I

Differential Equations of the First Order in One Unknown Function

PART A. THE CAUCHY-EULER APPROXIMATION METHOD

1. Definitions. Direction Fields

conexo

By a *domain D* in the plane we understand a connected open set of points; by a *closed domain* or *region* $\bar{D}$, such a set plus its boundary points. The most general differential equation of the first order in one unknown function is

$$F(x, y, y') = 0 \tag{1}$$

where F is a single-valued function in some domain of its arguments. A differentiable function $y(x)$ is a solution if for some interval of x, $(x, y(x), y'(x))$ is in the domain of definition of F and if further $F(x, y(x), y'(x)) = 0$. We shall in general assume that (1) may be written in the *normal form*:

$$y' = f(x, y) \tag{2}$$

where *$f(x, y)$ is a continuous function of both its arguments simultaneously* in some domain D of the x-y plane. It is known that this reduction may

$F(x, y, y') = x + y$

Como reescrever em (2)?

be carried out under certain general conditions. However, if the reduction is impossible, e.g., if for some (x_0, y_0, y_0') for which $F(x_0, y_0, y_0') = 0$ it is also the case that $[\partial F(x_0, y_0, y_0')]/\partial y' = 0$, we must for the present omit (1) from consideration.

A solution or integral of (2) over the interval $x_0 \leq x \leq x_1$ is a single-valued function $y(x)$ with a continuous first derivative $y'(x)$ defined on $[x_0, x_1]$ such that for $x_0 \leq x \leq x_1$:

(*a*) $(x, y(x))$ is in D, whence $f(x, y(x))$ is defined
(*b*) $y'(x) = f(x, y(x))$. (3)

Geometrically, we may take (2) as defining a *continuous direction field* over D; i.e., at each point P: (x, y) of D there is defined a line whose slope is $f(x, y)$; and an integral of (2) is a curve in D, one-valued in x and with a continuously turning tangent, whose tangent at P coincides with the direction at P. Equation (2) does not, however, define the most general direction field possible; for, if R is a bounded region in D, $f(x, y)$ is continuous in R and hence bounded

$$|f(x, y)| \leq M, \qquad (x, y) \text{ in } R \tag{4}$$

where M is a positive constant. If α is the angle between the direction defined by (2) and the x-axis, (4) means that α is restricted to such values that

$$|\tan \alpha| \leq M$$

The direction may approach the vertical as P: (x, y) approaches the boundary C of D, but it cannot be vertical for any point P of D.

This somewhat arbitrary restriction may be removed by considering the *system* of differential equations

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y) \tag{5}$$

where P and Q are the direction cosines of the direction at (x, y) to the x- and y-axes respectively and hence are continuous and bounded; and a solution of (5) is of the form

$$x = x(t), \qquad y = y(t), \qquad t_0 \leq t \leq t_1$$

which are the parametric equations of a curve L in D. We shall be able to solve (5) by methods similar to those which we shall develop for (2); hence we shall for the present consider only the theory of the equation (2).

2. Approximate Solutions of the Differential Equation

Since the physicist deals only with approximate quantities, an approximate solution of a differential equation is just as good for his purposes as an exact solution, provided the approximation is sufficiently close. The present section will formulate the idea of an approximate solution and prove that the approximation may be made arbitrarily close. The methods used will be practical even though rather crude. More important, the demonstration of the existence of approximate solutions will later lead to the proof of the existence and uniqueness of exact solutions.

Definition 1. *Let $f(x, y)$ be continuous, (x, y) in some domain D. Let $x_1 \leq x \leq x_2$ be an interval. Then a function $y(x)$ defined on $[x_1, x_2]$ is a* solution of $y' = f(x, y)$ up to the error ϵ *if:*

(*a*) *$y(x)$ is* admissible*; i.e., $(x, y(x))$ is in D, $x_1 \leq x \leq x_2$.*
(*b*) *$y(x)$ is continuous; $x_1 \leq x \leq x_2$.*
(*c*) *$y(x)$ has a piecewise continuous derivative on $[x_1, x_2]$ which may fail to be defined only for a finite number of points, say $\xi_1, \xi_2, \cdots, \xi_n$.*
(*d*) *$|y'(x) - f(x, y(x))| \leq \epsilon$; $x_1 \leq x \leq x_2$, $x \neq \xi_i$, $i = 1, \cdots, n$.*

Theorem 1. *Let (x_0, y_0) be a point of D, and let the points of a rectangle R: $|x - x_0| \leq a$, $|y - y_0| \leq b$ lie in D. Let $|f(x, y)| \leq M$, (x, y) in R. Then if $h = \min(a, b/M)$, there can be constructed an approximate solution $y(x)$ of*

$$y' = f(x, y) \tag{1}$$

over the interval $|x - x_0| \leq h$, such that $y(x_0) = y_0$, where the error ϵ may be an arbitrarily small positive number. Observe that h is independent of ϵ.

