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Finding n-th Roots of a 2×2 Real Matrix Using De Moivre's Formula

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Introduction

- ★ A matrix B is said to be an n -th root of a matrix A if $B^n = A$.
- ★ There are many studies in the literature giving different methods of finding roots of a matrix. These methods mainly depend on the Shur Theorem, the Cayley Hamilton Theorem or the Newton Method.
- ★ Denman described an algorithm for computing roots of a real matrix with the real part of eigenvalues not zero [Denman 1981].
- ★ Björk and Hammarling developed a method for calculating the square root of a matrix based upon the Schur factorization method [Björk 1983].
- ★ Higham described a generalization of the Schur factorization method for the real square root of a n by n matrix [Higham 1987].
- ★ The n -th root of a matrix A may not exist. In this case, A is called **rootless matrix**. If a matrix A is nonsingular and diagonalizable then A always has a root. If an $n \times n$ matrix has at least $n - 1$ nonzero eigenvalues, then this matrix has a square root.
- ★ For roots of nilpotent matrices see [Yuttanan 2005].
- ★ **Some of the recent studies related to the finding of the roots of 2×2 matrices are as follows** [Andrescu 2014], [Choudry 2004], [Jadhav 2017], [Mackinnon 1989], [Northshield 2010], [Sadeghi 2011], [Sambasiva 2013], [Scott 1990], [Sullivan 1993], [Choudry 2004].

Some Known Methods n-th Roots of a 2×2 Matrix

Some of the basic methods to find n-th roots of a 2×2 matrix as follows :

1. **Basic Algebraic Method**
2. **Diagonalization**
3. **Cayley Hamilton Method**
4. **Schur Decompositon Method**
5. **Abel-Mobius Method**
6. **Newton Method**
7. **Using Dual, Hyperbolic and Complex Numbers**

The first sixt methods to find square roots of a 2×2 matrix are summarized in the Nortshield's paper [Northshield 2010].

★ First, we will give briefly about how these methods are used.

★ After, we will define a new numbers system called **Hybrid numbers**, and we will give a different method using these numbers. (De Moivre's formula).

8. **Using De Moivre's Formula for 2×2 Matrices**

Basic Algebraic Method

This method is based on the solution of four nonlinear equations. In this method, we assume that

$$B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

is a square root of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a, b, c, d \in \mathbb{R}$. In this case, we obtain the nonlinear system of equation. For example. fifth roots of A can be found solving the system of equation

$$\begin{cases} t^3yz + 2t^2xyz + 3tx^2yz + 2ty^2z^2 + x^5 + 4x^3yz + 3xy^2z^2 = a \\ y(t^4 + t^3x + t^2x^2 + 3t^2yz + tx^3 + 4txyz + x^4 + 3x^2yz + y^2z^2) = b \\ z(t^4 + t^3x + t^2x^2 + 3t^2yz + tx^3 + 4txyz + x^4 + 3x^2yz + y^2z^2) = c \\ t^5 + 4t^3yz + 3t^2xyz + 2tx^2yz + 3ty^2z^2 + x^3yz + 2xy^2z^2 = d \end{cases}.$$

But solving this equation is not always easy.

Special Cases (n-th Roots of Diagonal and Triangular Matrices)

The number of solutions depends on whether n is odd or even.

Fact

If $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and $a \neq d$, then we obtain

$$\sqrt[n]{A} = \pm \begin{bmatrix} \sqrt[n]{a} & 0 \\ 0 & \sqrt[n]{d} \end{bmatrix}.$$

Fact

If $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ and $a = d \neq 0$, then we obtain

$$\sqrt[n]{A} = \pm \begin{bmatrix} \sqrt[n]{a} & \frac{b}{n \sqrt[n]{a^{n-1}}} \\ 0 & \sqrt[n]{a} \end{bmatrix},$$

n-th Roots of a Triangular Matrix

The number of solutions depends on whether n is odd or even.

Theorem

If $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a real **triangular matrix**, we get

$$\sqrt[n]{A} = \begin{cases} \pm \begin{bmatrix} \sqrt[n]{a} & \frac{(\sqrt[n]{a} - \epsilon \sqrt[n]{d}) b}{(a - d)} \\ 0 & \epsilon \sqrt[n]{d} \end{bmatrix} & n \text{ is even and } a, d \in \mathbb{R}^+ \\ \begin{bmatrix} \sqrt[n]{a} & \frac{(\sqrt[n]{a} - \sqrt[n]{d}) b}{(a - d)} \\ 0 & \sqrt[n]{d} \end{bmatrix} & n \text{ is odd} \end{cases} \quad (1)$$

where $\epsilon = \pm 1$. In this case, we have four square roots.

Example

Sixth roots of $A = \begin{bmatrix} 64 & 665 \\ 0 & 729 \end{bmatrix}$ are

$$\sqrt[6]{A} = \pm \begin{bmatrix} \sqrt[6]{64} & \frac{(\sqrt[6]{64} - \sqrt[6]{729}) 665}{(64 - 729)} \\ 0 & \sqrt[6]{729} \end{bmatrix} = \pm \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

and

$$\sqrt[6]{A} = \pm \begin{bmatrix} \sqrt[6]{64} & \frac{(\sqrt[6]{64} + \sqrt[6]{729}) 665}{(64 - 729)} \\ 0 & -\sqrt[6]{729} \end{bmatrix} = \pm \begin{bmatrix} 2 & -5 \\ 0 & -3 \end{bmatrix}.$$

Diagonalization Method

The well-known procedure for computing the square roots of a 2×2 matrix A is to diagonalize A . This method can be used for a diagonalizable matrix.

★ We know that a square matrix A is called diagonalizable if it is similar to a diagonal matrix. To diagonalize a real matrix is we need to find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. P is the matrix constituting of eigenvectors and

$$P^{-1}AP = D$$

is the matrix of by writing eigenvalues to principal diagonal respectively.

★ In this method, even if eigenvalues are complex, A may has real square roots.

Theorem

Let A be a 2×2 matrix such that $P^{-1}AP = D$. We know that $\Delta_A = (\text{tr } A)^2 - 4 \det A$ is the discriminant of the characteristic equation of A . If $D^{1/n}$ is a n th root of D , then $\pm PD^{1/n}P^{-1}$ are the n th root of A where P and D are

$$P = \begin{bmatrix} a - d - \sqrt{\Delta_A} & a - d + \sqrt{\Delta_A} \\ 2c & 2c \end{bmatrix}$$

and

$$D = \frac{1}{2} \begin{bmatrix} \text{tr } A - \sqrt{\Delta_A} & 0 \\ 0 & \text{tr } A + \sqrt{\Delta_A} \end{bmatrix}.$$

Proof.

We have

$$\left(\pm PD^{1/n}P^{-1} \right)^n = \left(\pm PD^{1/n}P^{-1} \right) \cdots \left(\pm PD^{1/n}P^{-1} \right) = PDP^{-1} = A.$$

It means that, $PD^{1/n}P^{-1}$ is a square root of A . □

Example

Let's find third roots of the matrix $A = \begin{bmatrix} 25 & 26 \\ 39 & 38 \end{bmatrix}$, using diagonalization. We can find

$$P = \begin{bmatrix} -78 & 52 \\ 78 & 78 \end{bmatrix}, \quad D = \begin{bmatrix} 64 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -\frac{1}{130} & \frac{1}{195} \\ \frac{1}{130} & \frac{1}{130} \end{bmatrix}.$$

where $P^{-1}AP = D$. Thus, we obtain,

$$\begin{aligned} \sqrt[3]{A} &= PD^{1/3}P^{-1} \\ &= \begin{bmatrix} -\frac{1}{130} & \frac{1}{195} \\ \frac{1}{130} & \frac{1}{130} \end{bmatrix} \begin{bmatrix} \sqrt[3]{-1} & 0 \\ 0 & \sqrt[3]{64} \end{bmatrix} \begin{bmatrix} -78 & 52 \\ 78 & 78 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}. \end{aligned}$$

Cayley-Hamilton Method

The Cayley-Hamilton method **is one of the most useful method to find square roots of a matrix.** See [Sullivan 1993] for how the Cayley-Hamilton theorem may be used to calculate for all the square roots of 2×2 matrices. **Cayley-Hamilton Theorem for a 2×2 matrix can be stated as $A^2 - (\text{tr } A) A + (\det A) I = 0$. The following formula is one of the most useful method to find square roots of a matrix.**

Theorem

Let A be a 2×2 real matrix such that $(\text{tr } A)^2 \neq 4 \det A$, then

$$\sqrt{A} = \pm \frac{A + \epsilon (\det A)^{1/2} I}{\sqrt{\text{tr } A + 2\epsilon \sqrt{\det A}}} \quad (2)$$

where $\epsilon = \text{sign}(\text{tr } A)$.

How the Cayley - Hamilton Theorem can be used to calculating the n-th roots of a 2x2 matrix can be found in Choudry's article. [Choudry 2004] and It can be seen that this method is **not an easy and useful method** for finding n -th roots of 2×2 matrix.

Schur Decomposition Method

A matrix that is similar to a triangular matrix is referred to as triangularizable. This method is similar to diagonalization method.

Every 2×2 matrix A is similar to an upper triangular matrix. We know that a matrix U is unitary if $U^* = U^{-1}$ where U^* shows that conjugate transpose of U .

Schur Theorem states that if A is a square matrix, then A is unitarily similar to an upper triangular matrix whose diagonal entries are the eigenvalues of A .

That is, the equality $T = U^*AU$ ($A = UTU^*$) satisfies for some unitary matrix U and upper triangular matrix T . The Schur decomposition is not unique. However, the eigenvalues of A will always appear on the diagonal of T , since A is similar to T . **The theorem does not guarantee that U and T will be real matrices, even if we start with a real matrix A . In the case A is a square real matrix with real eigenvalues, then there is an orthogonal matrix Q and an upper triangular matrix T such that $A = QTQ^{-1}$.** That is, for each matrix A having real eigenvalues, there is an orthogonal matrix Q such that $Q^{-1}AQ$ is a upper triangular matrix. So, $(Q^{-1}AQ)_{21} = 0$.

