



Recap and exercises

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SUSTAINABLE DEVELOPMENT GOAL 3:
Ensure healthy lives and promote well-being for all
at all ages

<https://sdgs.un.org/goals/goal3>



QUESTIONS & ANSWERS
‘Any time’ on TELEGRAM
Friday with YAGO
@2pm !!



Updating the schedule...

HOMEWORK 1 - PROJECT WORK (TASK 1)

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INSTRUÇÕES E PRAZOS

O objetivo do HW1 e da primeira tarefa do trabalho de projeto (PW) é revisar e se familiarizar com os conceitos de modelagem e representá-los na linguagem de programação de sua escolha. Você pode se referir aos slides das aulas (de L1 a L7) e os capítulos 2 e 3 de Nise para o embasamento teórico [The objective of HW1 and of the first task of the project work (PW) is to revise and familiarise with the modelling concepts and to represent them in the programming language of your choice. You might refer to the lecture slides (from L1 to L7) and Nise's chapter 2 and 3 for the theoretical background].

GUILDELINEs: Resolver cada exercício usando cálculos teóricos e relatando as principais etapas de sua resolução. Compare, em seguida, os resultados no ambiente de programação de sua preferência. O máximo de cada questão é de 25 pontos. O resolução tem que ser enviada em um único documento e tem que deve incluir [You must solve each exercise using theoretical calculations and reporting the main steps of your resolution. Then, you compare the results on the programming environment of your choice. Each exercise pays max 25 points. The resolution must be sent in a single document and it must include]:

o **REPORT:** O relatório tem que conter (i) uma breve explicação da base teórica associada a cada exercício; (ii) os passos relevantes dos cálculos, os resultados, os gráficos (quando necessário) e, principalmente, seus comentários sobre os resultados e/ou gráficos. [The report must include (i) a brief explanation of the theoretical background associated to each exercise; (ii) the relevant steps of your calculations, the results, the graphs (when required) and especially, your comments on the results and/or graphs].

HOMEWORK

Check
the updated
version !!

TUESDAY	THURSDAY
Feb. 27	Feb. 29 COURSE INTRO BACKGROUND SURVEY
Mar. 5	Mar. 7
SYSTEMS AND MODELS	TRANSFER FUNCTIONS
Mar. 12	Mar. 14
TRANSFER FUNCTIONS	SIMULATION OF DYNAMIC SYSTEMS
Mar. 19	Mar. 21
FERIADO ESTADUAL DIA DE SÃO JOSÉ	STATE SPACE MODELS
Mar. 26	Mar. 28
TRANSFER FUNCTION TO/FROM STATE SPACE	RECESSO ESCOLAR SEMANA SANTA
Apr. 2	Apr. 4
1ST ORDER SYSTEM	2ND ORDER SYSTEM
Apr. 9 AP1	Apr. 11
Apr. 16 NO CLASS	Apr. 18 NO CLASS
BLOCK DIAGRAM REPRESENTATION	SIGNAL FLOW REPRESENTATION
Apr. 23	Apr. 25
STABILITY	STABILITY
Apr. 30	May. 2
ROOT-LOCUS	ROOT-LOCUS
May. 7	May. 9
DESIGN WITH RL	EXERCISES COURSE ASSISTANTS
May. 14 EXERCISES COURSE ASSISTANTS	May. 16 EXERCISES COURSE ASSISTANTS
May. 21 EXERCISES COURSE ASSISTANTS	May. 23
	REVISION EXERCISES



CHANGE of !!
PLANS ..

