Modelling dynamic systems

Michela Mulas

2 ZERO HUNGER

End hunger, achieve food security and improved nutrition and promote sustainable agriculture

SUSTAINABLE DEVELOPMENT GOALS

ONLY THIS WEEK!!

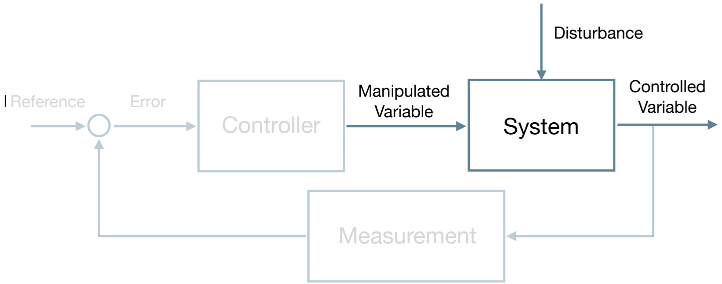
MICHELA
WED
16:00 – 18:00

YAGO
FRIDAY
16:00 – 18:00

Recap

We did...

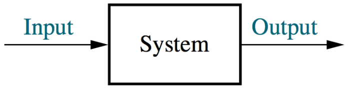
- Describe dynamic systems
- Describe and understand mathematical models



Recap

A **system** is an physical "object" in which variables of different kinds interact and produce observable signals.

If the system is **dynamic** these variables (or signals) evolve with time.



- **Outputs** are the observable signals of interest to us.
- **Inputs** are external signals that can be manipulated by the observer.
- **Disturbances** are external signals that can not be manipulated.
- **The choice between inputs and outputs depends on the control objectives.**

Dynamic systems can be described with:

- **State-variable** representation.
- **Input-output** representation.

We use ORDINARY DIFFERENTIAL EQUATIONS to represent system dynamics

$\dot{x}(t) = a x(t)$ $\quad a \in \mathbb{R}$ $\quad \dot{x} \triangleq dx/dt$

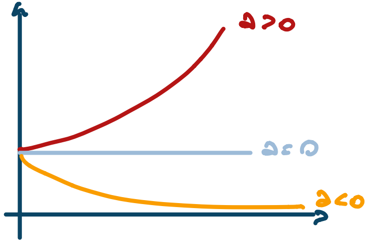
given $x(0) = 0$ $\quad x_0 \in \mathbb{R}$

It has a unique solution $\Longrightarrow$ $x(t) = e^{at} x_0$

Introducing a forcing signal

$\dot{x}(t) = a x(t) + b u(t)$ $\quad a, b \in \mathbb{R}$

$x(0) = x_0$ $\quad x_0 \in \mathbb{R}$



$$x(t) = \underbrace{e^{at} x_0}_{\text{NATURAL RESPONSE}} + \underbrace{\int_0^t e^{a(t-\tau)} b u(\tau)\, d\tau}_{\text{FORCED RESPONSE}}$$ LAGRANGE FORMULA

NATURAL RESPONSE: effect of the initial condition

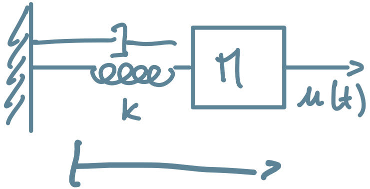
FORCED RESPONSE: it's the effect of the input signal

Recap – State-space representation

The notion of the state of a dynamic system is a fundamental notion in physics.

- The state of a dynamic system is a set of physical quantities, the specification of which (in the absence of external excitation) completely **determines the evolution of the system**.
- In analysis of dynamic systems such as mechanical systems, electric networks, etc. the differential equations typically relate the dynamic variables and their time derivatives of various orders.
- In the state-space approach, all the differential equations in the mathematical model of a system are first-order equations.

STATE VARIABLES provide the minimum set of variables that fully describes the system



How the system evolves ?

We know the constants k and M
we have an external force
acting on the mass M

We need to know the initial condition

~> Is the mass moving ? ~> Velocity

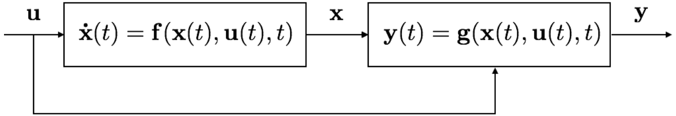
~> Is the spring pulling the mass ? ~> SPRING FORCE = k · DISTANCE (position)

2 state variables

Recap – State-space representation

The link between input and output is given by two equations:

- A differential equation, the **state equation**, that relates the inputs with the variables describing the states of the systems.
- An algebraic equation, the **output transformation**, that allows to determine the output at a specific time given the states and the input at the same time.