Proof. The rectangle

$$S\colon\ |x - x_0| \leq h, \qquad |y - y_0| \leq Mh \tag{2}$$

is contained in R by the definition of h. See Fig. 1.

Let the ϵ of the theorem be given. Since $f(x, y)$ is continuous in S, it is

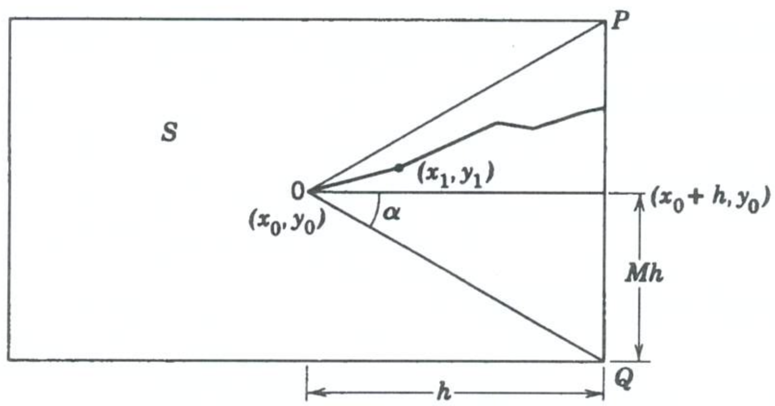

Fig. 1

uniformly so; i.e., given $\epsilon > 0$ (which we take to be the ϵ of the theorem) there exists $\delta > 0$ such that

$$|f(\tilde{x}, \tilde{y}) - f(x, y)| \leq \epsilon \tag{3}$$

for $(\tilde{x}, \tilde{y})$; (x, y) in S; $|\tilde{x} - x|$, $|\tilde{y} - y| \leq \delta$.

Let $x_1, \cdots, x_{n-1}$ be any set of points such that:

$$\begin{aligned} &(a)\ x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = x_0 + h; \\ &(b)\ x_i - x_{i-1} \leq \min(\delta, \delta/M), \qquad i = 1, \cdots, n. \end{aligned} \tag{4}$$

We shall construct the approximate solution on the interval $x_0 \leq x \leq x_0 + h$; a similar process will define it on the interval $x_0 - h \leq x \leq x_0$. The approximate solution will be a polygon constructed in the following fashion: from (x_0, y_0) we draw a segment to the right with slope of $f(x_0, y_0)$; this will intersect the line $x = x_1$ in a point (x_1, y_1). From (x_1, y_1) we draw a segment to the right with slope $f(x_1, y_1)$ intersecting $x = x_2$ at y_2; etc. The point (x_1, y_1) must lie in the triangle OPQ; for $\tan \alpha = M$, and $|f(x_0, y_0)| \leq M$. Likewise (x_2, y_2) lies in OPQ; etc. Hence the process may be continued up to $x_n = x_0 + h$, since the only way in which the process could stop would be for $f(x_k, y_k)$ to be undefined; in which case we should have $|y_k - y_0| > Mh$ contrary to construction. Analytically we may define $y(x)$ by the recursion formulas

$$y(x) = y_{i-1} + (x - x_{i-1})f(x_{i-1}, y(x_{i-1})) \tag{5}$$

where

$$y_{i-1} = y(x_{i-1}), \qquad x_{i-1} \leq x \leq x_i, \qquad i = 1, \cdots, n$$

Obviously by definition $y(x)$ is admissible, continuous, and has a piecewise continuous derivative

$$y'(x) = f(x_{i-1}, y(x_{i-1})), \qquad x_{i-1} < x < x_i, \qquad i = 1, \cdots, n$$

which fails to be defined only at the points x_i, $i = 1, \cdots, n - 1$. Furthermore, if $x_{i-1} < x < x_i$

$$|y'(x) - f(x, y(x))| = |f(x_{i-1}, y_{i-1}) - f(x, y(x))| \tag{6}$$

But by (4), $|x - x_{i-1}| < \min(\delta, \delta/M)$, and by (5)

$$|y - y_{i-1}| \leq M|x - x_{i-1}| \leq M\frac{\delta}{M} = \delta$$

Hence by (3)

$$|f(x_{i-1}, y_{i-1}) - f(x, y(x))| \leq \epsilon$$

and

$$|y'(x) - f(x, y(x))| \leq \epsilon \qquad x \neq x_i, \qquad i = 1, 2, \cdots, n - 1 \tag{7}$$

Hence $y(x)$ satisfies all the conditions of Def. 1, and the construction required by the theorem has been performed. This method of constructing an approximate solution is known as the Cauchy-Euler method.

