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COMPUTING EXPONENTIALS OF SKEW-SYMMETRIC MATRICES AND LOGARITHMS OF ORTHOGONAL MATRICES

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Abstract

The authors show that there is a generalization of Rodrigues' formula for computing the exponential map exp: $\mathfrak{so}(n) \to \mathbf{SO}(n)$ from skew-symmetric matrices to orthogonal matrices when $n \geq 4$, and give a method for computing some determination of the (multivalued) function log: $\mathbf{SO}(n) \to \mathfrak{so}(n)$. The key idea is the decomposition of a skew-symmetric $n \times n$ matrix B in terms of (unique) skew-symmetric matrices $B_1, \ldots, B_p$ obtained from the diagonalization of B and satisfying some simple algebraic identities. A subproblem arising in computing $\log R$, where $R \in \mathbf{SO}(n)$, is the problem of finding a skew-symmetric matrix B, given the matrix B^2, and knowing that B^2 has eigenvalues -1 and 0. The authors also consider the exponential map exp: $\mathfrak{se}(n) \to \mathbf{SE}(n)$, where $\mathfrak{se}(n)$ is the Lie algebra of the Lie group $\mathbf{SE}(n)$ of (affine) rigid motions. The authors show that there is a Rodrigues-like formula for computing this exponential map, and give a method for computing some determination of the (multivalued) function log: $\mathbf{SE}(n) \to \mathfrak{se}(n)$. This yields a direct proof of the surjectivity of exp: $\mathfrak{se}(n) \to \mathbf{SE}(n)$.

Key Words

Rotations, skew-symmetric matrices, exponentials, logarithms, rigid motions, interpolation

1. Introduction

Given a real skew-symmetric $n \times n$ matrix B, it is well known that $R = e^B$ is a rotation matrix, where:

$$e^B = I_n + \sum_{k=1}^{\infty} \frac{B^k}{k!}$$

is the exponential of B (for instance, see Chevalley [1], Marsden and Ratiu [2], or Warner [3]). Conversely, given any rotation matrix $R \in \mathbf{SO}(n)$, there is some skew-symmetric matrix B such that $R = e^B$. These two facts can be expressed by saying that the map exp: $\mathfrak{so}(n) \to \mathbf{SO}(n)$ from the Lie algebra $\mathfrak{so}(n)$ of skew-symmetric $n \times n$ matrices to the Lie group $\mathbf{SO}(n)$ is surjective (see Bröcker and tom Dieck [4]). The surjectivity of exp is an important property. Indeed, it implies the existence of a function log: $\mathbf{SO}(n) \to \mathfrak{so}(n)$ (only locally a function, log is really a multivalued function), and this has interesting applications. For example, exp and log can be used for *motion interpolation*, as illustrated in Kim, M.-J., Kim, M.-S and Shin [5, 6], and Park and Ravani [7, 8]. Motion interpolation and rational motions have also been investigated by Jüttler [9, 10], Jüttler and Wagner [11, 12], Horsch and Jüttler [13], and Röschel [14]. In its simplest form, the problem is as follows: given two rotation matrices $R_1, R_2 \in \mathbf{SO}(n)$, find a "natural" interpolating rotation $R(t)$, where $0 \leq t \leq 1$. Of course, it would be necessary to clarify what we mean by "natural," but note that we have the following solution:

$$R(t) = \exp((1-t)\log R_1 + t \log R_2)$$

In theory, the problem is solved. However, it is still necessary to compute $\exp(B)$ and $\log R$ effectively.

When $n = 2$, a skew-symmetric matrix B can be written as $B = \theta J$, where:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and it is easily shown that:

$$e^B = e^{\theta J} = \cos\theta I_2 + \sin\theta J$$

Given $R \in \mathbf{SO}(2)$, we can find $\cos\theta$ because $\mathrm{tr}(R) = 2\cos\theta$ (where $\mathrm{tr}(R)$ denotes the trace of R). Thus, the problem is completely solved.

When $n = 3$, a real skew-symmetric matrix B is of the form:

$$B = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

and letting $\theta = \sqrt{a^2 + b^2 + c^2}$, we have the well-known formula due to Rodrigues:

$$e^B = I_3 + \frac{\sin\theta}{\theta} B + \frac{(1-\cos\theta)}{\theta^2} B^2$$

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with $e^B = I_3$ when $B = 0$ (for instance, see Marsden and Ratiu [2], McCarthy [15], or Murray, Li, and Sastry [16]).

It turns out that it is more convenient to normalize B, that is, to write $B = \theta B_1$ (where $B_1 = B/\theta$, assuming that $\theta \neq 0$), in which case the formula becomes:

$$e^{\theta B_1} = I_3 + \sin\theta B_1 + (1 - \cos\theta)B_1^2$$

Also, given $R \in \mathbf{SO}(3)$, we can find $\cos\theta$ because $\mathrm{tr}(R) = 1 + 2\cos\theta$, and we can find B_1 by observing that:

$$\tfrac{1}{2}(R - R^{\mathsf{T}}) = \sin\theta B_1$$

Actually, the above formula cannot be used when $\theta = 0$ or $\theta = \pi$, as $\sin\theta = 0$ in these cases. When $\theta = 0$, we have $R = I_3$ and $B_1 = 0$, and when $\theta = \pi$, we need to find B_1 such that:

$$B_1^2 = \tfrac{1}{2}(R - I_3)$$

As B_1 is a skew-symmetric 3×3 matrix, this amounts to solving some simple equations with three unknowns. Again, the problem is completely solved.

What about the cases where $n \geq 4$? The reason why Rodrigues' formula can be derived is that:

$$B^3 = -\theta^2 B$$

or, equivalently, $B_1^3 = -B_1$. Unfortunately, for $n \geq 4$, given any non-null skew-symmetric $n \times n$ matrix B, it is generally false that $B^3 = -\theta^2 B$, and the reasoning used in the $3D$ case does not apply.

In this article, we show that there is a generalization of Rodrigues' formula for computing the exponential map exp: $\mathfrak{so}(n) \to \mathbf{SO}(n)$, when $n \geq 4$, and we give a method for computing some determination of the (multivalued) function log function log: $\mathbf{SO}(n) \to \mathfrak{so}(n)$. The key to the solution is that, given a skew-symmetric $n \times n$ matrix B, there are p unique skew-symmetric matrices $B_1, \ldots, B_p$ such that B can be expressed as:

$$B = \theta_1 B_1 + \cdots + \theta_p B_p$$

where:

$$\{i\theta_1, -i\theta_1, \ldots, i\theta_p, -i\theta_p\}$$

is the set of distinct eigenvalues of B, with $\theta_i > 0$ and where:

$$\begin{aligned} B_i B_j &= B_j B_i = 0_n \quad (i \neq j) \\ B_i^3 &= -B_i \end{aligned}$$

This reduces the problem to the case of 3×3 matrices. We also consider the exponential map exp $\mathfrak{se}(n) \to \mathbf{SE}(n)$, where $\mathfrak{se}(n)$ is the Lie algebra of the Lie group $\mathbf{SE}(n)$ of (affine) rigid motions. We show that there is a Rodrigues-like formula for computing this exponential map, and we give a method for computing some determination of the (multivalued) function log: $\mathbf{SE}(n) \to \mathfrak{se}(n)$.