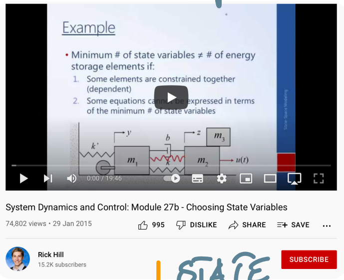


$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \end{cases}$$

CHOICE OF STATE VARIABLES to obtain the states are

state variable = # of energy storage elements

▷ As starting point



TYPE	ELEMENT	ENERGY	STATE
MECHANICAL	Mass	Kinetic energy: $\frac{1}{2} m v^2$	velocity
	Spring	Potential elastic energy $\frac{1}{2} k x^2$	position
ELECTRICAL	Inductor	Potential magnetic energy: $\frac{1}{2} L i^2$	current
	capacitor	Potential electric energy: $\frac{1}{2} C V^2$	voltage
HYDRAULIC	tank	Potential gravitational energy: $\rho g h$	height

...

Recap – State-space representation
Model classification

A **state-space model is said to be linear** if and only if the equations of state equation and the output transformation are linear equations

$$\begin{cases} \dot{x}_1(t) = a_{1,1}(t)x_1(t) + \cdots + a_{1,n}(t)x_n(t) + b_{1,1}(t)u_1(t) + \cdots + b_{1,r}(t)u_r(t) \\ \quad\vdots \\ \dot{x}_n(t) = a_{n,1}(t)x_1(t) + \cdots + a_{n,n}(t)x_n(t) + b_{1,1}(t)u_1(t) + \cdots + b_{n,r}(t)u_r(t) \\ y_1(t) = c_{1,1}(t)x_1(t) + \cdots + c_{1,n}(t)x_n(t) + d_{1,1}(t)u_1(t) + \cdots + d_{1,r}(t)u_r(t)(t) \\ \quad\vdots \\ y_m(t) = c_{m,1}(t)x_1(t) + \cdots + c_{n,n}(t)x_n(t) + d_{1,1}(t)u_1(t) + \cdots + d_{n,r}(t)u_r(t) \end{cases}$$

It is a system of

* FIRST ORDER DIFF. EQUATIONS and
* ALGEBRAIC OUTPUT EQUATIONS

In vectorial form:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \end{cases}$$

$$\mathbf{A} \in \mathbb{R}^{n\times n} = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) & \dots & a_{1,n}(t) \\ \vdots & & & \\ a_{n,1}(t) & a_{n,2}(t) & \dots & a_{n,n}(t) \end{pmatrix} \quad \mathbf{B} \in \mathbb{R}^{n\times r} = \begin{pmatrix} b_{1,1}(t) & b_{1,2}(t) & \dots & b_{1,r}(t) \\ \vdots & & & \\ b_{n,1}(t) & b_{n,2}(t) & \dots & b_{n,r}(t) \end{pmatrix}$$

$$\mathbf{C} \in \mathbb{R}^{p\times m} = \begin{pmatrix} c_{1,1}(t) & c_{1,2}(t) & \dots & c_{1,n}(t) \\ \vdots & & & \\ c_{p,1}(t) & c_{p,2}(t) & \dots & c_{p,n}(t) \end{pmatrix} \quad \mathbf{D} \in \mathbb{R}^{p\times r} = \begin{pmatrix} d_{1,1}(t) & d_{1,2}(t) & \dots & d_{1,r}(t) \\ \vdots & & & \\ d_{n,1}(t) & d_{n,2}(t) & \dots & d_{n,r}(t) \end{pmatrix}$$

The 'repacking' into matrix forms gives access to many mathematical tools

Recap – State-space representation

Model classification

A **state-space model is said to be time-invariant** if and only if the parameters of state equation and the output transformation equations are time independent:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases}$$

and for a linear system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

A **state-space model is said to be proper** if the output transformation equations are independent from the inputs:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), t) \end{cases}$$

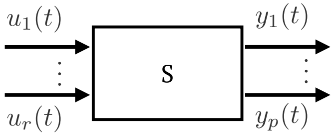
and for a linear system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

Recap – Input-output representation

A different view of dynamics emerged from electrical engineering: **input/output models**.

- The input/output framework is used in many engineering disciplines since it allows us to decompose a system into individual components connected through their inputs and outputs.
- Conceptually an input/output model can be viewed as a giant table of inputs and outputs.
- Given an input $u(t)$ over some interval of time, the model should produce the resulting output $y(t)$.



Inputs (causes) Outputs (effects)

- An **input-output model** describes the link between $\mathbf{y}(t)$ and the inputs $\mathbf{u}(t)$ as differential equations.
- The input-output model for a SISO system is given by one differential equation as:

$$h\Big(\underbrace{y(t), \dot{y}(t), \ldots, y^{(n)}(t)}_{\text{output}}, \underbrace{u(t), \dot{u}(t), \ldots, u^{(m)}(t)}_{\text{input}}\Big) = 0$$

