Equivalence relations.

Exercice 1 :
Can we do without one of the three properties required to define equivalence relations ?

Exercice 2 :
Let f be an involution (i.e. $f \circ f = Id$) on a finite set E.

1. Show that the cardinality of all the fixed points of E is of the same parity as the cardinal of E.
2. Deduce that if E is of an odd cardinal, f has (at least) a fixed point.

Exercice 3 :
A relationship is given by a matrix $T \in \{0,1\}^{N \times N}$. Where if $T[i,j] = 1$ then $i \mathcal{R} j$, and $T[i,j] = 0$, otherwise. What conditions does the matrix needs to exhibit in order for the relationship to be an equivalence relation.

Exercice 4 :
Show that the following relation $\sim$ on $(\mathbb{Z} \times (\mathbb{N} \setminus \{0\}))$:

$$(m,n) \sim (p,q) \Leftrightarrow mq = np$$

is an equivalence relation and show that the following operations respect the equivalence relation :

1. $(m,n) \oplus (p,q) = (mq + np, nq)$
2. $(m,n) \cdot (p,q) = (mp, nq)$

Exercice 5 (Équivalence de Nérode) :
Let $\mathcal{A} = (Q, \Sigma, \delta, i_o, F)$ a deterministic automaton, where Q represents the set of states, Σ the alphabet, δ the transition function, i_o the initial state and F the set of final states.
Let $\sim$ be an equivalence relation on Q s.t. :

- For all states $p, q \in Q$, if $p \sim q$, then for all a in Σ, $\delta(p,a) \sim \delta(q,a)$.
- If $p \in F$, for all q in $\mathcal{A}$ s.t. $p \sim q$, then q is also in F.

For all states p, denote their equivalence class by $C(p)$.

1. (a) Demonstrate that the following is a deterministic automaton :

   $$\mathcal{A}_\sim = (Q/\sim, \Sigma, \delta_\sim, C(i_o), F/\sim)$$

   where $Q/\sim$ is the set of the equivalence classes, $\delta_\sim(C(p), a) = C(\delta(p,a))$, $F/\sim$ is the set of equivalence classes of the states of F.

   (b) Demonstrate that $L(\mathcal{A}) = L(\mathcal{A}_\sim)$.

2. We define for all integers n, an equivalence relation on Q : For all states p and q of $\mathcal{A}$, $p \sim_n q$ if for all the words $|w| \leq n$, $\delta(p,w) \in F \Leftrightarrow \delta(q,w) \in F$. Show that :

   (a) For all the states p and q of $\mathcal{A}$, $p \sim_{n+1} q \Rightarrow p \sim_n q$.

   (b) There exists n s.t. $\sim_{n+1} = \sim_n$.

Order relations.

Exercice 6 :
Show that :

1. Every asymmetrical relationship($aRb \to \neg(bRa)$) is anti-symmetric and irreflexive.
2. Every strict order is asymmetrical.
3. Every trichotomous relation(one of the following holds xRy, yRx or $x = y$) is asymmetrical.

Let $(E, \leq)$ an ordered set, i.e. $\leq$ is an order on E.

- $x \in E$ is the **greatest element** of E if $\forall y \in E, y \leq x$.
- $x \in E$ is a **maximal element** of E if $\forall y \in E$, $x \leq y \Rightarrow x = y$.
- the **minimal element** and the **least element** are defined analogously.

Let $F \subseteq E$.

- $x \in E$ is a **upper bound** of F if $\forall y \in F,\ y \leq x$.
- $x \in E$ is a **lower bound** of F if $\forall y \in F,\ x \leq y$.

Exercice 7 :
Let $(E, \leq)$ an ordered set.
What is the upper bound of $\emptyset$? what is the lower bound of $\emptyset$?

Exercice 8 :

1. We consider the set of numbers whose representation in base 2 has exactly n numbers. Describe precisely using this representation the `Successeur` function.
2. Denote $\mathcal{S}_n$ the set of bijections of $\{1, ..., n\}$ to itself. Recall that the cardinality of $\mathcal{S}_n$ is $n!$.
   A bijection f of the set $\{1, ..., n\}$ to itself is coded by n numbers in base n : $f(1)f(2)\cdots f(n)$.
   The lexicographic order thus defines a total order on $\mathcal{S}_n$.
   (a) Show the entire order for the cases $n = 2$ and $n = 3$ exhaustively.
   (b) What is the smallest element of $\mathcal{S}_n$?
   (c) What is the biggest element of $\mathcal{S}_n$?
   (d) Give an algorithm that from the representation of a bijection, that calculates its position in the order.
   (e) Describe the precisely `Successeur` function.

Exercice 9 :
Let E be a set with a partial order $\leq$. Recall that an *antichain* is a subset of E in which all the elements are incomparable.

1. We consider $\mathbb{N}^2$ with the product order $((a, b) \leq (x, y) \iff a \leq x \land b \leq y)$
   (a) Show an antichain of cardinality n, for any $n > 1$.
   (b) Can we find an infinite antichain ?
2. Show that the set Σ^* with the sub-string order ($w_1 < w_2$ iff $\exists u, v \in \Sigma^*$ s.t. $w_2 = uw_1v$), has an infinite chain.

Exercice 10 :
Given $A, B \in \mathcal{P}([n])$ we say that $A < B$ iff $A \subset B$.

1. Assume that $n > 1$. Show that :

$$1 < \binom{n}{1} < \binom{n}{2} < \cdots < \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \cdots > \binom{n}{n-2} > \binom{n}{n-1} > 1$$

2. For $k \in \{1, ..., \frac{n}{2}\}$ find an antichain of cardinality $\binom{n}{k}$ in $\mathcal{P}([n])$.
3. Let A be an anti chain in $\mathcal{P}([n])$. For k in $[\![0, n]\!]$, we denote by a_k the number of sets of cardinality k in A. We will now show the Lubell-Yamamoto-Meshalkin inequality :

$$\sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \leq 1$$

   (a) Demonstrate that there are exactly $n!$ Strictly increasing sequences in $\mathcal{P}([n])$, of the form $X_0 = \emptyset \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$.
   (b) Let S be a subset of X of cardinality s. Show that there are exactly $s!(n-s)!$ strictly increasing sequences in $\mathcal{P}([n])$, of the form $X_0 = \emptyset \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$, where $X_s = S$.
   (c) Let $X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_r$ a strictly increasing sequence in $\mathcal{P}([n])$. Then there is at most one X_i in A(the antichain). By partitioning all the strictly increasing sequences $X_0 = \emptyset \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$, according to their possible intersection with A, demonstrate the Lubell-Yamamoto-Meshalkin inequality.
4. Deduce the maximal cardinality of an antichain in $\mathcal{P}([n])$.

Bonus question.

Exercice 11 :

Using Équivalence de Nérode deduce an algorithm for calculating the minimum automaton of $\mathcal{A}$.