→ EXTRA on FRIDAY

1 March 22
@ 2pm

→ EXTRA on FRIDAY

APRIL 5
@2pm

REVISION EXERCISES

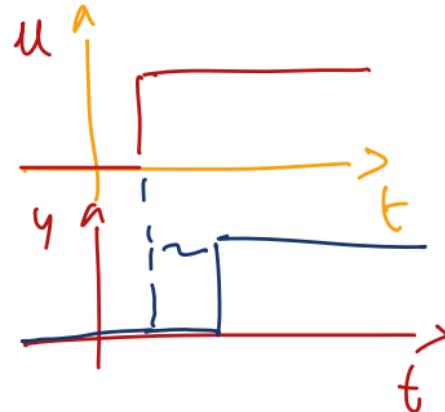
to be
rescheduled !!



Brief recap

So far, we did...

- ▶ Describe dynamic systems and their mathematical models as IO and SV
- ▶ Define the main properties of a mathematical model
 - ~~ Dynamic and instantaneous
 - ~~ Linear and non linear
 - ~~ Stationary and non stationary
 - ~~ Proper and improper
 - ~~ Lumped and distributed parameters
 - ~~ Time delay



I/O representation	↗ EXTERNAL DESCRIPTION
$b(y(t), \dot{y}(t), \dots, y^{(n)}(t), u(t), \dot{u}(t), \dots, u^{(m)}(t)) = 0 \quad n > m$	
S/V representation	↗ INTERNAL DESCRIPTION
$\begin{cases} \dot{x}(t) = f(x(t), u(t), t \mid \theta_x) \\ y(t) = g(x(t), u(t), t \mid \theta_y) \end{cases}$	

$$\begin{aligned}\theta &\triangleq \theta_x \\ \sigma &\triangleq \theta_y\end{aligned}$$



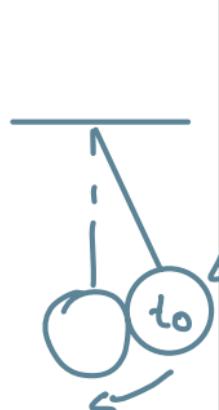
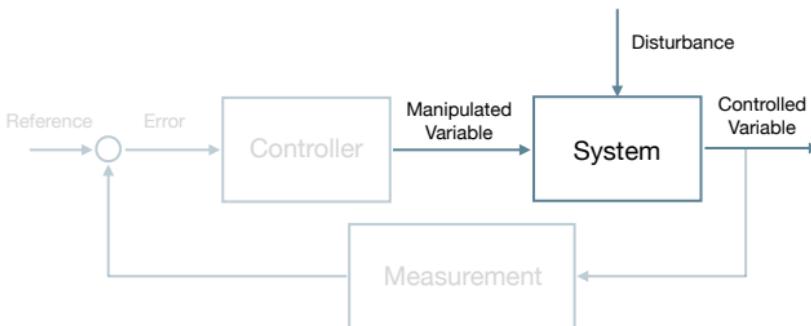
Check EXTRA NOTES on
LINEARIZATION ??



Brief recap

So far, we did...

- ▶ Describe dynamic systems and their mathematical models as IO and SV
- ▶ Define the main properties of a mathematical model



FUNDAMENTAL PROBLEM of system analysis consists of finding the behaviour of $y(t)$ as $t \geq t_0$

$$\begin{aligned} & \text{In } \frac{d^n y(t)}{dt^n} + \dots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = \\ & = b_m \frac{d^m u(t)}{dt^m} + \dots + b_1 \frac{du(t)}{dt} + b_0 u(t) \text{ given} \end{aligned}$$

- the initial conditions

$$y(t) \Big|_{t=t_0} = y_0 \quad \frac{dy(t)}{dt} \Big|_{t=t_0} = y'_0 \quad \dots$$

$$\frac{d^{n-1} y(t)}{dt^{n-1}} \Big|_{t=t_0} = y_0^{n-1}$$

- an input signal $u(t)$ for $t \geq t_0$

$$y(t) = y_{\text{free}}(t) + y_{\text{forced}}(t)$$



Brief recap

So far, we did...

- ▶ Review on the Laplace transform of time functions.
- ▶ Review on the inverse Laplace transform.
- ▶ Review on the partial-fraction expansion.
- ▶ Solve the differential equation using the Laplace transforms.