It is unnecessary to improve the value of h, since, in general, as we shall show, $y(x)$ is defined in a larger interval than $|x - x_0| \leq h$.

3. The Fundamental Inequality

With a certain additional restriction upon $f(x, y)$ we shall prove an inequality which will be the basis of our fundamental results.

Definition 2. *A function $f(x, y)$ defined on a (open or closed) domain D is said to satisfy Lipschitz conditions with respect to y for the constant $k > 0$ if for every x, y_1, y_2 such that $(x, y_1), (x, y_2)$ are in D*

$$|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2| \tag{1}$$

In connection with this definition we shall need two theorems of analysis:

Lemma 1. *If $f(x, y)$ has a partial derivative for y, bounded for all (x, y) in D, and D* is convex *(i.e., the segment joining any two points of D lies entirely in D), then $f(x, y)$ satisfies a Lipschitz condition for y where the constant k is given by*

$$k = \text{l.u.b.} \left|\frac{\partial f(x, y)}{\partial y}\right|, \qquad (x, y) \text{ in } D \tag{2}$$

Proof. By Rolle's theorem there exists a number ξ such that

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1)\frac{\partial f(x, \xi)}{\partial y}, \qquad y_1 \leq \xi \leq y_2$$

i.e.,

$$|f(x, y_2) - f(x, y_1)| \leq \text{l.u.b.} \left|\frac{\partial f(x, y)}{\partial y}\right| |y_2 - y_1|$$

since (x, ξ) is in D; whence the theorem.

Lemma 2. *If D is not convex, let D be imbedded in a larger domain D'. Let δ be the distance between the boundaries C and C' of D and D' respectively (i.e., $\delta = \min(\overline{PP'})$, P in C, P' in C'), and let $\delta > 0$. Then if $f(x, y)$ is continuous in D' (hence bounded by M, say, in D) and $\partial f/\partial y$ exists and is bounded by N in D', then $f(x, y)$ satisfies a Lipschitz condition in D with respect to y for a constant*

$$k = \max\left(N, \frac{2M}{\delta}\right)$$

Proof. Let P_1: (x, y_1) and P_2: (x, y_2) be in D.

(*a*) If $|y_1 - y_2| > \delta$, $\left|\dfrac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\right| \le \dfrac{2M}{\delta}$.

(*b*) If $|y_1 - y_2| \le \delta$, the segment $\overline{P_1P_2}$ lies wholly in D'.

Hence as in Lemma 1, $\left|\dfrac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\right| \le N$. Hence the lemma.

Theorem 2. Let (x_0, y_0) be a point of a region R in which $f(x, y)$ is continuous and satisfies the Lipschitz condition for k. Let $y(x)$, $\tilde{y}(x)$ be admissible functions for $|x - x_0| \le h$ (where h is any constant, not necessarily that of Theorem 1), satisfying

$$y' = f(x, y), \qquad |x - x_0| \le h$$

with errors ϵ_1 and ϵ_2 respectively. Set

$$p(x) = \tilde{y}(x) - y(x), \qquad \epsilon = \epsilon_1 + \epsilon_2 \tag{3}$$

Then

$$|p(x)| \le e^{k|x-x_0|}\,|p(x_0)| + \frac{\epsilon}{k}\left(e^{k|x-x_0|} - 1\right) \tag{4}$$

This is the fundamental inequality.

Proof. We give the proof only for $x_0 \le x \le x_0 + h$; a similar process will give the proof for $x_0 - h \le x \le x_0$. By Def. 1, except for a finite number of points

$$\left|\frac{dy}{dx} - f(x, y)\right| \le \epsilon_1, \qquad \left|\frac{d\tilde{y}}{dx} - f(x, \tilde{y})\right| \le \epsilon_2, \qquad x_0 \le x \le x_0 + h$$

Hence

$$\begin{aligned}\left|\frac{dy}{dx} - \frac{d\tilde{y}}{dx}\right| &\le |f(x, y) - f(x, \tilde{y})| + \epsilon \\ &\le k|\tilde{y} - y| + \epsilon\end{aligned}$$

by the Lipschitz condition; i.e.,

$$\left|\frac{dp}{dx}\right| \le k|p| + \epsilon, \qquad x_0 \le x \le x_0 + h \tag{5}$$

except for a finite number of points at which $[dp(x)]/dx$ fails to be defined.

