EXTRA READING – NOTES ON LINEARISATION

WHY LINEARISE?

- Many dynamical models are modelled by a set of NON-LINEAR first order differential equations that generally arise from first principle laws.
- Common analysis techniques are based on LINEAR SYSTEM THEORY
- Most control system design techniques are based on <u>linear models</u>

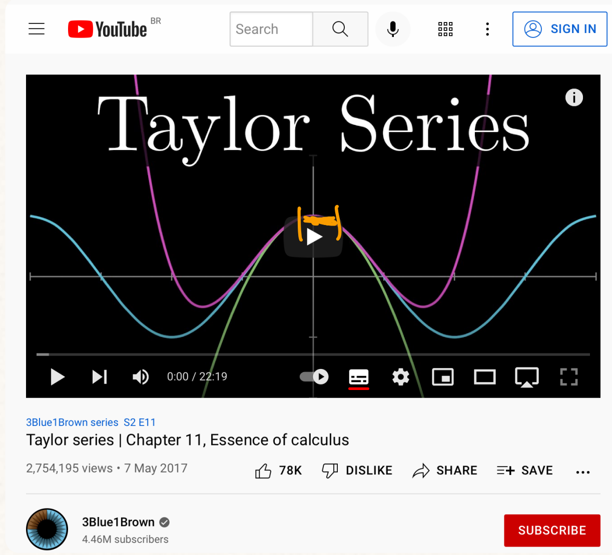
LINEARISATION OF NON LINEAR MODELS

Consider a function with ONE STATE VARIABLE and ONE INPUT VARIABLE

⇝ 1 state variable, x

⇝ 1 input variable, u

$$\dot{x}_{NL} = \frac{dx}{dt} = f(x, u, t)$$



The function $f(x, u, t)$ can be <u>APPROXIMATED</u> by a truncated <u>TAYLOR SERIES EXPANSION</u> around an <u>equilibrium (stationary)</u> operating point (x_s, u_s)

$$\boxed{\begin{aligned}\dot{x}_L = {} & f(x_s, u_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s,u_s}(x - x_s) + \left.\frac{\partial f}{\partial u}\right|_{x_s,u_s}(u - u_s) + \\ & + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_s,u_s}(x - x_s)^2 + \frac{1}{2}\left.\frac{\partial f^2}{\partial u^2}\right|_{x_s,u_s}(u - u_s)^2 + \\ & + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x \partial u}\right|_{x_s,u_s}(x - x_s)(u - u_s) + \text{higher order terms}\end{aligned}}$$

Remember that in CALCULUS:

A stationary point of a differentiable function of one variable $\frac{dx}{dt} = f(x(t))$ is a point on the graph of the function where the function derivative is $\emptyset$

$f(x_s) = 0$ ⇝ the function stops increasing.

Truncating after the linear term we have

$$\boxed{\dot{x}_L \approx f(x_s, u_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s,u_s} (x - x_s) + \left.\frac{\partial f}{\partial u}\right|_{x_s,u_s} (u - u_s)}$$

Indicates the partial derivative evaluated at the stationary point

$=0$ at the stationary point

Also note that, we can write $\frac{dx}{dt}$ as $\frac{d(x-x_s)}{dt}$ since the derivative of a constant is zero. The reason for this is that we are interested the deviation of the state from the stationary point

We call $(x-x_s)$, $(u-u_s)$ $\rightarrow$ DEVIATION VARIABLES

$$\frac{d(x-x_s)}{dt} \simeq \left.\frac{\partial f}{\partial x}\right|_{x_s,u_s} (x-x_s) + \left.\frac{\partial f}{\partial u}\right|_{x_s,u_s} (u-u_s)$$

Using the deviation variables $x' = x - x_s$, $u' = u - u_s$

$$\frac{dx'}{dt} = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x_s,u_s}}_{a} x' + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{x_s,u_s}}_{b} u' \longrightarrow \dot{x}' = ax' + bu'$$

these are constant values

If there is a single output y that is a function of the state and input $y = g(x,u)$. Again performing a Taylor Series expansion and truncating the quadratic and higher terms

$$y_L \approx \underbrace{g(x_s, u_s)}_{} + \left.\frac{\partial g}{\partial x}\right|_{x_s,u_s} (x - x_s) + \left.\frac{\partial g}{\partial u}\right|_{x_s,u_s} (u - u_s)$$

is the stationary value of the output (y_s)

We can write

$$y_L - y_s \approx \left.\frac{\partial g}{\partial x}\right|_{x_s,u_s} (x - x_s) + \left.\frac{\partial g}{\partial u}\right|_{x_s,u_s} (u - u_s)$$

y' — c — x' — d — u'

$$y' = cx' + du'$$

TWO-STATE SYSTEM

$$\begin{cases} \dot{x}_1 = \dfrac{dx_1}{dt} = f_1(x_1, x_2, u) \\ \dot{x}_2 = \dfrac{dx_2}{dt} = f_2(x_1, x_2, u) \\ y = g(x_1, x_2, u) \end{cases}$$

