Exercice 1 :

Let E be a set with a prtial order denoted by $\preccurlyeq$. We say that $\preccurlyeq$ is a well quasi-order(wqo) if from any sequence of elements of E, we can extract an infinite monotone increasing sequence. I.e. $\forall (x_i)_{i\in\mathbb{N}} \in E^{\mathbb{N}}$, there exists an increasing sub-sequence of indexes : $i_0 < i_1 < \cdots < i_n < \cdots$ for which the sequence $(x_{i_n})_{n\in\mathbb{N}}$ is increasing : $x_{i_0} \preccurlyeq x_{i_1} \preccurlyeq \cdots \preccurlyeq x_{i_n} \preccurlyeq \cdots$.

1. Let $E, \preccurlyeq$ be an ordered set, we call it well founded if there is no infinite decreasing sequence. Assume that E is countable and show that the order is wqo iff the set of all antichains is countable.
2. Dickson's Lemma : Let $(E_1, \preccurlyeq_1)$ and $(E_2, \preccurlyeq_2)$ be a wqo set. Show that $(\preccurlyeq_1, \preccurlyeq_2)$ is a wqo on the product $E_1 \times E_2$.
3. Higman's Lemma : Let $\preccurlyeq$ be a wqo on Σ. Define a relationship on Σ^* as follows :

$$a_1...a_m \leq_{sw} b_1b_2...b_n \Leftrightarrow \begin{cases} \exists 1 \leq i_1 < i_2 < \cdots < i_m \leq n \\ a_1 \preccurlyeq b_{i_1} \wedge a_2 \preccurlyeq b_{i_2} \cdots a_m \preccurlyeq b_{i_m} \end{cases}$$

   (a) Show that $\leq_{sw}$ is an order

   (b) Show that $\leq_{sw}$ is wqo. Recall that we've shown that $\leq_{sw}$ is wqo if for any sequence $(x_i)_{i\in\mathbb{N}}$, we can find $i < j$ s.t. $x_i \preccurlyeq x_j$.

4. Let $E, \leq$ be an ordered set, and $F \subset E$ s.t. : $\forall y \in E$, if there exists $x \in F$ s.t $x \preccurlyeq y$, then $y \in F$ (we say that the set F is *upward closed* or *upper*).

   (a) Let $E, \leq$ be wqo. Show that a that any increasing sequence of upward closed sets is stationary, i.e. $F_1 \subseteq F_2 \subseteq \cdots$ there exists i such that for any $j > i$ $F_i = F_j$.

   (b) Show that if F is an upward closed set, there exists a finte set of elements $x_1, ..., x_n$ in F s.t. $F = \cup_i \{y \in E, x_i \preccurlyeq y\}$.

Exercice 2 :

Let k a non-zero natural number. We equip $\mathbb{N}^k$ with the relation :

$$(x_1, ..., x_k) \leq (y_1, ..., y_k) \Leftrightarrow \forall i \in \{1, ..., k\}, x_i \leq y_i$$

1. Show that $\leq$ is a wqo $\mathbb{N}^k$.
2. A vectors addition system (VAS) on $\mathbb{N}^k$ is a the tuple $\langle v_0, \mathcal{T}\rangle$, where $v_0 \in \mathbb{N}_k$ (called the initial state), and $\mathcal{T}$ is a fintite subset of $\mathbb{Z}^k$(called a set of transitions). For any element $t \in \mathcal{T}$ we define a application("firing") of t on an element of $\mathbb{N}^k$ by $\xrightarrow{t}$: $v \xrightarrow{t} v'$ if $v' = v + t$, for all $v, v' \in \mathbb{N}^k$(note that for some t and $v \in \mathbb{N}^k$, $v + t \notin \mathbb{N}^k$, in this case the application of t is not defined on v). We say that $v \in \mathbb{N}^k$ is reachable from v_0 if there exists a finite sequence $t_1, t_2, \ldots, t_n$ such that $v_0 \xrightarrow{t_1} v_1 \xrightarrow{t_2} \cdots \xrightarrow{t_n} v_n = v$.

   (a) Let u_1, u_2, v_1 elements of $\mathbb{N}^k$ such that $u_1 \leq u_2$ and V_1 is reachable from u_1. Show that there exists v_2 reachable from u_1 such that $v_1 \leq v_2$

   (b) Assume there exist $u, v \in \mathbb{N}^k$ such that $u \leq v$, v is reachable from u, and that for the j component $u_j < v_j$. Show that there is an infinite sequence $u_0 = u \leq u_1 = v \leq u_2 \leq \cdots \leq u_n \leq \cdots$ where all u_i are reachable from u and such that the sequence of the j-th components is unbounded ($(u_n)_j \xrightarrow{n\to\infty} \infty$).

