Leftovers from last week :

1. Show that $\forall(n_r, n_b) \in \mathbb{N}^2, \exists N \in \mathbb{N}$ such that, for any 2 (edge) coloring $\{r, b\}$ of the complete graph K_N, there exists a color $c \in \{r, b\}$ for which there is a complete sub-graph K_{n_c} which is monochromatic in the color c.
(the smallest N for which this property holds is denoted by $R(n_r, n_b)$).
2. Show that $\forall k \in \mathbb{N}, \forall(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k, \exists N \in \mathbb{N}$ such that, for any k (edge) coloring of the complete graph K_N, there exists a color $c \in [\![1, k]\!]$ for which there is a complete sub-graph K_{n_c} which is monochromatic in the color c.
(the smallest N for which this property holds is denoted by $R(n_1, \ldots, n_k)$).

Exercice 0 :

1. Let E and F two countable sets. Show that $E \cup F$ is countable.
2. Let $E_1, \ldots, E_n$ be countable sets. Show that $\prod_{i=1}^n E_i$ is countable.

Combinatorial reasoning

Exercice 1 :
Demonstrate by combinatorial arguments the identity :

$$\forall n \in \mathbb{N},\ \binom{3n}{3} = 3\binom{n}{3} + 6n\binom{n}{2} + n^3$$

Applications of the pigeonhole principle

Exercice 2 :
Some arcs of a circle with a diameter 1 were colored. The sum of the lengths of the colored arcs is $> \pi/2$. Show that there exists a diameter of the circle, for which both ends are colored.

Exercice 3 :
Let $m_1, ..., m_{n+1}$ be $n+1$ numbers, chosen from the set $[2n]$.
Show that there exist a pair i, j, $1 \leq i \neq j \leq n+1$ such that m_i is divisible by m_j.

Cardinalities.

Exercice 4 :
Show that the family of all the finite sets of $\mathbb{N}$ is countable.

Exercice 5 :
Show that the set of decimal numbers is countable.

Exercice 6 :
Let Σ be a finite alphabet.

1. Show that the set of finite trees on Σ is countable.
2. Show that Σ^∞ isn't countable (unless it $|\Sigma| = 1$)

Exercice 7 :

Give an example of an uncountable family $F \subseteq P(\mathbb{N})$, such that for any $A \neq B \in F$, $A \cap B$ is finite.

Exercice 8 :

By using Cantor–Schröder–Bernstein theorem, show that $\{0,1\}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{R}$ are equipotent.

Exercice 9 :

Another way to show that $[0,1]$ is a uncountable set.
Let $\{x_n\}_{n\in\mathbb{N}}$ a series of real numbers in the interval $[0,1]$.

1. Construct recursively a sequence of closed intervals I_n of length > 0 such that :
   - $I_0 \subset [0,1]$,
   - $I_n \subset I_{n-1}$,
   - I_n are intervals of length > 0 that do not contain x_n.
2. Deduce that $[0,1]$ is uncountable.

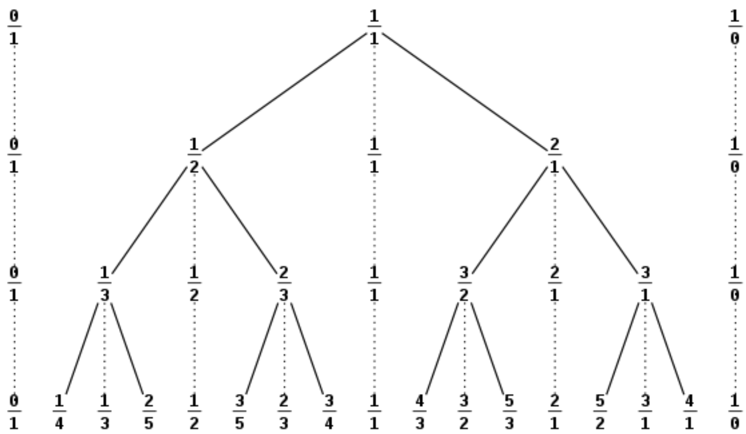
Exercise 10 (Stern–Brocot tree) :

In this exercise we'll show a way to represent strictly positive rational numbers in the form of an infinite binary tree.

We start we two imaginary vertices $0/1$ (which represents 0) and $1/0$ (which represent infinity). Each step we insert between each two of the consecutive fractions m_1/n_1 and m_2/n_2 the fraction $(m_1+m_2)/(n_1+n_2)$.

Thus, we obtain after four stages :

1) init : $[\frac{0}{1}, \frac{1}{0}]$
2) $[\frac{0}{1}, \frac{1}{1}, \frac{1}{0}]$
3) $[\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0}]$
4) $[\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}]$
5) $[\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{2}{1}, \frac{5}{2}, \frac{3}{1}, \frac{4}{1}, \frac{1}{0}]$



1. Verify that for $m_1/n_1 < m_2/n_2$, we have $m_1/n_1 < (m_1+m_2)/(n_1+n_2) < m_2/n_2$.
2. Show that for any step and for any two consecutive fractions $m_1/n_1 < m_2/n_2$, we have $n_2 m_1 - m_2 n_1 = \pm 1$.
3. Deduce that the fractions constructed by this process are in an irreducible form (i.e. the numerator and the denominator greatest common divisor is 1).
4. Let p/q be a strictly positive rational represented by an irreducible fraction. Suppose that $\frac{m_1}{n_1} < \frac{p}{q} < \frac{m_2}{n_2}$ and show that $m_1 + m_2 + n + n_2 \leq p + q$ (where $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$ are two consecutive fractions).
5. Let p/q be a strictly positive rational number represented by an irreducible fraction. Show that it appears uniquely in the construction.
6. Conclude that $\mathbb{Q}^{*+}$ is countable.