Case I. Suppose $p(x) \ne 0$, $x_0 < x \le x_0 + h$. Hence, being continuous, it has the same sign, say without loss of generality $p(x) > 0$. Then a fortiori we can write (5) in the form

$$\frac{dp}{dx} \le kp(x) + \epsilon \tag{5'}$$

supôs que $\frac{dp}{dx} > 0$ pois c/ a fçé unt, n c/ muda o final

This may be written as

$$e^{-kx}\,[p'(x) - kp(x)] \leq \epsilon e^{-kx}$$

or, integrating from x_0 to x, where $x_0 \leq x \leq x_0 + h$

$$\int_{x_0}^{x} e^{-kx}\,[p'(x) - kp(x)]\,dx \leq \epsilon \int_{x_0}^{x} e^{-kx}\,dx \tag{6}$$

The integrand on the left-hand side may have a finite number of simple discontinuities but it has a continuous indefinite integral. Hence we may write

$$[e^{-kx}\,p(x)]_{x_0}^{x} \leq \epsilon \left[-\frac{1}{k}e^{-kx}\right]_{x_0}^{x}$$

or

$$p(x) \leq e^{k(x-x_0)}\,p(x_0) + \frac{\epsilon}{k}\,[e^{k(x-x_0)} - 1] \tag{7}$$

which is the required inequality.

Case II. If for all x, $p(x) = 0$, the theorem is obvious.

Case III. If $p(\bar{x}) \neq 0$ where $\bar{x}$ is some fixed number $x_0 \leq \bar{x} \leq x_0 + h$, but $p(x) = 0$ for some value of x, $x_0 \leq x < \bar{x}$; since $p(x)$ is continuous, there exists a number x_1, $x_0 \leq x_1 < \bar{x} \leq x_0 + h$ such that $p(x_1) = 0$, but $p(x) \neq 0$, $x_1 < x < \bar{x}$. Applying Case I to the interval (x_1, x) we have

$$p(\bar{x}) \leq e^{k(\bar{x}-x_1)}\,p(x_1) + \frac{\epsilon}{k}\,[e^{k(\bar{x}-x_1)} - 1]$$

$$= \frac{\epsilon}{k}\,[e^{k(\bar{x}-x_1)} - 1] \tag{8}$$

which is an even stronger inequality than (4).

Hence the inequality (4) holds in all cases, since, if $p(x) < 0$, the same results follow by considering $|p(x)|$.

4. Uniqueness and Existence Theorems

Theorem 3. *If $f(x, y)$ is continuous and satisfies a Lipschitz condition for y in a domain D, and if (x_0, y_0) is in D and $y(x)$ and $\tilde{y}(x)$ are two exact solutions of $y' = f(x, y)$ in an interval $|x - x_0| \leq h$ such that $y(x_0) = \tilde{y}(x_0) = y_0$, then $y(x) \equiv \tilde{y}(x)$, $|x - x_0| \leq h$; i.e., there is at most one integral curve passing through any point of D.*

Proof. Applying Theorem 2, we have $p(x_0) = y_0 - y_0 = 0$, and $\epsilon = \epsilon_1 + \epsilon_2 = 0$; hence $p = y - \tilde{y} \equiv 0$.

We may state this in the form that under the above assumptions, two integral curves cannot meet or intersect at any point of D.

Observe that without the additional requirement of a Lipschitz condition uniqueness need not follow. For consider the differential equation

$$\frac{dy}{dx} = y^{\frac{1}{3}} \tag{1}$$

$f(x, y) = y^{\frac{1}{3}}$ is continuous at (0, 0). But there are two solutions passing through (0, 0), namely

$$\begin{aligned} &(a) \qquad y \equiv 0 \\ &(b) \qquad \begin{cases} y = (\frac{2}{3}x)^{3/2} & x \geq 0 \\ y = 0 & x \leq 0 \end{cases} \end{aligned} \tag{2}$$

Of course, $y^{\frac{1}{3}}$ does not satisfy the Lipschitz condition at $y = 0$. For, if $y_1 = \delta$ and $y_2 = -\delta$

$$\left| \frac{f(y_1) - f(y_2)}{y_1 - y_2} \right| = \frac{1}{\delta^{2/3}} \tag{3}$$

which is unbounded for δ arbitrarily small.