The general problem of computing the exponential of a matrix is discussed in Moler and Van Loan [17]. However, more general types of matrices are considered. The problem of computing the logarithm and the exponential of a matrix is also investigated in [18, 19].

The article is organized as follows. In Section 2 we give a Rodrigues-like formula for computing exp: $\mathfrak{so}(n) \to \mathbf{SO}(n)$. In Section 3 we show how to compute log: $\mathbf{SO}(4) \to \mathfrak{so}(4)$ in the special case of $\mathbf{SO}(4)$, which is simpler. In Section 4 we show how to compute some determination of the (multivalued) function log: $\mathbf{SO}(n) \to \mathfrak{so}(n)$ in general ($n \geq 4$). In Section 5 we give a Rodrigues-like formula for computing exp: $\mathfrak{se}(n) \to \mathbf{SE}(n)$. In Section 6 we show how to compute some determination of the (multivalued) function log: $\mathbf{SE}(n) \to \mathfrak{se}(n)$. Our method yields a simple proof of the surjectivity of exp: $\mathfrak{se}(n) \to \mathbf{SE}(n)$. In Section 7 we solve the problem of finding a skew-symmetric matrix B, given the matrix B^2, and knowing that B^2 has eigenvalues -1 and 0. Section 8 draws conclusions.

2. A Rodrigues-Like Formula for exp: $\mathfrak{so}(n) \to \mathbf{SO}(n)$

In this section, we give a Rodrigues-like formula showing how to compute the exponential e^B of a skew-symmetric $n \times n$ matrix B, where $n \geq 4$. We also show the uniqueness of the matrices $B_1, \ldots, B_p$ used in the decomposition of B mentioned in the introductory section. The following fairly well-known lemma plays a key role in obtaining the matrices $B_1, \ldots, B_p$ (see Horn and Johnson [20], Corollary 2.5.14, or Bourbaki [21]).

Lemma 2.1. Given any skew-symmetric $n \times n$ matrix B ($n \geq 2$), there is some orthogonal matrix P and some block diagonal matrix E such that:

$$B = PEP^{\mathsf{T}}$$

with E of the form:

$$E = \begin{pmatrix} E_1 & \cdots & & \\ \vdots & \ddots & \vdots & \\ & \cdots & E_m & \\ \cdots & & & 0_{n-2m} \end{pmatrix}$$

where each block E_i is a real two-dimensional matrix of the form:

$$E_i = \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix} = \theta_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{with } \theta_i > 0$$

Observe that the eigenvalues of B are $\pm i\theta_j$, or 0, reconfirming the well-known fact that the eigenvalues of a skew-symmetric matrix are purely imaginary, or null. We now prove the existence and uniqueness of the B_j's as well as the generalized Rodrigues' formula.

Theorem 2.2. Given any non-null skew-symmetric $n \times n$ matrix B, where $n \geq 3$, if:

$$\{i\theta_1, -i\theta_1, \ldots, i\theta_p, -i\theta_p\}$$

is the set of distinct eigenvalues of B, where $\theta_j > 0$ and each $i\theta_j$ (and $-i\theta_j$) has multiplicity $k_j \geq 1$, there are p unique skew-symmetric matrices $B_1, \ldots, B_p$ such that:

$$
\begin{aligned}
B &= \theta_1 B_1 + \cdots + \theta_p B_p && (1)\\
B_i B_j &= B_j B_i = 0_n \quad (i \neq j) && (2)\\
B_i^3 &= -B_i && (3)
\end{aligned}
$$

for all i, j with $1 \leq i, j \leq p$, and $2p \leq n$. Furthermore:

$$e^B = e^{\theta_1 B_1 + \cdots + \theta_p B_p} = I_n + \sum_{i=1}^{p} (\sin\theta_i B_i + (1 - \cos\theta_i) B_i^2)$$

and $\{\theta_1, \ldots, \theta_p\}$ is the set of the distinct positive square roots of the $2m$ positive eigenvalues of the symmetric matrix $-1/4(B - B^\intercal)^2$, where $m = k_1 + \cdots + k_p$.

Proof. By Lemma 2.1, the matrix B can be written as:

$$B = PEP^\intercal$$

where E is a block diagonal matrix consisting of m non-zero blocks of the form:

$$E_i = \theta_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{with } \theta_i > 0$$

If:

$$\{i\theta_1, -i\theta_1, \ldots, i\theta_p, -i\theta_p\}$$

is the set of distinct eigenvalues of B, where $\theta_j > 0$, for every j, there is a non-empty set:

$$S_j = \{i_1, \ldots, i_{k_j}\}$$

of indices (in the set $\{1, \ldots, m\}$) corresponding to all the blocks E_j in which θ_j occurs. Let F_j be the matrix obtained by zeroing from E the blocks E_k, where $k \notin S_j$. By factoring θ_j in F_j, we have:

$$F_j = \theta_j G_j$$

and we let:

$$B_j = PG_jP^\intercal$$

It is obvious by construction that the three equations (1)–(3) hold.

As B_i and B_j commute for all i, j, we have:

$$e^B = e^{\theta_1 B_1 + \cdots + \theta_p B_p} = e^{\theta_1 B_1} \cdots e^{\theta_p B_p}$$

However, using:

$$B_i^3 = -B_i$$

as in the 3×3 case, we can show that:

$$e^{\theta_i B_i} = I_n + \sin\theta_i B_i + (1 - \cos\theta_i) B_i^2$$

Indeed, $B_i^3 = -B_i$ implies that:

$$B_i^{4k+j} = B_i^j \quad \text{and} \quad B_i^{4k+2+j} = -B_i^j$$
$$\text{for } j = 1, 2 \quad \text{and} \quad \text{all } k \geq 0$$

and thus, we get:

$$
\begin{aligned}
e^{\theta_i B_i} &= I_n + \sum_{k \geq 1} \frac{\theta_i^k B_i^k}{k!} \\
&= I_n + \left(\frac{\theta_i}{1!} - \frac{\theta_i^3}{3!} + \frac{\theta_i^5}{5!} + \cdots \right) B_i \\
&\quad + \left(\frac{\theta_i^2}{2!} - \frac{\theta_i^4}{4!} + \frac{\theta_i^6}{6!} + \cdots \right) B_i^2 \\
&= I_n + \sin\theta_i B_i + (1 - \cos\theta_i) B_i^2
\end{aligned}
$$