- The input model for a MIMO system with p output and r inputs is given by p differential equations as:

$$\begin{cases} h_1\Big(\underbrace{y_1(t), \dot{y}_1(t), \ldots, y_1^{(n)}(t)}_{\text{output 1}}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_r(t), \dot{u}_r(t), \ldots, u_r^{(m)}(t)}_{\text{input r}}, t\Big) = 0 \\ h_2\Big(\underbrace{y_2(t), \dot{y}_2(t), \ldots, y_2^{(n)}(t)}_{\text{output 2}}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_r(t), \dot{u}_r(t), \ldots, u_r^{(m)}(t)}_{\text{input r}}, t\Big) = 0 \\ \vdots \quad \vdots \\ h_p\Big(\underbrace{y_p(t), \dot{y}_p(t), \ldots, y_p^{(n)}(t)}_{\text{output p}}, \underbrace{u_1(t), \dot{u}_1(t), \ldots, u_1^{(m)}(t)}_{\text{input 1}}, \ldots, \underbrace{u_r(t), \dot{u}_r(t), \ldots, u_r^{(m)}(t)}_{\text{input r}}, t\Big) = 0 \end{cases}$$

Recap – Input-output representation

Model classification

An **input-output model is said to be linear** if and only if the input-output relation is given by a linear differential equation.

That is, for a SISO input-output model:

$$a_0(t)y(t) + a_1\dot{y}(t) + \cdots + a_n(t)y^{(n)}(t) = b_0(t)u(t) + b_1\dot{u}(t) + \cdots + b_m(t)u^{(n)}(t)$$

where the linear combinations of the input-output coefficients are functions of time.

An **input-output model is said to be time-invariant** if and only if the input-output relation is (explicitly) time independent:

That is, for a SISO input-output model:

$$h\left(y(t), \dot{y}(t), \ldots, y^{(n)}, u(t), \dot{u}(t), \ldots, u^{(m)}(t)\right) = 0$$

and for a linear SISO:

$$a_0 y(t) + a_1(t)\dot{y}(t) + \cdots + a_n y^{(n)}(t) = b_0 u(t) + b_1(t)\dot{u}(t) + \cdots + b_n u^{(m)}(t)$$

An **input-output model is said to be a proper model** if and only if the causality principle applies:

That is, for a SISO input-output model:

$$h(\left(y(t), \dot{y}(t), \ldots, y^{(n)}, u(t), \dot{u}, \ldots, u^{(m)}(t), t\right) = 0$$

the output order is greater or equal to the input order ($n \geq m$).

If $n > m$ the system is said to be strictly proper.

A MIMO system is said to be proper if $n_i \geq \max_{j=1,\ldots,r} m_{i,j}$.

Recap – Input-output representation

Model classification

SYSTEM PROPERTIES

- DYNAMIC / STATIC $\rightsquigarrow h(y(t), u(t)) = 0$
  (memory / memoryless)

- NON LINEAR / LINEAR $\rightsquigarrow$ CAUSE $c_1 \rightsquigarrow$ EFFECT e_1
  CAUSE $c_2 \rightsquigarrow$ EFFECT e_2
  CAUSE $(\alpha c_1 + \alpha c_2) \rightsquigarrow$ EFFECT $(\alpha e_1 + \alpha e_2)$
  ADDITIVITY / HOMOGENEITY (SCALING)

- TIME-VARIANT / TIME-INVARIANT $\rightsquigarrow$ translation principle
  CAUSE $c(t) \rightsquigarrow$ EFFECT $e(t)$
  CAUSE $c(t-T) \rightsquigarrow$ EFFECT $e(t-T)$

- NON-CASUAL / CASUAL $\rightsquigarrow$ it does not depend on future inputs
  (improper) (proper)

- LUMPED / DISTRIBUTED
  All variables are function of time and one or more spatial variables
  PARTIAL DIFF. EQ.
  The variables are function of time alone (ORDINARY DIFF. EQUATIONS)

Today's goal

We are going to do...

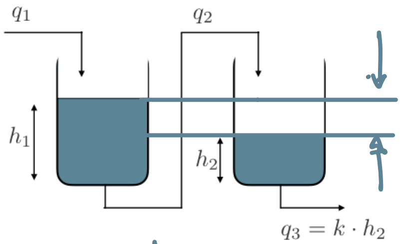
- Finalize L01: Review the the main properties of a mathematical model.
- Review Laplace transform:
  - Find the direct/inverse Laplace transform.
  - Find the partial fraction expansion.
  - Solve the differential equation using the Laplace transforms.
- Programming examples.

Reading list

- Nise, *Control Systems Engineering* (6th Edition)[1]
- Ogata and Severo, *Engenharia de Controle Moderno* (3rd Edition)

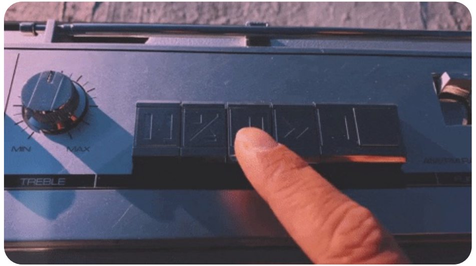
[1] Today's lecture is based mainly on **Ch.2** of Nise.
Same concepts can be found in **Ogata, Ch.2**