   (c) We extend $\mathbb{N}$ with an element ω which is greater then all elements in $\mathbb{N}$: $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$. We extend the addition of elements in $\mathbb{N}$ whrere $n + \omega = \omega + n = \omega$, $\forall n \in \mathbb{N}_\omega$ and the multiplication $n\omega = \omega n = \omega$. This allows us to extend our VAS on $\mathbb{N}_\omega$. We now build a *covering tree*, in the following way :

- The root of the tree is the vertex s_o labeled with the vector v_0.
- Given a branch of the tree $(s_0, v_0) \to (s_1, v_1) \to \cdots \to (s_n, v_n)$, and $t \in \mathcal{T}$ which is fire-able from $v_n \xrightarrow{t} v_{n+1}$, we extend the branch in the following way :

  R1 If there $\exists i \leq n$ such that $v_{n+1} \leq v_i$, we **do not** extend the branch.

  R2 If there $\exists i \leq n$ such that, $v_{n+1} > v_i$, We define the vector $\bar{v}_{n+1} = v_i + \omega(v_{n+1} - v_i)$ (if $(v_i)_j < (v_{n+1})_j$ then $(\bar{v}_{n+1})_j = \omega$). And we add the vertex $(s_{n+1}, \bar{v}_{n+1})$ to the end of the branch.

  R3 If $\forall i \leq n$, v_{n+1} and v_i are incomparable, we add the vertex (s_{n+1}, v_{n+1}) to the end of the branch.

Show that this tree is finite.

(d) Let $k = 3$, and the VAS with $V_0 = (1, 0, 1)$ and $\mathcal{T} = \{a = (1, 1, -1), b = (-1, 0, 1), c = (0, -1, 0)\}$. Show that the set or reachable vectors are infinite and build the covering tree.

(e) Show that the covering tree over-approximate the reachabilty set of a given VAS $(v_0, \mathcal{T})$, in the following way :

- $\forall v$ reachable from v_0, There is a vertex of the tree labeled by w such that $v \leq w$.
- The set of reachable vectors of the VAS is finite iff there are no vertices labeled with ω.

Exercice 3 (Dilworth's theorem) :

Recall Dilworth's theorem : Let k be the maximal cardinality of an antichain in E. Then E is a disjoint union of k chains(a set of comparable elements).

1. Let N be a non zero natural number.

   (a) Let $\mathcal{F} = \{n_i, i \in [N]\}$ a sequance of natural numbers. We equip $\mathcal{F}$ with the following relation :
   $$n_i \preceq n_j \text{ if } (i \leq j \wedge n_i \leq n_j)$$
   Show that $\preceq$ is a partial order. Describe the chains and antichains for the order $\preceq$.

   (b) Let m and n be two natural numbers such that $N - 1 = nm$. Show that $\mathcal{F}$ contains an increasing subsequence of length $n + 1$ or a decreasing one of length $m + 1$.

2. Let $\mathcal{I}$ be a family of $N \in \mathbb{N}$ closed intervals in $\mathbb{R}$. Let m and n be two natural numbers such that $N - 1 = nm$. Show that there are either $m+1$ disjoint intervals in $\mathcal{I}$ or there are $n + 1$ intervals with a non empty intersection.

Exercice 4 (Théorème de Cantor-Bernstein) :

Let A and B two sets and $f : A \to B$ and $g : B \to A$ two injective functions. Let $H : \mathcal{P}(A) \to \mathcal{P}(A)$ the map :
$$X \mapsto A \setminus g[B \setminus f[X]]$$

1. Using the Knaster-Tarski theorem, show that H has a fixed point
2. Deduce that A and B are equipotent.
3. What are the smalles and largest fixed points of H ? (Comparer avec la preuve du cours)

Exercise 5 :

Show that any function from $\mathbb{R}$ to $\mathbb{R}$, is Scott continuous iff it is left continuous and monotonically increasing.

Exercise 6 :

Let $R \in \mathcal{P}(E \times E)$, and t the map :

$$\left\{ \begin{array}{rcl} \mathcal{P}(E \times E) & \to & \mathcal{P}(E \times E) \\ Q & \mapsto & Q \cup R \cup Q^2 \end{array} \right.$$

where $Q^2 := \{(x, z) \in E \times E \mid \exists y \in E,\, xQy \wedge yQz\}$.

Show that t is Scott continuous for inclusion.