Monitoria de Revisão

Tuesday, May 19, 2020 11:11 AM

Sistemas Lineares

Propriedades Importantes

$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \rightarrow$ convergente

- Se $A = PDP^{-1} \Rightarrow e^A = Pe^DP^{-1}$
- Se $AS = SA \Rightarrow e^{A+S} = e^Ae^S$
- $e^{-S} = (e^S)^{-1}$

Matrizes essenciais

$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow e^{Bt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

$B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \Rightarrow e^{Bt} = \begin{bmatrix} e^{\lambda t} & t\cdot e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$

$B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \Rightarrow e^{Bt} = e^a \begin{bmatrix} \cos b & -\text{sen} b \\ \text{sen} b & \cos b \end{bmatrix}$

- A solução única para $X'(t) = AX$ e $X(0) = X_0$ é:

$$X(t) = e^{At}X_0$$

Alternativa apresentada no livro:

Se $\lambda_1 \neq \lambda_2$ (ou v_1 e v_2 LI) $\Rightarrow X(t) = c_1v_1e^{\lambda_1 t} + c_2v_2e^{\lambda_2 t}$

Se $\lambda_1 = \lambda_2 \Rightarrow X(t) = c_1v_1e^{\lambda_1 t} + c_2v_2te^{\lambda_1 t}$

onde $(A - \lambda_1 I)v_2 = v_1$

Exercício: Sejam $\begin{cases} x' = 2x - y \\ y' = 3x - 2y \end{cases}$ e $\begin{matrix} x(0) = x_0 \\ y(0) = y_0 \end{matrix}$. Encontre $x(t)$ e $y(t)$

$A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$

Passo 1: Autovalores

$p\lambda = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

Passo 2: Qual dos tipos?

1: Autovalores reais e diferentes
2: Autovalores reais e iguais
3: Autovalores com parte imaginária diferente de 0.

Passo 3: Autovetores (Caso 1)

Se $\lambda_1 = 1 \Rightarrow A - \lambda_1 I = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \Rightarrow v_1 = (1,1)$

Se $\lambda_2 = -1 \Rightarrow A - \lambda_2 I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \Rightarrow v_2 = (1,3)$

Passo 4: Diagonalizar Matriz

$A = PDP^{-1} = \begin{array}{c} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \\ v_1 \quad v_2 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \cdot \frac{1}{2}$

Passo 5: Exponencial e Solução

$X(t) = e^{At}X_0 = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \cdot \frac{1}{2}$

$= \begin{bmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{bmatrix} \begin{bmatrix} 3x_0 - y_0 \\ -x_0 + y_0 \end{bmatrix} \frac{1}{2}$

$= \frac{1}{2} \begin{bmatrix} 3x_0e^t - y_0e^t - x_0e^{-t} + y_0e^{-t} \\ 3x_0e^t - y_0e^t - 3x_0e^{-t} + 3y_0e^{-t} \end{bmatrix}$

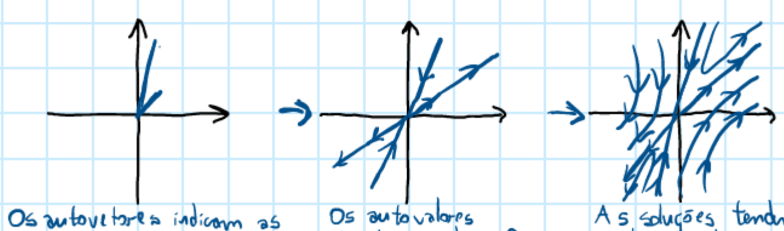
Livro:

$x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$

$X(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \cdot \frac{1}{2}$

Passo 6: Desenho



Os autovetores indicam as direções "invariantes". Os autovalores negativos tendem a 0. Os positivos ao infinito. As soluções tendem para a direção do maior autovalor.