Modelling in the frequency domain

S-domain //



LAPLACE TRANSFORM

- Is an integral transform that converts a function of a real variable t (time) to a complex variable $s = \sigma + j\omega$
- It transforms differential equations into algebraic equations
and convolutions into multiplication

$$\tilde{f}(s) = \mathcal{L}[f(t)] \iff f(t) = \mathcal{L}^{-1}[\tilde{f}(s)]$$

$$\tilde{f}(s) = \int_0^{\infty} f(t) e^{-st} dt \sim f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \tilde{f}(s) e^{st} dt$$



Brief recap

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Modelling in the frequency domain

S-domain //

USEFUL PROPERTIES of LAPLACE TRANSFORM

1. LINEARITY : Given $f(t) = c_1 f_1(t) + c_2 f_2(t)$

$$\hat{f}(s) = c_1 \hat{f}_1(s) + c_2 \hat{f}_2(s)$$

2. TIME DIFFERENTIATION $\mathcal{L}\left[\frac{d}{dt} f(t)\right] = s\hat{f}(s) - f(0)$

generalising $\mathcal{L}\left[f^{(n)}(t)\right] = s^n \hat{f}(s) - \sum_{i=0}^{n-1} s^{n-i-1} f^{(i)}(0)$

3. TIME-INTEGRATION : $\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{\hat{f}(s)}{s}$

4. TIME-SHIFTING : $\mathcal{L}[f(t-\tau)] = e^{-\tau s} \hat{f}(s)$

5. SCALING $\mathcal{L}[e^{\alpha t} f(t)] = \hat{F}(s-\alpha)$



Brief recap

So far, we did...

- ▶ Review on the Laplace transform of time functions.
- ▶ Review on the inverse Laplace transform.
- ▶ Review on the partial-fraction expansion.
- ▶ Solve the differential equation using the Laplace transforms.

Modelling in the frequency domain

S-domain !!

USEFUL PROPERTIES of LAPLACE TRANSFORM

6. TIME convolution Given $f(t)$ and $g(t)$ such that
 $f(t) \cdot g(t) = 0$ for $t < 0$

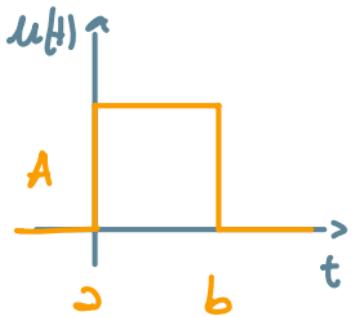
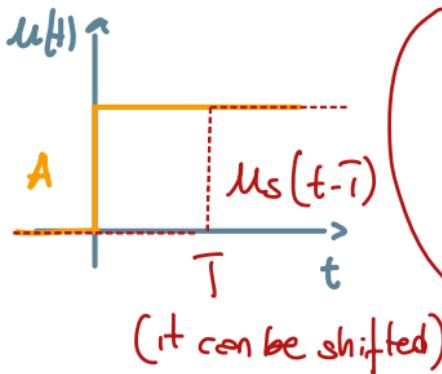
$$h(t) = f(t) \cdot g(t) \rightsquigarrow H(s) = \bar{F}(s) G(s)$$

• FINAL VALUE THEOREM. $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s\bar{f}(s)$

