Convex Optimization - HW1

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Disclaimer

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Problem 1

1.1 - Let $T = \{1, \dots, 6\}$ be the set of beginning of years, where the element 6 corresponds to the beginning of Year 6 (end of Year 5).

Now, let x_t denote the cash available at the beginning of period $t \in T$, and let $y_{i,t}$ represent the amount invested at the beginning of period t in an i-year deposit, where $i \in \{1, 2, 3\}$. With these definitions, the problem of maximizing the cash available at the end of Year 5 can be formulated as the following linear program:

$$
\begin{aligned}
\max\ & x_6 && && \text{(1a)}\\
\text{s.t. } & x_1 \leq 5000 && && \text{(1b)}\\
& x_2 \leq 1.04 y_{1,1} && && \text{(1c)}\\
& x_t \leq 1.04 y_{1,t-1} + 1.09 y_{2,t-2} && \forall t \in \{3, 4\} && \text{(1d)}\\
& x_t \leq 1.04 y_{1,t-1} + 1.09 y_{2,t-2} + 1.15 y_{3,t-3} && \forall t \in \{5, 6\} && \text{(1e)}\\
& y_{1,t} + y_{2,t} + y_{3,t} \leq x_t && \forall t \in T \setminus \{6\} && \text{(1f)}\\
& x_t \geq 0 && \forall t \in T &&\\
& y_{i,t} \geq 0 && \forall t,\ i \in \{1, 2, 3\}. &&
\end{aligned}
$$

The objective function (1a) maximizes the cash balance at the beginning of period 6, which corresponds to the end of Year 5. Constraint (1b) sets the initial cash available for investment. Constraints (1c)–(1e) bound the cash available in each period by the total matured value of the investments made in earlier periods, depending on which deposit types could have matured by that time. Finally, constraint (1f) ensures that the total amount allocated to deposits at the beginning of each period does not exceed the cash on hand.

1.2 - The results of the model solution are presented in Table 1.

Table 1: Optimal investment plan and cash balances

Period t	Cash x_t	$y_{1,t}$ (1-year)	$y_{2,t}$ (2-year)	$y_{3,t}$ (3-year)
1	5000.0	0.0	5000.0	0.0
2	0.0	0.0	0.0	0.0
3	5450.0	0.0	0.0	5450.0
4	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0
6	6267.5	0.0	0.0	0.0

From Table 1, we observe that the optimal strategy consists of investing the entire initial endowment of $\$5,000$ in a two-year deposit at $t = 1$. When this matures at $t = 3$, the proceeds of $\$5,450$ are reinvested in a three-year certificate. This investment then matures at $t = 6$, yielding a final cash balance of $\$6,267.5$ at the end of Year 5.

Problem 2

2.1 - Let F denote the set of factories, D the set of distribution centers, S the set of stores, and E the set of all transportation routes. Let su_f represent the supply capacity of factory $f \in F$, and de_s the demand requirement of store $s \in S$. For each route $e \in E$, let x_e denote the number of units transported along that route. With these definitions, the problem of minimizing the total transportation cost can be formulated as follows:

$$
\begin{aligned}
\min \ & \sum_{e \in E} c_e x_e && && \text{(2a)}\\
\text{s.t. } & x_e \le \mu_e && \forall e \in E && \text{(2b)}\\
& \sum_{f \in F} x_{(f,d)} - \sum_{s \in S} x_{(d,s)} = 0 && \forall d \in D && \text{(2c)}\\
& \sum_{d \in D} x_{(f,d)} \le su_f && \forall f \in F && \text{(2d)}\\
& \sum_{d \in D} x_{(d,s)} \ge de_s && \forall s \in S && \text{(2e)}\\
& x_e \in \mathbb{N} && \forall e \in E.
\end{aligned}
$$

The objective function (2a) minimizes the total cost of transporting goods across all routes. Constraint (2b) enforces the capacity limit μ_e on each route. Constraint (2c) ensures flow conservation at every distribution center, requiring that the total inflow equals the total outflow. Constraint (2d) limits the total shipments from each factory to its supply capacity. Finally, Constraint (2e) guarantees that the demand of every store is met.

2.2 - Let $\bar{c}_e$ and $\bar{\mu}_e$ denote the nominal cost and capacity for route e, and let

$$C_e = [(1-\delta)\bar{c}_e,\ (1+\delta)\bar{c}_e], \qquad M_e = [(1-\delta)\bar{\mu}_e,\ (1+\delta)\bar{\mu}_e],$$

with $\delta = 0.01$. The worst-case formulation for the objective function and capacity constraint is

$$
\begin{aligned}
\min_{x_e} \max_{c_e \in C_e} \ & \sum_{e \in E} c_e x_e \\
& x_e \le \min_{\mu_e \in M_e} \{\mu_e\} \qquad \forall e \in E.
\end{aligned}
$$

Which is equivalent to:

$$
\begin{aligned}
\min_{x_e} \ & \sum_{e \in E} 1.01 \bar{c}_e x_e \\
& x_e \le 0.99 \bar{\mu}_e \qquad \forall e \in E.
\end{aligned}
$$