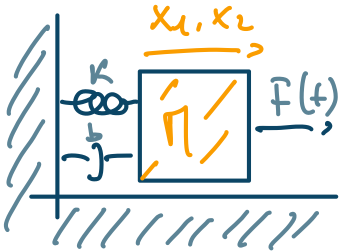
Examples of Input-output representation
Hydraulic systems



$d = h_1 - h_2$

LO1





MECHANICAL SYSTEM

$\dot{x}_1(t) = x_2(t)$ velocity = derivative of the travelled system

$F(t) - kx_1(t) - bx_2(t) = m\dot{x}_2(t)$

$$\begin{cases} \dfrac{dx_1}{dt} = x_2(t) \\ \dfrac{dx_2}{dt} = -\dfrac{k}{m}x_1(t) - \dfrac{b}{m}x_2(t) + \dfrac{1}{m}u(t) \end{cases}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

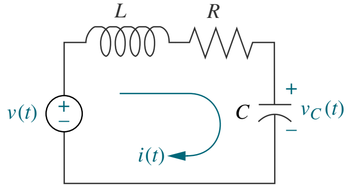
$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} u(t)$$

$y(t) = x_2(t)$ C=1 D=0

y?

Input-output representation

Simple circuits via mesh analysis



Problem: Find the IO model relating the capacitor voltage, $v_C(t)$, to the input voltage, $v(t)$.

We must first decide what the input and output should be.

- In this network, we could choose several variables to be the output: inductor voltage, capacitor voltage, resistor voltage, or current.
- In this case, the problem statement is clear:
  - The **capacitor voltage is the output**
  - The **applied voltage is the input**

- Our guiding principles are the **Kirchhoff's laws**.
  - We sum voltages around loops or currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and then equate the result to zero.
- From these relationships we write the differential equations for the circuit.

This yields the integro-differential equation for this network as:

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau=v(t)$$

Changing variables from current to charge using $i(t)=dq(t)/dt$ yields

$$L\frac{d^2q(t)}{dt^2}+R\frac{dq(t)}{dt}+\frac{1}{C}q(t)=v(t)$$

From the voltage-charge relationship for a capacitor in the table $q(t)=Cv_c(t)$:

$$LC\frac{d^2v_C(t)}{dt^2}+RC\frac{dv_C(t)}{dt}+v_c(t)=v(t)$$

Laplace transform review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Pierre-Simon Laplace

From Wikipedia, the free encyclopedia

"Laplace" redirects here. For other uses, see Laplace (disambiguation).

Pierre-Simon, marquis de Laplace (/ləˈplɑːs/; French: [pjɛʁ simɔ̃ laplas]; 23 March 1749 – 5 March 1827) was a French scholar and polymath whose work was important to the development of engineering, mathematics, statistics, physics, astronomy, and philosophy. He summarized and extended the work of his predecessors in his five-volume *Mécanique céleste* (*Celestial Mechanics*) (1799–1825). This work translated the geometric study of classical mechanics to one based on calculus, opening up a broader range of problems. In statistics, the Bayesian interpretation of probability was developed mainly by Laplace.[2]

Laplace formulated Laplace's equation, and pioneered the Laplace transform which appears in many

Pierre-Simon Laplace

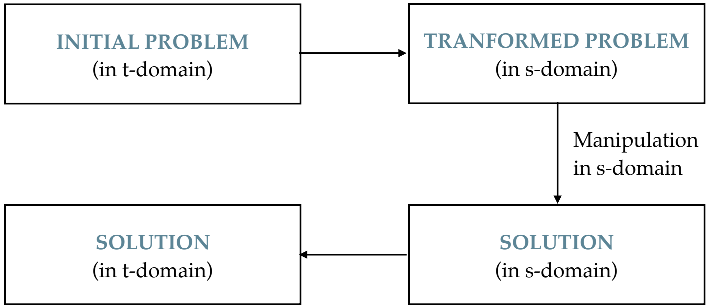
Pierre-Simon Laplace as chancellor of the Senate under the First French Empire

Born	23 March 1749 Beaumont-en-Auge, Normandy, Kingdom of France
Died	5 March 1827 (aged 77) Paris, Kingdom of France
Nationality	French
Alma mater	University of Caen

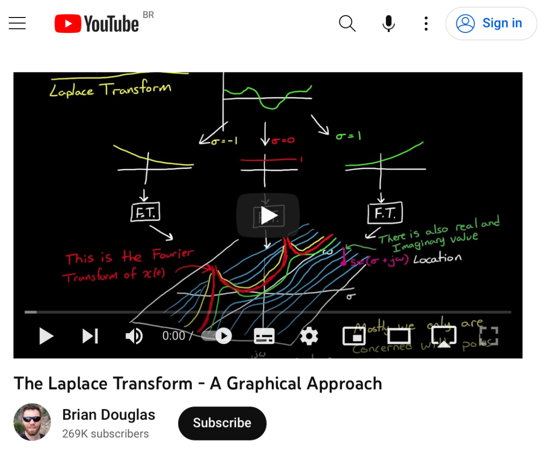
Laplace transform definition

Laplace transform is a technique that, among other things, converts linear, ordinary differential equations into relatively easier-to-handle algebraic equations.