Exercício 2: $X' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} X$

1) $p\lambda = \lambda^2 + 9 = 0 \Rightarrow \lambda = \pm \boxed{3i}$ Caso 3

2) Caso 3

3) $(A - \lambda_1 I) = \begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix} \Rightarrow \begin{matrix} v_1 = (-2, 1-3i) \\ v_2 = (-2, 1+3i) \end{matrix}$

Passo 3.5: Encontrar parte real dos autovetores

$u = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \rightsquigarrow$ em relação a λ_1

Livro:

$x(t) = e^{at}(u\cos bt - v\,\text{sen}\,bt)c_1 + e^{at}(u\,\text{sen}\,bt + v\cos bt)\cdot c_2$

$= \left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}\cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix}\text{sen}\,3t\right)c_1 + \left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}\text{sen}\,3t + \begin{pmatrix} 0 \\ -3 \end{pmatrix}\cos 3t\right)c_2$

$= \begin{pmatrix} -2\cos 3t & -2\,\text{sen}\,3t \\ \cos 3t - 3\,\text{sen}\,3t & \text{sen}\,3t - 3\cos 3t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

4) $\lambda_1 = 0 + 3i$

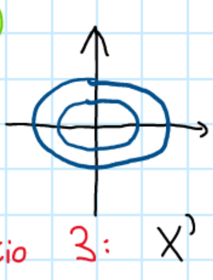
$A = [v \ u] \begin{bmatrix} a & -b \\ b & a \end{bmatrix} [v \ u]^{-1} X_0$

$= \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} [v \ u]^{-1} X_0$

5) $X(t) = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix} e^0 \begin{bmatrix} \cos 3t & -\text{sen}\,3t \\ \text{sen}\,3t & \cos 3t \end{bmatrix} [v \ u]^{-1} X_0$

$= \begin{bmatrix} -2\cos 3t & 2\,\text{sen}\,3t \\ \cos 3t - 3\,\text{sen}\,3t & -\text{sen}\,3t - 3\cos 3t \end{bmatrix} [v \ u]^{-1} X_0$

6)



Como $a = 0 \Rightarrow$ soluções elípticas
Se $a < 0 \Rightarrow$ espirais tendem a 0
Se $a > 0 \Rightarrow$ espirais tendem ao infinito.

Exercício 3: $X' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

1) $p\lambda = \lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda_1 = \lambda_2 = \boxed{-3}$ autovalores iguais

2) Caso 2

3) $(A - \lambda_1 I) = \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \Rightarrow v_1 = (1,1)$

3.5 Encontrar v_2 LI com v_1, tal que

$A[v_1 \ v_2] = [v_1 \ v_2] \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} = [-3v_1 \quad v_1 - 3v_2]$

Temos que resolver $Av_2 = v_1 - 3v_2 \Rightarrow (A + 3I)v_2 = v_1$

$\left(\begin{array}{cc|c} 4 & -4 & 1 \\ 4 & -4 & 1 \end{array}\right) \Rightarrow \begin{pmatrix} 4 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow 4x - 4y = 1 \Rightarrow x - \frac{1}{4} = y$

Tome $x = 1/4$ e $y = 0$.

$v_2 = (1/4, 0)$

4) $A = [v_1 \ v_2] \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix} [v_1 \ v_2]^{-1} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$= \begin{bmatrix} 1 & 1/4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & -1/4 \\ -1 & 1 \end{bmatrix} (-4) \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

5) $X(t) = \begin{bmatrix} 1 & 1/4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-3t} & t\cdot e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 & -1/4 \\ -1 & 1 \end{bmatrix} \overset{\text{mult.}}{(-4)} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$= e^{-3t} \begin{bmatrix} 1 & t + 1/4 \\ 1 & t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 4 & -4 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$= e^{-3t} \begin{bmatrix} 4t + 1 & -4t \\ 4t & 1 - 4t \end{bmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$= e^{-3t} \begin{bmatrix} 3 + 4t \\ 2 + 4t \end{bmatrix}$

Livro:

$x(t) = c_1v_1e^{\lambda_1 t} + c_2(v_1te^{\lambda_1 t} + v_2e^{\lambda_1 t})$

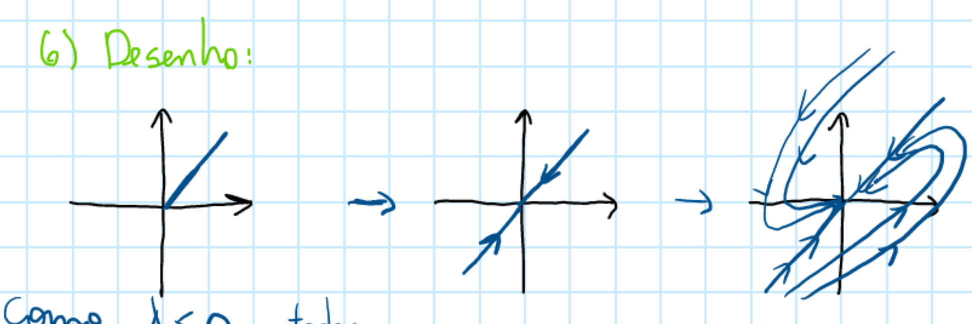
$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t}\right)$