• INITIAL VALUE THEOREM $f(0^+) = \lim_{s \rightarrow \infty} s\bar{f}(s)$



Brief recap

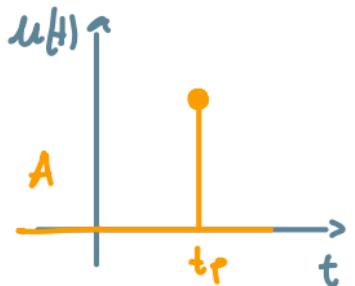


STEP FUNCTION might be realised by implementing a sudden change in the position of the actuator

$$u_s = \begin{cases} 0 & t < 0 \\ A & t \geq 0 \end{cases}$$

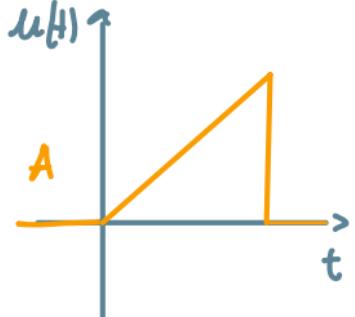
$$u_{RP} = \begin{cases} 0 & t < 0 \\ h & a \leq t < b \\ 0 & t \geq b \end{cases}$$

RECTANGULAR PULSE might be realised by making an instantaneous change in the actuator position at time a and keeping it until time b



$\delta(t)$ function

$$\int_{-\infty}^{\infty} \delta(t) dt = u_s(t) \Big|_{-\infty}^{+\infty} = 1$$



IMPULSE is physically impossible to implement the pulse function exactly. To obtain an impulse input, it is necessary to inject a finite amount of energy or material into the system in an infinitesimal length of time (impractical!!)

$$u_I = \begin{cases} P & t = t_p \\ 0 & \text{otherwise} \end{cases}$$

RAMP might be realised with a gradual change upward or downward for some period of time with a roughly constant slope

$$u_R = \begin{cases} 0 & t < 0 \\ at & t \geq 0 \end{cases}$$

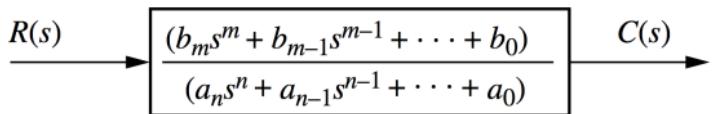
NEXT ... SINEWAVE WAVES!!



Brief recap

Now, we finalise from last lecture...

- ▶ The concept of transfer function.
 - ~~ We can formulate the **system block representation** by establishing a viable definition for a function that algebraically relates a system's output to its input.
 - ~~ This function will allow separation of the input, system and output into three separate and distinct parts.
 - ~~ Transfer function also allows us to algebraically combine mathematical representations of subsystems to yield a total system representation.



RATIONAL FUNCTIONS \leadsto particularly important in system analysis
 $f(t)$ can be written as a linear combination of exponential ramps

$$f(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0 s + a_0} \quad \leadsto \text{ causal system}$$

$n > m$
(or $n > m$)

$D(s) \leadsto n$ roots $\leadsto p_1, \dots, p_n \rightarrow$ POLES

$N(s) \leadsto m$ roots $\leadsto z_1, \dots, z_n \rightarrow$ ZEROS

We can factorize $D(s)$ and $N(s)$ as

$$N(s) = b_m (s - z_1)(s - z_2) \dots (s - z_m)$$

$$D(s) = a_n (s - p_1)(s - p_2) \dots (s - p_n)$$

$$\rightarrow f(s) = k' \frac{(s - z_1) \dots (s - z_m)}{(s - p_1) \dots (s - p_n)}$$

$$k' = b_m / a_n$$

STATIC GAIN

INITIAL REALIZATION



Brief recap

1. $\tilde{F}(s)$ has poles of **UNIARY MULTIPLICITY**

$$\tilde{F}(s) = \sum_{i=1}^n \frac{R_i}{s-p_i} = \frac{R_1}{s-p_1} + \frac{R_2}{s-p_2} + \dots + \frac{R_n}{s-p_n}$$

$$f(t) = \mathcal{L}^{-1}[\tilde{F}(s)] = \sum_{i=1}^n \mathcal{L}\left[\frac{R_i}{s-p_i}\right] = \sum_{i=1}^n R_i e^{p_i t}$$