Performing a Taylor series expansion of the nonlinear functions, and neglecting the quadratic and higher terms

$$f_1(x_1, x_2, u) \simeq \cancelto{0}{f_1(x_{1,s}, x_{2,s}, u_s)} + \left.\frac{\partial f_1}{\partial x_1}\right|_{ss} (x_1 - x_{1s}) + \left.\frac{\partial f_1}{\partial x_2}\right|_{ss} (x_2 - x_{2s}) +$$

$$+ \left.\frac{\partial f_1}{\partial u_1}\right|_{ss} (u_1 - u_{1s}) + \text{higher order terms}$$

$$f_2(x_1, x_2, u) = \cancelto{0}{f_2(x_{1,s}, x_{2,s}, u_s)} + \left.\frac{\partial f_2}{\partial x_1}\right|_{ss} (x_1 - x_{1s}) + \left.\frac{\partial f_2}{\partial x_2}\right|_{ss} (x_2 - x_{2s}) +$$

fsolve

$$+ \left.\frac{\partial f_2}{\partial u_1}\right|_{ss} (u_1 - u_{1s}) + \text{higher order terms}$$

$$g(x_1, x_2, u) = g(x_{1,s}, x_{2,s}, u_s) + \left.\frac{\partial g}{\partial x_1}\right|_{ss} (x_1 - x_{1s}) + \left.\frac{\partial g}{\partial x_2}\right|_{ss} (x_2 - x_{2s}) +$$

$$+ \left.\frac{\partial g}{\partial u_1}\right|_{ss} (u_1 - u_{1s}) + \text{higher order terms}$$

y_s

$$\frac{dx_1}{dt} = \frac{d(x_1 - x_{1s})}{dt} \qquad \frac{dx_2}{dt} = \frac{d(x_2 - x_{2s})}{dt} \qquad ss = (x_{1s}, x_{2s}, u_s)$$

$$\begin{bmatrix} \dfrac{d(x_1 - x_{1s})}{dt} \\ \dfrac{d(x_2 - x_{2s})}{dt} \end{bmatrix} = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial x_1}\right|_{ss} & \left.\dfrac{\partial f_1}{\partial x_2}\right|_{ss} \\ \left.\dfrac{\partial f_2}{\partial x}\right|_{ss} & \left.\dfrac{\partial f_2}{\partial /}\right|_{ss} \end{bmatrix} \begin{bmatrix} x_1 - x_{1s} \\ x_2 - x_{2s} \end{bmatrix} + \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial u}\right|_{ss} \\ \left.\dfrac{\partial f_2}{\partial u}\right|_{ss} \end{bmatrix} [u - u_s]$$

$$\dot{x}' = A \quad x' + B \quad u'$$

$$y - y_s = \begin{bmatrix} \left.\dfrac{\partial g}{\partial x_1}\right|_{ss} & \left.\dfrac{\partial g}{\partial x_2}\right|_{ss} \end{bmatrix} \begin{bmatrix} x_1 - x_{1s} \\ x_2 - x_{2s} \end{bmatrix} + \left.\frac{\partial g}{\partial u}\right|_{ss} [u - u_s]$$

$$y' = C \quad x' + D \quad u'$$

GENERALIZATION

Now consider the general non linear model where

- x is a vector of n state variables
- u is a vector of m input variables
- y is a vector of r output variables

$$\dot{x}_1 = f_1(x_1, \dots, x_n, u_1, \dots, u_m)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, \dots, x_n, u_1, \dots, u_m)$$
$$y_1 = g_1(x_1, \dots, x_n, u_1, \dots, u_m)$$
$$y_r = g_r(x_1, \dots, x_n, u_1, \dots, u_m)$$

$$\xrightarrow{\text{in vector form}} \quad \begin{aligned} \dot{x} &= f(x, u) \\ y &= g(x, u) \end{aligned}$$

Elements of the linearization matrices are defined in the following fashion

$$a_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x_s, u_s} \qquad b_{ij} = \left.\frac{\partial f_i}{\partial u_j}\right|_{x_s, u_s}$$

$$c_{ij} = \left.\frac{\partial g_i}{\partial x_j}\right|_{x_s, u_s} \qquad d_{ij} = \left.\frac{\partial g_i}{\partial u_j}\right|_{x_s, u_s}$$

$$\underbrace{\begin{bmatrix} \frac{d(x_1 - x_s)}{dt} \\ \vdots \\ \frac{d(x_n - x_s)}{dt} \end{bmatrix}}_{\dot{x}'} = \underbrace{\begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_{x_s, u_s} & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_{x_s, u_s} \\ \left.\frac{\partial f_n}{\partial x_1}\right|_{x_s, u_s} & \cdots & \left.\frac{\partial f_n}{\partial x_n}\right|_{x_s, u_s} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 - x_s \\ \vdots \\ x_n - x_s \end{bmatrix}}_{x} +$$