It also converts partial differential equations in into ordinary ones.



The Laplace transform is often thought in terms of a **mapping** from the t-domain (or **time domain**) to the s-domain (or **Laplace domain**)



https://www.youtube.com/watch?v=ZGPtPkTft8g

Laplace transform definition

With the Laplace transform we can represent the input, output, and system as separate entities. Further, **their interrelationship will be simply algebraic**.

The **Laplace trasform is defined as**:

$$L[f(t)] = \underbrace{F(s)}_{\text{Laplace transform of } f(t)} = \int_{0-}^{\infty} f(t)e^{-st}dt$$

It is a complex exponential

where $s = \sigma + j\omega$ is a **complex variable**; knowing $f(t)$ and that the integral exists, we can find the function $F(s)$.

- Using differential equations, we have to solve for the initial conditions after the discontinuity knowing the initial conditions before the discontinuity.
- Using the Laplace transform we need only know the initial conditions before the discontinuity.

It is an integral transform that converts a function of a real variable t (time) to a complex variable ($s=\sigma+j\omega$)

$\rightsquigarrow$ Consider a function $f(t)$, $f: \mathbb{R} \to \mathbb{R}$

$f(t)=0$ for all $t<0$

The LAPLACE TRANSFORM $\mathcal{L}[f]$ of f is the function

$F: \mathbb{C} \to \mathbb{C}$ of the complex variable $s \in \mathbb{C}$

$F(s) = \int_0^{+\infty} f(t)\, e^{-st} dt$

for all $s \in \mathbb{C}$ for which the integral exists

Inverse Laplace transform

The inverse of the Laplace transform (we can find $f(t)$ given $F(s)$)

$$L^{-1}[f(t)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{-st}ds = f(t)u(t)$$

where

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

is the is the unit step function.

- Multiplication of $f(t)$ by $u(t)$ yields a time function that is zero for $t < 0$.

Common Laplace transforms

Exponential function

Exponential functions appear often in the solution of linear differential equations:

$$f(t) = e^{-at}$$

The transform is defined for $t > 0$:

$$\begin{aligned} L[f(t)] = L[e^{-at}] = F(s) &= \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt = \\ &= -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty \\ &= -\frac{1}{s+a}\left[0-1\right] = \frac{1}{s+a} \end{aligned}$$

$$L[e^{-at}] = \frac{1}{s+a}$$

Common Laplace transforms

Derivatives

This will be important in transforming the derivative term in a dynamic equation to the Laplace domain (using integration per parts):

$$\begin{aligned} L\left[\frac{df(t)}{dt}\right] &= \int_0^\infty \frac{df(t)}{dt} e^{-st} dt = \left[e^{-st} f(t)\right]_0^\infty + \int_0^\infty f(t) s e^{-st} dt \\ &= \left[0 - f(0)\right] + s\int_0^\infty f(t) e^{-st} dt = -f(0) + sF(s) \end{aligned}$$

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

For an nth derivative, we can derive:

$$L\left[\frac{d^{(n)} f(t)}{dt}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2}\frac{df(0)}{dt} - \ldots - \frac{f^{(n-1)}(0)}{dt}$$

Common Laplace transforms

Time delays

Time delays often occurs owing to fluid transport through pipes, or measurement sample delays.

- We use θ to represent the time delay.

If $f(t)$ represents a particular function of time, then $f(t-\theta)$ represents the value of the function θ time units in the past.

$$L\Big[f(t-\theta)\Big] = \int_0^\infty f(t-\theta)e^{-st}dt = \int_0^\infty f(t-\theta)e^{-s(t-\theta+\theta)}dt =$$
$$= \int_0^\infty f(t-\theta)e^{-s(t-\theta)}e^{-s\theta}dt = e^{-s\theta}\int_0^\infty f(t-\theta)e^{-s(t-\theta)}d(t-\theta)$$

We can use a change of variables: $t^* = t-\theta$, to integrate a function. Note that the lower limit of integration does not change, because the function is defined as $f(t)=0$ for $t<0$.

$$L\Big[f(t-\theta)\Big] = e^{-s\theta}\int_0^\infty f(t^*)e^{-st^*}dt^* = e^{-s\theta}F(s)$$

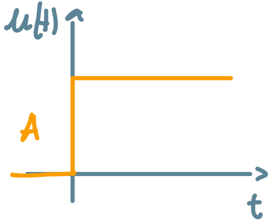
Common Laplace transforms

Step functions

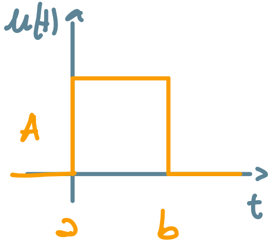
The analysis of the dynamic system behaviour is carried out by investigating the system response forced by the application of certain well-characterised input functions.

