Let Σ a finite alphabet.

Exercice 1 (Syntactic monoid) :
Let $L \subset \Sigma^*$ be a language. This defines the equivalence relation on Σ^* : $w \sim_L w' \Leftrightarrow \forall u, v \in \Sigma^*,\ uwv \in L \Leftrightarrow uw'v \in L$. Justify that $\sim_L$ is a congruence on Σ^* ($x \sim y$ iff $uxv \sim uyv$ $\forall v, u \in M$). We define the Syntactic monoid M_L as the quotient $\Sigma^*|_{\sim_L}$.

Exercice 2 (Star-free languages) :
Let Σ be a finite alphabet. The Star-free family of languages is the smallest family containing empty language, singletons and stable by union, passage to complement and concatenation.

1. Show that Σ^* is star-free.
2. Demonstrate that the intersection of two star-free languages is star-free.
3. Let $a, b \in \Sigma$ where $a \neq b$. Show that $(ab)^*$ is star-free.

We call a finite monoid aperiodic if the only group it contains is the trivial group $\{1\}$.

4. Let M be a finite monoid. Show that the following are equivalent :
   (a) The monoid M is aperiodic.
   (b) For all m in M, there exists a nonzero natural number n such that $m^{n+1} = m^n$,
   (c) There exists a non zero natural number n such that for all m in M, $m^{n+1} = m^n$.
5. Let L be a regular language and let M_L be its syntactic monoid. By the definition of a syntactic monoid, we deduce by the previous question that M_L is aperiodic if and only if, for all words u, there exists a non-zero natural number n s.t. for all words v, w, $vu^n w \in L \Leftrightarrow vu^{n+1}w \in L$. In this case we denote it by $i(L)$ the smallest natural non-zero number n such that for all words u, v, w, $vu^n w \in L \Leftrightarrow vu^{n+1}w \in L$.
   (a) Show the following(for languages L with finite $i(L)$) :
      i. $i(\{a\}) = 2$,
      ii. $i(L_1 \cup L_2) \leq \max(i(L_1), i(L_2))$,
      iii. $i(L_1 L_2) \leq i(L_1) + i(L_2) + 1$,
      iv. $i(\Sigma^* \setminus L) = i(L)$.
   (b) Deduce that a syntactic monoid of a star-free language is aperiodic.

Exercice 3 (3) :
If M is a monoid and K, L two subsets of M, we denote by $L^{-1}K = \{x \in M \mid \exists y \in L, yx \in K\}$ and $KL^{-1} = \{x \in M \mid \exists y \in L, xy \in K\}$.

1. Let L a sub monoid of Σ^*. Show that L is a free monoid iff $L^{-1}L \cap LL^{-1} = L$.
2. Let L be a sub-monoid of Σ^*. We define by recursion :
   - $M_0 = L$
   - $M_{n+1} = \langle {M_n}^{-1} M_n \cap M_n {M_n}^{-1} \rangle$

   Demonstrate that this is a well defined increasing sequence and that $\cup_N M_n$ is the smallest free sub-monoid containing L.

Groupes

Exercice 4 (4) :
Let p be a prime number. We denote by $(1, \ldots, p)$ the p-cycle (the permutation of the symmetrical group that sends 1 on 2, 2 on 3, ..., $p-1$ on p and p on 1). We denote by G the group generated by the p-cycle $(1, \ldots, p)$. Assuming that Σ a finite alphabet.

1. Show that G is a group of order p.
2. What are the orders of the elements in G ? For each d, we will specify how many G elements are of order d.
3. We take G to act on Σ^p, the set of words of length p written with the letters of Σ as follows :

$$\begin{array}{lll} G \times \Sigma^p & \to & \Sigma^p \\ (\tau, a_1 a_2 \dots a_p) & \to & \tau \cdot (a_1 a_2 \dots a_p) = a_{\tau^{-1}(1)} a_{\tau^{-1}(2)} \dots a_{\tau^{-1}(p)} \end{array}$$

   (a) Demonstrate that this is a group operation.
   (b) Determine the fixer of $(1, \dots, p)$. Deduce the fixer of $(1, \dots, p)^i$, for any integer i co prime with $p(\text{Fix}(g) = \{x \in X | gx = x\})$.
   (c) Show that the number of orbits, r, of this operation are : $r = \frac{1}{p}(|\Sigma|^p + (p-1)|\Sigma|)$
   (d) Retrieve Fermat's little theorem ($a^p \equiv_p a$).

Probability

Exercice 5 (5) :
Show that for all sequences $(A_n)_{n \in \mathbb{N}}$ of events we have $\mathrm{P}(\cup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} \mathrm{P}(A_n)$ where the right sum can diverge.

Exercice 6 (6) :
Let $(\Omega, \mathcal{T}, \mathrm{P})$ be a probability space.

1. Show that for all sequences $(A_n)_{n \in \mathbb{N}}$ of growing events by inclusion, the sequence of $\mathrm{P}(A_n)$ converges and $\lim_{n \to \infty} \mathrm{P}(A_n) = \mathrm{P}(\cup_{n \in \mathbb{N}} A_n)$
2. Show for all decreasing sequences (by inclusion) $(A_n)_{n \in \mathbb{N}}$ of events, the sequence $\mathrm{P}(A_n)$ converges and $\lim_{n \to \infty} \mathrm{P}(A_n) = \mathrm{P}(\cap_{n \in \mathbb{N}} A_n)$

Exercice 7 (7) :
Let $\{B_1, \dots, B_n\}$ be a partition of Ω such that for all i, $\mathrm{P}(B_i) > 0$. Show that for all events A we have $\mathrm{P}(A) = \sum_{i=1}^{n} \mathrm{P}(A \mid B_i)\mathrm{P}(B_i)$. Keeping B_i's disjoint, what condition condition can we add for this to remain true ?

Exercice 8 (8) :
An urn contains b black balls, w white balls and r red balls. We pick two balls, what is the probability of the event "the second ball drawn is black" ?

Exercice 9 (9) :
Let $\{B_1, \dots, B_n\}$ be a partition of Ω such that $\mathrm{P}(B_i) > 0$ for all i. Then for every event A such that $\mathrm{P}(A) > 0$ we have

$$\mathrm{P}(B_i \mid A) = \frac{\mathrm{P}(A \mid B_i)\mathrm{P}(B_i)}{\sum_{j=1}^{n} \mathrm{P}(A \mid B_j)\mathrm{P}(B_j)}$$

Exercice 10 (10) :
We consider two six-sided dice, one is balanced, the other is rigged(loaded dice). We denote p_i the probability that the rigged die falls on the face i ($i \in \{1, 2, 3, 4, 5, 6\}$).

1. Describe the probability space.
2. (a) What is the probability of rolling a double ?
   (b) What is the probability that the sum of the dice is equal to 7 ?