Since:

$$B_i B_j = B_j B_i = 0_n \quad (i \neq j)$$

we get:

$$
\begin{aligned}
e^B &= \prod_{i=1}^{p} e^{\theta_i B_i} = \prod_{i=1}^{m} (I_n + \sin\theta_i B_i + (1 - \cos\theta_i) B_i^2) \\
&= I_n + \sum_{i=1}^{p} (\sin\theta_i B_i + (1 - \cos\theta_i) B_i^2)
\end{aligned}
$$

The matrix $1/4(B - B^\intercal)^2$ is of the form $PE^2P^\intercal$, where:

$$E_i^2 = \begin{pmatrix} -\theta_i^2 & 0 \\ 0 & -\theta_i^2 \end{pmatrix}$$

Thus, the eigenvalues of $-1/4(B - B^\intercal)^2$ are:

$$(\theta_1^2, \theta_1^2, \ldots, \theta_m^2, \theta_m^2, \underbrace{0, \ldots, 0}_{n-2m})$$

and thus $(\theta_1, \ldots, \theta_m)$ are the positive square roots of the eigenvalues of the symmetric matrix $-1/4(B - B^\intercal)^2$.

We now prove the uniqueness of the B_j's. If we assume that matrices B_j's with the required properties exist, using the properties of the B_j's, we get the system:

$$
\begin{aligned}
B &= \sum_{i=1}^{p} \theta_i B_i \\
B^3 &= -\sum_{i=1}^{p} \theta_i^3 B_i \\
B^5 &= \sum_{i=1}^{p} \theta_i^5 B_i && (4)\\
&\vdots \\
B^{2p-1} &= (-1)^{p-1} \sum_{i=1}^{p} \theta_i^{2p-1} B_i
\end{aligned}
$$

The determinant of this system is:

$$\delta_n = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_p \\ -\theta_1^3 & -\theta_2^3 & \cdots & -\theta_p^3 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{p-1}\theta_1^{2p-1} & (-1)^{p-1}\theta_2^{2p-1} & \cdots & (-1)^{p-1}\theta_p^{2p-1} \end{bmatrix}$$

Observe that the above matrix is the product of the diagonal matrix:

$$\mathrm{diag}(1, -1, 1, -1, \ldots, 1, (-1)^{p-1})$$

by the matrix:

$$\left(\prod_{i=1}^{p} \theta_i\right) V(\theta_1^2, \ldots, \theta_p^2)$$

where $V(\theta_1^2, \ldots, \theta_p^2)$ is a Vandermonde matrix. Therefore, the determinant δ_n can be immediately computed, and we get:

$$\delta_n = (-1)^{p(p-1)/2} \prod_{i=1}^{p} \theta_i \prod_{1 \le i < j \le p} (\theta_j^2 - \theta_i^2)$$

Since the θ_i's are positive and all distinct, $\delta_n \neq 0$. Thus, $B_1, \ldots, B_p$ are uniquely determined from B and its non-null eigenvalues. □

Given a skew-symmetric $n \times n$ matrix B, we can compute $\theta_1, \ldots, \theta_p$ and $B_1, \ldots, B_p$ as follows. By Theorem 2.2 $\theta_1^2, \ldots, \theta_p^2$ are the distinct non-null eigenvalues of the symmetric matrix $-1/4(B - B^\intercal)^2$, and there are several numerical methods for computing eigenvalues of symmetric matrices (see Golub and Van Loan [22] or Trefethen and Bau [23]). Then, we find $B_1, \ldots, B_p$ by solving the linear system (4) used in the proof of Theorem 2.2.

Note that B_j has the eigenvalues $i, -i$, each with multiplicity k_j, and 0 with multiplicity $n - 2k_j$. Now recall the following structure lemma for rotations in $\mathbf{SO}(n)$ (e.g., see Berger [24] or Horn and Johnson [20], Corollary 2.5.14).

Lemma 2.3. For every rotation matrix $R \in \mathbf{SO}(n)$, there is a block diagonal matrix D and an orthogonal matrix P such that:

$$R = PDP^\intercal$$

where D is a block diagonal matrix of the form:

$$D = \begin{pmatrix} D_1 & \cdots & & \\ \vdots & \ddots & \vdots & \\ & \cdots & D_m & \\ \cdots & & & I_{n-2m} \end{pmatrix}$$

where the first m blocks D_i are of the form:

$$D_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \quad \text{with } 0 < \theta_i \le \pi$$

Using the surjectivity of the exponential map $\exp\colon \mathfrak{so}(n) \to \mathbf{SO}(n)$, which easily follows from Lemma 2.1, Lemma 2.3 and the fact that if:

$$E_i = \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix}$$

then

$$e^{E_i} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

and we obtain the following characterization of rotations in $\mathbf{SO}(n)$, where $n \ge 3$:

Lemma 2.4. Given any rotation matrix $R \in \mathbf{SO}(n)$, where $n \ge 3$, if:

$$\{e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_p}, e^{-i\theta_p}\}$$

is the set of distinct eigenvalues of R different from 1, where $0 < \theta_i \le \pi$, there are p skew symmetric matrices $B_1, \ldots, B_p$ such that:

$$\begin{aligned} B_i B_j &= B_j B_i = 0_n \quad (i \neq j) \\ B_i^3 &= -B_i \end{aligned}$$

for all i, j with $1 \le i, j \le p$, and $2p \le n$, and furthermore:

$$R = e^{\theta_1 B_1 + \cdots + \theta_p B_p} = I_n + \sum_{i=1}^{p} (\sin\theta_i B_i + (1 - \cos\theta_i) B_i^2)$$

Lemma 2.4 implies that:

$$\{\cos\theta_1, \ldots, \cos\theta_p\}$$

is the set of eigenvalues of the symmetric matrix $1/2(R + R^\intercal)$ that are different from 1. However, the matrices $B_1, \ldots, B_p$ are not necessarily unique. This has to do with the fact that we may have $\sin\theta_i = 0$ when $\theta_i = \pi$. Nevertheless, it is possible to find $B_1, \ldots, B_p$ from R. We begin with the case $n = 4$, as it is simpler.

