

Orders.

Exercise 1 :

Let E be a set with a partial order \leq . Recall that an *antichain* is a subset of E in which all the elements are incomparable.

1. We consider \mathbb{N}^2 with the product order $((a, b) \leq (x, y) \iff a \leq x \wedge b \leq y)$
 - (a) Show an antichain of cardinality n , for any $n > 1$.
 - (b) Can we find an infinite antichain?
2. Show that the set Σ^* with the sub-string order ($w_1 < w_2$ iff $\exists u, v \in \Sigma^*$ s.t. $w_2 = uw_1v$), has an infinite chain.

Exercise 2 :

Given $A, B \in \mathcal{P}([n])$ we say that $A < B$ iff $A \subset B$.

1. Assume that $n > 1$. Show that :

$$1 < \binom{n}{1} < \binom{n}{2} < \dots < \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq \dots > \binom{n}{n-2} > \binom{n}{n-1} > 1$$

2. For $k \in \{1, \dots, \frac{n}{2}\}$ find an antichain of cardinality $\binom{n}{k}$ in $\mathcal{P}([n])$.
3. Let A be an anti chain in $\mathcal{P}([n])$. For k in $\llbracket 0, n \rrbracket$, we denote by a_k the number of sets of cardinality k in A . We will now show the Lubell-Yamamoto-Meshalkin inequality :

$$\sum_{k=0}^n \frac{a_k}{\binom{n}{k}} \leq 1$$

- (a) Demonstrate that there are exactly $n!$ Strictly increasing chains in $\mathcal{P}([n])$, of the form $X_0 = \emptyset \subsetneq X_1 \subsetneq X_2 \subsetneq \dots \subsetneq X_n = X$.
 - (b) Let S be a subset of X of cardinality s . Show that there are exactly $s!(n-s)!$ strictly increasing chains in $\mathcal{P}([n])$, of the form $X_0 = \emptyset \subsetneq X_1 \subsetneq X_2 \subsetneq \dots \subsetneq X_n = X$, where $X_s = S$.
 - (c) Let $X_1 \subsetneq X_2 \subsetneq \dots \subsetneq X_r$ a strictly increasing chains in $\mathcal{P}([n])$. Then there is at most one X_i in A (the antichain). By partitioning all the strictly increasing sequences $X_0 = \emptyset \subsetneq X_1 \subsetneq X_2 \subsetneq \dots \subsetneq X_n = X$, according to their possible intersection with A , demonstrate the Lubell-Yamamoto-Meshalkin inequality.
4. Deduce the maximal cardinality of an antichain in $\mathcal{P}([n])$.

Exercise 3 :

Show 2 of the following :

Let $(R_i)_{i \in I}$ a family of binary relationships on the set E . Let $R = \bigcap_{i \in I} R_i$. In other words, xRy if xR_iy for all $i \in I$.

- If one of the R_i 's is irreflexive/asymmetric/antisymmetric, then R is too.
- If all R_i 's are reflexive/symmetric/transitives, then R is too
- The intersection does not preserve totality or trichotomy.

Exercise 4 :

Show 2 of the following :

Let R be a relation on the set E , and $F \subseteq E$. If R is reflexive / symmetric / transitive / total / antisymmetric / irreflexive / asymmetric / trichotomous, then R_F is the same.

Exercise 5 :

Show 2 of the following :

If R is reflexive / symmetric / transitive / total / antisymmetric / irreflexive / asymmetric / trichotomous, Then R^{-1} is the same.

Exercise 6 :

Show the following propositions :

- If (E, \leq) is a finite ordered set and $x \in E$. There is a maximal element y in E s.t. $x \leq y$.
- If any finite part of an ordered set has a greatest element, then it is a total order.

Exercise 7 :

Show that the lexicographic product of (total) orders is a (total) order.

For all $x, y \in \prod_{i \in I} (E_i, \leq_i)$, $x \leq_{lex} y$ iff given $k := \min\{j \in I \mid x_j \neq y_j\}$ we have $x_k \leq_k y_k$.