Theorem

Let A be a 2×2 matrix and $U^*AU = T$ where T is an upper triangular matrix and U is a unitary matrix. If, $T^{1/n}$ is a square root of T , then $\pm UT^{1/n}U^{-1}$ are the square roots of A .

Proof.

Since U is unitary, we have $U^* = U^{-1}$. Therefore, we get

$$\left(\pm UT^{1/n}U^*\right)^n = UT^{1/n}U^* \dots UT^{1/n}U^* = UTU^* = A.$$

It means that, $\pm UT^{1/n}U^*$ are the square roots of A . Now, let's triangularize a matrix A , by using a rotation matrix Q . Then, from

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \lambda_1 & . \\ 0 & \lambda_2 \end{bmatrix},$$

and, we obtain, $c \cos^2 \theta - (a - d) \cos \theta \sin \theta - b \sin^2 \theta = 0$. Assume that $\sin \theta \neq 0$, then we have $cx^2 - (a - d)x - b = 0$ where $\cot \theta = x$.

Solving this equation, we obtain θ . Thus, we can find Q and T .



Example

Let's find 4-th roots of the matrix $A = \begin{bmatrix} 103 & 174 \\ 261 & 538 \end{bmatrix}$, by using Schur decomposition. We have $261x^2 - (103 - 538)x - 174 = 0 \Rightarrow x = -2$ or $x = 1/3$. If $x = -2$, then $\cot \theta = -2$, $\cos \theta = -2/\sqrt{5}$ and $\sin \theta = 1/\sqrt{5}$. Hence, we find

$$\frac{1}{5} \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 103 & 174 \\ 261 & 538 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 16 & -87 \\ 0 & 625 \end{bmatrix}.$$

Therefore, square roots of A are

$$\sqrt[4]{A} = \pm Q T^{1/4} Q^{-1} = \frac{\pm 1}{5} \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 16 & -87 \\ 0 & 625 \end{bmatrix}^{1/4} \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}.$$

By using (1), we find $\begin{bmatrix} 16 & -87 \\ 0 & 625 \end{bmatrix}^{1/4} = \pm \frac{1}{7} \begin{bmatrix} 14 & 2 - 5\epsilon \\ 0 & 35\epsilon \end{bmatrix}$. So,

$$\sqrt[4]{A} = \pm \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} \text{ or } \frac{\pm 1}{7} \begin{bmatrix} 17 & 6 \\ 9 & 32 \end{bmatrix}$$

Abel-Mobius Method

Every invertible complex matrix $A = [a_{ij}]_{2 \times 2}$ can associate the Möbius transformation $f(x)$. Remember that Möbius transformation is given by

$$\varphi_A(x) = \frac{a_{11}x + a_{12}}{a_{21}x + a_{22}},$$

where a, b, c, d are any complex numbers satisfying $ad - bc \neq 0$. Also, we have $\varphi_A \circ \varphi_B = \varphi_{AB}$ for invertible matrices A and B . Finding square roots of a 2×2 real matrix by using Abel-Mobius method can be found Nortshield's paper [Northshield 2010]. Let's consider the equation

$$cx^2 - (a - d)x - b = 0.$$

It can be written as $\frac{ax + b}{cx + d} = x$. Therefore, we have the following system of equation for $\lambda \in \mathbb{R}$,

$$\begin{cases} ax + b = \lambda x \\ cx + d = \lambda \end{cases} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

It means that, λ is eigenvalue and $\vec{u} = \begin{bmatrix} x \\ 1 \end{bmatrix}$ is eigenvector of A .

Theorem

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real matrix and $R(x) = cx^2 + (d - a)x - b$ is a polynomial. If $\varphi_A(x)$ is the Möbius transformation associated with A , then the function

$$F(x) = \int \frac{dx}{R(x)}$$

satisfies the Abel functional equation,

$$F(\varphi_A(x)) = F(x) + m.$$

for $m \in \mathbb{R}$. Then, we have

$$\varphi_{A^n}(x) = F^{-1}(F(x) + nm)$$

where $m = \frac{\ln k}{\lambda_2 - \lambda_1}$ and $k = \left| -\frac{2b+a\lambda_1-d\lambda_2}{2b+a\lambda_2-d\lambda_1} \right|$. Moreover, $\sqrt[n]{A}$ is

$$\lambda \left(F^{-1} \left(F(x) + \frac{m}{n} \right) \right)$$

for a real number λ [Northshield 2010].

Proof.

Let λ_1 and λ_2 be the roots of $R(x) = cx^2 + (d - a)x - b$. If we choose

$$P = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix},$$

we have the diagonal matrix $D = P^{-1}AP$. So,

$$\varphi_P = \frac{\lambda_1 x + 1}{\lambda_2 x + 1}, \quad \varphi_{P^{-1}} = \frac{1}{\lambda_2 - \lambda_1} \left(\frac{x - \lambda_2}{x - \lambda_1} \right) \text{ and } \varphi_D(x) = kx$$

where $k = -\frac{2b+a\lambda_1-d\lambda_2}{2b+a\lambda_2-d\lambda_1} \in \mathbb{R}$. Then,

$$\begin{aligned} \varphi_{P^{-1}} \varphi_A \varphi_P(x) &= \varphi_D(x) \Rightarrow \varphi_{P^{-1}}(\varphi_A(x)) = \varphi_D(\varphi_{P^{-1}}(x)) \\ &\Rightarrow \varphi_{P^{-1}}(\varphi_A(x)) = k\varphi_{P^{-1}}(x). \end{aligned}$$

Now, let's consider the function

$$F(x) = \int \frac{dx}{R(x)} = \frac{1}{\lambda_2 - \lambda_1} \ln \left| \frac{x - \lambda_2}{x - \lambda_1} \right|.$$



Proof.

Since, $\ln \varphi_{P^{-1}}(x) = \ln \left(\frac{x-\lambda_2}{x-\lambda_1} \right) + \ln \frac{1}{\lambda_2-\lambda_1}$, we have

$$F(x) = \frac{1}{\lambda_2-\lambda_1} (\ln \varphi_{P^{-1}}(x) - \ln(\lambda_2 - \lambda_1))$$

Thus,

$$\begin{aligned} F(\varphi_A(x)) &= \frac{1}{\lambda_2 - \lambda_1} (\ln \varphi_{P^{-1}}(\varphi_A(x)) - \ln(\lambda_2 - \lambda_1)) \\ &= \frac{1}{\lambda_2 - \lambda_1} (\ln k \varphi_{P^{-1}}(x) - \ln(\lambda_2 - \lambda_1)) \\ &= \frac{1}{\lambda_2 - \lambda_1} (\ln \varphi_{P^{-1}}(x) - \ln(\lambda_2 - \lambda_1)) \\ &= F(x) + m \end{aligned}$$

Also, using $\varphi_A \circ \varphi_B = \varphi_{AB}$, we get

$F(\varphi_{A^2}(x)) = m + F(\varphi_A(x)) = 2m + F(x)$. So, by induction we have $\varphi_{A^n}(x) = F^{-1}(F(x) + nm)$ where $m = \frac{\ln k}{\lambda_2 - \lambda_1}$. □

Finding n -th roots of a 2×2 matrix by Abel-Mobius Theorem is not a useful method.

Example

Let's find square roots of the matrix $A = \begin{bmatrix} 11 & -3 \\ 1 & 7 \end{bmatrix}$. Then,

$$R(x) = x^2 - 4x + 3 \Rightarrow F(x) = \int \frac{dx}{x^2 - 4x + 3} = \ln \frac{\sqrt{x-3}}{\sqrt{x-1}}.$$

Then, we get $F\left(\frac{11x-3}{x+7}\right) = F(x) + \ln \frac{2}{\sqrt{5}}$. That is, $m = \ln \frac{2}{\sqrt{5}}$. On the other hand, we have $F^{-1}(X) = \frac{e^{2x}-3}{e^{2x}-1}$. Thus, we find $\varphi_{A^n}(x)$ as

$$F^{-1}(F(x) + mn) = F^{-1}\left(\ln \frac{2^n \sqrt{x-3}}{5^{n/2} \sqrt{x-1}}\right) = \frac{(4^n - 3 \cdot 5^n)x + (3 \cdot 5^n - 3 \cdot 4^n)}{(4^n - 5^n)x + (5^n - 3 \cdot 4^n)}.$$

Then, for $n = 1/3$, we can find $\lambda = \frac{\sqrt{2}}{2}$. Thus, square roots of A are

$$\sqrt{A} = \pm \frac{\sqrt{2}}{2} \begin{bmatrix} 2 - 3\sqrt{5} & 3\sqrt{5} - 6 \\ 2 - \sqrt{5} & \sqrt{5} - 6 \end{bmatrix}.$$

Newton's Method

Another method to computing an n-th roots of a matrix A is to apply Newton's method to the quadratic matrix equation $f(X) = X^n - A = 0$. Newton's method for the matrix square root can be found in the Higham's paper [Higham 1986]. Let $f(x)$ a function with an initial value x_0 . Let's define the sequence $a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$. This sequence converges to a root of $f(x)$. For the matrix equality $f(X) = X^n - A = 0$, we have

$$X_{k+1} - A = X_k - A - (X_k^n - A) (nX_k^{n-1})^{-1} \Rightarrow X_{k+1} = \frac{n-1}{n} (X_k + AX_k^{1-n}).$$

Then, the following Theorem can be expressed [Lannazzo 2006].

Theorem

Let $X_{k+1} = \frac{(n-1)X_k + AX_k^{1-n}}{n}$ be a matrix recurrence relation where the matrices A and X_0 commute, then X_k converges to $\sqrt[n]{A}$ or $-\sqrt[n]{A}$, according to whether the initial value $\det X_0 > 0$ or $\det X_0 < 0$, respectively.

