Exercice 1 (Dilworth's theorem) :
Recall Dilworth's theorem : Let k be the maximal cardinality of an antichain in E. Then E is a disjoint union of k chains(a set of comparable elements).

1. Let $\mathcal{I}$ be a family of $N \in \mathbb{N}$ closed intervals in $\mathbb{R}$. Let m and n be two natural numbers such that $N-1 = nm$. Show that there are either $m+1$ disjoint intervals in $\mathcal{I}$ or there are $n+1$ intervals with a non empty intersection.

Exercice 2 (Théorème de Cantor-Bernstein) :
Let A and B two sets and $f : A \to B$ and $g : B \to A$ two injective functions. Let $H : \mathcal{P}(A) \to \mathcal{P}(A)$ the map :

$$X \mapsto A \setminus g[B \setminus f[X]]$$

1. Using the Knaster-Tarski theorem, show that H has a fixed point
2. Deduce that A and B are equipotent.

Exercise 3 :
Show that any function from $\mathbb{R}$ to $\mathbb{R}$, is Scott continuous iff it is left continuous and monotonically increasing.

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Exercise 4 :
For all $n, k \in \mathbb{N}$ such that $k \leq n$, we denote by $\binom{n}{k}$ the number of subsets of $[n]$ with cardinality k. By using the following formula

$$(k+1)\binom{n+1}{k+1} = (n+1)\binom{n}{k}$$

for al $n, k \in \mathbb{N}$ such that $k \leq n$, show that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

for all $n, k \in \mathbb{N}$ such that $k \leq n$.

Exercise 5 :
A finite complete ternary tree is a finite tree such that any node has either zero or three children. More formally, a ternary tree is composed either of only a leaf, denoted [], or of a root having three children T_1, T_2, T_3, which is denoted by $[T_1, T_2, T_3]$. A childless node is a leaf, the others are internal nodes.

1. Conjecture a formula linking the number of leaves $\ell(T)$ and the number of internal nodes $i(T)$ in a finite ternary tree T.
2. Define recursively (on the structure of the tree) the functions ℓ and i.
3. Prove the formula conjectured previously by structural induction.

Exercise 6 :
Let $D \subseteq \mathbb{Z}$. A function $f : D \to \mathbb{Z}$ is convex (or concave) if for every $x, y, z \in D$ such that $x < y < z$, we have $\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(y)}{z-y}$ (resp. $\frac{f(y)-f(x)}{y-x} \geq \frac{f(z)-f(y)}{z-y}$). A function $f : D \to \mathbb{Z}$ is affine if it is convex and concave.

1. What is the cardinality of the set of affine functions from $\mathbb{Z}$ to $\mathbb{Z}$, countable or uncountable ?
2. Are the following two assertions equivalent ? (If not, does one imply the other ?)
   (a) $f : \mathbb{Z} \to \mathbb{Z}$ is convex.
   (b) For all $n \in \mathbb{Z}$ we have $f(n+1) - f(n) \leq f(n+2) - f(n+1)$.
3. What is the cardinality of the set of convex functions from $\mathbb{Z}$ to $\mathbb{Z}$, countable or uncountable ?
4. Let $f : \mathbb{Z} \to \mathbb{Z}$. Show that there exists an infinite subset $D \subseteq \mathbb{Z}$ such that $f|_D : D \to \mathbb{Z}$ (i.e. $f|_D(n) := f(n)$ for all $n \in D$) has the following two properties :
   - $f|_D$ is increasing or decreasing, and
   - $f|_D$ is convex or concave.

Exercise 7 : 1.

Let $(E, \leq)$ be a partially ordered set. Let $\mathcal{C}$ be the set of well-founded chains of $(E, \leq)$. We define a binary relation R on $\mathcal{C}$ as follows : for all $C_1, C_2 \in \mathcal{C}$ we put $C_1 \; R \; C_2$ if $C_1 \subseteq C_2$ and for all $x \in C_1$ and $y \in C_2 \setminus C_1$ we have $x \leq y$.

(a) Which type of relation is R ?
(b) Show that all the chains $\{C_i\}_{i \in I}$ in $(\mathcal{C}, R)$ have a least upper bound in $(\mathcal{C}, R)$.
(c) Deduce that there is a chain of $(E, \leq)$ which is a maximal element of $(\mathcal{C}, R)$.

2. Let $(E, \leq)$ a lattice for which any well-founded chain has an upper bound.
   (a) Show that $(E, \leq)$ has a least element $\perp$.
   (b) Let A be a subset of E. Let B be the set of all elements smaller that all elements in A . Show that if B has a maximum element b, then b is the greatest element of B.
   (c) Show that B has a maximal element b. (you may use question 7.1.c)
   (d) Show that $(E, \leq)$ is a complete lattice.