Some unctions we are concerned about are:

1. The step function
2. The rectangular pulse function
3. The impulse function
4. The ramp function



A STEP FUNCTION may be realized in practice by implementing a sudden change in the position of the actuator (e.g. a control valve) which will corresponds to a change of magnitude A in the input variable



A RECTANGULAR INPUT: the procedure for its realization involves making an instantaneous change in the actuator position maintaing the change for a time b

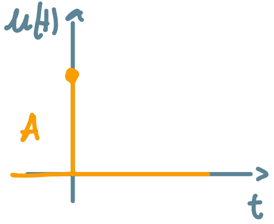
Common Laplace transforms

Step functions

The analysis of the dynamic system behaviour is carried out by investigating the system response forced by the application of certain well-characterised input functions.

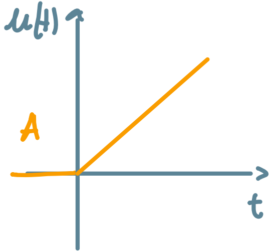
Some unctions we are concerned about are:

1. The step function
2. The rectangular pulse function
3. The impulse function
4. The ramp function



IMPULSE: It is physically impossible to implement the pulse function exactly

However, implementing it for a very short time gives a good approx. expecially for systems with slow dynamics.



Common Laplace transforms
Step function

Step functions are used to simulate the sudden change in an input variable. A step function is discontinuous a $t=0$. A **unit step function** is defined as:

$$u(t) = \begin{cases} 0 \text{ for } t < 0 \\ 1 \text{ for } t \geq 0 \end{cases}$$

Using the definition of the Laplace transform:

$$L\left[u(t)\right] = \int_0^\infty e^{-st} dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = -\frac{1}{s}\left[0-1\right] = \frac{1}{s}$$

For the **unit step**:

$$L\left[1\right] = \frac{1}{s}$$

For a **constant** A:

$$L\left[A\right] = \frac{A}{s}$$

- ⇝ The step function, as such, is idealised
- ⇝ It cannot, in general, be realised exactly in practice.

Common Laplace transforms
Rectangular function

A ideal rectangular pulse function of magnitude A, and duration b time units is a function whose values remains zero until it takes on the value of A instantaneously at the “starting time” $t=0$, maintains this new value for precisely b time units and returns to its initial value thereafter.

$$u(t) = \begin{cases} 0 \text{ for } t \leq 0 \\ A \text{ for } 0 < t < b \\ 0 \text{ for } t \geq b \end{cases}$$

In finding the Laplace transform of this function, it is more instructive to consider $u(t)$ as a combination of two step functions:

- One starting at $t=0$;
- The other one starting at $t=b$.

$$L\left[u(t)\right] = \frac{A}{s}\left(1-e^{-bs}\right)$$

Common Laplace transforms

Impulse function

The ideal impulse function of magnitude A is a function represented mathematically as

$$u(t) = A\delta(t)$$

- where $\delta(t)$ is the **Dirac delta function**.
- For our purposes, it is convenient to define it as a function whose value is zero elsewhere and infinite at the point $t = 0$.
- It has the property that the total area under the “curve” is 1.

The impulse function of area A has the Laplace transform:

$$L\left[u(t)\right] = A$$

Common Laplace transforms

Ramp function

An ideal ramp function is a function whose values starts at 0 (at $t = 0$) and increases linearly with a constant slope A.

$$u(t) = \begin{cases} 0 \text{ for } t < 0 \\ At \text{ for } t > 0 \end{cases}$$

Observe that this function can be obtained by integrating the step function of magnitude A.

The Laplace transform of this ramp function is given by:

$$L\left[u(t)\right] = \frac{A}{s^2}$$

Common Laplace transforms

$f(t)$	$F(s)$		$f(t)$	$F(s)$
$\delta(t)$	1	Unit impulse	$t^2/2e^{-at}$	$\frac{1}{(s+a)^3}$
$S(t)$	$\frac{1}{s}$	Unit step	$t^n/n!e^{-at}$	$\frac{1}{(s+a)^{n+1}}$
C	$\frac{C}{s}$	Constant	$(1-e^{t/\tau})$	$\frac{1}{s(\tau s+1)}$
$f(t-\delta)$	$e^{-\delta s}F(s)$		$\sin(\omega t)$	$\frac{\omega}{(s^2+\omega^2)}$
t	$\frac{1}{s^2}$		$\cos(\omega t)$	$\frac{s}{(s^2+\omega^2)}$
t^n	$\frac{n!}{s^{n+1}}$		$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$\frac{df}{dt}$	$sF(s)-f(0)$		$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s^2+a)^2+\omega^2}$
e^{-at}	$\frac{1}{s+a}$		te^{-at}	$\frac{1}{(s+a)^2}$

Laplace transform properties

Some properties of the Laplace transform are specifically important in control applications:

1. Not all functions $f(t)$ have a Laplace transform.
   - This is because the Laplace transform involves an indefinite integral whose values must be finite if the Laplace transform is to exist.
2. However, that all the functions that of interest in our study possess Laplace transform.
3. The Laplace transform $F(s)$ contains no information about the behaviour of $f(t)$ for $t < 0$ since the integration involved is from $t = 0$.
   - This is not a problem for our application since t usually represents time and we are normally interested in what happens at time $t > t_0$ where t_0 is the initial time that can be arbitrarily set to 0.

