Probability

Exercice 1 :

Let $(\Omega, \mathcal{T}, \mathrm{P})$ be a probability space.

Show that for all sequences $(A_n)_{n\in\mathbb{N}}$ of events we have $\mathrm{P}(\cup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \mathrm{P}(A_n)$ where the right sum can diverge.

Exercice 2 :

Let $(\Omega, \mathcal{T}, \mathrm{P})$ be a probability space.

1. Show that for all sequences $(A_n)_{n\in\mathbb{N}}$ of growing events by inclusion, the sequence of $\mathrm{P}(A_n)$ converges and $\lim_{n\to\infty} \mathrm{P}(A_n) = \mathrm{P}(\cup_{n\in\mathbb{N}} A_n)$
2. Show for all decreasing sequences (by inclusion) $(A_n)_{n\in\mathbb{N}}$ of events, the sequence $\mathrm{P}(A_n)$ converges and $\lim_{n\to\infty} \mathrm{P}(A_n) = \mathrm{P}(\cap_{n\in\mathbb{N}} A_n)$

Exercice 3 :

Let $\{B_1, \ldots, B_n\}$ be a partition of Ω such that for all i, $\mathrm{P}(B_i) > 0$. Show that for all events A we have $\mathrm{P}(A) = \sum_{i=1}^{n} \mathrm{P}(A \mid B_i)\mathrm{P}(B_i)$. Keeping B_i's disjoint but not a partition of Ω, what condition condition can we add for this to remain true ?

Exercice 4 :

An urn contains b black balls, w white balls and r red balls. We pick two balls, what is the probability of the event "the second ball drawn is black" ?

Exercice 5 :

We consider two six-sided dice, one is balanced, the other is rigged(loaded dice). We denote p_i the probability that the rigged die falls on the face i ($i \in \{1, 2, 3, 4, 5, 6\}$).

1. Describe the probability space.
2. (a) What is the probability of rolling a double ?
   (b) What is the probability that the sum of the dice is equal to 7 ?

Exercice 6 :

Let $\{B_1, \ldots, B_n\}$ be a partition of Ω such that $\mathrm{P}(B_i) > 0$ for all i. Then for every event A such that $\mathrm{P}(A) > 0$ we have

$$\mathrm{P}(B_i \mid A) = \frac{\mathrm{P}(A \mid B_i)\mathrm{P}(B_i)}{\sum_{j=1}^{n} \mathrm{P}(A \mid B_j)\mathrm{P}(B_j)}$$

Exercice 7 :

"Suppose you're on a game show, and you're given the choice of three doors : Behind one door is a car ; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2 ?" Is it to your advantage to switch your choice ?" -Parade magazine in 1990

Exercice 8 :

We throw three balanced dice.

1. What is the most likely sum of the three dice ?
2. What is the expected value of the sum ?

Exercice 9 :
How big a group of people needs to be in order for the probability of two of them having the same birth day is greater then half ?

Exercice 10 :
Let $s, n \in \mathbb{N}$, $s < n$ and K_n the complete graph on n vertices. We randomly and independently color all the edges by either red or blue with the probability 1/2.

1. What is the probability for of a complete subgraph on s vertices to be monochromatic.
2. Show that for $n = \lfloor 2^{(s-1)/2} \rfloor$ the probability for K_n to have a monochromatic complete subgraph of size s is less then 1.
3. Conclude that the Ramsey number $R(s, s) > \lfloor 2^{(s-1)/2} \rfloor$.

Exercice 11 :
Let $f : E \to F$, show that

1. $f^{-1}[F] = E$ and $f^{-1}[\emptyset] = \emptyset$.
2. $f^{-1}[F \setminus B] = E \setminus f^{-1}[B]$ for all $B \subseteq F$.
3. $f^{-1}[\cup_n B_n] = \cup_n f^{-1}[B_n]$ and $f^{-1}[\cap_n B_n] = \cap_n f^{-1}[B_n]$.

Exercice 12 :
Let U and V two independent random variables on the same probability space with values in $\{-1, 1\}$ and the same probability distribution defined by :

$$P(U = -1) = \frac{1}{3} \text{ and } P(U = 1) = \frac{2}{3}$$

The X and Y the random variables defined by : $X = U$ et $Y = UV$

1. What is the distribution of the random variable (X, Y) ?
2. Are X and Y independent ?
3. Are the variables X^2 and Y^2 independent ?

Exercice 13 :
Let X and Y be two independent random variables defined on the same probability space of the same geometric distribution p ($P(X = k) = p(1-p)^k$).

1. Calculate $P(Y \geq X)$, for $p = \frac{1}{2}$.
2. Calculate $P(Y = X)$, for $p = \frac{1}{2}$.

We define the random variables U and V by

$$U = \max(X, Y) \text{ and } V = \min(X, Y)$$

3. For all $u \leq v \in \mathbb{N}$ calculate $P(U \leq u, V \geq v)$.
4. Calculate the distribution of the random variable U.

Exercice 14 :
Turán's theorem : Let G be any graph with n vertices, such that G is K_{t+1}-free. Then the number of edges in G is at most :

$$\left(1 - \frac{1}{t}\right) \frac{n^2}{2}$$

Denote by $d(v)$ the degree of the vertex v.

