Probability

Exercice 1 :

Let X and Y be two independent random variables defined on the same probability space of the same geometric distribution p ($P(X = k) = p(1-p)^k$).

1. Calculate $P(Y \geq X)$, for $p = \frac{1}{2}$.
2. Calculate $P(Y = X)$, for $p = \frac{1}{2}$.

We define the random variables U and V by

$$U = \max(X, Y) \text{ and } V = \min(X, Y)$$

3. For all $u \leq v \in \mathbb{N}$ calculate $P(U \leq u, V \geq v)$.
4. Calculate the distribution of the random variable U.

Exercice 2 :

Turán's theorem : Let G be any graph with n vertices, such that G is K_{t+1}-free. Then the number of edges in G is at most :

$$\left(1 - \frac{1}{t}\right)\frac{n^2}{2}$$

Denote by $d(v)$ the degree of the vertex v.

1. Let $<$ be a uniformly chosen total order on V. Define :

$$I = \{v \in V \mid \{v, u\} \notin E \Rightarrow v < u\}$$

   For $v \in V$, what is the probability that $v \in I$.
2. What is the expected value of $|I|$?
3. Denote by $\omega(G)$ the size of largest complete sub graph of G. Show that $\omega(G) \geq \sum_{v \in V} \frac{1}{n - d(v)}$.
4. Recall, Cauchy–Schwarz inequality $(\sum a_i b_i)^2 \leq (\sum a_i^2)(\sum b_i^2)$. Show that :

$$n^2 \leq \omega(G) \sum_{v \in V}(n - d(v))$$

   and conclude the theorem.

Exercice 3 :

We suppouse that $P : \mathbb{N} \to [0, 1]$ is a probability distribution. This equates to the having a sequence of positive numbers $(p_n)_{n \in \mathbb{N}}$ such that $\sum_n p_n = 1$, with $p_n = P(n)$.

1. Prove that there is no probability distribution on $\mathbb{N}$ that makes the drawing of an integer equiprobable.
2. Suppose $p_n = \frac{1}{2^{n+1}}$.
   (a) Demonstrate that we have a probability distribution on $\mathbb{N}$.
   (b) Calculate the probability of the set of even numbers and the set of odd numbers.
3. Assume we have a probability distribution on $\mathbb{N}$ satisfying the following properties :

$$\forall a \in \mathbb{N} \setminus \{0\}, P(a\mathbb{N}) = \frac{1}{a}.$$

   Prove that if a and b are co-prime, the events "all multiples of a" and the event "all multiple of b" are independent.

Exercice 4 :

Information is transmitted in a population. Each individual transmits the correct information with probability p, and the negation of the information with probability $1-p$. Let p_n be the probability that the information is correct after n repetitions.

1. Calculate value p_n as a function of p and n.
2. Calculate $\lim_n p_n$.

Exercice 5 :

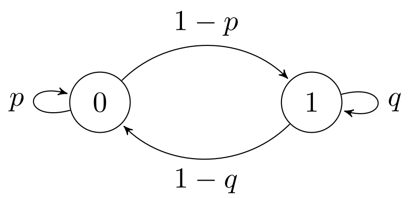
We consider the Markov chain on 5 states $1, 2, 3, 4, 5$ with a transition matrix P :

$$P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 0 & 1/5 & 4/5 \end{pmatrix}.$$

1. Draw the Markov Chain.
2. Classify the states.
3. What is the probability to start at state 1 and after 3 steps to arrive to the state 5 ?

Exercice 6 :

We consider the Markov chain with two states :



with p and q in $[0, 1]$

1. What is the transition matrix for this chain.
2. Is it aperiodic ? irreducible ? Given $v \geq 0$ what is $\lim_{n\to\infty} v \cdot P^n$? What are the stationary distributions ?

Exercice 7 :

We have a biased coin that returns heads with the probability $p \in (0, 1)$.

1. We can simulate an unbiased die with an unbiased coin in the following way :
   The coin is first thrown : heads denotes $1, 2, 3$, and tail denotes $4, 5, 6$.
   Then we throw the coin twice : if we get heads, heads, then we say 1 (or 4), if we get heads, tail we say 2 (or 5), if we get tail, heads we say 3 (or 6) and if we get tail, tail we start again from the first throw.
   (a) Draw the underlying Markov chain and justify that it simulates a balanced die.
   (b) What is the average number of coin rolls for this simulation of a die ?
2. Find in the same spirit an algorithm to simulate an unbiased die by casting the biased coins several times. Present it in the form of a Markov chain.

Exercice 8 :
A rectangular image is formed of $m \times n$ square pixels, m representing the number of pixels in width and n in length. We consider the following algorithm : At each step, a pixel is chosen uniformly, and then an immediate neighbor is chosen (if it is not on an edge, it has 8 immediate neighbors) with a uniform probability and it takes his color. Demonstrate that with probability 1, the image becomes monochromatic.

Exercice 9 :
Two players A and B play heads or tails, where head occurs with probability $0 \leq p \leq 1$. Each time head(resp. tail) occurs player A(resp. B) gives 1 coin to player B(resp .A). They play till one of them runs out of money. Given that A starts with a coins and B with b, what is the probability that A wins ?

Exercice 10 :
Let $\mathcal{C}$ be a Markov chain on N states with a transition matrix P. We assume that P is Doubly stochastic, i.e. $\forall i \in [\![1, N]\!], \sum_j p_{i,j} = \sum_j p_{j,i} = 1$ (the sum of each column and each row is 1).

1. Give meaning to $\sum_j p_{j,i}$ in general.
2. We suppose that the chain $\mathcal{C}$ is irreducible and aperiodic. Given $v \geq 0$, what is the value of $\lim_{n\to\infty} v \cdot P^n$

Exercice 11 :
A board game consists of a ring of N squares numbered from 0 to $N-1$. At each stage, the player rolls a balanced die and advances the corresponding number of squares. We denote by X_n the player's position at the nth step.

1. Draw the Markov chain, give $N = 7$.
2. What are the properties of this chain ? Justify that the chain has a stationary distribution.
3. Determine without calculation this stationary distribution.

Exercice 12 :
N objects are evenly distributed in cereal boxes. A collector seeks to get them all and is interested in the average number of cereal boxes he needs to purchase on order to get the entire collection. We denote this number by e_N.
Let $i \in [\![1, N]\!]$. N_i is the number of boxes purchased between the acquisition of the $i-1$-th object (if $i > 0$) until the i-th object is obtained.

1. What is the probability of getting a new object in a cereal box, knowing that we already have $i-1$ of them.
2. What is the distribution of N_i.
3. Calculate $E(N_i)$
4. Deduce the exact expression of e_N for any N, and show that $e_N \sim N \ln N$.

We now assume that the distribution of objects is no longer uniform. We suppose that there is an object which is rarer than the others. We denote ε its probability of occurrence. The others are distributed evenly.

5. Justify that $\lim_{\varepsilon\to 0} e_n = +\infty$.
6. Model the problem with a Markov chain.