Numerical Analysis - Project 2

Tej Sai Kakumanu

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1 Lagrange Interpolation

To find a polynomial that interpolates the following points using lagrange interpolation:

i	x_i	$f(x_i)$
0	-1	0
1	0	-1
2	1	0
3	2	15

Firstly, we define the lagrange interpolates for 4 points using:

$$L_i(x) = \frac{\prod_{j=0,j\neq i}^{3} x - x_i}{\prod_{j=0,j\neq i}^{3} x_i - x_j}$$

Therefore, the fours lagrange interpolants would be:

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

Since we know the final polynomial generated by Lagrange interpolants is given by

$$P(x) = \sum_{i=0}^{3} L_i(x) \times f(x_i)$$

By solving this summation and substituting the $f(x_i)$ and $L_i(x)$ values we get the following polynomial:

$$P(x) = -\frac{(x+1)(x-1)(x-2)}{2} + \frac{5x(x+1)(x-1)}{2}$$

$$P(x) = -\frac{(x+1)(x-1)(x-2) + 5x(x+1)(x-1)}{2}$$

$$P(x) = \frac{(x+1)(x-1)(-x+2+5x)}{2}$$

$$P(x) = \frac{(x+1)(x-1)2(2x+1)}{2}$$

$$P(x) = (x+1)(x-1)(2x+1)$$

$$P(x) = (x^2-1)(2x+1)$$

$$P(x) = 2x^3 + x^2 - 2x - 1$$

2 Divided Difference

To find the polynomial that interpolates the same points, we use the following formulae for calculating each divided difference:

$$f[x_i] = y$$

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

$$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] = \frac{f[x_{i+1}, x_{i+2}, x_{i+3}] - f[x_i, x_{i+1}, x_{i+2}]}{x_{i+3} - x_i}$$

Now we calculate the required divided differences:

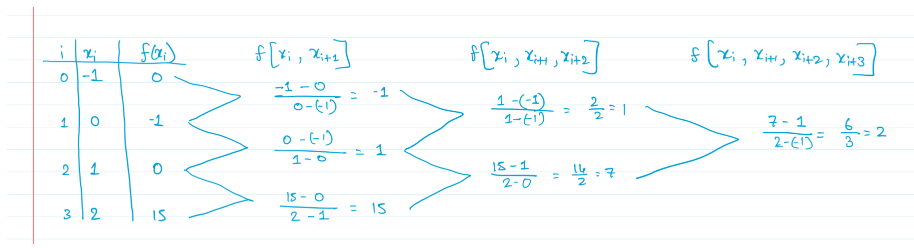

Figure 1: Newton's Divided Differences

Then, we can substitute the values found in the divided difference into the following polynomial formula:

$$P(x) = f[x_0]+f[x_0, x_1](x-x_0)+f[x_0, x_1, x_2](x-x_0)(x-x_1)+f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$$

After substitution we get,

$$P(x) = 0 + (-1)(x - (-1)) + 1(x - (-1))(x - 0) + 2(x - (-1))(x - 0)(x - 1)$$

And by simplifying further, we get:

$$P(x) = 2x^3 + x^2 - 2x - 1$$

3 Linear Algebra Technique

Since we have 4 points, We can assume that the polynomial we are looking for is of at least 3^{rd} degree and we can express it as:

$$P(x) = a_1 + a_2x + a_3x^2 + a_4x^3$$

By plugging in the nodes and function values in $P(x)$ we get the following linear equations:

For $x = -1, f(x) = 0$,

$$a_1 - a_2 + a_3 - a_4 = 0$$

For $x = 0, f(x) = -1$,

$$a_1 = -1$$

For $x = 1, f(x) = 0$,

$$a_1 + a_2 + a_3 + a_4 = 0$$

For $x = 2, f(x) = 15$,

$$a_1 + 2a_2 + 4a_3 + 8a_4 = 15$$

Now we can represent these equations in the following matrix form:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 15 \end{bmatrix} \tag{1}$$

By solving the augmented matrix associated with this matrix equation we get the following values:

$$a_1 = -1, a_2 = -2, a_3 = 1, a_4 = 2$$

And by substituting these values into the polynomial equation we get:

$$P(x) = 2x^3 + x^2 - 2x - 1$$

Therefore, as this polynomial matches with the two polynomials found in the parts (1) and (2), we can conclude that they satisfy each other.