Theorem 4. *If $f(x, y)$ is continuous and satisfies the Lipschitz condition for y in a domain D, then for (x_0, y_0) in D there exists an exact solution of $y' = f(x, y)$ for $|x - x_0| \leq h$, where h is defined as in Theorem* 1, *such that*

$$y(x_0) = y_0$$

Proof. Given a monotone positive sequence $\{\epsilon_n\}$ approaching 0 as a limit, by Theorem 1 there exists a sequence $\{y_n(x)\}$ of functions satisfying

$$y_n'(x) = f(x, y_n(x)) \text{ up to } \epsilon_n, \qquad y_n(x_0) = y_0$$

over $|x - x_0| \leq h$; i.e.

$$|y_n'(x) - f(x, y_n(x))| \leq \epsilon_n, \qquad |x - x_0| \leq h \tag{4}$$

except for the finite set of points $x_i^{(n)}$, $i = 1, \cdots, m_n$.

Part I. The sequence $\{y_n(x)\}$ converges uniformly over $|x - x_0| \leq h$ to a (continuous) function $y(x)$.

For let n and p be positive integers; and apply Theorem 2 to $y_n(x)$, $y_{n+p}(x)$

$$\begin{aligned} |y_n(x) - y_{n+p}(x)| &\leq \frac{\epsilon_n + \epsilon_{n+p}}{k} [e^{k(x-x_0)} - 1] \\ &\leq \frac{2\epsilon_n}{k} [e^{kh} - 1] \end{aligned} \tag{5}$$

whence the assertion.

Part II. The sequence $\left\{\int_{x_0}^{x} f(t, y_n(t))\,dt\right\}$ approaches $\int_{x_0}^{x} f(t, y(t))\,dt$ uniformly for $|x - x_0| \le h$.

Let R be the rectangle $|x - x_0| \le h$, $|y - y_0| \le Mh$, by hypothesis in D. Since $f(x, y)$ is continuous in the closed domain R, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \le \epsilon, \qquad (x, y_1), (x, y_2) \text{ in } R, \qquad |y_1 - y_2| \le \delta \tag{6}$$

Likewise for this $\delta > 0$ there exists N such that

$$|y_n(x) - y(x)| \le \delta, \qquad n > N, \qquad |x - x_0| \le h \tag{7}$$

in virtue of Part I. Hence if $n > N$

$$|f(x, y_n(x)) - f(x, y(x))| \le \epsilon, \qquad n > N, \qquad |x - x_0| \le h \tag{8}$$

Therefore the sequence $\{f(x, y_n(x))\}$ converges uniformly to $f(x, y(x))$. Hence, by a well-known theorem we may reverse the order of integration and pass to the limit; which completes the proof of Part II.

Part III. $y(x)$ is differentiable, $y(x_0) = y_0$, and

$$y'(x) = f(x, y(x)), \qquad |x - x_0| \le h$$

Integrating each side of (4) from x_0 to x_1 we have

$$\left|\int_{x_0}^{x}\left[\frac{dy_n(t)}{dt} - f(t, y_n(t))\right] dt\right| \le \epsilon_n|x - x_0| \le \epsilon_n h \tag{9}$$

But since $y_n(t)$ is continuous

$$|y_n(x) - y_0 - \int_{x_0}^{x} f(t, y_n(t))\,dt\,| \le \epsilon_n h \tag{10}$$

Approaching the limit by Part II, we have

$$y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))\,dt = 0 \tag{11}$$

from which the assertion follows by an elementary theorem on functions defined by definite integrals.

Theorem 5. *Let $y(x)$ be an exact solution of $y' = f(x, y(x))$ under the assumptions of Theorem* 4, *and $\tilde{y}(x)$ an approximate solution up to the error ϵ, for $|x - x_0| \le h$, such that $\tilde{y}(x_0) = y_0$. Then there exists a constant N* independent of ϵ *such that $|\tilde{y}(x) - y(x)| \le \epsilon N$, $|x - x_0| \le h$.*

Proof. By Theorem 2

$$|\tilde{y} - y| \le \frac{\epsilon}{k}(e^{kh} - 1); \qquad \text{i.e., } N = \frac{e^{kh} - 1}{k} \tag{12}$$

Theorem 5 justifies our description of the function of Theorem 1 as an "approximate solution," for its actual value differs from that of the exact solution by a multiple of ϵ; whereas in our definition we only required it *to satisfy the differential equation* to within an error ϵ. To recapitulate our results: If $f(x, y)$ is continuous in D and satisfies a Lipschitz condition with constant k, and (x_0, y_0) is in D, then there exist quantities h and N such that

(*a*) There is a unique exact solution $y(x)$ of $y'(x) = f(x, y(x))$ passing through (x_0, y_0) and defined for $|x - x_0| \leq h$.