3. Computing log: $\mathbf{SO}(4) \to \mathfrak{so}(4)$

By Theorem 2.2, a rotation matrix for $n = 4$ is given by:

$$R = I_4 + \sin\theta_1 B_1 + (1 - \cos\theta_1) B_1^2$$

or

$$\begin{aligned} R = I_4 &+ \sin\theta_1 B_1 + \sin\theta_2 B_2 + (1 - \cos\theta_1) B_1^2 \\ &+ (1 - \cos\theta_2) B_2^2 \end{aligned}$$

where B_1 and B_2 are all 4×4 skew-symmetric matrices, and:

$$\begin{aligned} B_1 B_2 &= B_2 B_1 = 0 \\ B_1^3 &= -B_1 \\ B_2^3 &= -B_2 \end{aligned}$$

The first case in which $i\theta_1$ has multiplicity 2 is analogous to the case of a rotation in $\mathbf{SO}(3)$. We can compute $\cos\theta_1$ easily because:

$$\mathrm{tr}(R) = 4\cos\theta_1$$

The case $\theta_1 = \pi$ requires computing B_1 from B_1^2. This subproblem is solved in Section 7.

In the second case, $\theta_1 \neq \theta_2$, with $0 < \theta_i \leq \pi$. This is analogous to the case of a rotation in $\mathbf{SO}(3)$.

In all cases, we know that $\cos\theta_1$ and $\cos\theta_2$ are double eigenvalues of $1/2(R + R^\mathsf{T})$, but we can easily compute $\cos\theta_1 + \cos\theta_2$ and $\cos\theta_1\cos\theta_2$, and $\cos\theta_1$ and $\cos\theta_2$ are the roots of a quadratic equation that will be found explicitly.

The properties of the B_i's immediately imply that:

$$\begin{aligned}R^2 = I_4 &+ \sin 2\theta_1 B_1 + \sin 2\theta_2 B_2 + (1-\cos 2\theta_1)B_1^2\\ &+ (1-\cos 2\theta_2)B_2^2\end{aligned}$$

As B_1 and B_2 are skew-symmetric, we get:

$$\begin{aligned}\tfrac{1}{2}(R - R^\mathsf{T}) &= \sin\theta_1 B_1 + \sin\theta_2 B_2\\ \tfrac{1}{2}(R^2 - R^{2\mathsf{T}}) &= \sin 2\theta_1 B_1 + \sin 2\theta_2 B_2\\ \mathrm{tr}(R) &= 2\cos\theta_1 + 2\cos\theta_2\end{aligned}$$

We first look at the special cases in which $\sin\theta_1 = 0$ or $\sin\theta_2 = 0$. Assume that $\theta_1 = \pi$ and $\theta_2 \neq \pi$, the case where $\theta_1 \neq \pi$ and $\theta_2 = \pi$ being similar. Then we get:

$$\tfrac{1}{2}(R - R^\mathsf{T}) = \sin\theta_2 B_2$$

from which we can compute B_2. We can now compute B_1^2 from:

$$\tfrac{1}{2}(R + R^\mathsf{T}) = 2B_1^2 + (1-\cos\theta_2)B_2^2$$

Finally, we have to compute B_1 from B_1^2. This subproblem is solved in Section 7.

We may now assume that $\sin\theta_i \neq 0$, for $i = 1, 2$. We show the following proposition:

Proposition 3.1. The numbers $\cos\theta_1$ and $\cos\theta_2$ are solutions of the equation $x^2 - px + q = 0$, where:

$$\begin{aligned}p &= \cos\theta_1 + \cos\theta_2 = \tfrac{1}{2}\mathrm{tr}(R)\\ q &= \cos\theta_1\cos\theta_2 = \tfrac{1}{8}\mathrm{tr}(R)^2 - \tfrac{1}{16}\mathrm{tr}((R - R^\mathsf{T})^2) - 1\end{aligned}$$

Proof. We know that:

$$\tfrac{1}{2}(R - R^\mathsf{T}) = \sin\theta_1 B_1 + \sin\theta_2 B_2$$

and:

$$\mathrm{tr}(B_1^2) = -2 \quad \mathrm{tr}(B_2^2) = -2$$

Therefore, some algebra yields:

$$\tfrac{1}{4}\mathrm{tr}((R - R^\mathsf{T})^2) = 2\cos^2\theta_1 + 2\cos^2\theta_2 - 4$$

As we also know that:

$$\mathrm{tr}(R) = 2\cos\theta_1 + 2\cos\theta_2$$

we easily get the desired expression for $p = \cos\theta_1 + \cos\theta_2$ and $q = \cos\theta_1\cos\theta_2$.

Note in passing that we also have:

$$\cos^2\theta_1\cos^2\theta_2 = \det\big(\tfrac{1}{2}(R + R^\mathsf{T})\big)$$

which is the product of the eigenvalues.

Consider the system:

$$\begin{aligned}\tfrac{1}{2}(R - R^\mathsf{T}) &= \sin\theta_1 B_1 + \sin\theta_2 B_2\\ \tfrac{1}{2}(R^2 - R^{2\mathsf{T}}) &= \sin 2\theta_1 B_1 + \sin 2\theta_2 B_2\end{aligned}$$

The determinant of the above system is:

$$2\sin\theta_1\sin\theta_2(\cos\theta_2 - \cos\theta_1)$$

As we assumed that $\sin\theta_i \neq 0$ and $0 < \theta_i < \pi$ for $i = 1, 2$, we have $\cos\theta_2 \neq \cos\theta_1$, and the system has a unique solution for B_1 and B_2. ☐

4. Computing log: $\mathbf{SO}(n) \to \mathfrak{so}(n)$

Given an orthogonal matrix $R \in \mathbf{SO}(n)$, we would like to find a logarithm of R, that is, some skew-symmetric matrix B such that $R = e^B$. By Theorem 2.2 and Lemma 2.4, we know that we can look for a matrix:

$$B = \theta_1 B_1 + \cdots + \theta_p B_p$$

where:

$$\{i\theta_1, -i\theta_1, \ldots, i\theta_p, -i\theta_p\}$$

is the set of distinct eigenvalues of B, with $0 < \theta_i \leq \pi$, and $B_1, \ldots, B_p$ are skew matrices such that:

$$\begin{aligned}B_iB_j &= B_jB_i = 0_n \quad (i \neq j)\\ B_i^3 &= -B_i\end{aligned}$$

for all i, j with $1 \leq i, j \leq p$ and $2p \leq n$. Then, we have:

$$R = e^{\theta_1 B_1 + \cdots + \theta_p B_p} = I_n + \sum_{i=1}^{p}(\sin\theta_i B_i + (1-\cos\theta_i)B_i^2)$$

As we observed earlier:

$$\{\cos\theta_1, \ldots, \cos\theta_p\}$$

is the set of eigenvalues of the symmetric matrix $1/2\,(R + R^\mathsf{T})$ that are different from 1. Furthermore, $\{\cos\theta_1, \ldots, \cos\theta_p\}$ can be computed as the set of eigenvalues of the symmetric matrix $1/2\,(R + R^\mathsf{T})$ that are different from 1. The question is, how can we compute $B_1, \ldots, B_p$?