1. Let $<$ be a uniformly chosen total order on V. Define :

$$I = \{v \in V \mid \{v, u\} \notin E \Rightarrow v < u\}$$

For $v \in V$, what is the probability that $v \in I$.

2. What is the expected value of $|I|$?
3. Denote by $\omega(G)$ the size of largest complete sub graph of G. Show that $\omega(G) \geq \sum_{v \in V} \frac{1}{n-d(v)}$.
4. Recall, Cauchy–Schwarz inequality $(\sum a_i b_i)^2 \leq (\sum a_i^2)(\sum b_i^2)$. Show that :

$$n^2 \leq \omega(G) \sum_{v \in V} (n - d(v))$$

and conclude the theorem.

Exercice 15 :

We suppouse that $P : \mathbb{N} \to [0, 1]$ is a probability distribution. This equates to the having a sequence of positive numbers $(p_n)_{n \in \mathbb{N}}$ such that $\sum_n p_n = 1$, with $p_n = P(n)$.

1. Prove that there is no probability distribution on $\mathbb{N}$ that makes the drawing of an integer equiprobable.
2. Suppose $p_n = \frac{1}{2^{n+1}}$.
   (a) Demonstrate that we have a probability distribution on $\mathbb{N}$.
   (b) Calculate the probability of the set of even numbers and the set of odd numbers.
3. Assume we have a probability distribution on $\mathbb{N}$ satisfying the following properties :

$$\forall a \in \mathbb{N} \setminus \{0\}, P(a\mathbb{N}) = \frac{1}{a}.$$

Prove that if a and b are co-prime, the events "all multiples of a" and the event "all multiple of b" are independent.

Exercice 16 :

Information is transmitted in a population. Each individual transmits the correct information with probability p, and the negation of the information with probability $1 - p$. Let p_n be the probability that the information is correct after n repetitions.

1. Calculate value p_n as a function of p and n.
2. Calculate $\lim_n p_n$.

Exercice 17 :

A smoker has two boxes of matches with the same capacity in each of his two pockets. When he needs a match, he chooses equally one of his pockets and takes the match in the corresponding box. We denote by n the number of matches initially in each box.

1. Let r be a number between 0 and n. Denote by p_r the probability that when for the first time the smoker finds that one box is empty, the other contains exactly r matches. Calculate p_r.
2. For r between 0 and n, q_r is the probability that one box will be emptied, while the other contains exactly r matches. Calculate the probability q_r.
3. What is the probability q that the first empty box is not the first box that the smoker finds empty.

Exercice 18 :
If M is a monoid and K, L two subsets of M, we denote by $L^{-1}K = \{x \in M \mid \exists y \in L, yx \in K\}$ and $KL^{-1} = \{x \in M \mid \exists y \in L, xy \in K\}$.

1. Let L a sub monoid of Σ^*. Show that L is a free monoid iff $L^{-1}L \cap LL^{-1} = L$.

   Solution :
   Suppose that L is free on the set B. Let $m \in L^{-1}L \cap LL^{-1}$. There exist p, q in L s.t. $mq, pm \in L$. Moreover since $(pm)q = p(mq) \in L$, we get that $m \in L$

   Conversely, suppose that $L^{-1}L \cap LL^{-1} = L$. Let B be the minimal generating parts of L (The elements of L which are not a product of two distinct elements, which are not 1). Let $u_1 \dots u_m = v_1 \dots v_n$ with u_i and v_j in B. Assume for example that in Σ^*, $u_m = wv_n$, that $u_1 \dots u_{m-1}w = v_1 \dots v_{n-1}$, then $w \in L^{-1}L \cap LL^{-1} = L$. By the minimialty of elements in B of L, $w = 1$. we conclude by recurrence.

2. Let L be a sub-monoid of Σ^*. We define by recursion :
   - $M_0 = L$
   - $M_{n+1} = \langle {M_n}^{-1}M_n \cap M_n{M_n}^{-1} \rangle$

   Demonstrate that this is a well defined increasing sequence and that $\cup_N M_n$ is the smallest free sub-monoid containing L.

   Solution :
   We note that for any L, $L \subset M^{-1}M \cap MM^{-1}$ ($\forall u \in M, 1u \in M$ and $u1 \in M$) therefore $(M_n)_{n=0}^{\infty}$ is well defined increasing sequence of monoids. Hence $M = \cup_n M_n$ is a monoid.

   We show that $M^{-1}M \cap MM^{-1} \subset M$: let $u \in \Sigma^*$ s.t. there exists v and w in M s.t. $vu \in M$ and $uv \in M$. $M = \cup_n M_n$, hence there exists an integer l and m s.t. $v \in M_l$ and $w \in M_m$. For $n = \max(l, m)$, v and w are in M_n, hence $u \in M_n^{-1}M_n \cap M_nM_n^{-1} \subset M_{n+1} \subset M$. Therefore M is free.

   Finally, if $N \subset P$ is an inclusion of sub-monoids, with P free, we have $N^{-1}N \cap NN^{-1} \subset P^{-1}P \cap PP^{-1} = P$, so $\langle N^{-1}N \cap NN^{-1} \rangle \subset \langle P \rangle = P$,hence if P contains L, it contains all M_n and therefore M : M is the smallest free sub monoid containing L.