Example

Let's find square roots of the matrix $A = \begin{bmatrix} 1 & -18 \\ 18 & 19 \end{bmatrix}$. If we take initial matrix as $X_0 = I$ where A and X_0 commute. Then, from $X_{k+1} = \frac{1}{3} (2X_k + AX_k^{-2})$, we get

$$k=0 \Rightarrow X_1 = \frac{1}{3} \left(2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -18 \\ 18 & 19 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & -6 \\ 6 & 7 \end{bmatrix},$$

$$k=1 \Rightarrow X_2 = \frac{1}{3} \left(2 \begin{bmatrix} 1 & -6 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & -18 \\ 18 & 19 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ 6 & 7 \end{bmatrix}^{-2} \right) = \begin{bmatrix} \frac{230}{129} & -\frac{8}{143} \\ \frac{8}{143} & \frac{254}{129} \end{bmatrix}$$

$$k=2 \Rightarrow \dots\dots\dots X_3 = \begin{bmatrix} 0.91586 & -2.3245 \\ 2.3245 & 3.2403 \end{bmatrix}$$

...

$$k=7 \Rightarrow \dots\dots\dots X_7 = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}.$$

If we would take the initial matrix as $X_0 = -I$, we would find $\begin{bmatrix} -2 & 1 \\ -1 & -3 \end{bmatrix}$.

Finding n-th Roots by using De Moivre's Formula for Dual, Hyperbolic and Complex Numbers

Complex, dual and hyperbolic numbers can be represented by 2×2 matrices. Let's define the matrix sets,

$$M_i = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}; \text{ (FIELD)}$$

$$M_{\varepsilon} = \left\{ \begin{bmatrix} a+b & -b \\ b & a-b \end{bmatrix} : a, b \in \mathbb{R} \right\}; \text{ (RING)}$$

$$M_{\mathbf{h}} = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\} \quad (\text{RING}).$$

If we define a map $\varphi_1 : \mathbb{C} \rightarrow M_i$ by

$$\varphi_1(z) = \varphi_1(\rho e^{i\theta}) = \begin{bmatrix} \rho \cos \theta & -\rho \sin \theta \\ \rho \sin \theta & \rho \cos \theta \end{bmatrix},$$

we have the equalities,

$$\varphi_1(z + w) = \varphi_1(z) + \varphi_1(w) \text{ and } \varphi_1(z \times w) = \varphi_1(z) \cdot \varphi_1(w).$$

That is, φ_1 is a field isomorphism and $M_{\mathbb{I}}$ and \mathbb{C} are isomorphic fields. That is, a complex number $z = a + ib = \rho (\cos \theta + i \sin \theta) = \rho e^{i\theta}$ corresponds to a matrix

$$z = a + \mathbf{i}b \leftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \rho \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

where $\rho = \sqrt{a^2 + b^2}$ and $\theta = \arctan \frac{b}{a}$. **Also, each unit complex number corresponds to a 2D rotation matrix in the Euclidean space.**

Similarly, the maps $\varphi_2 : \mathbb{C} \rightarrow M_{\varepsilon}$ and $\varphi_3 : \mathbb{C} \rightarrow M_{\mathbf{h}}$ defined as

$$\begin{aligned} \varphi_2 (z) &= \varphi_2 \left(\rho e^{\varepsilon \theta} \right) = \rho \begin{bmatrix} \theta + 1 & -\theta \\ \theta & 1 - \theta \end{bmatrix}, \\ \varphi_3 (z) &= \varphi_3 \left(\rho e^{h\theta} \right) = \rho \begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix} \end{aligned}$$

are ring isomorphism.

Therefore, a dual number $z = a + \varepsilon b = \rho (1 + \theta \varepsilon) = ae^{\varepsilon \theta}$ can be represented by a matrix as

$$z = a + \varepsilon b \leftrightarrow \begin{bmatrix} a+b & -b \\ b & a-b \end{bmatrix} = \rho \begin{bmatrix} \theta+1 & -\theta \\ \theta & 1-\theta \end{bmatrix}$$

where $\rho = |a|$ and $\theta = \frac{b}{|a|}$. **Each unit dual number corresponds to a 2D rotation matrix in the Galilean space.**

Finally, a hyperbolic number

$z = a + \mathbf{h}b = k\rho (\cosh \theta + \mathbf{h} \sinh \theta) = k\rho e^{\mathbf{h}\theta}$ corresponds to a matrix

$$z = k(a + \mathbf{h}b) \leftrightarrow k \begin{bmatrix} a & b \\ b & a \end{bmatrix} = k\rho \begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix}$$

where $k \in \{1, -1, \mathbf{h}, -\mathbf{h}\}$, $\rho = \sqrt{a^2 - b^2}$ and $\theta = \ln \frac{|a+b|}{\sqrt{|a^2 - b^2|}}$.

Lorentzian rotations can be interpreted by hyperbolic numbers.

Example

Let's find 3-rd roots of the matrices

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}.$$

These matrices correspond to the numbers

$$z_A = 3 - \mathbf{i} = \sqrt{10} (\cos \theta_A + \mathbf{i} \sin \theta_A), \quad \theta_A = \arctan(-1/3)$$

$$z_B = 3 + 2\varepsilon = 3(1 + \theta_B \varepsilon), \quad \theta_B = 2/3,$$

$$z_C = 4 + 3\mathbf{h} = \sqrt{7} (\cosh \theta_C + \mathbf{h} \sinh \theta_C), \quad \theta_C = \ln \sqrt{7}.$$

Third roots of these numbers can be found easily.

$$\sqrt[3]{z_A} = 10^{1/6} \left(\cos \frac{\theta_A}{3} + \mathbf{i} \sin \frac{\theta_A}{3} \right),$$

$$\sqrt[3]{z_B} = \sqrt[3]{3} \left(1 + \frac{\theta_B}{3} \varepsilon \right) = \sqrt[3]{3} \left(1 + \frac{2}{9} \varepsilon \right)$$

$$\sqrt[3]{z_C} = 7^{1/6} \left(\cosh \frac{\theta_C}{3} + \mathbf{h} \sinh \frac{\theta_C}{3} \right).$$

Example

Then, third roots of A , B and C are

$$\sqrt[3]{A} = 10^{1/6} \begin{bmatrix} \cos \frac{\theta_A}{3} & -\sin \frac{\theta_A}{3} \\ \sin \frac{\theta_A}{3} & \cos \frac{\theta_A}{3} \end{bmatrix} \simeq \begin{bmatrix} 1.4594 & 0.15712 \\ -0.15712 & 1.4594 \end{bmatrix},$$

$$\sqrt[3]{B} = 3^{1/3} \begin{bmatrix} 11/9 & -2/9 \\ 2/9 & 7/9 \end{bmatrix}^3 = \begin{bmatrix} \frac{11}{9} \sqrt[3]{3} & -\frac{2}{9} \sqrt[3]{3} \\ \frac{2}{9} \sqrt[3]{3} & \frac{7}{9} \sqrt[3]{3} \end{bmatrix},$$

$$\sqrt[3]{C} = 7^{1/6} \begin{bmatrix} \cosh \frac{\theta_A}{3} & \sinh \frac{\theta_A}{3} \\ \sinh \frac{\theta_A}{3} & \cosh \frac{\theta_A}{3} \end{bmatrix} \simeq \begin{bmatrix} 1.4565 & 0.45647 \\ 0.45647 & 1.4565 \end{bmatrix}.$$

The aim of this paper, this method to all 2x2 matrices.

As seen as, the above method can be used for only the matrices in the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{via complex numbers}$$

$$\begin{bmatrix} a+b & -b \\ b & a-b \end{bmatrix} \quad \text{via dual numbers}$$

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad \text{via hyperbolic numbers}$$

For this aim, we need to define

a new set of numbers (HYBRID NUMBERS)

which are isomorphic to the set of 2x2 matrices. Thus, we give a new method which can be used for all 2x2 matrices.

Hybrid Numbers

Hybrid numbers are a new generalization of complex, hyperbolic and dual numbers. It is a noncommutative ring.

Definition

The set of hybrid numbers \mathbb{K} , defined as

$$\mathbb{K} = \left\{ a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h} : a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \boldsymbol{\varepsilon}^2 = 0, \mathbf{h}^2 = 1, \mathbf{i}\mathbf{h} = -\mathbf{h}\mathbf{i} = \boldsymbol{\varepsilon} + \mathbf{i} \right\}.$$

Multiplication table of hybrid numbers as follows.

\cdot	\mathbf{i}	$\boldsymbol{\varepsilon}$	\mathbf{h}
\mathbf{i}	-1	$\mathbf{1} - \mathbf{h}$	$\boldsymbol{\varepsilon} + \mathbf{i}$
$\boldsymbol{\varepsilon}$	$\mathbf{h} + \mathbf{1}$	0	$-\boldsymbol{\varepsilon}$
\mathbf{h}	$-\boldsymbol{\varepsilon} - \mathbf{i}$	$\boldsymbol{\varepsilon}$	1

Multiplication operation in the hybrid numbers is associative and not commutative. The conjugate of a hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}$, denoted by $\overline{\mathbf{Z}}$, is defined as $\overline{\mathbf{Z}} = S(\mathbf{Z}) - V(\mathbf{Z}) = a - b\mathbf{i} - c\boldsymbol{\varepsilon} - d\mathbf{h}$ as in quaternions.

Character of a Hybrid Number

Definition

(Character of a Hybrid Number) Let $\mathbf{Z} = a + bi + c\varepsilon + d\mathbf{h}$ be a hybrid number. The real number

$$\mathcal{C}(\mathbf{Z}) = \mathbf{Z}\bar{\mathbf{Z}} = \bar{\mathbf{Z}}\mathbf{Z} = a^2 + (b - c)^2 - c^2 - d^2 \quad (3)$$

is called the characteristic number of \mathbf{Z} . We say that a hybrid number;

$$\begin{cases} \mathbf{Z} \text{ is spacelike} & \text{if } \mathcal{C}(\mathbf{Z}) < 0; \\ \mathbf{Z} \text{ is timelike} & \text{if } \mathcal{C}(\mathbf{Z}) > 0; \\ \mathbf{Z} \text{ is lightlike} & \text{if } \mathcal{C}(\mathbf{Z}) = 0. \end{cases}$$

These are called the **characters of the hybrid numbers**.

Each complex and dual number different from zero is a spacelike number. But, a hyperbolic number can be spacelike, lightlike or timelike.