$A \in \mathbb{R}^{n\times n}$

$B \in \mathbb{R}^{n\times m}$

$$+ \underbrace{\begin{bmatrix} \left.\frac{\partial f_1}{\partial u_1}\right|_{x_s,u_s} & \cdots & \left.\frac{\partial f_1}{\partial u_m}\right|_{x_s,u_s} \\ \left.\frac{\partial f_1}{\partial u_1}\right|_{x_s,u_s} & \cdots & \left.\frac{\partial f_1}{\partial u_m}\right|_{x_s,u_s} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1 - u_s \\ u_m - u_s \end{bmatrix}}_{u}$$

$$\underbrace{\begin{bmatrix} y - y_s \end{bmatrix}}_{y'} = \underbrace{\begin{bmatrix} \left.\partial g_1/\partial x_1\right|_{x_s,u_s} & \left.\partial g/\partial x_n\right|_{x_s,u_s} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 - x_s \\ \vdots \\ x_n - x_s \end{bmatrix}}_{x'}$$

$C \in \mathbb{R}^{r\times n}$

$D \in \mathbb{R}^{r\times m}$

$$+ \underbrace{\begin{bmatrix} \left.\partial g/\partial u_1\right|_{x_s,u_s} & \cdots & \left.\partial g/\partial u_m\right|_{x_s,u_s} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} u - u_s \end{bmatrix}}_{u}$$

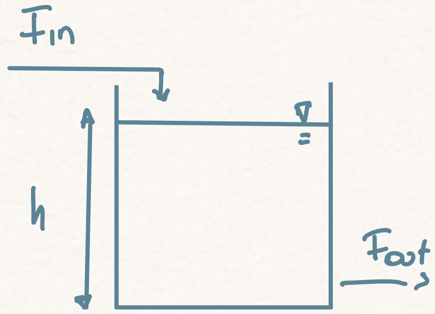
The states in this model are in deviation (perturbation) variable form

→ that is, the state are perturbation from a nominal steady-state

→ A state-space model provides a good approximation to the physical system when the operating point is close to the linearization point (nominal steady-state)

Remember this when comparing the nonlinear Vs linear models

EXAMPLE (one state variable and one input)



A = cross-sectional area of the tank (constant)

Consider a single water level tank

h is the level of water in the tank

F is the influent flow rate

F_{out} is the outlet flow rate $= \alpha\sqrt{2gh}$

α is the orifice area

g is the acceleration constant

Assuming a incompressible fluid, the MASS CONSERVATION for the tank is

$$\frac{dV}{dt} = F_{in} - F_{out}$$

MASS BALANCE:

$$\begin{bmatrix}\text{rate of mass}\\ \text{accumulation in}\\ \text{the system}\end{bmatrix} = \begin{bmatrix}\text{rate of mass}\\ \text{entering the}\\ \text{system}\end{bmatrix} - \begin{bmatrix}\text{rate of mass}\\ \text{leaving the}\\ \text{system}\end{bmatrix}$$

Note that $V = A \cdot h$

$$A\frac{dh}{dt} = F_{in} - F_{out} \rightsquigarrow \frac{dh}{dt} = \underbrace{\frac{1}{A}F - \frac{1}{A}\cdot\alpha\sqrt{2gh}}_{f(x,u)}$$

The right-hand side is

$$f(h,F) = \frac{F}{A} - \frac{\alpha}{A}\sqrt{2gh} \rightsquigarrow \text{non linear function}$$

Using a truncated Taylor series expansion, we find: (h_s, F_s)

$$f(h,F) \approx f(h_s,F_s) + \left.\frac{\partial f}{\partial h}\right|_{h_s,F_s}(h-h_s) + \left.\frac{\partial f}{\partial F}\right|_{h_s,F_s}(F-F_s) =$$

$$= \left[\frac{F}{A} - \frac{\alpha}{A}\sqrt{h_s}\right] + \frac{1}{A}\left[F-F_s\right] - \frac{\alpha}{2A\sqrt{h_s}}\left[h-h_s\right]$$

0 at steady state

$$\frac{d\overbrace{(h-h_s)}^{x'}}{dt} = \overbrace{-\frac{\alpha}{2A\sqrt{h_s}}}^{a}\overbrace{(h-h_s)}^{x'} + \overbrace{\frac{1}{A}}^{b}\overbrace{(F-F_s)}^{u'}$$

$$\boxed{\frac{dx'}{dt} = a x' + b u'}$$