Since

$$R = e^{\theta_1 B_1 + \cdots + \theta_p B_p}$$

we get:

$$R^j = e^{j\theta_1 B_1 + \cdots + j\theta_p B_p}$$

and thus:

$$R^j = I_n + \sum_{i=1}^{p}\sin j\theta_i B_i + \sum_{i=1}^{p}(1-\cos j\theta_i)B_i^2$$

Then, we get the system:

$$
\begin{aligned}
\tfrac{1}{2}(R - R^{\mathsf{T}}) &= \sum_{i=1}^{p} \sin\theta_i B_i \\
\tfrac{1}{2}(R^2 - R^{2\mathsf{T}}) &= \sum_{i=1}^{p} \sin 2\theta_i B_i \\
\tfrac{1}{2}(R^3 - R^{3\mathsf{T}}) &= \sum_{i=1}^{p} \sin 3\theta_i B_i \\
\vdots \qquad & \qquad \vdots \\
\tfrac{1}{2}(R^p - R^{p\mathsf{T}}) &= \sum_{i=1}^{p} \sin p\theta_i B_i
\end{aligned}
$$

As we will prove shortly, the determinant:

$$
\delta'_p = \begin{bmatrix} \sin\theta_1 & \sin\theta_2 & \cdots & \sin\theta_p \\ \sin 2\theta_1 & \sin 2\theta_2 & \cdots & \sin 2\theta_p \\ \vdots & \vdots & \ddots & \vdots \\ \sin p\theta_1 & \sin p\theta_2 & \cdots & \sin p\theta_p \end{bmatrix} \tag{5}
$$

of this system is given by the formula:

$$
\delta'_p = 2^{p(p-1)/2} \prod_{i=1}^{p} \sin\theta_i \prod_{1\le i<j\le p} (\cos\theta_j - \cos\theta_i)
$$

When $0 < \theta_i < \pi$ for $i = 1, \ldots, p$, the determinant δ'_p is non-null. On the other hand, -1 is an eigenvalue of R iff $\theta_j = \pi$ for some j. Without loss of generality, we may assume that $\theta_p = \pi$ iff -1 is an eigenvalue of R, and we get the following theorem.

Theorem 4.1. Given any rotation matrix $R \in \mathbf{SO}(n)$, where $n \ge 3$, let:

$$
\{e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_p}, e^{-i\theta_p}\}
$$

be the set of distinct eigenvalues of R different from 1, where $0 < \theta_i \le \pi$. Then, there are p skew-symmetric matrices $B_1, \ldots, B_p$ such that:

$$
\begin{aligned}
B_i B_j = B_j B_i = 0_n \quad (i \neq= j) \\
B_i^3 = -B_i
\end{aligned}
$$

for all i, j, with $1 \le i, j \le p$, and $2p \le n$, and:

$$
R = e^{\theta_1 B_1 + \cdots + \theta_p B_p}
$$

so that:

$$
B = \theta_1 B_1 + \cdots + \theta_p B_p
$$

is a logarithm of R. Furthermore, if -1 is not an eigenvalue of R, the matrices $B_1, \ldots, B_p$ are unique, and if -1 is an eigenvalue of R, the matrices $B_1, \ldots, B_{p-1}$ are unique and the skew-symmetric square root of B_p^2 can be determined using the method of Section 7.

Proof. First, assume that -1 is not an eigenvalue of R, so that $\theta_p \neq \pi$. We observed earlier that the determinant of the system determining $B_1, \ldots, B_p$ is:

$$
\delta'_p = \begin{bmatrix} \sin\theta_1 & \sin\theta_2 & \cdots & \sin\theta_p \\ \sin 2\theta_1 & \sin 2\theta_2 & \cdots & \sin 2\theta_p \\ \vdots & \vdots & \ddots & \vdots \\ \sin p\theta_1 & \sin p\theta_2 & \cdots & \sin p\theta_p \end{bmatrix}
$$

Thus, we need to compute δ'_n.

From the identity:

$$
(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta
$$

we get:

$$
\begin{aligned}
\sin n\theta = \sin\theta \Bigg(\binom{n}{1} \cos^{n-1}\theta - \binom{n}{3} \cos^{n-3}\theta \sin^2\theta \\
+ \binom{n}{5} \cos^{n-5}\theta \sin^4\theta + \cdots \Bigg)
\end{aligned}
$$

As all the powers of $\sin\theta$ in the sum are even, using the fact that $\cos^2\theta + \sin^2\theta = 1$, we can express the sum within the parentheses in terms of $\cos\theta$ only, so that:

$$
\sin n\theta = \sin\theta(a_{n-1}\cos^{n-1}\theta + a_{n-3}\cos^{n-3}\theta + \cdots)
$$

Similarly:

$$
\begin{aligned}
\cos n\theta = \cos^n\theta - \binom{n}{2} \cos^{n-2}\theta \sin^2\theta \\
+ \binom{n}{4} \cos^{n-4}\theta \sin^4\theta + \cdots
\end{aligned}
$$

so that $\cos n\theta$ can be expressed in terms of $\cos\theta$ only, and we get:

$$
\cos n\theta = b_n \cos^n\theta + b_{n-2}\cos^{n-2}\theta + \cdots
$$

We claim that:

$$
a_{n-1} = b_n = 2^{n-1}
$$

This is easily shown by induction using the identities:

$$
\sin(n+1)\theta = \sin n\theta\cos\theta + \cos n\theta\sin\theta
$$

and:

$$
\cos(n+1)\theta = \cos n\theta\cos\theta - \sin n\theta\sin\theta
$$

Now, if we look at the determinant:

$$
\delta'_p = \begin{bmatrix} \sin\theta_1 & \sin\theta_2 & \cdots & \sin\theta_p \\ \sin 2\theta_1 & \sin 2\theta_2 & \cdots & \sin 2\theta_p \\ \vdots & \vdots & \ddots & \vdots \\ \sin p\theta_1 & \sin p\theta_2 & \cdots & \sin p\theta_p \end{bmatrix}
$$

and express each $\sin j\theta_k$ using:

$$\sin j\theta_k = \sin\theta_k \left(2^{j-1}\cos^{j-1}\theta_k + s_j(\cos\theta_k)\right)$$

where $s_j(X)$ is a polynomial of degree $j-3$, we can factor $\sin\theta_k$ from each column, and we get a determinant where the jth row is of the form:

$$\begin{matrix} 2^{j-1}\cos^{j-1}\theta_1 + s_j(\cos\theta_1) \cdots \\ 2^{j-1}\cos^{j-1}\theta_p + s_j(\cos\theta_p) \end{matrix}$$

and where the first row is:

$$1 \quad \cdots \quad 1$$

Then, we can cancel all constant terms in rows $2,\ldots,p$ by subtracting some appropriate multiple of the first row; every term of degree 1 in rows $3,\ldots,p$ by subtracting some appropriate multiple of the second row; every term of degree 2 in rows $4,\ldots,p$ by subtracting some appropriate multiple of the third row; and so on, so that in the end we get the product of the Vandermonde determinant $V(\cos\theta_1,\ldots,\cos\theta_p)$ by the determinant of the diagonal matrix:

$$\mathrm{diag}(1,2,2^2,\ldots,2^{p-1})$$

The result is indeed:

$$\delta'_p = 2^{p(p-1)/2}\prod_{i=1}^{p}\sin\theta_i \prod_{1\le i<j\le p}(\cos\theta_j - \cos\theta_i)$$

Under the assumptions of the theorem, namely, $0<\theta_j<\pi$ and $\theta_i \neq \theta_j$ for $i\neq j$, we have $\delta'_p \neq 0$.