Type of a Hybrid Number

Definition

(Type of a Hybrid Number) Let $\mathbf{Z} = a + bi + c\varepsilon + d\mathbf{h}$ be a hybrid number. The real number

$$\Delta(\mathbf{Z}) = -(b - c)^2 + c^2 + d^2$$

is called the type number of \mathbf{Z} . We say that a hybrid number;

$$\begin{cases} \mathbf{Z} \text{ is elliptic} & \text{if } \Delta(\mathbf{Z}) < 0; \\ \mathbf{Z} \text{ is hyperbolic} & \text{if } \Delta(\mathbf{Z}) > 0; \\ \mathbf{Z} \text{ is parabolic} & \text{if } \Delta(\mathbf{Z}) = 0. \end{cases}$$

These are called the **types of the hybrid numbers**. Also, the vector $\mathcal{E}_{\mathbf{Z}} = (b - c, c, d)$ is called hybridian vector of \mathbf{Z} .

Each complex number is elliptic, each hyperbolic number is hyperbolic, each dual number is parabolic.

Character and Type of A Hybrid Number

	THE CHARACTER		
	Spacelike	Lightlike	Timelike
T Y P E	Hyperbolic	Hyperbolic	Hyperbolic
		Parabolic	Parabolic
			Elliptic

Norms of Hybrid Numbers

Definition

(Norms of Hybrid Numbers) Let $\mathbf{Z} = a + b\mathbf{i} + c\mathbf{\varepsilon} + d\mathbf{h}$ be a hybrid number. The real number

$$\|\mathbf{Z}\| = \sqrt{|\mathcal{C}(\mathbf{Z})|} = \sqrt{|a^2 + (b - c)^2 - c^2 - d^2|}$$

is called **norm of \mathbf{Z}** . Besides, the real number

$$\mathcal{N}(\mathbf{Z}) = \sqrt{|\Delta|} = \sqrt{|-(b - c)^2 + c^2 + d^2|}$$

will be called the norm of the hybrid vector of \mathbf{Z} .

Fact

This norm definition is a generalized norm definition that overlaps with the definitions of norms in complex, hyperbolic and dual numbers.

Actually,

1. If Z is a complex number ($c = d = 0$), then

$$\|Z\| = \sqrt{|Z\bar{Z}|} = \sqrt{a^2 + b^2}$$

2. If Z is a hyperbolic number ($b = c = 0$), then

$$\|Z\| = \sqrt{|Z\bar{Z}|} = \sqrt{|a^2 - d^2|}$$

3. If Z is a dual number ($b = d = 0$), then $\|Z\| = \sqrt{a^2} = |a|$.

The inverse of the number of the hybrid number $Z = a + bi + c\varepsilon + d\mathbf{h}$, $\|Z\| \neq 0$ is defined as $Z^{-1} = \frac{\bar{Z}}{C(Z)}$. Accordingly, lightlike hybrid numbers have no inverse.

Classification of 2×2 Matrices Using Hybrid Numbers

Just as we classify a hybrid numbers, we can classify a 2×2 matrix. Any 2×2 matrix is classified as spacelike, timelike or lightlike and sorted as hyperbolic, elliptic, or parabolic, taking into account the isomorphisms and relations between hybrid numbers and 2×2 matrices.

Theorem

(Özdemir 2018) The ring of hybrid numbers \mathbb{K} is isomorphic to the ring of real 2×2 matrices $\mathbb{M}_{2 \times 2}$ with the map $\varphi : \mathbb{K} \rightarrow \mathbb{M}_{2 \times 2}$ where

$$\varphi(a + b\mathbf{i} + c\varepsilon + d\mathbf{h}) = \begin{bmatrix} a + c & b - c + d \\ c - b + d & a - c \end{bmatrix} \quad (4)$$

for $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \in \mathbb{K}$.

The matrix $\varphi(\mathbf{Z}) \in \mathbb{M}_{2 \times 2}(\mathbb{R})$ is called the hybrid matrix corresponding to the hybrid number \mathbf{Z} . Also, we have

$$\varphi^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left(\frac{a+d}{2} \right) + \left(\frac{a+b-c-d}{2} \right) \mathbf{i} + \left(\frac{a-d}{2} \right) \varepsilon + \left(\frac{b+c}{2} \right) \mathbf{h}. \quad (5)$$

Theorem

(Özdemir 2018) Let A be a 2×2 real matrix corresponding to the hybrid number \mathbf{Z} , then there are the following relations.

i. $\rho = \|\mathbf{Z}\| = \sqrt{|\det A|}$,

ii. $\Delta(\mathbf{Z}) = \left(\frac{\text{tr} A}{2}\right)^2 - \det A$,

iii. $P(\lambda) = \lambda^2 - (\text{tr} A)\lambda + \det A$, $\Delta_A = (\text{tr} A)^2 - 4 \det A = 4\Delta(\mathbf{Z})$ is discriminant of the characteristic polynomial of A .

iv. \mathbf{Z}^{-1} exists if and only if $\det(A) \neq 0$.

As a conclusion of this Theorem, we had shown that the classification of hybrid numbers depends entirely on the determinant and the trace of the 2×2 corresponding matrices.

Definition

Let A be a 2×2 real matrix. Then,

$$\begin{cases} \mathbf{Z} \text{ is spacelike} & \text{if } \det A < 0; \\ \mathbf{Z} \text{ is timelike} & \text{if } \det A > 0; \\ \mathbf{Z} \text{ is lightlike} & \text{if } \det A = 0. \end{cases} \quad (6)$$

Definition

Let A be a 2×2 real matrix where λ_1 and λ_2 are the eigenvalues of A . Then,

$$\begin{cases} A \text{ is called elliptic} & \text{if } \lambda_1, \lambda_2 \text{ are complex numbers;} \\ A \text{ is called hyperbolic} & \text{if } \lambda_1, \lambda_2 \text{ are real numbers;} \\ A \text{ is called parabolic} & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Moreover, they can be defined as

$$\begin{cases} A \text{ is elliptic} & \text{if } \Delta_A < 0; \\ A \text{ is hyperbolic} & \text{if } \Delta_A > 0; \\ A \text{ is parabolic} & \text{if } \Delta_A = 0. \end{cases} \quad (7)$$

where $\Delta_A = (\text{tr} A)^2 - 4 \det A$.

Corollary

Norm of a 2×2 real matrix, defined as follows :

$$\rho = \|A\| = \sqrt{|\det A|}, \text{ when } A \text{ is spacelike or timelike matrix,}$$

$$\rho = \|A\| = \operatorname{tr} A, \text{ when } A \text{ is lightlike matrix.}$$

So, classification 2×2 matrices can be given with the following table.

A	$\det A > 0$	$\det A = 0$	$\det A < 0$
$(\operatorname{tr} A)^2 < 4\det A$	Timelike Elliptic	\emptyset	\emptyset
$(\operatorname{tr} A)^2 = 4\det A$	Timelike Parabolic	Null Parabolic	\emptyset
$(\operatorname{tr} A)^2 > 4\det A$	Timelike Hyperbolic	Null Hyperbolic	Spacelike Hyperbolic

Polar Representations of 2×2 Matrices

Definition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a real matrix. Argument of A defined as follows :

- i. If A is elliptic such that $\text{tr}A < 0$, then $\arg A = \theta = \pi - \arctan \frac{\sqrt{-\Delta_A}}{|\text{tr}A|}$;
- ii. If A is elliptic such that $\text{tr}A > 0$, then $\arg A = \theta = \arctan \frac{\sqrt{-\Delta_A}}{|\text{tr}A|}$;
- iii. If A is hyperbolic, then $\arg A = \theta = \ln \left| \frac{\text{tr}A + \sqrt{\Delta_A}}{2\rho} \right|$,
- iv. If A is parabolic, then $\arg A = \theta = \frac{a-d}{|a+d|}$

where $\rho = \sqrt{|\det A|}$, $\Delta_A = (\text{tr}A)^2 - 4 \det A$.

After that, throughout the paper, we will use the above formulas for the argument of elliptic, hyperbolic and parabolic matrices.

Polar Representation of an Elliptic 2x2 Matrix

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an elliptic matrix, then A can be written in the polar form as

$$A = \rho \begin{bmatrix} \cos \theta + \left(\frac{a-d}{\sqrt{-\Delta_A}} \right) \sin \theta & \frac{2b}{\sqrt{-\Delta_A}} \sin \theta \\ \frac{2c}{\sqrt{-\Delta_A}} \sin \theta & \cos \theta - \left(\frac{a-d}{\sqrt{-\Delta_A}} \right) \sin \theta \end{bmatrix} \quad (8)$$

where $\Delta_A = (\text{tr} A)^2 - 4 \det A$.

Proof.

If A is elliptic, then we have $\Delta_A < 0$ and $\det A = ad - bc > 0$. The hybrid number $\varphi^{-1}(A)$ corresponding to the matrix A is

$$\mathbf{W} = \left(\frac{a+d}{2} \right) + \left(\frac{a+b-c-d}{2} \right) \mathbf{i} + \left(\frac{a-d}{2} \right) \varepsilon + \left(\frac{b+c}{2} \right) \mathbf{h}.$$

Therefore, according to Theorem ??, we can write

$\mathbf{W} = \rho (\cos \theta + \mathbf{U} \sin \theta)$, where

$$\mathbf{U} = \frac{1}{\sqrt{-\Delta_A}} ((a+b-c-d) \mathbf{i} + (a-d) \varepsilon + (b+c) \mathbf{h}).$$

since $\rho = \sqrt{\det A}$ and $\mathcal{N}(\mathbf{W}) = \sqrt{-4\Delta_A}$. Thus, using the isomorphism (4), we obtain (8). □

Example

Let's find the polar form of the elliptical matrix

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}.$$

Since $\Delta_A = -8$, $\det A = 3$, $\rho = \sqrt{3}$, we obtain the polar form

$$A = \sqrt{3} \begin{bmatrix} \cos \theta + \frac{\sqrt{2}}{2} \sin \theta & \frac{3\sqrt{2}}{2} \sin \theta \\ \frac{-\sqrt{2}}{2} \sin \theta & \cos \theta - \frac{\sqrt{2}}{2} \sin \theta \end{bmatrix}$$

where $\theta = \arctan \sqrt{2}$.