Hence, we set up the robust optimization formulation as follows.

$$
\begin{aligned}
\min \ & \sum_{e \in E} 1.01 \bar{c}_e x_e \\
\text{s.t. } & x_e \le 0.99 \bar{\mu}_e && \forall e \in E\\
& \sum_{f \in F} x_{(f,d)} - \sum_{s \in S} x_{(d,s)} = 0 && \forall d \in D\\
& \sum_{d \in D} x_{(f,d)} \le su_f && \forall f \in F\\
& \sum_{d \in D} x_{(d,s)} \ge de_s && \forall s \in S\\
& x_e \in \mathbb{N} && \forall e \in E.
\end{aligned}
$$

Now consider the vector of random capacities $\mu = (\mu_e)_{e \in E}$ and the vector of random costs $c = (c_e)_{e \in E}$. The average-case formulation for the objective function and capacity constraint would be

$$\min_{x_e} \mathbb{E}_c \, [\sum_{e \in E} c_e x_e]$$
$$x_e \leq \mathbb{E}_\mu[\mu_e] \qquad \forall e \in E.$$

Which is equivalent to:

$$\min_{x_e} \sum_{e \in E} \mathbb{E}_c[c_e] x_e$$
$$x_e \leq \mathbb{E}_\mu[\mu_e] \qquad \forall e \in E.$$

Hence, we set up the stochastic programming formulation as follows.

$$\begin{aligned}
\min \; & \sum_{e \in E} \mathbb{E}_c[c_e] x_e \\
\text{s.t. } & x_e \leq \mathbb{E}_\mu[\mu_e] && \forall e \in E \\
& \sum_{f \in F} x_{(f,d)} - \sum_{s \in S} x_{(d,s)} = 0 && \forall d \in D \\
& \sum_{d \in D} x_{(f,d)} \leq su_f && \forall f \in F \\
& \sum_{d \in D} x_{(d,s)} \geq de_s && \forall s \in S \\
& x_e \in \mathbb{N} && \forall e \in E.
\end{aligned}$$

2.3 - Let P denote the set of product lines. For each $p \in P$, let su_{pf} denote the supply capacity of factory $f \in F$ for product p, de_{ps} denote the demand requirement of store $s \in S$ for product p, and c_{pe} denote the shipping cost associated with route $e \in E$ for product p. Define the decision variable x_{pe} as the number of units of product p transported along route e. With these definitions, we obtain the following multi-product extension of the base formulation for minimizing the total transportation cost:

$$\begin{aligned}
\min \; & \sum_{p \in P} \sum_{e \in E} c_{pe} x_{pe} \\
\text{s.t. } & \sum_{p \in P} x_{pe} \leq \mu_e && \forall e \in E \\
& \sum_{p \in P} \sum_{f \in F} x_{p(f,d)} - \sum_{p \in P} \sum_{s \in S} x_{p(d,s)} = 0 && \forall d \in D \\
& \sum_{d \in D} x_{p(f,d)} \leq su_{pf} && \forall p \in P, f \in F \\
& \sum_{d \in D} x_{p(d,s)} \geq de_{ps} && \forall p \in P, s \in S \\
& x_e \in \mathbb{N} && \forall e \in E.
\end{aligned}$$

Problem 3

3.1 - To solve the Sudoku puzzle, we introduce the following decision variable:

$$x_{ijn} = \begin{cases} 1, \text{ if number } n \text{ is selected for cell } (i, j), \\ 0, \text{ otherwise.} \end{cases}$$

Let B_k denote the set of cells in sub-block k, and let a_{ijn} be a parameter representing the initial state of the puzzle. Using these definitions, we can formulate a mixed-integer programming (MIP) model for the Sudoku problem as follows:

$$\begin{aligned}
&\min 0 && && \text{(3a)} \\
&\text{s.t.} \sum_{n \in [9]} x_{ijn} = 1 && \forall i \in [9], j \in [9] && \text{(3b)} \\
&\sum_{i \in [9]} x_{ijn} \leq 1 && \forall j \in [9], n \in [9] && \text{(3c)} \\
&\sum_{j \in [9]} x_{ijn} \leq 1 && \forall i \in [9], n \in [9] && \text{(3d)} \\
&\sum_{(i,j) \in B_k} x_{ijn} \leq 1 && \forall k \in [9], n \in [9] && \text{(3e)} \\
&x_{ijn} \geq a_{ijn} && \forall i \in [9], j \in [9], n \in [9] && \text{(3f)} \\
&x_{ijn} \in \{0, 1\} && \forall i \in [9], j \in [9], n \in [9]. &&
\end{aligned}$$

The objective function (3a) is set to zero, since Sudoku is purely a feasibility problem: any valid solution is acceptable, and no optimization criterion is required. Constraint (3b) ensures that each cell (i, j) receives exactly one digit from 1 to 9. Constraints (3c)–(3e) enforce that in each row, column and sub-block, respectively, distinct numbers appears at most once. Finally, constraint (3f) incorporates the initial clues of the puzzle by fixing the variables corresponding to the given digits.