When -1 is an eigenvalue of R, we have $\theta_p = \pi$. In this case, $\sin\theta_p = 0$, and the above system involves only $B_1,\ldots,B_{p-1}$, which are uniquely determined because the determinant δ'_{p-1} is non-null. Finally, because:

$$\tfrac{1}{2}(R+R^{\mathsf{T}}) = I_n + \sum_{i=1}^{p}(1-\cos\theta_i)B_i^2$$

with $\theta_p = \pi$, we get:

$$B_p^2 = \tfrac{1}{4}(R+R^{\mathsf{T}}) - \tfrac{1}{2}\left(I_n + \sum_{i=1}^{p-1}(1-\cos\theta_i)B_i^2\right)$$

and we can compute B_p given B_p^2 using the method presented in Section 7. Thus:

$$B = \theta_1 B_1 + \cdots + \theta_p B_p$$

is a logarithm of R. □

5. A Rodrigues-Like Formula for $\exp\colon \mathfrak{se}(n) \to \mathbf{SE}(n)$

In this section, we give a Rodrigues-like formula showing how to compute the exponential e^{Ω} of an element Ω of the Lie algebra $\mathfrak{se}(n)$ of the Lie group $\mathbf{SE}(n)$ of (affine) rigid motions, where $n \geq 3$.

First, we review the usual way of representing affine maps of $\mathbb{R}^n$ in terms of $(n+1)\times(n+1)$ matrices.

Definition 5.1. The set of affine maps ρ of $\mathbb{R}^n$ defined such that:

$$\rho(X) = RX + U$$

where R is a rotation matrix ($R \in \mathbf{SO}(n)$) and U is some vector in $\mathbb{R}^n$, is a group under composition called the group of *direct affine isometries*, or *rigid motions*, denoted as $\mathbf{SE}(n)$.

Every rigid motion can be represented by the $(n+1)\times(n+1)$ matrix:

$$\begin{pmatrix} R & U \\ 0 & 1 \end{pmatrix}$$

in the sense that:

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} R & U \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = RX + U$$

Definition 5.2. The vector space of real $(n+1)\times(n+1)$ matrices of the form:

$$\Omega = \begin{pmatrix} B & U \\ 0 & 0 \end{pmatrix}$$

where B is a skew-symmetric matrix and U is a vector in $\mathbb{R}^n$ is denoted as $\mathfrak{se}(n)$.

The group $\mathbf{SE}(n)$ is a Lie group, and $\mathfrak{se}(n)$ is its Lie algebra. In order to give a Rodrigues-like formula for computing the exponential map $\exp\colon \mathfrak{se}(n) \to \mathbf{SE}(n)$, we need the following key lemma.

Lemma 5.3. Given any $(n+1)\times(n+1)$ matrix of the form:

$$\Omega = \begin{pmatrix} B & U \\ 0 & 0 \end{pmatrix}$$

where B is any matrix and $U \in \mathbb{R}^n$, we have:

$$e^{\Omega} = \begin{pmatrix} e^B & VU \\ 0 & 1 \end{pmatrix}$$

where:

$$V = I_n + \sum_{k\geq 1}\frac{B^k}{(k+1)!}$$

Proof. A trivial induction on k. □

Observing that:

$$V = I_n + \sum_{k\geq 1}\frac{B^k}{(k+1)!} = \int_0^1 e^{Bt}\,dt$$

we can now prove our main result.

Theorem 5.4. Given any $(n+1)\times(n+1)$ matrix of the form:

$$\Omega = \begin{pmatrix} B & U \\ 0 & 0 \end{pmatrix}$$

where B is a non-null skew-symmetric matrix and $U \in \mathbb{R}^n$, with $n \geq 3$, if:

$$\{i\theta_1, -i\theta_1, \ldots, i\theta_p, -i\theta_p\}$$

is the set of distinct eigenvalues of B, where $\theta_i > 0$, there are p unique skew-symmetric matrices $B_1, \ldots, B_p$ such that the three equations (1)–(3) hold. Furthermore:

$$e^{\Omega} = \begin{pmatrix} e^B & VU \\ 0 & 1 \end{pmatrix}$$

where:

$$e^B = I_n + \sum_{i=1}^{p}(\sin\theta_i B_i + (1-\cos\theta_i)B_i^2)$$

and:

$$V = I_n + \sum_{i=1}^{p}\left(\frac{(1-\cos\theta_i)}{\theta_i}B_i + \frac{(\theta_i - \sin\theta_i)}{\theta_i}B_i^2\right)$$

Proof. The existence and uniqueness of $B_1, \ldots, B_p$ and the formula for e^B come from Theorem 2.2. Since:

$$V = I_n + \sum_{k\geq 1}\frac{B^k}{(k+1)!} = \int_0^1 e^{Bt}\,dt$$

we have:

$$\begin{aligned} V &= \int_0^1 \left[I_n + \sum_{i=1}^{p}\left(\sin t\theta_i B_i + (1-\cos t\theta_i)B_i^2\right)\right]dt \\ &= \left[tI_n + \sum_{i=1}^{p}\left(-\frac{\cos t\theta_i}{\theta_i}B_i + \left(t - \frac{\sin t\theta_i}{\theta_i}\right)B_i^2\right)\right]_0^1 \\ &= I_n + \sum_{i=1}^{p}\left(\frac{(1-\cos\theta_i)}{\theta_i}B_i + \frac{(\theta_i-\sin\theta_i)}{\theta_i}B_i^2\right) \end{aligned}$$