4 Comparing Lagrange and Divided Difference Interpolation

Lagrange Interpolation starts from an n^{th} order polynomial and does not require any memorization of previous calculations. On the other hand, Newton's divided difference depends on the previous calculation of the divided differences and requires fewer calculations than Lagrange, it is also beneficial when dealing with dynamic nodes since there would be not many calculations to be modified and added to re construct the interpolation. However, Lagrange is beneficial when we are dealing with static nodes and small set of points since it is a more straightforward approach than the Newton's Divided Difference.

5 Errors of Polynomial Interpolation

5.1 Error

For polynomial $P_n(x)$ of degree n with points $x_0, x_1, x_2, ..., x_n$ and corresponding function values $f_0, f_1, f_2, ..., f_n$, the error can be represented using the following expression.

$$f(x) - P_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

If f were an $n^t h$ order polynomial, the first term in the error $\frac{f^{n+1}(\xi)}{(n+1)!}$ indicates some constant value since $(n+1)^{th}$ derivative of n^{th} order polynomial must be a constant.

Furthermore, the prod function indicate that whenever $x = x_i$ the error of the polynomial interpolation is zero.

5.2 The function to prove

To prove this error expression, we can setup a function of y for each x:

$$\Omega(y; x) = f(y) - P_n(y) - \Phi(x)w(y)$$

where,

$$w(y) = \prod_{i=0}^{n}(y - x_i)$$

$$\Phi(x) = \frac{f(x) - P_n(x)}{w(x)}$$

5.3 Proving that x nodes are all zeroes

And by substituting them back into the original function we get,

$$\Omega(y; x) = f(y) - P_n(y) - \frac{f(x) - P_n(x)}{w(x)} \prod_{i=0}^{n}(y - x_i)$$

when $y = x_i$, i=0,1,2,...n

$$\Omega(x_i; x) = f(x_i) - P_n(x_i) - \frac{f(x) - P_n(x)}{w(x)} \prod_{i=0}^{n}(x_i - x_i)$$

$$\Omega(y; x) = f(x_i) - P_n(x_i) - 0$$

Now we know from the theorem that $f(x_i) - P_n(x_i) = 0$ for i=0,1,2,...,n

$$\Omega(y; x) = 0$$

, Hence all the n+1 nodes are zeroes of

$$\Omega(y; x)$$

5.4 Proving that y=x is a zero

$$\Omega(x;x) = f(x) - P_n(x) - \frac{f(x) - P_n(x)}{w(x)} w(x)$$

$$\Omega(x;x) = f(x) - P_n(x) - (f(x) - P_n(x))$$

$$\Omega(x;x) = 0$$

y = x is a zero of $\Omega(y;x)$

5.5 Derivation

Since $\Omega(y;x)$ has n+ 2 zeroes, we know (by Rolle's theorem) that the n+ 1 derivative of $\Omega(y;x)$ has at least one zero. Denote the zero of $\frac{d^{n+1}\Omega}{dy^{n+1}}$ by ξ.

$$0 = \frac{d^{n+1}\Omega}{dy^{n+1}}\xi = f(\xi) - P_n(\xi) - \frac{f(x) - P_n(x)}{w(x)} \frac{d^{n+1}}{dy^{n+1}} w(y)$$

$$f(\xi) - P_n(\xi) - \frac{f(x) - P_n(x)}{w(x)} \frac{d^{n+1}}{dy^{n+1}} \prod_{i=0}^{n} (\xi - x_i)$$

Since $P_n(x)$ is an n^{th} order polynomial $P_n^{n+1}(\xi) = 0$

$$\frac{d^{n+1}\Omega}{dy^{n+1}}\xi = f(\xi) - 0 - \frac{f(x) - P_n(x)}{w(x)} (n+1)!$$