$X(0) = \begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2/4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Rightarrow \begin{matrix} c_1 = 2 \\ c_2 = 4 \end{matrix}$

$\Rightarrow x(t) = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + 4\begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t} + 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-3t}$

$= e^{-3t} \begin{pmatrix} 3 + 4t \\ 2 + 4t \end{pmatrix}$

6) **Desenho:**



Como $\lambda < 0$, todas as soluções tendem a 0! Desenhar o único autovetor

Todos as outras soluções crescem com t, mas em dado momento, e^{-3t} puxa para 0!

Sistemas Lineares Não Homogêneos

Considere $x' = Ax + g(t) \Rightarrow$ Encontrar autovalores e autovetores de A

Exercício: $x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} x + \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix}$

Autovalores: $p\lambda = \lambda^2 + \lambda - 6 \Rightarrow \lambda_1 = 2, \ \lambda_2 = -3$

Autovetores: $v_1 = (1,1), \ v_2 = (1,-4)$

$P = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \left(\frac{1}{5}\right)$

- Se $x = Py \Rightarrow y' = Dy + P^{-1}g(t) = \begin{pmatrix} 2y_1 \\ -3y_2 \end{pmatrix} + \begin{pmatrix} 4e^{-2t} - 2e^t \\ e^{-2t} + 2e^t \end{pmatrix} \frac{1}{5}$

$\begin{cases} y_1' - 2y_1 = 1/5(4e^{-2t} - 2e^t) \\ y_2' + 3y_2 = 1/5(e^{-2t} + 2e^t) \end{cases}$

Variação de Parâmetros:

Nesse caso $x = e^{At}u(t)$ por hipótese é solução particular.

Assim $x' = Ax + g(t) \Rightarrow \cancel{Ae^{At}u(t)} + e^{At}u'(t) = A\cancel{e^{At}u(t)} + g(t)$

$\Rightarrow e^{At}u'(t) = g(t)$

$\Rightarrow u'(t) = e^{-At}g(t)$

$\Rightarrow u(t) = \int e^{At}g(t)\,dt + c$

Isto é: $X(t) = e^{At}c + e^{At}\int_0^t e^{-As}g(s)\,ds, \quad x(0) = x_0$

$c = e^{-At_0}x_0$

$$\boxed{X(t) = e^{A(t-t_0)}x_0 + e^{At}\int_0^t e^{-As}g(s)\,ds}$$

Exercício: $X' = \overbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}^{A} x + \overbrace{\begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}}^{g(t)}$

Autovalores de A: $\lambda^2 + 4\lambda + 3 = 0 \Rightarrow \lambda_1 = -1, \ \lambda_2 = -3$

Autovetores: $v_1 = (1,1), \quad v_2 = (1,-1)$

- $e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left(\frac{1}{2}\right) = \begin{pmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$

$u(t) = \int e^{-At}g(t)\,dt + c$

$= \int \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}^{-1} \begin{pmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{pmatrix}^{-1} \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} dt + c$

$= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}^{-1} \int \begin{pmatrix} e^{-3t} & e^{-3t} \\ e^{-t} & -e^{-t} \end{pmatrix} \begin{pmatrix} 3e^{-t} \\ 3t \end{pmatrix} 2e^{4t}\,dt + c$

$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \int \begin{pmatrix} 6 + 6te^t \\ 6e^{2t} - 6te^{3t} \end{pmatrix} dt + c$

$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 6t + 6te^t - 6e^t \\ 3e^{2t} - 2te^{3t} + 2e^{3t} \end{pmatrix} + c$

- Com $u(t) \Rightarrow x_p(t) = e^{At}u(t)$

E $X(t) = x_h(t) + x_p(t)$