Problem 4

4.1 - We begin by eliminating the free variables. Each free component of x can be written as the difference of two nonnegative variables:

$$x = x^+ - x^-, \qquad x^+, x^- \geq 0.$$

Substituting this expression into the original formulation gives

$$\begin{aligned} \max\ & \langle c,\, x^+ - x^- \rangle \\ \text{s.t. } & A_1(x^+ - x^-) + B_1 y \leq d_1 \\ & A_2(x^+ - x^-) \geq d_2 \\ & A_3(x^+ - x^-) + B_3 y = d_3 \\ & x^+, x^-, y \geq 0. \end{aligned}$$

Next, note that maximizing $\langle c, x^+ - x^- \rangle$ is equivalent to minimizing its negative. Thus the problem can be expressed as

$$\min\ \langle -c,\, x^+ - x^- \rangle$$

which expands to

$$\min\ \langle -c, x^+ \rangle + \langle c, x^- \rangle.$$

To eliminate inequalities, introduce decision variables $s_1, s_2 \geq 0$ so that

$$\begin{aligned} & A_1(x^+ - x^-) + B_1 y + s_1 = d_1 \\ & A_2(x^+ - x^-) - s_2 = d_2 \\ & A_3(x^+ - x^-) + B_3 y = d_3 \\ & x^+, x^-, y, s_1, s_2 \geq 0. \end{aligned}$$

Now we are able to write the formulation to a minimization problem with only nonnegative variables and equalities in the constraints as follows.

$$\begin{aligned} \min\ & \langle -c, x^+ \rangle + \langle c, x^- \rangle \\ \text{s.t. } & A_1(x^+ - x^-) + B_1 y + s_1 = d_1 \\ & A_2(x^+ - x^-) - s_2 = d_2 \\ & A_3(x^+ - x^-) + B_3 y = d_3 \\ & x^+, x^-, y, s_1, s_2 \geq 0. \end{aligned}$$

4.2 - To prove the two formulations are equivalent, we prove that any solution is feasible in the original if, and only if, it is feasible in the tranformed formulation.

Proof.

($\Rightarrow$)

Let (x, y) be any feasible solution to the original problem. Define nonnegative variables x^+ and x^-, so that $x = x^+ - x^-$ with $x^+, x^- \geq 0$. Next set

$$s_1 := d_1 - \big(A_1 x + B_1 y\big) \geq 0, \qquad s_2 := A_2 x - d_2 \geq 0.$$

Then $(x^+, x^-, y, s_1, s_2) \geq 0$ satisfies

$$\begin{aligned} A_1(x^+ - x^-) + B_1 y + s_1 &= d_1, \\ A_2(x^+ - x^-) - s_2 &= d_2, \\ A_3(x^+ - x^-) + B_3 y &= d_3, \end{aligned}$$

hence it is feasible for the transformed formulation. Moreover,

$$\langle -c, x^+\rangle + \langle c, x^-\rangle = \langle -c, x\rangle,$$

so maximizing $c^\top x$ in the original problem is equivalent to minimizing the new objective.

($\Leftarrow$)

Conversely, let $(x^+, x^-, y, s_1, s_2) \geq 0$ be feasible for the transformed problem. Define x such that $x := x^+ - x^-$. This equalities imply

$$\begin{aligned} A_1 x + B_1 y &= d_1 - s_1 \leq d_1, \\ A_2 x &= d_2 + s_2 \geq d_2, \\ A_3 x + B_3 y &= d_3, \end{aligned}$$

with $y \geq 0$ and x free. Thus (x, y) is feasible for the original problem. Finally,

$$\langle -c, x^+\rangle + \langle c, x^-\rangle = \langle -c, x\rangle,$$

so the objective values are consistent.

Therefore the two formulations are equivalent: feasibility is preserved in both directions and the optimal values agree.

$\square$

4.3 - Now collect all decision variables into a single nonnegative vector

$$\bar{x} = \begin{bmatrix} x^+ \\ x^- \\ y \\ s_1 \\ s_2 \end{bmatrix}.$$

The new cost vector is

$$c' = \begin{bmatrix} -c \\ c \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and the left-hand side constraint matrix is

$$A' = \begin{bmatrix} A_1 & -A_1 & B_1 & I_{m_1} & 0 \\ A_2 & -A_2 & 0 & 0 & -I_{m_2} \\ A_3 & -A_3 & B_3 & 0 & 0 \end{bmatrix},$$

where m_1 and m_2 denote the number of rows of A_1 and A_2, respectively. Here I_{m_1} and I_{m_2} are identity matrices of dimensions $m_1 \times m_1$ and $m_2 \times m_2$, while the zero blocks have compatible sizes.

The right-hand side vector is

$$d' = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Thus the final LP is

$$\begin{aligned} \min\ & \langle c', \bar{x}\rangle \\ \text{s.t. } & A'\bar{x} = d' \\ & \bar{x} \geq 0. \end{aligned}$$