General Problem Setting

The *Pendulum-Cart* system defines a classic problem in dynamics and control theory and is widely used as a benchmark for testing control algorithms. A simplified variation of this system is illustrated in Figure 2 where a mathematical pendulum is connected to a pair of only-rolling wheels with one translation degree of freedom. This models the *Wheeled Inverted Pendulum* system. An example for that is the Robot *Suricate* (Figure 1) built by the students at LRS / University of Kaiserslautern.

In the simplified model, the point mass m_p is connected to the cart with the mass m_c via a massless arm with the length L. q_1 is the displacement of the cart and q_2 is the angle of the pendulum. As input, u is assumed to be a force.

Abbildung 1: Suricate (JIM2)

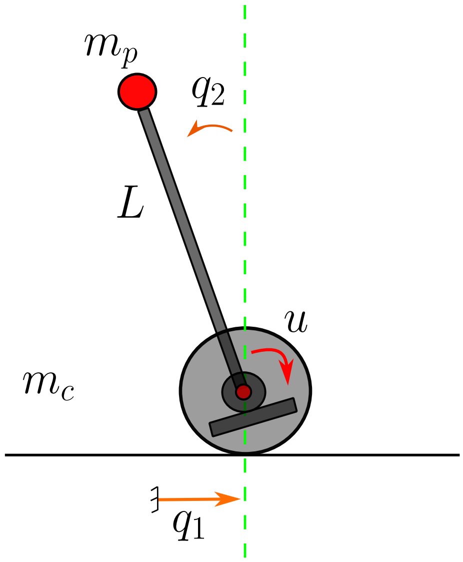

Abbildung 2: Pendulum-Cart System

In our exercises, we will handle the stabilization problem around the upper position of the pendulum. We will only consider the linearised model. The overall objective will be broken down into the following tasks:

- Creating the dynamical system using MATLAB.
- System Analysis, controller design using MATLAB tools and simulation.
- Design of state-space and optimal controllers and simulation.
- Design of state observers and investigating the closed loop including an observer.
- Time-discrete controller and observer design.

The results will be verified by simulation using SIMULINK.

Dynamic Model of the System

The equations of motion for the described system can be derived using different methods such as Lagrange-Formalism or Newton-Euler. The result is in any case a nonlinear system in the following form:

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{g}_q\,\boldsymbol{u}$$

with

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_c + m_p & -L\,m_p\,\cos q_2 \\ -L\,m_p\,\cos q_2 & L^2\,m_p \end{pmatrix}$$

$$\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} L\,m_p\,\dot{q}_2^2\,\sin q_2 \\ -L\,g\,m_p\,\sin q_2 \end{pmatrix} + \begin{pmatrix} d_1\,\dot{q}_1 \\ d_2\,\dot{q}_2 \end{pmatrix}\ ,\ \boldsymbol{g}_q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where d_1 and d_2 are the so-called *damping factors* representing friction in the cart displacement and the joint respectively.

A state-space representation can be derived using the equations above defining $\boldsymbol{x} = \begin{pmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{pmatrix}$

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\,u = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ -\boldsymbol{M}^{-1}(\boldsymbol{q})\,\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \boldsymbol{M}^{-1}(\boldsymbol{q})\,\boldsymbol{g}_q \end{pmatrix} u$$

Linearising about the operation point $q_1^* = q_2^* = 0$ yields the linear state-space equation for the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{B}\,u \tag{1}$$

with

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{g\,m_p}{m_c} & -\dfrac{d_1}{m_c} & -\dfrac{d_2}{L\,m_c} \\ 0 & \dfrac{g\,(m_c+m_p)}{L\,m_c} & -\dfrac{d_1}{L\,m_c} & -\dfrac{d_2\,(m_c+m_p)}{L^2\,m_c\,m_p} \end{pmatrix}\ ,\ \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{m_c} \\ \dfrac{1}{L\,m_c} \end{pmatrix}.$$

We the following values:

$$m_c = 1.5,\ m_p = 0.5,\ g = 9.82,\ L = 1,\ d_1 = d_2 = 0.01$$