Exercise 8 :

Let E be a set with a partial order denoted by \preceq . We say that \preceq is a well quasi-order (wqo) if from any sequence of elements of E , we can extract an infinite monotone increasing sequence. I.e. $\forall (x_i)_{i \in \mathbb{N}} \in E^{\mathbb{N}}$, there exists an increasing sub-sequence of indexes : $i_0 < i_1 < \dots < i_n < \dots$ for which the sequence $(x_{i_n})_{n \in \mathbb{N}}$ is increasing : $x_{i_0} \preceq x_{i_1} \preceq \dots \preceq x_{i_n} \preceq \dots$.

1. Show that if the order \preceq is total, then it is wqo iff any non-empty subset of E has a least element.
2. Give an example of a total ordered set which is not wqo.
3. Show that the following are equivalent :
 (def1) The ordered set (E, \preceq) , is wqo.
 (def2) For any sequence $(x_i)_{i \in \mathbb{N}}$, we can find $i < j$ s.t. $x_i \preceq x_j$.
 (def3) (i) There is no infinite sequence strictly decreasing in E ,
 (ii) There is no infinite antichain.
4. Let E, \preceq be an ordered set, we call it well founded if there is no infinite decreasing sequence. Assume that E is countable and show that the order is wqo iff the set of all antichains is countable.
5. Dickson's Lemma : Let (E_1, \preceq_1) and (E_2, \preceq_2) be a wqo set. Show that (\preceq_1, \preceq_2) is a wqo on the product $E_1 \times E_2$.
6. Higman's Lemma : Let \preceq be a wqo on Σ . Define a relationship on Σ^* as follows :

$$a_1 \dots a_m \leq_{sw} b_1 b_2 \dots b_n \Leftrightarrow \begin{cases} \exists 1 \leq i_1 < i_2 < \dots < i_m \leq n \\ a_1 \preceq b_{i_1} \wedge a_2 \preceq b_{i_2} \dots a_m \preceq b_{i_m} \end{cases}$$

- (a) Show that \leq_{sw} is an order
- (b) Show that \leq_{sw} is wqo.
7. Let E, \leq be an ordered set, and $F \subset E$ s.t. : $\forall y \in E$, if there exists $x \in F$ s.t. $x \preceq y$, then $y \in F$ (we say that the set F is *upward closed* or *upper*).
 (a) Let E, \leq be wqo. Show that a that any increasing sequence of upward closed sets is stationary, i.e. $F_1 \subseteq F_2 \subseteq \dots$ there exists i such that for any $j > i$ $F_i = F_j$.
 (b) Show that if F is an upward closed set, there exists a finite set of elements x_1, \dots, x_n in F s.t. $F = \cup_i \{y \in E, x_i \preceq y\}$.

Exercise 9 (Dilworth's theorem) :

Let (E, \leq) be a finite ordered set..

Let F be a non-empty subset of E . Denote by $\text{Max}(F)$ all of its maximal elements.

1. if F is a non-empty subset of E , show that $\text{Max}(F)$ is a non-empty subset of F .
2. If F is a non-empty subset of E , show that $\text{Max}(F)$ is an antichain and that it is maximal(inclusion-wise) of the antichains in F

We now show by induction on $|E|$, **Dilworth's theorem** :

Let k be the maximal cardinality of an antichain in E . Then E is a disjoint union of k chains(a set of comparable elements).

3. Demonstrate the result when $k = 1$.
4. Suppose that $k > 1$. Let $z \in \text{Max}(E)$ and $F = E \setminus \{z\}$. Let l the maximal cardinality of an antichain in F ; by induction assumption F is a disjoint union of chains C_1, \dots, C_l .
 - (a) Give a bound on k as a function of l .
 - (b) Let \mathcal{C} be an antichain of E . What can we say about $\mathcal{C} \cap C_i$, for $i \in [l]$?
 - (c) Let $i \in [l]$. We denote by D_i the set of elements of C_i that are in an antichain of cardinality l in F .
 - i. Show that D_i isn't empty.
Denote by y_i the maximal element of D_i .
 - ii. Show that $\{y_1, \dots, y_l\}$ is an antichain of F .
 - iii. Assume that for every i , z is incomparable to y_i . Conclude Dilworth's theorem for this case.
 - iv. Assume that there exists an i s.t. z is comparable to y_i . Let $C' = \{z\} \cup \{x \in C_i \mid x \leq y_i\}$. Show that C' is a chain and that $F \setminus C'$ does not contain an antichain of cardinality l . Conclude Dilworth's theorem.