(*b*) There may be constructed by the Cauchy–Euler method approximate solutions $\tilde{y}(x)$ such that, if $\epsilon > 0$

$$\begin{gathered} |\tilde{y}'(x) - f(x, \tilde{y}(x))| \leq \epsilon, \qquad |x - x_0| \leq h, \qquad x \neq \xi_1, \xi_2, \cdots, \xi_n \\ |\tilde{y}(x) - y(x)| \leq N\epsilon, \qquad |x - x_0| \leq h \\ \tilde{y}(x_0) = y(x_0) = y_0 \end{gathered} \tag{13}$$

Appendix to §4. Extension of the Existence Theorem

We have seen that without the Lipschitz condition on $f(x, y)$ the solution need not be unique. The condition is, however, superfluous for the proof of existence of a solution. This may be proved directly; we give a proof based upon Theorem 4 by the use of Weierstrass' polynomial approximation theorem.

Theorem 6. *If $f(x, y)$ is continuous in D, there is a solution of $y' = f(x, y)$ such that $y(x_0) = y_0$ where (x_0, y_0) lies in D, defined for $|x - x_0| \leq h = \min (a, b/M)$.*

Proof. We use the following two theorems of analysis:

Lemma 1 (Weierstrass). *If $f(x, y)$ is continuous in the region R, given $\epsilon > 0$, there exists a polynomial $P(x, y)$ such that*

$$|P(x, y) - f(x, y)| \leq \epsilon, \qquad (x, y) \text{ in } R$$

and

$$P(x, y) \leq f(x, y), \qquad (x, y) \text{ in } R$$

Definition 3. *A set S of functions $f(x)$ is* equicontinuous *over an interval $[a, b]$ if given $\epsilon > 0$ there exists $\delta > 0$ such that if x_1, x_2 in $[a, b]$, $|x_1 - x_2| \leq \delta$, then $|f(x_1) - f(x_2)| \leq \epsilon$, for* all f in S.

Lemma 2. *An infinite set S of functions uniformly bounded and equicontinuous over a closed interval $[a, b]$ contains a sequence converging uniformly over $[a, b]$.*

By virtue of Lemma 1, let $P_n(x, y)$ be a sequence of polynomials such that

$$|P_n(x, y) - f(x, y)| \leq \epsilon_n, \qquad |x - x_0| \leq h, \qquad |y - y_0| \leq Mh$$

where h and M are defined as usual, and the sequence $\{\epsilon_n\}$ approaches 0. Then $\{P_n(x, y)\}$ is uniformly bounded; and it may be assumed that M is the common bound of f and P_n. Since $P_n(x, y)$ satisfies Lipschitz conditions in y, by Theorem 4 there exists a sequence of functions $\{y_n(x)\}$ defined over $|x - x_0| \leq h$, satisfying $y_n'(x) = P_n\{x, y_n(x)\}$, and such that $y_n(x_0) = y_0$. By Theorem 4

$$|y_n(x) - y_n(x_0)| \leq Mh, \qquad |x - x_0| \leq h$$

i.e., $\{y_n(x)\}$ *is uniformly bounded over* $|x - x_0| \leq h$.

Furthermore, $\{y_n(x)\}$ *is equicontinuous over* $|x - x_0| \leq h$ for

$$y_n(x_2) - y_n(x_1) = y_n'(\xi)(x_2 - x_1), \qquad x_1 < \xi < x_2$$

for all x_1, x_2 in $[a, b]$. But

$$|y_n'(\xi)| = |P_n(\xi, y_n(\xi))| \leq M$$

Hence

$$|y_n(x_2) - y_n(x_1)| \leq |x_2 - x_1| M$$

Given $\epsilon > 0$, if $\delta = \epsilon/M$, then if $|x_2 - x_1| \leq \delta$; x_1, x_2 in $|x - x_0| \leq h$, then $|y_n(x_2) - y_n(x_1)| \leq \epsilon$ all n. Hence by Lemma 2, $\{y_n(x)\}$ has a subsequence converging uniformly over $|x - x_0| \leq h$ to a (continuous) function $y(x)$. It will do no harm if we denote the subsequence by the same symbol $\{y_n(x)\}$.