5.6 Rearrange to get the theorem expression

$$0 = f(\xi) - 0 - \frac{f(x) - P_n(x)}{w(x)} (n+1)!$$

$$\frac{f(x) - P_n(x)}{w(x)} (n+1)! = f^{n+1}(\xi)$$

$$f(x) - P_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!} w(x)$$

Substituting back the $w(x)$ value we get,

$$f(x) - P_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

6 Evaluating the difference between Polynomial Interpolation and cubic splines

6.1 Observing Runge Phenomenon

The function $f(x) = \frac{1}{1+25x^2}$ with 100 points in spcae as a constant, shows the runge phenomenon as the degree of the interpolation polynomial or the number of equidistance points of the function increase. It is also observed when the points in space are equal to the number of nodes of the function, the runge phenomenon disappears as the ends of the interval get stabilized but a spike in the middle part of the function is created. The figure below shows the interpolating polynomial generated using Lagrange technique when the points in space and the nodes are equal to 100 when compared to the original function $f(x) = \frac{1}{1+25x^2}$.

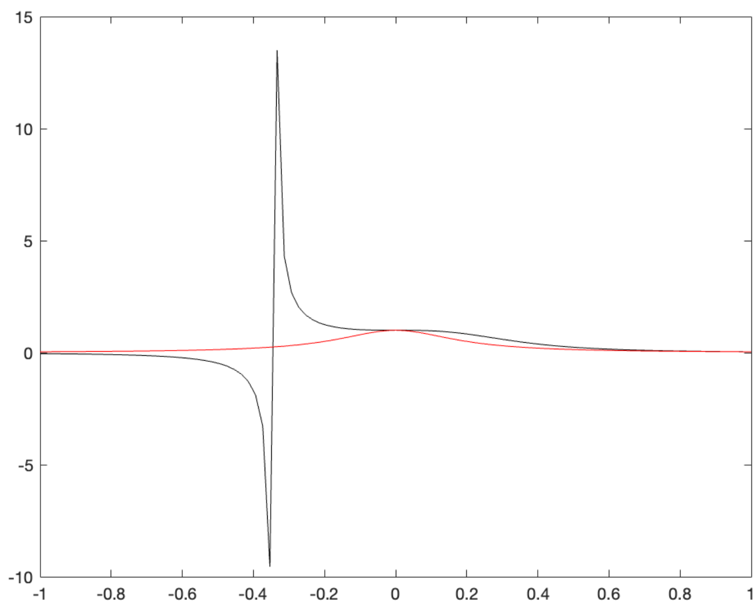

Figure 2: Lagrange Interpolation Polynomial

6.2 Comparing the Interpolation with cubic splines

When used the Matlab function to graph the polynomial generated through cubic splines in comparison to the polynomial generated through Lagrange interpolation, it is apparent that cubic spline polynomial does not imitate the same runge phenomenon observed in the other one and gives a smoother function.

The following are the figures as the number of nodes increases with black indicating Lagrange polynomial interpolation and green indicating the cubic spline polynomial:

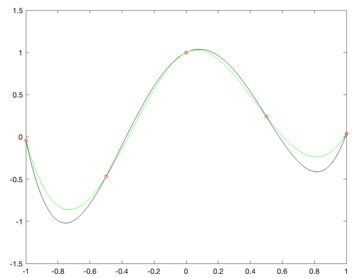

Figure 3: 5 Interpolation nodes

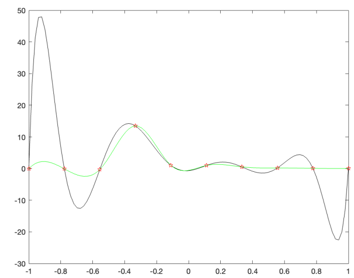

Figure 4: 10 Interpolation nodes

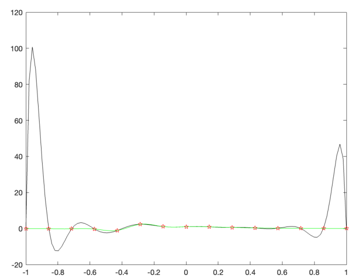

Figure 5: 15 Interpolation nodes

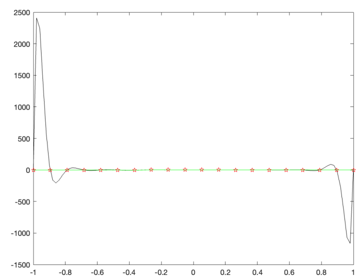

Figure 6: 20 Interpolation nodes