R_i is the POLE RESIDUAL $\Rightarrow R_i = \lim_{s \rightarrow p_i} (s-p_i)\tilde{F}(s)$

EXAMPLE

$$\tilde{F}(s) = \frac{s+8}{s^2+2s} = \frac{s+8}{s(s+2)} = \frac{R_1}{s} + \frac{R_2}{s+2}$$

$$R_1 = s \cdot \tilde{F}(s) \Big|_{s \rightarrow 0} = \frac{s+8}{s+2} \Big|_{s \rightarrow 0} = 4$$

$$R_2 = (s+2) \tilde{F}(s) \Big|_{s \rightarrow -2} = \frac{s+8}{s} \Big|_{s \rightarrow -2} = -3$$

$$\tilde{F}(s) = \frac{4}{s} - \frac{3}{s+2} \Rightarrow f(t) = 4 - 3e^{-2t}$$



Brief recap

1. $F(s)$ has poles of **UNITARY MULTIPLICITY**

$$F(s) = \sum_{i=1}^n \frac{R_i}{s-p_i} = \frac{R_1}{s-p_1} + \frac{R_2}{s-p_2} + \dots + \frac{R_n}{s-p_n}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = \sum_{i=1}^n \mathcal{L}\left[\frac{R_i}{s-p_i}\right] = \sum_{i=1}^n R_i e^{p_i t}$$

R_i is the **POLY RESIDUAL** $\Rightarrow R_i = \lim_{s \rightarrow p_i} (s-p_i) F(s)$

```
clear; clc; close all
```

% Example

```
syms s
F=(s+8)/(s*(s+2))
pretty(ilaplace(F))
```



```
1 # import sympy as sym
2 from sympy.abc import A,a,s,t
3 from sympy.integrals import laplace_transform
4 from sympy.integrals import inverse_laplace_transform
5
6 # Example
7 F2 = (s+8)/(s*(s+2))
8 f2 = sym.inverse_laplace_transform(F2,s,t)
9 f2
```





Brief recap

2. $\tilde{F}(s)$ has poles with **MULTIPlicity > 1**

$$\tilde{F}_i(s) = \frac{R_{i,0}}{s-p_i} + \frac{R_{i,1}}{(s-p_i)^2} + \dots + \frac{R_{i,v_{i-1}}}{(s-p_i)^{v_i}} = \sum_{k=0}^{v_{i-1}} \frac{R_{ik}}{(s-p_i)^{k+1}}$$

$$\tilde{F}(s) = \sum_{i=1}^r \tilde{F}_i(s) = \sum_{i=1}^r \sum_{k=0}^{v_{i-1}} \frac{R_{ik}}{(s-p_i)^{k+1}}$$

$$\mathcal{L}^{-1}[\tilde{F}_i(s)] = R_{ik} \frac{t^k}{k!} e^{p_i t}$$

$$\mathcal{L}^{-1}[\tilde{F}(s)] = \sum_{i=1}^r \sum_{k=0}^{v_{i-1}} R_{ik} \frac{t^k}{k!} e^{p_i t}$$

$$P_{i,v_{i-1}} = (s-p_i)^{v_i} \tilde{F}(s) \Big|_{s=p_i}$$

$$P_{i,v_{i-2}} = \frac{d}{ds} [(s-p_i)^{v_i} \tilde{F}(s)] \Big|_{s=p_i}$$

$$P_{i,v_{i-3}} = \frac{d^2}{ds^2} [(s-p_i)^{v_i} \tilde{F}(s)] \Big|_{s=p_i}$$



Brief recap

2. $\tilde{F}(s)$ has poles with multiplicity > 1 - EXAMPLE

$$\tilde{F}(s) = \frac{2}{(s+1)(s+2)^2} = \frac{R_1}{s+1} + \frac{R_2}{(s+2)^2} + \frac{R_3}{s+2}$$

