Modelling dynamic systems

Transfer functions

Michela Mulas

3 GOOD HEALTH AND WELL-BEING

SUSTAINABLE DEVELOPMENT GOAL 3:
Ensure healthy lives and promote well-being for all at all ages

https://sdgs.un.org/goals/goal3

QUESTION & ANSWERS

"Any" time on Telegram

Laplace transform of time functions

Exercise L3E0: Find the inverse Laplace transform of

$$F(s) = \frac{s-6}{s^2(s+3)}$$

$$F(s) = \frac{s-6}{s^2(s+3)} = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_3}{s+3}$$

$$K_1 = s^2 \cdot F(s)\Big|_{s\to 0} = \frac{s-6}{s+3}\Big|_{s\to 0} = -2$$

$$K_2 = \frac{d}{ds}\left[s^2 \cdot F(s)\right]\Big|_{s\to 0} = \frac{d}{ds}\left(\frac{s-6}{s+3}\right)\Big|_{s\to 0} = \frac{9}{(s+3)^2}\Big|_{s\to 0} = 1$$

$$K_3 = (s+3)\cdot F(s)\Big|_{s\to -3} = \frac{s-6}{s^2}\Big|_{s\to -3} = -1$$

$$F(s) = \frac{1}{s} - \frac{2}{s^2} - \frac{1}{s+3}$$

$$f(t) = 1 - 2t - e^{-3t}$$

Laplace transform of time functions

Exercise L3E0: Find the inverse Laplace transform of

$$F(s) = \frac{20}{s(s^2+2s+5)}$$

$$F(s) = \frac{20}{s(s^2+2s+5)} = \frac{20}{s(s+1-j2)(s+1+j2)} = \frac{k_1}{s} + \frac{K}{s+1-j2} + \frac{K'}{s+1-j2}$$

$$K_1 = sF(s)\Big|_{s\to 0} = \frac{20}{s^2+2s+5}\Big|_{s\to 0} = 4$$

$$K = (s+1-j2)F(s)\Big|_{s\to -1+j2} = \frac{20}{s(s+1-j2)}\Big|_{s\to -1+j2} = \frac{20}{-8+j4} =$$

$$= \frac{20(-8+j4)}{(-8-j4)(-8+j4)} = \frac{1}{4}(-8+j4) = -2+j = u+jv$$

$$f(t) = 4 + Be^{\alpha t}\cos\omega t + Ce^{\alpha t}\sin\omega t$$

$$B = 2u = -4 \qquad C = -2v = 2$$

Exercises

Exercise L3E1: It is required to

1. Find the transfer function represented by:

$$\frac{dy(t)}{dt} + 2y(t) = u(t)$$

2. Find the response, $y(t)$ to an unit step input, assuming zero initial conditions.

- In Matlab/Octave

```
num=1;
den=[1 2];

model=tf(num,den);
step(model);
```

- In Python

```
den = np.array([1.0, 2.0])
sys_1st = (1, den)
t, y = signal.step(sys_1st)
```

$$sY(s) + 2Y(s) = U(s)$$

$$(s+2)Y(s) = U(s) \rightsquigarrow Y(s) = \frac{1}{s+2}U(s)$$

$$Y(s) = \frac{1}{s+2} \cdot \frac{1}{s} = \frac{A}{s+2} + \frac{B}{s}$$

$$A = (s+2) \cdot Y(s)\Big|_{s \to -2} = -\frac{1}{2}$$

$$B = s\,Y(s)\Big|_{s \to 0} = \frac{1}{2}$$

$$y(t) = -\frac{1}{2}e^{-2t} + \frac{1}{2}$$

Exercises

Exercise L3E2: For a system with the following transfer function:

$$G(s) = \frac{s}{(s+4)(s+8)}$$

1. Find the impulse response of the system.
2. Find the ramp response of the system. $\rightsquigarrow U(s) = 1/s^2$

- In Matlab/Octave

```
% Construct the transfer function
num=[1 0];   den=[1 12 32]; G=tf(num,den);
% Impulse response
impulse(G)

% Construct the input ramp
t=0:0.1:10; alpha=1;
ramp=alpha*t;

% Simulate and plot the output
[y,t]=lsim(G,ramp,t);
figure; plot(t,y)
```

IMPULSE $U(s) = 1$

STEP $U(s) = 1/s$

RAMP $U(s) = 1/s^2$

1. $Y(s) = G(s)U(s) = \frac{s}{(s+4)(s+8)} \cdot 1 = \frac{K_1}{s+4} + \frac{K_2}{s+8}$

$K_1 = (s+4)Y(s)\Big|_{s\to-4} = \frac{s}{s+8}\Big|_{s\to-4} = -1$

$K_2 = (s+8)Y(s)\Big|_{s\to-8} = \frac{s}{s+4}\Big|_{s\to-8} = 2$

$Y(s) = -\frac{1}{s+4} + \frac{2}{s+8}$

$y(t) = -e^{-4t} + 2e^{-8t}$

2. $Y(s) = \frac{s}{(s+4)(s+8)} \cdot \frac{1}{s^2} = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$

$y(t) = \frac{1}{32}e^{-t} - \frac{1}{16}e^{-4t} + \frac{1}{16}e^{-8t}$

Exercises

Exercise L3E3: For each of the following transfer functions, write the corresponding differential equations.

a. $G(s) = \dfrac{7}{s^2+5s+10} = \dfrac{Y(s)}{X(s)}$ $\quad Y(s)(s^2+5s+10) = 7X(s) \rightsquigarrow s^2Y(s) + 5sY(s) + 10Y(s) = 7X(s)$

$\ddot{y}(t) + 5\dot{y}(t) + 10y(t) = 7u(t)$

b. $G(s) = \dfrac{15}{(s+10)(s+11)}$

c. $G(s) = \dfrac{s+3}{s^3+11s^2+12s+18}$

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt}\right] = s^2F(s) - sF(0) - \frac{df(0)}{dt}$$

Exercises

Exercise L3E4: Consider the following model:

$$2\frac{d^2y(t)}{dt}+6\frac{dy(t)}{dt}+4y(t)=\frac{du(t)}{dt}+3u(t)$$

1. Find the free response of the system, assuming the following initial conditions:

$$y(t)\Big|_{t=0}=2,\qquad \frac{dy(t)}{dt}\Big|_{t=0}=1$$

2. Find the transfer function for the system and its response to an input

$$u(t)=12e^{-4t}\delta(t)$$

$$U(s)=\frac{12}{s+4}$$

$$2\left[s^2Y(s)-2sy_0-2\dot{y}_0\right]+$$

$$+6\left[sY(s)-y_0\right]+4Y(s)=sU(s)+3U(s)$$

$$(2s^2+6s+4)Y(s)-(2sy_0+2\dot{y}_0+6y_0)=(s+3)U(s)$$

$$Y(s)=\underbrace{\frac{2sy_0+6y_0+2\dot{y}_0}{2(s^2+3s+2)}}_{Y_{free}(s)}+\underbrace{\frac{s+3}{2(s^2+3s+2)}U(s)}_{Y_{forced}(s)}$$

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