□

Remark. Given:

$$\Omega = \begin{pmatrix} B & U \\ 0 & 0 \end{pmatrix}$$

where $B = \theta_1 B_1 + \cdots + \theta_p B_p$, if we let:

$$\Omega_i = \begin{pmatrix} B_i & U/\theta_i \\ 0 & 0 \end{pmatrix}$$

using the fact that $B_i^3 = -B_i$ and:

$$\Omega_i^k = \begin{pmatrix} B_i^k & B_i^{k-1}U/\theta_i \\ 0 & 0 \end{pmatrix}$$

it is easily verified that:

$$e^{\Omega} = I_{n+1} + \Omega + \sum_{i=1}^{p}\left((1-\cos\theta_i)\Omega_i^2 + (\theta_i - \sin\theta_i)\Omega_i^3\right)$$

6. Computing log: $\mathbf{SE}(n) \to \mathfrak{se}(n)$

Given an element:

$$M = \begin{pmatrix} R & U \\ 0 & 1 \end{pmatrix}$$

of $\mathbf{SE}(n)$, because R is a rotation matrix, we know from Lemma 2.4 that if:

$$\{e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_p}, e^{-i\theta_p}\}$$

is the set of distinct eigenvalues of R different from 1, where $0 < \theta_i \leq \pi$, there are p skew-symmetric matrices $B_1, \ldots, B_p$ such that:

$$\begin{aligned} B_iB_j &= B_jB_i = 0_n \quad (i \neq j) \\ B_i^3 &= -B_i \end{aligned}$$

for all i, j with $1 \leq i, j \leq p$, and $2p \leq n$, and furthermore:

$$R = e^{\theta_1 B_1 + \cdots + \theta_p B_p} = I_n + \sum_{i=1}^{p}\left(\sin\theta_i B_i + (1-\cos\theta_i)B_i^2\right)$$

We can also compute $B_1, \ldots, B_p$ from R, as shown in Section 4. Thus, if V is invertible, we have a method to compute a log of M.

Using Theorem 5.4 we can prove that V is invertible. This yields a fairly direct proof of the surjectivity of the exponential map exp: $\mathfrak{se}(n) \to \mathbf{SE}(n)$, and gives a method for computing some determination of the (multivalued) function the log function.

Theorem 6.1. The matrix:

$$V = I_n + \sum_{i=1}^{p}\left(\frac{(1-\cos\theta_i)}{\theta_i}B_i + \frac{(\theta_i-\sin\theta_i)}{\theta_i}B_i^2\right)$$

from Theorem 5.4 is invertible.

Proof. Since:

$$V = I_n + \sum_{i=1}^{p}\left(\frac{(1-\cos\theta_i)}{\theta_i}B_i + \frac{(\theta_i-\sin\theta_i)}{\theta_i}B_i^2\right)$$

Let us assume that the inverse of V is of the form:

$$W = I_n + \sum_{i=1}^{p}\left(\alpha_i B_i + \beta_i B_i^2\right)$$

The condition $VW = I_n$ is expressed as:

$$
\begin{aligned}
I_n = I_n &+ \sum_{i=1}^{p} \left(\frac{(1-\cos\theta_i)}{\theta_i} B_i + \frac{(\theta_i - \sin\theta_i)}{\theta_i} B_i^2 \right) \\
&+ \sum_{i=1}^{p} \left(\alpha_i B_i + \beta_i B_i^2 \right) \\
&+ \sum_{i=1}^{p} \Bigg(\frac{(1-\cos\theta_i)\alpha_i}{\theta_i} B_i^2 - \frac{(1-\cos\theta_i)\beta_i}{\theta_i} B_i \\
&\qquad - \frac{(\theta_i - \sin\theta_i)\alpha_i}{\theta_i} B_i - \frac{(\theta_i - \sin\theta_i)\beta_i}{\theta_i} B_i^2 \Bigg) \\
= I_n &+ \sum_{i=1}^{p} \left(\frac{\sin\theta_i \alpha_i}{\theta_i} - \frac{(1-\cos\theta_i)\beta_i}{\theta_i} + \frac{(1-\cos\theta_i)}{\theta_i} \right) B_i \\
&+ \sum_{i=1}^{p} \left(\frac{(1-\cos\theta_i)\alpha_i}{\theta_i} + \frac{\sin\theta_i \beta_i}{\theta_i} + \frac{(\theta_i - \sin\theta_i)}{\theta_i} \right) B_i^2
\end{aligned}
$$

Thus, we just have to solve the p systems of equations:

$$
\begin{aligned}
\sin\theta_i \alpha_i - (1-\cos\theta_i)\beta_i &= \cos\theta_i - 1 \\
(1-\cos\theta_i)\alpha_i + \sin\theta_i \beta_i &= \sin\theta_i - \theta_i
\end{aligned}
$$

Since the determinant of the above matrix is:

$$
\sin^2\theta_i + (1-\cos\theta_i)^2 = 2(1-\cos\theta_i)
$$

and $0 < \theta_i \leq \pi$, the matrix is invertible and the system has a unique solution. In fact, α_i and β_i are given by:

$$
\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \frac{1}{2(1-\cos\theta_i)} \begin{pmatrix} \sin\theta_i & (1-\cos\theta_i) \\ -(1-\cos\theta_i) & \sin\theta_i \end{pmatrix} \times \begin{pmatrix} \cos\theta_i - 1 \\ \sin\theta_i - \theta_i \end{pmatrix}
$$

That is:

$$
\begin{aligned}
\alpha_i &= -\frac{\theta_i}{2} \\
\beta_i &= 1 - \frac{\theta_i \sin\theta_i}{2(1-\cos\theta_i)}
\end{aligned}
$$

Therefore, the inverse of V is:

$$
V^{-1} = I_n + \sum_{i=1}^{p} \left(-\frac{\theta_i}{2} B_i + \left(1 - \frac{\theta_i \sin\theta_i}{2(1-\cos\theta_i)} \right) B_i^2 \right) \quad \square
$$

Remark. This formula is equivalent to the formula given in the Appendix of Murray, Li, and Sastry [16] in the special case of $\mathbf{SE}(3)$. This is because:

$$
\frac{\theta \sin\theta}{2(1-\cos\theta)} = \frac{\theta \sin\theta (1+\cos\theta)}{2(1-\cos\theta)(1+\cos\theta)} = \frac{\theta(1+\cos\theta)}{2\sin\theta}
$$

and thus:

$$
1 - \frac{\theta \sin\theta}{2(1-\cos\theta)} = \frac{2\sin\theta - \theta(1+\cos\theta)}{2\sin\theta}
$$

which is the expression found in Murray, Li, and Sastry [16], except that our B_i's are normalized. Note that this expression is not well defined for $\theta = \pi$. Our expression does not suffer from this minor problem.