As in Theorem 4, Part II, we can prove that

$$\lim_{n \to \infty} \int_{x_0}^{x} f(t, y_n(t))\, dt \to \int_{x_0}^{x} f(t, y(t))\, dt$$

uniformly for $|x - x_0| \leq h$. By definition

$$y_n(x) - y_0 = \int_{x_0}^{x} P_n(t, y_n(t))\, dt$$

Hence we write

$$\left|y_n(x) - y_0 - \int_{x_0}^{x} f(t, y_n(t))\, dt\right| \leq \int_{x_0}^{x} |P_n(t, y_n(t)) - f(t, y_n(t))|\, dt$$

But if $n > N$

$$|P_n(x, y) - f(x, y)| \leq \epsilon_n, \qquad |x - x_0| \leq h, \qquad |y - y_0| \leq Mh$$

Hence for $n > N$

$$\left|y_n(x) - y_0 - \int_{x_0}^{x} f(t, y_n(t))\right| dt \leq \epsilon_n h$$

Letting $n \to \infty$ the above is valid with $y_n(x)$ replaced by $y(x)$ and so

$$\left| y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))\, dt \right| = 0, \qquad |x - x_0| \le h$$

since ϵ_n tends to 0. As in Theorem 4, this last formula proves that $y(x)$ actually is the required solution.

5. Solutions Containing Parameters

We consider next changes in the solution of a differential equation caused by either a small change in the initial conditions or a small variation in the function $f(x, y)$. Our general results will be that such small changes will bring about only small changes in the solution. This again will be of interest to the physicist, since it will assure him that small errors in the statement of a problem cannot change the answer too greatly.

We have seen already that the solution $y(x)$ of

$$y' = f(x, y), \qquad y(x_0) = y_0$$

contains the initial value y_0 as parameter. We determine first a minimum range of values of x and y_0 for which we can be assured that $y(x, y_0)$ will exist and be unique.

Theorem 7. *If $f(x, y)$ is continuous and satisfies a Lipschitz condition on y in $|x - x_0| \le a$, $|y - \tilde{y}_0| \le b$, $(x_0, \tilde{y}_0)$ some fixed point, there exists a unique solution $y(x, y_0)$ of*

$$y' = f(x, y), \qquad y(x_0) = y_0$$

in the region

$$|x - x_0| \le h', \qquad |y_0 - \tilde{y}_0| \le b/2, \qquad |y - y_0| \le Mh' \tag{1}$$

where $h' = \min\,[a, (b/2M)]$, M being the bound of $f(x, y)$ in $|x - x_0| \le a$, $|y - \tilde{y}_0| \le b$.

Proof. The proof follows immediately upon applying Theorem 4 to the point (x_0, y_0), $|y_0 - \tilde{y}_0| \le b/2$. Observe that the h' of this theorem is not less than half of the h for which the original existence theorem was proved.

Theorem 8. *If in some domain D, $f(x, y)$ is continuous and satisfies a Lipschitz condition on y for some k, and if a solution $y(x, y_0)$ of*

$$y' = f(x, y), \qquad y(x_0) = y_0$$

exists for some rectangle R, $|x - x_0| \le h$, $|y_0 - \tilde{y}_0| \le l$, then $y(x, y_0)$ is continuous in x and y_0 simultaneously in R.

Proof. Consider the solutions $y(x, y_0^{(1)})$, $y(x, y_0^{(2)})$, $y_0^{(1)}$, and $y_0^{(2)}$ satisfying $|y_0^{(i)} - \tilde{y}_0| \leq l$, $i = 1, 2$. Applying Theorem 2 to these two functions over $|x - x_0| \leq h$, we obtain

$$|y(x, y_0^{(1)}) - y(x, y_0^{(2)})| \leq |y_0^{(1)} - y_0^{(2)}|e^{kh}, \qquad |x - x_0| \leq h \tag{2}$$

This means that in R, $y(x, y_0)$ is continuous in y_0 *uniformly in* x. Since $y(x, y_0)$ is continuous in x for any value of y_0, it follows by a general theorem of real variables that in R, $y(x, y_0)$ is continuous in x and y_0 simultaneously.

Geometrically (2) means that two integral curves close enough together for *one value of* x stay close together for $|x - x_0| \leq h$.