Polar Representation of a Hyperbolic 2x2 Matrix

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a hyperbolic real matrix, then A can be written in the polar form as follows :

i. if A is timelike, then

$$A = \epsilon \rho \begin{bmatrix} \cosh \theta + \frac{(a-d)}{\sqrt{\Delta_A}} \sinh \theta & \frac{2b}{\sqrt{\Delta_A}} \sinh \theta \\ \frac{2c}{\sqrt{\Delta_A}} \sinh \theta & \cosh \theta - \frac{(a-d)}{\sqrt{\Delta_A}} \sinh \theta \end{bmatrix}, \quad (9)$$

ii. if A is spacelike, then

$$A = \rho \begin{bmatrix} \sinh \theta + \frac{(a-d)}{\sqrt{\Delta_A}} \cosh \theta & \frac{2b}{\sqrt{\Delta_A}} \cosh \theta \\ \frac{2c}{\sqrt{\Delta_A}} \cosh \theta & \sinh \theta - \left(\frac{a-d}{\sqrt{\Delta_A}} \right) \cosh \theta \end{bmatrix}. \quad (10)$$

where $\epsilon = \text{sign}(\text{tr}A)$.

Proof.

If A is hyperbolic, then we have $\Delta > 0$. Let the hybrid number $\varphi^{-1}(A)$ corresponding to A be \mathbf{W} . The matrix A can be spacelike, timelike or lightlike according to sign of $\det A$. So, from Theorem (??), we can write

$$\mathbf{W} = \begin{cases} \pm \rho (\cosh \theta + \mathbf{U} \sinh \theta), & \text{when } \mathbf{W} \text{ is timelike;} \\ \rho (\sinh \theta + \mathbf{U} \cosh \theta), & \text{when } \mathbf{W} \text{ is spacelike;} \end{cases}$$

where

$$\mathbf{V} = \frac{1}{\sqrt{\Delta_A}} ((a + b - c - d)\mathbf{i} + (a - d)\mathbf{e} + (b + c)\mathbf{h}).$$

Scalar parts of these hybrid numbers depend on the

$$S(\varphi^{-1}(A)) = \frac{a + d}{2} = \frac{\text{tr} A}{2}.$$

On the other hand, scalar part of \mathbf{W} is $S(\mathbf{W}) = \pm \rho \cosh \theta$. So, if $\text{tr} A > 0$, namely $\epsilon = \text{sign}(\text{tr} A) = 1$, then we have to write $\rho \cosh \theta$, on the other case, we write $-\rho \cosh \theta$. Thus, using the isomorphism (4), we obtain the equalities (9) and (10). □

Example

Let's find polar form of the timelike and spacelike hyperbolic matrices

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

A and B are timelike and spacelike matrices, respectively. Since $\epsilon_A = -1$, $\epsilon_B = 1$, $\rho_A = \sqrt{5}$, $\rho_B = 1$ and $\Delta_A = 16$ and, $\Delta_B = 20$, we find

$$A = -\sqrt{5} \begin{bmatrix} \cosh \theta_A + \frac{1}{2} \sinh \theta_A & \frac{3}{2} \sinh \theta_A \\ \frac{1}{2} \sinh \theta_A & \cosh \theta_A - \frac{1}{2} \sinh \theta_A \end{bmatrix}$$

$$B = \begin{bmatrix} \sinh \theta_B + \frac{\sqrt{5}}{5} \cosh \theta_B & \frac{2\sqrt{5}}{5} \cosh \theta_B \\ \frac{2\sqrt{5}}{5} \cosh \theta_B & \sinh \theta_B - \frac{\sqrt{5}}{5} \cosh \theta_B \end{bmatrix}$$

where $\theta_A = \ln \sqrt{5}/5$ and $\theta_B = \ln(\sqrt{5} + 2)$, respectively.

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a lightlike hyperbolic matrix, then A can be written in the polar form as

$$A = \text{tr}A \begin{bmatrix} \frac{a}{\text{tr}A} & \frac{b}{\text{tr}A} \\ \frac{c}{\text{tr}A} & \frac{d}{\text{tr}A} \end{bmatrix}. \quad (11)$$

Proof.

If the matrix A is lightlike, hyperbolic, then $\det A = 0$ and $(\text{tr}A)^2 > 4 \det A = 0$. So, $\text{tr}A \neq 0$ and the polar form of the hybrid number $\varphi^{-1}(A)$ corresponding to A can be written as

$$\mathbf{W} = \text{tr}A \left(\frac{1}{2} + \left(\frac{a+b-c-d}{2\text{tr}A} \right) \mathbf{i} + \left(\frac{a-d}{2\text{tr}A} \right) \varepsilon + \left(\frac{b+c}{2\text{tr}A} \right) \mathbf{h} \right).$$

Therefore, using the isomorphism (5) we get the polar form of A as (11). □

Example

The polar representation of the lightlike hyperbolic matrix

$$C = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$$

is

$$C = 8 \begin{bmatrix} 3/4 & 3/8 \\ 1/2 & 1/4 \end{bmatrix}.$$

Polar Representations of a Parabolic 2×2 Matrix

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a timelike parabolic matrix such that $a \neq d$, then A can be written in the polar form as

$$A = \frac{\text{tr}A}{2} \begin{bmatrix} 1 + \epsilon\theta & \frac{\epsilon 2b\theta}{a-d} \\ \frac{2\epsilon c\theta}{a-d} & 1 - \epsilon\theta \end{bmatrix} \quad (12)$$

where $\theta = \frac{a-d}{|a+d|}$ and $\epsilon = \text{sign}(\text{tr}A)$.

Proof.

The hybrid number corresponding to A is (5). If A is parabolic, then we have $(\text{tr}A)^2 = 4 \det A$. Therefore, we get

$$\|\mathbf{W}\| = \sqrt{|\det A|} = \frac{\epsilon \text{tr}A}{2}$$

where $\epsilon = \text{sign}(\text{tr}A)$. Thus, according to the Theorem (??), the polar form of the \mathbf{W} is

$$\mathbf{W} = \frac{\epsilon \text{tr}A}{2} \left(\epsilon + \left(\frac{a+b-c-d}{\epsilon \text{tr}A} \right) \mathbf{i} + \left(\frac{a-d}{\epsilon \text{tr}A} \right) \epsilon + \left(\frac{b+c}{\epsilon \text{tr}A} \right) \mathbf{h} \right).$$

Since the argument of \mathbf{W} is

$$\theta = \frac{(a-d)/2}{(\epsilon \text{tr}A)/2} = \frac{a-d}{\epsilon \text{tr}A}.$$

we have

$$\mathbf{W} = \frac{\epsilon \text{tr}A}{2} \left(\epsilon + \left(1 + \frac{(b-c)\theta}{a-d} \right) \mathbf{i} + \theta \epsilon + \left(\frac{(b+c)\theta}{a-d} \right) \mathbf{h} \right)$$

Thus, using (4) and $\text{tr}A = a + d = \frac{a-d}{\theta \epsilon}$, we obtain, (12).



Example

The polar form of the parabolic matrix

$$A = \begin{bmatrix} 5 & 9 \\ -1 & -1 \end{bmatrix}$$

is

$$A = \frac{4}{2} \begin{bmatrix} 1 + \theta & 3\theta \\ -\theta/3 & 1 - \theta \end{bmatrix}$$

where $\theta = \frac{a - d}{|a + d|} = \frac{3}{2}$.

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a lightlike parabolic matrix, then $\det A = 0$ and $\text{tr} A = 0$. So, A can be written as

$$\begin{bmatrix} a & -\frac{a^2}{c} \\ c & -a \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix},$$

depending on whether $c \neq 0$ or $c = 0$, respectively. A parabolic lightlike matrix is a nilpotent matrix. That is, $A^n = 0$, for all $n \in \mathbb{N}$.

Proof.

If A is a parabolic null matrix, then $\det A = 0$ and $\text{tr} A = 0$. It means that $a + d = 0$ and $ad - bc = 0$. So, $d = -a$, $bc = -a^2$. If $c = 0$, then $a = d = 0$. In the case $c \neq 0$, we obtain $b = -a^2/c$. \square

Theorem

If the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a timelike parabolic matrix with $a = d$.
So, A can be written as

$$A = a \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix}, \quad A = a \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad A = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (13)$$

whether if $b = 0$, $c = 0$ or $b = c = 0$, respectively.

Proof.

If A is parabolic, then $(\operatorname{tr} A)^2 = 4 \det A$. So, in the case $a = d$, we find that $a^2 = a^2 - bc$ and $bc = 0$.

De Moivre's formula for Elliptic Matrices

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an elliptic real matrix whose polar representation is

$$A = \rho \begin{bmatrix} \cos \theta + \left(\frac{a-d}{\sqrt{-\Delta_A}} \right) \sin \theta & \frac{2b}{\sqrt{-\Delta_A}} \sin \theta \\ \frac{2c}{\sqrt{-\Delta_A}} \sin \theta & \cos \theta - \left(\frac{a-d}{\sqrt{-\Delta_A}} \right) \sin \theta \end{bmatrix},$$

then A^n has the form

$$A^n = \rho^n \begin{bmatrix} \cos(n\theta) + \left(\frac{a-d}{\sqrt{-\Delta_A}} \right) \sin(n\theta) & \frac{2b}{\sqrt{-\Delta_A}} \sin(n\theta) \\ \frac{2c}{\sqrt{-\Delta_A}} \sin(n\theta) & \cos(n\theta) - \left(\frac{a-d}{\sqrt{-\Delta_A}} \right) \sin(n\theta) \end{bmatrix} \quad (14)$$

for $n \in \mathbb{Z}$.

Proof.

