNR_{MISO}} = 2P/N_0$. Thus, the transmit array gain is still $3\,\mathrm{dB}$.

c) With $h_1 = 1$, $h_2 = -1$ and $\mathbf{x} = a[s,s]^T$, the received signal is $y = n$. The signal to noise ratio is 0, and thus array gain is $0 = -\infty\,\mathrm{dB}$. Not only do we get no transmit array gain, we are actually worse than the SISO system. This particular choice of transmit signal leads to cancellation of the desired signal.

With $\mathbf{x} = s[a_1, a_2]^T$, the received signal is $y = (a_1 - a_2)s + n$ with signal to noise ratio $(a_1 - a_2)^2/N_0$. The power constraints evaluates to $a_1^2 + a_2^2 \leq P$. To find the optimal $[a_1, a_2]$, solve the SNR maximization

$$\begin{array}{rl} \text{maximize} & (a_1 - a_2)^2/N_0 \\ \text{subject to} & a_1^2 + a_2^2 \leq P. \end{array}$$

This is a variant of the matched filtering problem. An optimal solution is $a_1 = \sqrt{P/2}$ and $a_2 = -\sqrt{P/2}$, which achieves the optimal value $\mathrm{SNR_{MISO}} = 2P/N_0$, ths recovering the array gain $2 = 3\,\mathrm{dB}$. Hence, transmit array gain is possible, but it requires channel knowledge at the transmitter. Only channel knowledge allows us to avoid signal cancellation and achieve coherent signal superposition.

Problem 10 *(Singular Values of* $\mathbf{H}$*)* The code for this problem is given:

```
clear all;
close all;

M = 10000;
```

```
for m = 1:M,
    H = (randn(4,4)+1i*randn(4,4))/sqrt(2);
    l_min(m) = min(eig(H*H'));
end

[pdf,X] = hist(l_min,100)
pdf = pdf/sum(pdf*(X(2)-X(1)));
pdf_theoretical = 4*exp(-4*X);
plot(X,pdf,'.')
hold on
plot(X,pdf_theoretical,'r');
```

Fig. 5 shows the distribution of the smallest eigenvalue of $\mathbf{HH}^H$ using Monte Carlo simulations as well as using the theoretical result.

One can derive a lower bound on the capacity of a MIMO channel which is a function of the smallest eigenvalue only. The distribution of this lower bound thus depends on the pdf of the smallest eigenvalue of $\mathbf{HH}^H$.

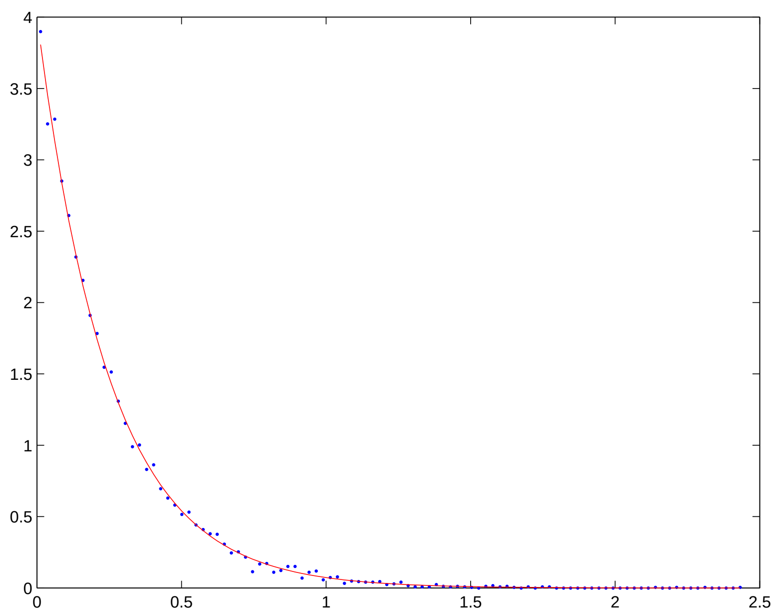

Figure 5: Distribution of minimum eigenvalue of a 4×4 $\mathbf{HH^H}$ using both the formula (solid red) and simulation (blue dots)