The following are some properties that are of importance in control applications:

4. The function $f(t)$ and its corresponding transform $F(s)$ are said to form a **transform pair**.
   - A most important property is that **transform pair are unique**.
   - This implies that no two distinct functions $f(t)$ and $g(t)$ have the same Laplace transform.
5. The Laplace transform operation is a linear one, by which we mean that for two constants c_1 and c_2 and two (Laplace transformable) functions $f_1(t)$ and $f_2(t)$, the following is true:

$$L\left[c_1 f_1(t) + c_2 f_2(t)\right] = c_1 L\left[f_1(t)\right] + c_2 L\left[f_2(t)\right]$$

Laplace transform properties

Property	Name
$L\left[f(t)+g(t)\right] = F_1(s)+F_2(s)$	Linearity
$L\left[kf(t)\right] = \alpha F(s)$	Linearity
$L\left[f(t-\tau)\right] = e^{-s\tau}F(s)$	Time shift
$L\left[e^{-\alpha t}\right] = F(s+\alpha)$	Frequency shift
$L\left[\frac{df}{dt}\right] = sF(s)-f(0-)$	Differentiation
$L\left[\frac{d^2f}{dt^2}\right] = s^2F(s)-sf(0-)-f^{'}(0-)$	Differentiation
$L\left[\frac{d^nf}{dt^n}\right] = s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation
$L\left[\int_{0-}^{\infty}\right] = \frac{F(s)}{s}$	Integration
$L\left[f(at)\right] = \frac{1}{a}F(\frac{s}{a})$	Scaling

Laplace transform theorems

The Final and Initial value theorems are very useful for determining limiting values in dynamics and control studies.

Theorem	Name
$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	Final value
$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$	Initial value

- **Final value theorem**: The long-term behaviour of a time-domain function can be found by analysing the Laplace domain behaviour in the limit as the s variable approaches zero.
- **Initial value theorem**: The initial value of a time-domain function can be found by analysing the Laplace domain behaviour in the limit as s approaches infinity.

Laplace transform exercises

Exercise L2E1: Find the Laplace transform of $f(t) = Ae^{-at}$

- In Matlab/Octave

```
syms A a t y

f1 = A* exp(-a*t);
F1 = laplace(f1);

>> F1
F1 = A/(a + s)
```

- In Python

```
import sympy as sym
from sympy.abc import A,a,s,t

f1 = A*sym.exp(-a*t)
F1 = laplace_transform(f1,t,s)

print(F1)
# (A/(a + s), 0, Abs(arg(a)) <= pi/2)
```


Exercise L2E2: Find the inverse Laplace transform $F_1(s) = \frac{1}{(s+3)^2}$

- In Matlab/Octave

```
syms s
F2=1/(s+3)^2;
f2=ilaplace(F2);

>> f2

f2 =
t*exp(-3*t)
```

- In Python

```
import sympy as sym
from sympy.abc import s,t

F2 = 1/(s+3)**2
f2 = sym.inverse_laplace_transform(F2,s,t)

print(f2)
# t*exp(-3*t)*Heaviside(t)
```

Application to the solution of differential equations

Consider the problem of finding the solution to the simple differential equation:

$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$$

for the situation where $u(t)$ is a step function and $y(0) = 0$.

- Because of the uniqueness of transform pairs, equality of two functions implies equality of their transforms.
- We can take Laplace transform of both sides of the equation to obtain an equivalent equation in the Laplace variable.

$$\tau s Y(s) + Y(s) = \frac{K}{s}$$

$$Y(s) = \frac{K}{s(\tau s + 1)}$$

The next step is obtaining $y(t)$, the inverse transform of $Y(s)$.

Application to the solution of differential equations

We then have that:

$$Y(s) = \frac{K}{s(\tau s+1)}, \text{ which can be written as: } Y(s) = \frac{K}{s} - \frac{K\tau}{\tau s+1}$$

That is, the right-hand side has been broken into **two partial fractions**.

- By recalling the shift property, and noting that the second function can be expressed as: $\dfrac{K}{s+1/\tau}$.

We obtain that:

$$y(t) = \left(K - Ke^{-t/\tau}\right)u(t) \quad \text{or}$$
$$y(t) = K\left(1 - e^{-t/\tau}\right)u(t)$$

$$Y(s) = \frac{K}{s(\tau s+1)} = \frac{A}{s} + \frac{B}{\tau s+1}$$

$$A(\tau s+1) + Bs = K$$

$$(A\tau + B)s + A = K$$

$$A = K$$

$$B = -\tau K$$

$$Y(s) = \frac{K}{s} - \frac{K}{s+1/\tau}$$

$$y(t) = \left(K - Ke^{-t/\tau}\right)u(t) = K\left(1 - e^{t/\tau}\right)u(t)$$

Application to the solution of differential equations

Regardless of the complexity of the general differential equation in the variable y as unknown function of t

- ⇝ As long as it is linear, and of the ordinary type, with constant coefficients

the procedure for solving the previous example may be generalised:

1. Take the Laplace transform of both sides of the equation.
   - ⇝ Note that by the definition of the Laplace transform of a derivative of any order, the initial conditions are automatically included.
2. Solve the resulting algebraic equation for the Laplace transform of the unknown function, $Y(s)$.
3. Obtain $y(t)$ by the inversion of the solution obtained for $Y(s)$ in Step 2.
   - ⇝ This step usually involves breaking up $Y(s)$ into simpler functions, whose inverse transforms are more easily recongnizeble, by a procedure known as **partial fraction expansion**.