It can be proved by induction. It is true for $n = 1$. Assume that (14) holds for $n = k$. Then, using $\Delta_A = 4\Delta = a^2 - 2ad + d^2 + 4bc$, we obtain A^{k+1} as

$$\rho^{k+1} \begin{bmatrix} \cos(k+1)\theta + \frac{a-d}{\sqrt{-\Delta_A}} \sin(k+1)\theta & \frac{2b}{\sqrt{-\Delta_A}} \sin(k+1)\theta \\ \frac{2c}{\sqrt{-\Delta_A}} \sin(k+1)\theta & \cos(k+1)\theta - \frac{a-d}{\sqrt{-\Delta_A}} \sin(k+1)\theta \end{bmatrix}.$$



Proof.

Moreover, since

$$A^{-1} = \rho^{-1} \begin{bmatrix} \cos \theta - \left(\frac{a-d}{\sqrt{-\Delta_A}} \right) \sin \theta & \frac{-2b}{\sqrt{-\Delta_A}} \sin \theta \\ \frac{-2c}{\sqrt{-\Delta_A}} \sin \theta & \cos \theta + \left(\frac{a-d}{\sqrt{-\Delta_A}} \right) \sin \theta \end{bmatrix},$$

we have

$$A^{-n} = \rho^{-n} \begin{bmatrix} \cos(n\theta) - \left(\frac{a-d}{\sqrt{-\Delta_A}} \right) \sin(n\theta) & \frac{-2b}{\sqrt{-\Delta_A}} \sin(n\theta) \\ \frac{-2c}{\sqrt{-\Delta_A}} \sin(n\theta) & \cos(n\theta) + \left(\frac{a-d}{\sqrt{-\Delta_A}} \right) \sin(n\theta) \end{bmatrix}.$$

So, the formula holds for all integers. □

Corollary

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an elliptic real matrix, then A^n is also an elliptic matrix.

Proof.

A^n is elliptic if and only if $(\operatorname{tr} A^n)^2 < 4 \det A^n$. So, using (14), we obtain

$$\operatorname{tr} A^n = 2\rho^n \cos^2 n\theta \text{ and } \det A^n = \rho^{2n}.$$

Therefore, we have

$$\operatorname{tr} A^n = 4 \left(\cos^2 n\theta \right) \det A^n \text{ and } (\operatorname{tr} A^n)^2 < 4 \det A^n,$$

since $0 < \cos \theta \leq 1$.



Example

Let's calculate A^{-n} for the elliptic matrix

$$A = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}.$$

Using the above theorem we find that

$$A^{-n} = \rho^{-n} \begin{bmatrix} \cos(n\theta) - \sin(n\theta) & -2 \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) + \sin(n\theta) \end{bmatrix}$$

where, $\rho = \sqrt{5}$ and $\theta = \arctan 2$.

De Moivre's formula for Hyperbolic Matrices

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a timelike hyperbolic real matrix whose polar representation is (9), then

$$A^n = \epsilon^n \rho^n \begin{bmatrix} \cosh(n\theta) + \frac{(a-d)}{\sqrt{\Delta_A}} \sinh(n\theta) & \frac{2b}{\sqrt{\Delta_A}} \sinh(n\theta) \\ \frac{2c}{\sqrt{\Delta_A}} \sinh(n\theta) & \cosh(n\theta) - \frac{(a-d)}{\sqrt{\Delta_A}} \sinh(n\theta) \end{bmatrix} \quad (15)$$

for $n \in \mathbb{Z}$, where $\epsilon = \text{sign}(\text{tr}A)$.

Proof.

It can be proved by induction similar to proof of Theorem 5.1 and using the equality $\Delta_A = 4\Delta = a^2 - 2ad + d^2 + 4bc$. □

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a spacelike hyperbolic real matrix whose polar representation is (10).

i. If n is an even integer, then A^n is a timelike matrix and

$$A^n = \epsilon^n \rho^n \begin{bmatrix} \cosh(n\theta) + \frac{a-d}{\sqrt{\Delta_A}} \sinh(n\theta) & \frac{2b}{\sqrt{\Delta_A}} \sinh(n\theta) \\ \frac{2c}{\sqrt{\Delta_A}} \sinh(n\theta) & \cosh(n\theta) - \frac{a-d}{\sqrt{\Delta_A}} \sinh(n\theta) \end{bmatrix} \quad (16)$$

ii. If n is an odd integer, then A^n is a spacelike matrix and

$$A^n = \epsilon^n \rho^n \begin{bmatrix} \sinh(n\theta) + \frac{a-d}{\sqrt{\Delta_A}} \cosh(n\theta) & \frac{2b}{\sqrt{\Delta_A}} \cosh(n\theta) \\ \frac{2c}{\sqrt{\Delta_A}} \cosh(n\theta) & \sinh(n\theta) - \frac{a-d}{\sqrt{\Delta_A}} \cosh(n\theta) \end{bmatrix} \quad (17)$$

where $\epsilon = \text{sign}(\text{tr} A)$, $\rho = \sqrt{|\det A|}$, $\theta = \ln \left| \frac{\text{tr} A + \sqrt{\Delta_A}}{2\sqrt{|\det A|}} \right|$ and

$$\Delta_A = (\text{tr} A)^2 - 4 \det A.$$

Proof.

Hybrid product of two spacelike hybrid numbers is a timelike hybrid number. So, if A is a spacelike matrix and n is an even number, A^n will be a timelike matrix. Also, if n is an odd number, then A^n will be a spacelike matrix. Let polar form of the spacelike hyperbolic matrix A be

$$\epsilon\rho \begin{bmatrix} \sinh \theta + \frac{a-d}{\sqrt{\Delta_A}} \cosh \theta & \frac{2b}{\sqrt{\Delta_A}} \cosh \theta \\ \frac{2c}{\sqrt{\Delta_A}} \cosh \theta & \sinh \theta - \frac{a-d}{\sqrt{\Delta_A}} \cosh \theta \end{bmatrix}.$$

Then, using the equality $4\Delta = a^2 - 2ad + d^2 + 4bc$, we obtain

$$A^2 = \rho^2 \begin{bmatrix} \cosh(2\theta) + \frac{a-d}{\sqrt{\Delta_A}} \sinh(2\theta) & \frac{2b}{\sqrt{\Delta_A}} \sinh(2\theta) \\ \frac{2c}{\sqrt{\Delta_A}} \sinh(2\theta) & \cosh(2\theta) - \frac{a-d}{\sqrt{\Delta_A}} \sinh(2\theta) \end{bmatrix},$$

$$A^3 = \epsilon\rho^3 \begin{bmatrix} \sinh 3\theta + \frac{a-d}{\sqrt{\Delta_A}} \cosh 3\theta & \frac{2b}{\sqrt{\Delta_A}} \cosh 3\theta \\ \frac{2c}{\sqrt{\Delta_A}} \cosh 3\theta & \sinh 3\theta - \frac{a-d}{\sqrt{\Delta_A}} \cosh 3\theta \end{bmatrix}.$$



Proof.

Therefore, it can be proved by induction. Let k is an even number and (16) is true for $n = k$. Then, $k + 1$ is odd. So, we get A^{k+1} as,

$$\epsilon \rho^{k+1} \begin{bmatrix} \sinh(k+1)\theta + \frac{a-d}{\sqrt{\Delta_A}} \cosh(k+1)\theta & \frac{2b}{\sqrt{\Delta_A}} \cosh(k+1)\theta \\ \frac{2c}{\sqrt{\Delta_A}} \cosh(k+1)\theta & \sinh(k+1)\theta - \frac{a-d}{\sqrt{\Delta_A}} \cosh(k+1)\theta \end{bmatrix}.$$

Similarly, if k is an odd number, $k + 1$ is even and A^{k+1} is

$$\rho^{k+1} \begin{bmatrix} \cosh(k+1)\theta + \frac{(a-d)}{\sqrt{\Delta_A}} \sinh(k+1)\theta & \frac{2b}{\sqrt{\Delta_A}} \sinh(k+1)\theta \\ \frac{2c}{\sqrt{\Delta_A}} \sinh(k+1)\theta & \cosh(k+1)\theta - \frac{(a-d)}{\sqrt{\Delta_A}} \sinh(k+1)\theta \end{bmatrix}.$$



Example

Let's find A^n and B^n for the hyperbolic matrices

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

Considering the Example 4.5, we find

$$A^n = (-1)^n 5^{n/2} \det \begin{bmatrix} \cosh n\theta + \frac{1}{2} \sinh n\theta & \frac{3}{2} \sinh n\theta \\ \frac{1}{2} \sinh n\theta & \cosh n\theta - \frac{1}{2} \sinh n\theta \end{bmatrix}$$

$$B^n = \begin{bmatrix} \sinh n\theta + \frac{\sqrt{5}}{5} \cosh n\theta & \frac{2\sqrt{5}}{5} \cosh n\theta \\ \frac{2\sqrt{5}}{5} \cosh n\theta & \sinh n\theta - \frac{\sqrt{5}}{5} \cosh n\theta \end{bmatrix}$$

where $\theta_A = \ln \frac{\sqrt{5}}{5}$ and $\theta_B = \ln(\sqrt{5} + 2)$, respectively.

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a spacelike hyperbolic real matrix such that $\text{tr}A = 0$, then A^n is

$$A^n = \begin{cases} \rho^{n-1} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} & \text{when } n \text{ is odd;} \\ \rho^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{when } n \text{ is even.} \end{cases} \quad (18)$$

where A^n is a parabolic matrix.

Proof.

Let A be a spacelike hyperbolic matrix such that $\text{tr}A = 0$, then we have $\rho = \sqrt{-\det A}$ and $\Delta = -\det A > 0$. So, we get

$$\theta = \ln \left| \frac{\text{tr}A + \sqrt{\Delta A}}{2\sqrt{|\det A|}} \right| = \ln \left| \frac{2\sqrt{-\det A}}{2\sqrt{-\det A}} \right| = \ln 1 = 0.$$

Thus, according to (16) and (17), we obtain

$$A^n = \rho^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } A^n = \rho^{n-1} \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

for n is even or odd, respectively. □

Example

Let's find A^{11} for

$$A = \begin{bmatrix} 7 & 3 \\ -8 & -7 \end{bmatrix}.$$

A is a spacelike hyperbolic matrix such that $\text{tr}A = 0$. Since $\rho = 5$, we find $A^{11} = 5^{10}A$.

