Exercice 1 : 1.

Let $(E, \leq)$ be a partially ordered set. Let $\mathcal{C}$ be the set of well-founded chains of $(E, \leq)$. We define a binary relation R on $\mathcal{C}$ as follows : for all $C_1, C_2 \in \mathcal{C}$ we put $C_1 \; R \; C_2$ if $C_1 \subseteq C_2$ and for all $x \in C_1$ and $y \in C_2 \setminus C_1$ we have $x \leq y$.

(a) Which type of relation is R ?

(b) Show that all the chains $\{C_i\}_{i \in I}$ in $(\mathcal{C}, R)$ have a least upper bound in $(\mathcal{C}, R)$.

(c) Deduce that there is a chain of $(E, \leq)$ which is a maximal element of $(\mathcal{C}, R)$.

2. Let $(E, \leq)$ a lattice for which any well-founded chain has an upper bound.

(a) Show that $(E, \leq)$ has a least element $\perp$.

(b) Let A be a subset of E. Let B be the set of all elements smaller that all elements in A . Show that if B has a maximum element b, then b is the greatest element of B.

(c) Show that B has a maximal element b. (you may use question 1.1.c)

(d) Show that $(E, \leq)$ is a complete lattice.

Monoids.

Exercice 2 :

1. Show that $(\mathcal{P}(E \times E), \circ, id_E)$, where $R \circ S := RS := \{(x, z) \in E \times E \mid \exists y \in E, \, xRySz\}$ is a monoid.
2. Under what condition a lattice is a monoid, if we take the superior of two elements as the rule of composition ?
3. Show that the product of two monoids is a monoid, where the binary operation is just the operation on each of the elements separately.
4. $(\mathbb{Z}/6\mathbb{Z}, \cdot, 1)$ is a monoid. Show that $(\{0, 2, 4\}, \cdot, 4)$ is a monoid.

Exercice 3 :

Prove the following statements :

1. The composition of two morphisms of monoid is a morphism.
2. The inverse of a bijective monoid morphism is a monoid morphism.
3. The image of a sub-monoid is a sub-monoid.
4. The inverse image of a sub-monoid is a sub-monoid.

Exercice 4 : 1.

An equivalence relation $\sim$ on a monoid M is congruence ($x \sim y$ iff $uxv \sim uyv \; \forall v, u \in M$) iff for all $x \sim x' \wedge y \sim y' \Rightarrow xy \sim x'y'$.

2. Let $f : M \to N$ be a monoid morphism. Show that if $x \sim y \Leftrightarrow f(x) = f(y)$, then $\sim$ is a congruence.

Let Σ a finite alphabet.

Exercice 5 :

Let u and v two words in Σ^*. show by induction on $|u|+|v|$ that $uv = vu \Rightarrow \exists w \in \Sigma^*, \{u, v\} \subseteq w^*$.

Exercice 6 :
Let m and n natural numbers > 0. Solve in Σ^* the equation $u^m = v^n$.

Exercice 7 :
Let u and v be two words in Σ^* , we say that they are conjugate if there exist x and y such that $u = xy$ and $v = yx$. Show that the words u and v are conjugate iff there exists a word z such that $uz = zv$.

Exercice 8 :
Consider the three words x, y, z in Σ^* such that $x^2y^2 = z^2$. Show that there exists a word w in Σ^* and numbers p and q such that $x = w^p$, $y = w^q$ and $z = w^{p+q}$.

Exercice 9 :
Let M be a finite monoid and let $x \in M$.

1. Show that there two natural numbers m and n such that $m < n$ and $x^m = x^n$.
2. We choose a minimal l from all the numbers n for which there exists $m < n$ such that $x^m = x^n$.
   (a) Show that $1, x, ..., x^{l-1}$ are all distinct.
   (b) Show that the monoid $< x >$ is of cardinality l.
   (c) Let $k < l$ such that $x^k = x^l$. Let r be the unique integer between k and $l - 1$ divisible by $l - k$. Show that $x^k, ..., x^{l-1}$ is a cyclic group of order $l - k$ where x^r is the natural element.
   (d) Show that there exists n such that $x^n = (x^n)^2$ i.e. idempotent. Are there several ?

Exercice 10 (Syntactic monoid) :
Let $L \subset \Sigma^*$ be a language. This defines the equivalence relation on Σ^* :

$$w \sim_L w' \Leftrightarrow \forall u, v \in \Sigma^*,\ uwv \in L \Leftrightarrow uw'v \in L$$

Justify that $\sim_L$ is a congruence on Σ^*. We define the Syntactic monoid M_L as the quotient $\Sigma^*|_{\sim_L}$.

Exercice 11 (Language recognized by a monoid) :
Let $L \subset \Sigma^*$ a language. Let M be a monoid. We say that a language L is recognizable by M if there exists a monoid morphism φ of Σ^* to M and a set X of M such that $L = \varphi^{-1}(X)$.

1. Show that a language recognized by a finite monoid is regular.
2. Show that a language L is recognized by its syntactic monoid.
3. Show that a language L is recognised by a monoid M iff M_L is isomorphic to a sub monoid of M.
4. Deduce the characterization of regular languges relating to their syntactic monoid.

Exercice 12 (12) :
Let A be a set. We consider the free monoid $(A^*, \cdot, \epsilon)$ defined on the alphabet A by concatenation and the empty word. The set $C \subseteq A^*$ is a code if the following condition holds : for all $c_1, \dots, c_n, d_1, \dots, d_p \in C$, if $c_1 \dots c_n = d_1 \dots d_p$ then $n = p$ and $c_i = d_i$ for all $i \in [\![1, n]\!]$. Let $X \subseteq A^*$. Which of these assertion imply which assertions ?

1. X is a code.
2. For all sets B and a morphisme $\varphi : B^* \to A^*$ such that $\varphi|_B : B \to X$ (i.e. the restriction of φ to B) is bijective, φ is injective.
3. there exists a set B and an injective morphism $\varphi : B^* \to A^*$ such that $\varphi[B] = X$.