With a little stronger restriction upon $f(x, y)$ we now prove a further property of $y(x, y_0)$, namely:

Theorem 9. *Under the assumptions of Theorem* 8 *and if in addition* $(\partial f/\partial y)\,(x, y)$ *exists in* D *and is continuous in* x *and* y *simultaneously, then* $[\partial y(x, y_0)]/\partial y_0$ *exists for* (x, y_0) *in* R *and is continuous in* x *and* y_0 *simultaneously.*

Proof. Let $\bar{y}_0$ be an arbitrary value of y_0 which is to remain fixed until the end of the argument, satisfying $|\bar{y}_0 - \tilde{y}_0| \leq l$. Let y_0 be a variable in the same interval. For convenience we write

$$y(x, \bar{y}_0) = \bar{y}(x), \qquad y(x, y_0) = y(x) \tag{3}$$

We shall prove that the function

$$p(x, y_0) = \frac{y(x) - \bar{y}(x)}{y_0 - \bar{y}_0}, \qquad y_0 \neq \bar{y}_0 \tag{4}$$

approaches a limit as $y_0 \to \bar{y}_0$, which limit must be $[\partial y(x, y_0)]/\partial y_0$ at $y_0 = \bar{y}_0$; and that this limit has the required properties.

In the first place we write

$$\frac{d}{dx}[y(x) - \bar{y}(x)] = f(x, y(x)) - f(x, \bar{y}(x))$$

or by the mean-value theorem

$$= [y(x) - \bar{y}(x)]\left[\frac{\partial}{\partial y} f(x, \bar{y}(x)) + \delta\{y(x), \bar{y}(x)\}\right] \tag{5}$$

where as $|y(x) - \bar{y}(x)| \to 0$, $\delta\{y(x), \bar{y}(x)\} \to 0$. Since by Theorem 8

$$\lim_{y_0 \to \bar{y}_0} y(x) = \bar{y}(x) \text{ uniformly for } x, \qquad |x - x_0| \leq h$$

it follows that

$$\lim_{y_0 \to \bar{y}_0} \delta\{y(x), \bar{y}(x)\} = 0 \tag{6}$$

uniformly in x.

By (4) and (5) we may write

$$\frac{\partial p(x, y_0)}{\partial x} = p(x, y_0)\left[\frac{\partial}{\partial y} f(x, \bar{y}(x)) + \delta\{y(x), \bar{y}(x)\}\right] \tag{7}$$

Since $p \neq 0$ for $y_0 \neq \bar{y}_0$ by the general uniqueness theorem, we may divide (7) by p and integrate explicitly with respect to x

$$p(x, y_0) = \exp\left\{\int_{x_0}^{x} [f_y(x, \bar{y}(x)) + \delta\{y(x), \bar{y}(x)\}]\, dx\right\} \tag{8}$$

where the constant of integration is determined by the fact that $p(x_0, y_0) = 1$. Equation (8) is valid for all y_0. But $\lim_{y_0 \to \bar{y}_0} p(x, y_0)$ exists; for by (6) and (8)

$$\frac{\partial y(x, \bar{y}_0)}{\partial y_0} = \lim_{y_0 \to \bar{y}_0} p(x, y_0) = \exp\left\{\int_{x_0}^{x} f_y(x, \bar{y}(x))\, dx\right\} \tag{9}$$

The right-hand side of (9) is continuous in x and $\bar{y}_0$ simultaneously; for $(\partial f/\partial y)\,(x, y)$ was continuous by hypothesis, and $\bar{y}(x) = y(x, \bar{y}_0)$ is continuous by Theorem 8, $\bar{y}_0$ now being considered the variable point. This completes the proof.

By similar (but considerably more complicated) reasoning we can prove under the same assumptions that if $f(x, y)$ in addition has a continuous first derivative with respect to some parameter μ in a certain interval, the solution $y(x)$ is also continuous and differentiable in μ. We omit the proof since this will appear as a special case of the general theory of Chapter 2.

We finally prove that the solution is only slightly changed if $f(x, y)$ is only slightly changed; precisely:

Theorem 10. If in some domain D,

(a) $F(x, y)$ and $f(x, y)$ are continuous,

(b) $f(x, y)$ satisfies a Lipschitz condition on y for some k,

(c) $|F(x, y) - f(x, y)| \leq \epsilon, \qquad (x, y)$ *in D,* (10)

(d) for some point (x_0, y_0) in D, $y(x)$ and $\tilde{y}(x)$ are admissible in $|x - x_0| \leq h$ and satisfy

$$y'(x) = f(x, y), \qquad \tilde{y}'(x) = F(x, \tilde{y}), \qquad y(x_0) = \tilde{y}(x_0) = y_0$$

then

$$|y(x) - \tilde{y}(x)| \leq \frac{\epsilon}{k}(e^{kh} - 1), \qquad |x - x_0| \leq h \tag{11}$$

Proof. By (10), $\tilde{y}(x)$ is an approximate solution of $y' = f(x, y)$ with error ϵ; applying Theorem 5 we immediately obtain (11).