$$R_1 = (s+1) \tilde{F}(s) \Big|_{s \rightarrow -1} = \frac{2}{(s+2)^2} \Big|_{s \rightarrow -1} = 2$$

$$R_2 = \frac{d}{ds} \left[(s+2)^2 \tilde{F}(s) \right] \Bigg|_{s \rightarrow -2} = \frac{d}{ds} \left[\frac{2}{s+2} \right] \Bigg|_{s \rightarrow -2} = 2$$

$$R_3 = (s+2)^2 \tilde{F}(s) \Big|_{s \rightarrow -2} = \frac{2}{s+1} \Big|_{s \rightarrow -2} = -2$$

$$F(s) = \frac{2}{s+1} + \frac{2}{(s+2)^2} - \frac{2}{s+2}$$

$$f(t) = 2e^{-t} + 2te^{-2t} - 2e^{-2t}$$



Brief recap

3. $F(s)$ has complex conjugate poles

$$\rho, \rho' = \alpha \pm j\omega$$

R, R' are the residues

$$M = 2|R| \text{ and } \phi = \arg R$$

$$\alpha^{-1} \left[\frac{R}{s-\rho} + \frac{R'}{s-\rho'} \right] = M e^{\alpha t} \cos(\omega t + \phi)$$

The polar representation of the residuals is

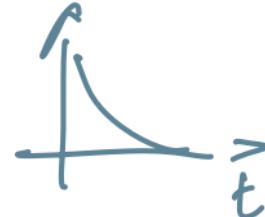
$$R = |R| e^{j\phi} \text{ and } R' = |R| e^{-j\phi}$$

$$\begin{aligned} \alpha^{-1} \left[\frac{R}{s-\rho} + \frac{R'}{s-\rho'} \right] &= R e^{\rho t} + R' e^{-\rho t} = \\ &= |R| \left[e^{\alpha t + j(\omega t + \phi)} + e^{\alpha t - j(\omega t + \phi)} \right] = \\ &= |R| e^{\alpha t} \left[e^{j(\omega t + \phi)} + e^{-j(\omega t + \phi)} \right] \times \frac{2}{2} = \\ &= 2|R| e^{\alpha t} \cos(\omega t + \phi) = M e^{\alpha t} \cos(\omega t + \phi) \end{aligned}$$

EULER FORMULA

$$\cos \phi = \frac{e^{j\phi} + e^{-j\phi}}{2}$$

$$\sin \phi = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$





Brief recap

3. $\tilde{F}(s)$ has complex conjugate poles

$$p, p' = \alpha \pm j\omega$$

$$R, R' = \mu + j\nu$$

With $B=2\mu$ and $C=-2\nu$

$$\mathcal{L}^{-1} \left[\frac{R}{s-p} + \frac{R'}{s-p'} \right] = Be^{\alpha t} \cos \omega t + Ce^{\alpha t} \sin \omega t$$

NOTE: Given a pair of complex conjugate poles: $p, p' = \alpha \pm j\omega$ and complex conjugate residuals $R, R' = \mu + j\nu$

$$\begin{aligned} \text{PROOF: } & \mathcal{L}^{-1} \left[\frac{R}{s-p} + \frac{R'}{s-p'} \right] = Re^{\alpha t} + R'e^{\alpha t} = \\ & = (\mu + j\nu) e^{\alpha t + j\omega t} + (\mu - j\nu) e^{\alpha t - j\omega t} = \\ & = \mu e^{\alpha t} (e^{j\omega t} + e^{-j\omega t}) + j\nu e^{\alpha t} (e^{j\omega t} - e^{-j\omega t}) = \\ & = 2\mu e^{\alpha t} \cos \omega t - 2j\nu e^{\alpha t} \sin \omega t = \\ & = Be^{\alpha t} \cos \omega t + Ce^{\alpha t} \sin \omega t \end{aligned}$$