Corollary

Let A be a 2×2 spacelike hyperbolic matrix such that $\text{tr}A = 0$. A^n is a parabolic matrix if and only if n is an even number.

Proof.

A^n is a parabolic matrix if and only if $\Delta(A^n) = 0$. According to (18), $\Delta(A^n) = -\rho^{2(n-1)} \det A$ for n is odd number and $\Delta(A^n) = 0$ for n is even number. We know that $\det A < 0$ for a spacelike matrix, then $\det A \neq 0$ and $\Delta(A^n) \neq 0$ for n is odd. So, A^n is a parabolic matrix if and only if n is an even number. □

Corollary

Let A be a 2×2 lightlike parabolic matrix such that $\text{tr}A = 0$. A^n is a parabolic matrix if and only if n is an even number.

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a lightlike hyperbolic real matrix whose polar representation is

$$A = \text{tr}A \begin{bmatrix} \frac{a}{\text{tr}A} & \frac{b}{\text{tr}A} \\ \frac{c}{\text{tr}A} & \frac{d}{\text{tr}A} \end{bmatrix}.$$

Then, we have

$$A^n = (\text{tr}A)^n \begin{bmatrix} \frac{a}{\text{tr}A} & \frac{b}{\text{tr}A} \\ \frac{c}{\text{tr}A} & \frac{d}{\text{tr}A} \end{bmatrix}. \quad (19)$$

for $n \in \mathbb{Z}^+$.

Proof.

We know that $\text{tr}A \neq 0$ for a lightlike hyperbolic matrix A . Let's prove it by induction. Assume that (19) is true for $n = k$. Then,

$$A^{k+1} = (\text{tr}A)^{n+1} \begin{bmatrix} \frac{a}{\text{tr}A} & \frac{b}{\text{tr}A} \\ \frac{c}{\text{tr}A} & \frac{d}{\text{tr}A} \end{bmatrix}^2 = \frac{(\text{tr}A)^{n+1}}{(\text{tr}AA)^2} \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{bmatrix}.$$

On the other hand, we know that $\det A = 0$, so, $ad = bc$ for the lightlike matrix A . If we use this equality, we get

$$A^{k+1} = \frac{(\text{tr}A)^{n+1}}{(\text{tr}AA)^2} \begin{bmatrix} a\text{tr}A & b\text{tr}A \\ c\text{tr}A & d\text{tr}A \end{bmatrix} = (\text{tr}A)^{n+1} \begin{bmatrix} \frac{a}{\text{tr}A} & \frac{b}{\text{tr}A} \\ \frac{c}{\text{tr}A} & \frac{d}{\text{tr}A} \end{bmatrix}.$$



Corollary

If A is a lightlike hyperbolic matrix, then A^n is also a lightlike hyperbolic matrix.

Proof.

If A is a lightlike hyperbolic matrix, then $\det A = 0$ and $\operatorname{tr} A \neq 0$. So, using (19), we obtain $\det A^n = 0$. That is, A^n is also a lightlike hyperbolic matrix. \square

Example

Let's find A^n for the lightlike hyperbolic matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$. The polar form of A is $A = 4 \begin{bmatrix} 1/2 & 1 \\ 1/4 & 1/2 \end{bmatrix}$. Then, $A^n = 4^n \begin{bmatrix} 1/2 & 1 \\ 1/4 & 1/2 \end{bmatrix}$.

Corollary

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a hyperbolic real matrix such that $\text{tr} A \neq 0$, then A^n is also an hyperbolic matrix.

Proof.

A^n is hyperbolic if and only if $(\text{tr}(A^n))^2 > 4 \det A^n$. If A is hyperbolic, A^n can be one of the equalities (15), (16), (17) and, (19). If A^n is (15) or (16), we obtain

$$\text{tr}(A^n) = 2\epsilon^n \rho^n \cosh n\theta \text{ and } \det A^n = \rho^{2n}.$$

Therefore, we have $(\text{tr}(A^n))^2 = 4 \cosh^2 n\theta \det A^n$. If A^n is (17), we have

$$\text{tr}(A^n) = 2\epsilon^n \rho^n \sinh n\theta \text{ and } \det A^n = -\rho^{2n}.$$

and $(\text{tr}(A^n))^2 = -4 \sinh^2 n\theta (\det A^n)$. Finally, if A^n is in the form (19), then $\text{tr}(A^n) = (\text{tr} A)^n$ and $\det A^n = 0$. Thus, we see that the inequality $(\text{tr} A)^2 > 4 \det A^n$ is true for all cases of A^n , if $\text{tr} A \neq 0$. \square

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a \neq d$ is a timelike parabolic matrix whose polar representation is

$$A = \frac{\text{tr}A}{2} \begin{bmatrix} \epsilon\theta + 1 & \frac{2b\epsilon\theta}{a-d} \\ \frac{2c\epsilon\theta}{a-d} & 1 - \epsilon\theta \end{bmatrix},$$

then

$$A^n = \left(\frac{\text{tr}A}{2}\right)^n \begin{bmatrix} \epsilon n\theta + 1 & \frac{2bn\epsilon\theta}{a-d} \\ \frac{2cn\epsilon\theta}{a-d} & 1 - \epsilon n\theta \end{bmatrix} \quad (20)$$

for $n \in \mathbb{Z}$ where $\theta = \frac{a-d}{|a+d|}$ and $\epsilon = \text{sign}(\text{tr} A)$.

Proof.

Assume that (20) is true for $n = k$. Then, we get

$$A^{k+1} = \left(\frac{\text{tr}A}{2} \right)^{k+1} \begin{bmatrix} \epsilon\theta + 1 & \frac{2b\epsilon\theta}{a-d} \\ \frac{2c\epsilon\theta}{a-d} & 1 - \epsilon\theta \end{bmatrix} \begin{bmatrix} \epsilon k\theta + 1 & \frac{2b\epsilon k\theta}{a-d} \\ \frac{2c\epsilon k\theta}{a-d} & 1 - \epsilon k\theta \end{bmatrix}.$$

On the other hand, in a parabolic timelike matrix, we know that the equality $(\text{tr}A)^2 = 4 \det A \neq 0$ satisfies. So, we have $(a-d)^2 = -4bc$ and we obtain

$$A^{k+1} = \left(\frac{\text{tr}A}{2} \right)^{k+1} \begin{bmatrix} 1 + (k+1)\theta\epsilon & \frac{2b\epsilon(k+1)\theta}{a-d} \\ \frac{2c\epsilon(k+1)\theta}{a-d} & 1 - (k+1)\theta\epsilon \end{bmatrix}.$$



Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a timelike parabolic matrix with $a = d$, then A^n is

$$a^n \begin{bmatrix} 1 & 0 \\ nc/a & 1 \end{bmatrix}, \quad a^n \begin{bmatrix} 1 & nb/a \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad a^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with respect to whether $b = 0$, $c = 0$ or $b = c = 0$, respectively.

Proof.

If A is parabolic, then $(\text{tr}A)^2 = 4 \det A$. So, in the case $a = d$, we find that $a^2 = a^2 - bc$ and $bc = 0$. Thus, it is clear from the Theorem 4.11. □

Corollary

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a timelike parabolic matrix, then A^n is also a timelike parabolic matrix.

Proof.

We know that, A is a timelike parabolic matrix if and only if $(\operatorname{tr} A)^2 = 4 \det A \neq 0$. Therefore, we obtain

$$\det A^n = \left(\frac{\operatorname{tr} A}{2}\right)^{2n} \text{ and } \operatorname{tr} A^n = 2 \left(\frac{\operatorname{tr} A}{2}\right)^n.$$

using (20), Thus, we see that the equality $(\operatorname{tr} A^n)^2 = 4 \det A^n$ satisfies. It means that A^n is also a timelike parabolic matrix. \square

Example

Let's find n -th power of the matrix $A = \begin{bmatrix} 5 & 9 \\ -1 & -1 \end{bmatrix}$.

Polar form of the matrix A is $2 \begin{bmatrix} 1+\theta & 3\theta \\ -\theta/3 & 1-\theta \end{bmatrix}$ where $\theta = 3/2$. So, we obtain, $A^n = 2^n \begin{bmatrix} 1+n\theta & 3n\theta \\ -n\theta/3 & 1-n\theta \end{bmatrix}$ for $n \in \mathbb{Z}$.

Theorem

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a lightlike parabolic real matrix, then $A^n = 0$ for all $n \in \mathbb{Z}^+$.

Example

Let's find n -th power of the matrix $A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$. This matrix is a lightlike parabolic matrix, since $\text{tr}A = 0$ and $\det A = 0$. So, $A^n = 0$ for all $n \in \mathbb{Z}^+$.

n-th Roots of an Elliptic Matrix

In this section we study n -th roots of a 2 by 2 real matrix, considering the De Moivre's formulas given above.

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an elliptic real matrix whose polar representation is

$$A = \rho \begin{bmatrix} \cos \theta + \frac{(a-d)}{\sqrt{-\Delta_A}} \sin \theta & \frac{b}{\sqrt{-\Delta}} \sin \theta \\ \frac{c}{\sqrt{-\Delta}} \sin \theta & \cos \theta - \frac{(a-d)}{\sqrt{-\Delta_A}} \sin \theta \end{bmatrix},$$

then the n -th roots of $\sqrt[n]{A}$ the matrix A are

$$\rho^{1/n} \begin{bmatrix} \cos \frac{\theta+2\pi k}{n} + \frac{(a-d)}{\sqrt{-\Delta_A}} \sin \frac{\theta+2\pi k}{n} & \frac{b}{\sqrt{-\Delta}} \sin \frac{\theta+2\pi k}{n} \\ \frac{c}{\sqrt{-\Delta}} \sin \frac{\theta+2\pi k}{n} & \cos \frac{\theta+2\pi k}{n} - \frac{(a-d)}{\sqrt{-\Delta_A}} \sin \frac{\theta+2\pi k}{n} \end{bmatrix} \quad (21)$$

where $k = 0, 1, 2, \dots, n-1$.

Proof.