7. A Method for Computing B Given B^2

As we saw in Section 4, in order to compute a logarithm of an orthogonal matrix, it may be necessary to compute a skew-symmetric matrix B given its square B^2. Actually, the eigenvalues of B's are $\pm i$, and this simplifies the problem. We need to solve the following problem: find a skew-symmetric matrix B such that $A = B^2$ is a given non-null symmetric matrix with eigenvalues -1 or 0, with an even number of -1. It is slightly more convenient to look for a skew-symmetric B, given $A = -B^2$, as A is then a non-null symmetric matrix with eigenvalues $+1$ and 0, with an even number of $+1$. Since A is a symmetric matrix whose eigenvalues are known, the problem can be solved by diagonalizing A. Then, if $A = PDP^\intercal$, with P orthogonal, as D has an even number of $+1$'s, we form E from D by replacing every 2×2-identity block I_2 in D by:

$$
J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

and we let $B = PEP^\intercal$. Since $J^2 = -I_2$, we get:

$$
E^2 = -D
$$

and then:

$$
B^2 = PEP^\intercal PEP^\intercal = PE^2P^\intercal = -PDP^\intercal = -A
$$

Therefore, $A = -B^2$, as desired. In principle, the problem is solved. Actually, becuase the eigenvalues of A are special ($+1$ and 0), a simple method based on the Gram–Schmidt orthonormalization procedure can be designed, as we now explain. As $A = PDP^\intercal$, where D is a diagonal matrix consisting or 0's and 1's, we have $A^2 = A$. As a consequence, every non-null column U of A is an eigenvector of A for the eigenvalue 1, that is, $AU = U$. Thus, we use the following inductive method to diagonalize A.

If $A = 0$ (the null matrix), then $B = 0$. Otherwise, proceed as follows. Let $(e_1, \ldots, e_n)$ be any basis of $\mathbb{R}^n$, for instance, the canonical basis (where the ith entry of e_i is 1, and all other entries are 0).

Let U_1 be any non-null column of A (for instance, the left-most non-null column). As U_1 is non-null, let i be the index of some non-null entry in U_1 (for instance, the least index i, or the least index such that the ith entry is maximum). We now form the new basis:

$$
(U_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n)
$$

obtained from $(e_1, \ldots, e_n)$ by replacing e_i by U_1 and reordering the vectors so that U_1 is now the first vector. This new basis is generally not orthonormal, and we apply Gram–Schmidt (or any of its variants, such as modified Gram–Schmidt; see Golub and Van Loan [22] or Trefethen and Bau [23]) to get an orthonormal basis:

$$
(U_1', e_1', \ldots, e_{i-1}', e_{i+1}', \ldots, e_n')
$$

This basis defines an orthogonal matrix Q_1, and we compute:

$$
A_1 = Q_1^\intercal A Q_1
$$

As U_1' is just U_1 normalized to unit length, U_1' is an eigenvector of A for the eigenvalue 1, and A_1 is of the form:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & A_1' \end{pmatrix}$$

We can now repeat the above procedure inductively on A_1', which is an $(n-1)\times(n-1)$ matrix. This will yield an orthogonal $(n-1)\times(n-1)$ matrix Q_2' such that:

$$D' = Q_2'^{\mathsf{T}} A_1' Q_2'$$

where D' is a diagonal $(n-1)\times(n-1)$ matrix of 0's and 1's. Then:

$$A_1' = Q_2' D' Q_2'^{\mathsf{T}}$$

and we form the orthogonal matrix:

$$Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & Q_2' \end{pmatrix}$$

and the diagonal matrix:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & D' \end{pmatrix}$$

so that:

$$A_1 = Q_2 D Q_2^{\mathsf{T}}$$

and we finally get:

$$A = QDQ^{\mathsf{T}}$$

where $Q = Q_1 Q_2$.

In forming the matrix E, instead of using the matrix:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we can use the matrix $K = -J$, since we also have $K^2 = -I_2$. This is the reason why B is not unique. In fact, if A has the eigenvalue 1 with multiplicity $2q$, there are $2q$ possibilities for B (recall that we are looking for B such that $B^2 = -A$, where A is a non-null symmetric matrix with eigenvalues $+1$ and 0, with an even number of $+1$).

8. Conclusion

In this work, we have given a generalization of Rodrigues' formula for computing the exponential map exp: $\mathfrak{so}(n) \to \mathbf{SO}(n)$ when $n \geq 4$, and we have also given a method for computing some determination of the (multivalued) function log function log: $\mathbf{SO}(n) \to \mathfrak{so}(n)$. A subproblem arising in computing $\log R$, where $R \in \mathbf{SO}(n)$, is the problem of finding a skew-symmetric matrix B, given the matrix B^2, and knowing that B^2 has eigenvalues -1 and 0. Technically, the key result is the decomposition of a skew-symmetric $n \times n$ matrix B in terms of some skew-symmetric matrices having some special properties. We also showed that there is a Rodrigues-like formula for computing this exponential map exp: $\mathfrak{se}(n) \to \mathbf{SE}(n)$, and we gave a method for computing some determination of the (multivalued) function log: $\mathbf{SE}(n) \to \mathfrak{se}(n)$. As a corollary we obtained a direct proof of the surjectivity of exp: $\mathfrak{se}(n) \to \mathbf{SE}(n)$. The method for computing log: $\mathbf{SO}(4) \to \mathfrak{so}(4)$ has been implemented. It has applications to a locomotion problem, where the parameter space is modelled by $\mathbb{R}^4$ (see Sun, [25]). The problem of interpolating between two rotations $R_1, R_2 \in \mathbf{SO}(4)$ comes up naturally. Our methods can be used to perform motion interpolation in $\mathbf{SO}(n)$ or $\mathbf{SE}(n)$ for fairly large n, but we are unaware of practical applications for $n \geq 5$. We are hoping that such problems will arise in the future, perhaps in robotics or even physics.

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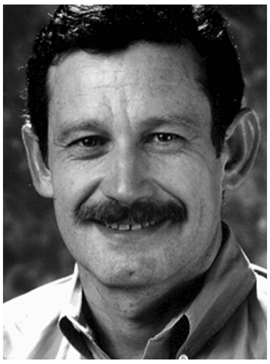
Biographies



Jean Gallier received his Ph.D. in computer science from the University of California, Los Angeles, in 1978. Dr. Gallier joined the University of Pennsylvania in 1978 as Assistant Professor in the Department of Computer and Information Science, where he has been a Professor since 1990. He has also had a secondary appointment in the Department of Mathematics since 1994. Dr. Gallier has worked in various areas, ranging from program correctness and automated theorem proving, incremental compilation, typed lambda calculi, and proof theory to geometric modelling, computer graphics, and algebraic geometry. The author of over 60 original papers, he has written three books: *Logic for Computer Science* (1986), *Curves and Surfaces in Geometric Modeling* (1999), and *Geometric Methods and Applications* (2000). He serves on the editorial board of several professional journals, including *Theoretical Computer Science* since 1992 and the *Journal of Symbolic Computation* since 1988.

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