Brief recap

3. $\tilde{F}(s)$ has complex conjugate poles

EXAMPLE

$$\tilde{F}(s) = \frac{20}{s(s^2+4s+5)} = \frac{20}{s(s+2+j)(s+2-j)} =$$

$$= \frac{R_1}{s} + \frac{R}{s+2+j} + \frac{R'}{s+2-j}$$

$$R_1 = s \cdot \tilde{F}(s) \Big|_{s=0} = \frac{20}{s^2+4s+5} \Big|_{s=0} = \frac{20}{5} = 4$$

$$\begin{aligned} R' &= (s+2-j) \tilde{F}(s) \Big|_{s=-2+j} = \frac{20}{s(s+2+j)} \Big|_{s=-j} = \frac{20}{(-2+j)(-2+j+2+j)} \\ &= \frac{20 (-2+j4)}{(-2-j4)(-2+j4)} = \frac{40 (-2+j4)}{20} = -2+j4 = u+jv \end{aligned}$$

$$R = -2 + j4$$

$$M = 2|R| = 2\sqrt{2^2+4^2} = 2\sqrt{20}$$

$$\phi = \arg |R| = \arctan \left(\frac{4}{-2} \right) \approx -2.03$$

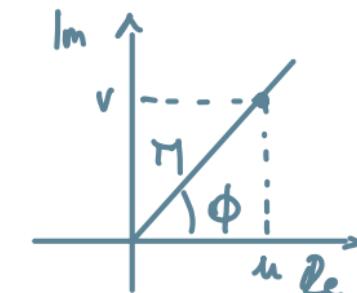
$$f(t) = 4 + 2\sqrt{20} e^{-2t} \cos(t - 2.03)$$

Alternatively: $B = 2u = 4 \quad C = -2v = -B$

$$f(t) = 4 - 4e^{-2t} (\cos t + 2 \sin t)$$

$$\text{Note: } M = \sqrt{B^2 + C^2} \quad \phi = \arctan \left(\frac{C}{B} \right)$$

$$B = M \cos \phi \quad C = M \sin \phi$$





Exercises

Exercise L4E1: Determine the Laplace transform of the following:

a. $\delta(t)$

b. $u(t) \rightsquigarrow u(t) = 1 \text{ for all } t \geq 0$

c. $\cos \omega t u(t)$

$\rightsquigarrow = \frac{1}{2} [e^{j\omega t} + e^{-j\omega t}] u(t)$

2. $\mathcal{L}[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = 1 \forall s$

b. $\mathcal{L}[u(t)] = \int_0^\infty u(t) e^{-st} dt = \int_0^\infty e^{-st} dt =$
 $= -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s}$

c. $\mathcal{L}[\cos \omega t u(t)] = \frac{1}{2} \mathcal{L}[e^{j\omega t} u(t) + e^{-j\omega t} u(t)]$
 $= \frac{1}{2} \left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega} \right] = \frac{s}{s^2 + \omega^2}$



Exercises

Exercise L4E2: Determine the inverse Laplace transform of the following:

a. $\frac{7s - 6}{s^2 - s - 6} = \mathcal{F}(s)$

$$f(t) = 4e^{-2t} + 3e^{3t}$$

To Do !!



Exercises

Exercise L4E2: Determine the inverse Laplace transform of the following:

b. $\frac{2s^2+5}{s^2+3s+2} = \mathcal{F}(s)$

$$f(t) = 2\delta(t) + 7e^{-t} - 13e^{2t}$$

To Do!!



Exercises

Exercise L4E2: Determine the inverse Laplace transform of the following:

c. $\frac{6(s+34)}{s(s^2+10s+34)} \underset{\mathcal{L}}{\sim} f(t)$

$$f(t) = 6 + 10e^{-5t} (\cos(3t) + 2.21) \quad \text{or}$$

$$f(t) = 6 - 6e^{-5t} \left(\cos(3t) + \frac{4}{3} \sin(3t) \right)$$

To do!!