Markov chains

Exercice 1 :

We have a biased coin that returns heads with the probability $p \in (0,1)$.

1. We can simulate an unbiased die with an unbiased coin in the following way :
   The coin is first thrown : heads denotes $1, 2, 3$, and tail denotes $4, 5, 6$.
   Then we throw the coin twice : if we get heads, heads, then we say 1 (or 4), if we get heads, tail we say 2 (or 5), if we get tail, heads we say 3 (or 6) and if we get tail, tail we start again from the first throw.
   (a) Draw the underlying Markov chain and justify that it simulates a balanced die.
   (b) What is the average number of coin rolls for this simulation of a die ?
2. Find in the same spirit an algorithm to simulate an unbiased die by casting the biased coins several times. Present it in the form of a Markov chain.

Exercice 2 :

A rectangular image is formed of $m \times n$ square pixels, m representing the number of pixels in width and n in length. We consider the following algorithm : At each step, a pixel is chosen uniformly, and then an immediate neighbor is chosen (if it is not on an edge, it has 8 immediate neighbors) with a uniform probability and it takes his color. Demonstrate that with probability 1, the image becomes monochromatic.

Exercice 3 :

Two players A and B play heads or tails, where head occurs with probability $0 \leq p \leq 1$. Each time head(resp. tail) occurs player A(resp. B) gives 1 coin to player B(resp .A). They play till one of them runs out of money. Given that A starts with a coins and B with b, what is the probability that A wins ?

Exercice 4 :

Let $\mathcal{C}$ be a Markov chain on N states with a transition matrix P. We assume that P is Doubly stochastic, i.e. $\forall i \in [\![1, N]\!], \sum_j p_{i,j} = \sum_j p_{j,i} = 1$ (the sum of each column and each row is 1).

1. We suppose that the chain $\mathcal{C}$ is irreducible and aperiodic. Given an initial distribution v, what is the value of $\lim_{n\to\infty} v \cdot P^n$

Exercice 5 :

A board game consists of a ring of N squares numbered from 0 to $N-1$. At each stage, the player rolls a balanced die and advances the corresponding number of squares. We denote by X_n the player's position at the nth step.

1. Draw the Markov chain, give $N = 7$.
2. What are the properties of this chain ? Justify that the chain has a stationary distribution.
3. Determine without calculation the stationary distribution.

Social choice

Let there be a society with N *voters* and a set $\mathcal{C} = \{A, B, C, \ldots\}$ of finite *candidates*. Each voter $i \in [N]$ has a *preferences* $\succ_i$ on the candidates (a total order). We call the set of all personal preferences of the voters in our society a *profile* (N-tuple of preferences). A *Social Choice Correspondence*(SCC) f, is a function from every profile to a candidate (or a set of candidates in case of ties).

Exercice 6 :

Given 2 candidates $A, B \in \mathcal{C}$ Condorcet's majority principle states that if most voters prefer A to B, then B should not be elected.

1. Given 3 candidates $\{A, B, C\}$, can you build a profile in which no candidate can be elected according to Condorcet's majority principle ?
2. Let there be N voters, a profile P, and candidates $\mathcal{C}$. We say that a candidate $A \in \mathcal{C}$ *weakly beats* or *weakly dominates* a candidate B if there exists $H \subset [N]$ such that $|H| \geq N/2$ and $\forall i \in H \; A \succ_i B$. Denote by :

$$CW(P) = \{A \in \mathcal{C} \mid \forall B \in \mathcal{C}, A \text{ weakly dominates } B\}$$

the set of *Condorcet winners*. We say that a SCC f is Condorcet if

$$CW(P) \neq \emptyset \Rightarrow f(P) \subseteq CW(P)$$

Build a SCC which is Condorcet.

A *social welfare function*(SWF) f is a function from every profile to a preference $\succ_s$ which is called the *social preference.*

We say that a SWF respects *unanimity* if it puts candidate A above B when every voter puts A above B.

We say that a SWF is *independent of irrelevant alternatives*(IIA) if the position of A compared to B on the social choice depends only on their relative ranking in every voter's preference. I.e f is IIA if given a profile $P = \{\succ_i\}_{i \in [N]}$, candidates A, B, C, and a social preference $f(P) = \{\succ_s\}$ which ranks $A \succ_s B$, then changing the profile $\succ_i$ to $\succ'_i$ where :

$$\forall i \in [N], \text{ if } A \succ_i B \Rightarrow A \succ'_i B; \text{ and if } B \succ_i A \Rightarrow B \succ'_i A$$

Then for the new profile P' we get that the social choice $f(P') = \{\succ'_s\}$ has that $A \succ'_s B$.

Exercice 7 :

Build a SWF which respects unanimity and is IIA.

We call a SWF *dictatorship* if there exists a voter $k \in [N]$ such that for any profile P $f(P) = \{\succ_k\}$.

Exercice 8 :

We will proof the following theorem by Arrow, which was originally proven by him in 1951-1963 and for which (among other works) he received a Nobel prize in economics.

Theorem 1 *Any SWF on $|A| \geq 3$ that respects unanimity and IIA is a dictatorship.*

The original proof was very long and complicated, so instead we follow a proof by John Geanakoplos 96 (in his paper he gives 3 different proofs for this theorem) :

1. For any candidate $A \in \mathcal{C}$ if every voter i ranks A at the top or the bottom of his preferences then $\succ_s$ must also rank A at either the top or the bottom.

Pick a profile that puts the candidate D in the last place in every voter's preference (which gives us that D is at the bottom of the social choice). Going from voter 1 till N move D from the bottom to the top of the preference. At some voter $v \in [N]$(for clarity we will call him Vlad) the candidate will also move to the top of the social choice(by unanimity).

2. Show that if Vlad prefers the candidate A over $B(A, B \neq D)$ then the social choice will have the same preference (i.e $A \succ_v B \Rightarrow A \succ_s B$).
3. Show that Vlad is a dictator.