Partial fraction expansion

With the **partial fraction expansion**, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term and then we find the inverse Laplace transform of a complicated function.

- If $F_1(s) = N(s)/D(s)$, where the order of $N(s)$ is less than the order of $D(s)$, then a partial fraction expansion can be made.
- If the order of $N(s)$ is greater than or equal to the order of $D(s)$, then $N(s)$ must be divided by $D(s)$ successively until the result has a remainder whose numerator is of order less than its denominator.

Example: The following function:

$$F_1(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

can be written as:

$$F_1(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

and the last term can be further expanded into partial fractions.

1	2	6	7
	-1	-1	-5
1	1	5	2

Partial fraction expansion

Case 1. Roots of the denominator of $F(s)$ are real and distinct

An example of an $F(s)$ with real and distinct roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)}$$

- The roots of the denominator are distinct.
- The partial fraction expansion can be written as the sum of terms:
  - Each factor of the original denominator forms the denominator of each term.
  - Constants (the **residues**) form the numerators.

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

The goal is to find the residues K_1 and K_2 and then $f(t)$ as the sum of the inverse Laplace transform of each term

In general terms, given an $F(s)$ whose denominator has real and distinct roots, a partial fraction expansion:

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\ldots(s+p_n)} = \\ = \frac{K_1}{s+p_1} + \frac{K_1}{s+p_1} + \ldots + \frac{K_n}{s+p_n}$$

can be made if the order of $N(s)$ is less than the order of $D(s)$.

- To evaluate each residue, K_i, we multiply the equation by the denominator of the corresponding partial fraction.
- Thus, if we want to find Km, we multiply the equation by $s+p_m$.
- and then ...

Partial fraction expansion

Case 1. Roots of the denominator of $F(s)$ are real and distinct

Exercise L2E3: Using the Laplace transform, solve the following differential equation

- $y(t)$ if all initial conditions are zero
- $u(t)$ is a step function

$$\frac{d^2y(t)}{dt^2}+12\frac{dy(t)}{dt}+32y(t)=32u(t)$$

ToDo!!

Partial fraction expansion

Case 2. Roots of the denominator of $F(s)$ are real and repeated

An example of an $F(s)$ with real and repeated roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

- The denominator root at -2 is multiple root of **multiplicity** 2.
- We can write the partial fraction expansion as a sum of terms, where each factor of the denominator forms the denominator of each term.
- In addition, each multiple root generates additional terms consisting of denominator factors of reduced multiplicity.

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

Again, the goal is to find the residues and then $f(t)$ as the sum of the inverse Laplace transform of each term.

In general, then, given an F(s) whose denominator has real and repeated roots, a partial fraction expansion,

$$\begin{aligned} F(s) &= \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)^r(s+p_2)\ldots(s+p_n)} = \\ &= \frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + \ldots + \frac{K_r}{(s+p_1)} + \\ &+ \frac{K_{r+1}}{(s+p_2)} + \ldots + \frac{K_n}{(s+p_n)} \end{aligned}$$

It can be shown that the general expression for K_1 through K_r for the multiple roots is:

$$K_i = \frac{1}{(i-1)!} \left. \frac{d^{i-1}F_1(s)}{ds^{i-1}} \right|_{s\to -p_1} \quad i = 1, 2, \ldots, r; \quad 0! = 1$$

Partial fraction expansion

Case 3. Roots of the denominator of $F(s)$ are complex or imaginary

An example of $F(s)$ with complex roots in the denominator is:

$$F(s) = \frac{3}{s(s^2+2s+5)} = \text{which can be expanded as} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2s+5}$$

- K_1 is found in the usual way to be 3/5.
- K_2 and K_3 can be found by first multiplying the previous equation by the lowest common denominator, $s(s^2+2s+5)$, and clearing the fractions.
- and then ...

Laplace transform
Exercises

Exercise L2E4: Find the Laplace transform of $f(t) = te^{-5t}$.

Exercise L2E5: Find the inverse Laplace transform of

$$F(s) = \frac{10}{s(s+2)(s+3)^2}$$

$$F(s) = \frac{s-10}{(s+2)(s+5)}$$

$$F(s) = \frac{100}{(s+1)(s^2+4s+13)}$$

$$F(s) = \frac{s+18}{s(s+3)^2}$$

To Do!!