Let the matrix X be an n -th root of A . Then we have $X^n = A$. Considering the Theorem (5.1) and the fact that positive integer power of an elliptic matrix is an elliptic matrix, X is in the form

$$X = \rho_x \begin{bmatrix} \cos \beta + \frac{(a-d)}{\sqrt{-\Delta_A}} \sin \beta & \frac{b}{\sqrt{-\Delta}} \sin \beta \\ \frac{c}{\sqrt{-\Delta}} \sin \beta & \cos \beta - \frac{(a-d)}{\sqrt{-\Delta_A}} \sin \beta \end{bmatrix}.$$

So, we have

$$X^n = \rho_x^n \begin{bmatrix} \cos n\beta + \frac{(a-d)}{\sqrt{-\Delta_A}} \sin n\beta & \frac{b}{\sqrt{-\Delta}} \sin n\beta \\ \frac{c}{\sqrt{-\Delta}} \sin n\beta & \cos n\beta - \frac{(a-d)}{\sqrt{-\Delta_A}} \sin n\beta \end{bmatrix}.$$

Therefore, from the equality $X^n = A$ and equality of matrices, we obtain, $\beta = \frac{\theta+2k\pi}{n}$ and $\rho_x = \sqrt[n]{\rho}$ for $k = 0, 1, 2, \dots, n-1$. □

Example

Let's find n -th roots of the elliptic matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$. According to Theorem (6.1), we obtain

$$\sqrt[n]{A} = 3^{1/2n} \begin{bmatrix} \cos \frac{\theta + 2\pi k}{n} & \sqrt{2} \sin \frac{\theta + 2\pi k}{n} \\ -\frac{\sqrt{2}}{2} \sin \frac{\theta + 2\pi k}{n} & \cos \frac{\theta + 2\pi k}{n} \end{bmatrix}$$

where $k = 0, 1, 2, \dots, n-1$ and $\theta = \arctan \sqrt{2}$. For instances, square roots of A are

$$A_k = \sqrt{A} = 3^{1/4} \begin{bmatrix} \cos \frac{\theta + 2\pi k}{2} & \sqrt{2} \sin \frac{\theta + 2\pi k}{2} \\ -\frac{\sqrt{2}}{2} \sin \frac{\theta + 2\pi k}{2} & \cos \frac{\theta + 2\pi k}{2} \end{bmatrix}, \text{ for } k = 0, 1.$$

Example

Thus, we find square roots of A as

$$A_1 = 3^{1/4} \begin{bmatrix} \cos \frac{\arctan \sqrt{2}}{2} & \sqrt{2} \sin \frac{\arctan \sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \sin \frac{\arctan \sqrt{2}}{2} & \cos \frac{\arctan \sqrt{2}}{2} \end{bmatrix}$$

$$\approx \begin{bmatrix} 1.1688 & 0.8556 \\ -0.4278 & 1.1688 \end{bmatrix},$$

$$A_2 = 3^{1/4} \begin{bmatrix} \cos \frac{2\pi + \arctan \sqrt{2}}{2} & \sqrt{2} \sin \frac{2\pi + \arctan \sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \sin \frac{2\pi + \arctan \sqrt{2}}{2} & \cos \frac{2\pi + \arctan \sqrt{2}}{2} \end{bmatrix}$$

$$\approx \begin{bmatrix} -1.1688 & -0.8556 \\ 0.4278 & -1.1688 \end{bmatrix}.$$

Example

Similarly, third roots $\sqrt[3]{A}$ are

$$\sqrt[3]{A} = 3^{1/6} \begin{bmatrix} \cos \frac{\theta+2\pi k}{3} & \sqrt{2} \sin \frac{\theta+2\pi k}{3} \\ -\frac{\sqrt{2}}{2} \sin \frac{\theta+2\pi k}{3} & \cos \frac{\theta+2\pi k}{3} \end{bmatrix}, \text{ for } k = 0, 1, 2.$$

So, the roots $\sqrt[3]{A}$ are

$$\begin{aligned} A_1 &= 3^{1/6} \begin{bmatrix} \cos \frac{(\arctan \sqrt{2})}{3} & \sqrt{2} \sin \frac{(\arctan \sqrt{2})}{3} \\ -\frac{\sqrt{2}}{2} \sin \frac{(\arctan \sqrt{2})}{3} & \cos \frac{(\arctan \sqrt{2})}{3} \end{bmatrix} \\ &\simeq \begin{bmatrix} 1.1406 & 0.53174 \\ -0.26587 & 1.1406 \end{bmatrix}. \end{aligned}$$

Example

$$A_2 = 3^{1/6} \begin{bmatrix} \cos \frac{2\pi + \arctan \sqrt{2}}{3} & \sqrt{2} \sin \frac{2\pi + \arctan \sqrt{2}}{3} \\ -\frac{\sqrt{2}}{2} \sin \frac{2\pi + \arctan \sqrt{2}}{3} & \cos \frac{2\pi + \arctan \sqrt{2}}{3} \end{bmatrix}$$

$$\approx \begin{bmatrix} -0.8959 & 1.131 \\ -0.56551 & -0.8959 \end{bmatrix}.$$

and

$$A_3 = 3^{1/6} \begin{bmatrix} \cos \frac{4\pi + \arctan \sqrt{2}}{3} & \sqrt{2} \sin \frac{4\pi + \arctan \sqrt{2}}{3} \\ -\frac{\sqrt{2}}{2} \sin \frac{4\pi + \arctan \sqrt{2}}{3} & \cos \frac{4\pi + \arctan \sqrt{2}}{3} \end{bmatrix}$$

$$\approx \begin{bmatrix} -0.24466 & -1.6628 \\ 0.83138 & -0.24466 \end{bmatrix}.$$

Corollary

If A is elliptic matrix, then there are n matrices X satisfying the equality

$$X^n = A.$$

So, an elliptic matrix has 2 square roots.

For example, the elliptic matrix $A = \begin{bmatrix} 4 & -7 \\ 1 & 5 \end{bmatrix}$ has 7 seventh roots.

n-th Roots of a Hyperbolic Matrix

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a timelike hyperbolic real matrix whose polar representation is

$$A = \epsilon_A \rho \begin{bmatrix} \cosh \theta + \frac{(a-d)}{\sqrt{\Delta_A}} \sinh \theta & \frac{2b}{\sqrt{\Delta_A}} \sinh \theta \\ \frac{2c}{\sqrt{\Delta_A}} \sinh \theta & \cosh \theta - \frac{(a-d)}{\sqrt{\Delta_A}} \sinh \theta \end{bmatrix}.$$

i. If n is even number and $\epsilon_A = 1$, then n -th roots of A are

$$\sqrt[n]{A} = \pm \rho^{1/n} \begin{bmatrix} \cosh \frac{\theta}{n} + \frac{(a-d)}{\sqrt{\Delta_A}} \sinh \frac{\theta}{n} & \frac{2b}{\sqrt{\Delta_A}} \sinh \frac{\theta}{n} \\ \frac{2c}{\sqrt{\Delta_A}} \sinh \frac{\theta}{n} & \cosh \frac{\theta}{n} - \frac{(a-d)}{\sqrt{\Delta_A}} \sinh \frac{\theta}{n} \end{bmatrix} \quad (22)$$

and

Theorem

$$\sqrt[n]{A} = \pm \rho^{1/n} \begin{bmatrix} \sinh \frac{\theta}{n} + \frac{(a-d)}{\sqrt{\Delta_A}} \cosh \frac{\theta}{n} & \frac{2b}{\sqrt{\Delta_A}} \cosh \frac{\theta}{n} \\ \frac{2c}{\sqrt{\Delta_A}} \cosh \frac{\theta}{n} & \sinh \frac{\theta}{n} - \frac{(a-d)}{\sqrt{\Delta_A}} \cosh \frac{\theta}{n} \end{bmatrix}. \quad (23)$$

ii. If n is odd number, then n -th roots of A are

$$\sqrt[n]{A} = \epsilon_A \rho^{1/n} \begin{bmatrix} \cosh \frac{\theta}{n} + \frac{(a-d)}{\sqrt{\Delta_A}} \sinh \frac{\theta}{n} & \frac{2b}{\sqrt{\Delta_A}} \sinh \frac{\theta}{n} \\ \frac{2c}{\sqrt{\Delta_A}} \sinh \frac{\theta}{n} & \cosh \frac{\theta}{n} - \frac{(a-d)}{\sqrt{\Delta_A}} \sinh \frac{\theta}{n} \end{bmatrix}. \quad (24)$$

Proof.

Let the matrix X be an n -th root of A . Then we have $X^n = A$. X can be one of the forms (9), (10) or (11). If X in the form (9), that is X is a timelike hyperbolic matrix, we can write as

$$X = \epsilon_X \rho_X \begin{bmatrix} \cosh \beta + \frac{(a-d)}{\sqrt{\Delta_A}} \sinh \beta & \frac{2b}{\sqrt{\Delta_A}} \sinh \beta \\ \frac{2c}{\sqrt{\Delta_A}} \sinh \beta & \cosh \beta - \frac{(a-d)}{\sqrt{\Delta_A}} \sinh \beta \end{bmatrix}.$$

So, considering the Theorem (16), we have

$$X^n = \epsilon_X^n \rho_X^n \begin{bmatrix} \cosh(n\beta) + \frac{(a-d)}{\sqrt{\Delta_A}} \sinh(n\beta) & \frac{2b}{\sqrt{\Delta_A}} \sinh(n\beta) \\ \frac{2c}{\sqrt{\Delta_A}} \sinh(n\beta) & \cosh(n\beta) - \frac{(a-d)}{\sqrt{\Delta_A}} \sinh(n\beta) \end{bmatrix}.$$



Proof.

Therefore, from the equality $X^n = A$ and equality of matrices, we obtain,

$$\beta = \frac{\theta}{n} \text{ and } \epsilon_X^n \rho_X^n = \epsilon \rho.$$

If n is even, then we have $\rho_X^n = \epsilon \rho$ and it has a solution if and only if $\epsilon = 1$. If n is odd number, then $\epsilon_X = \epsilon$ and $\rho_X = \sqrt[n]{\rho}$. If X in the form (10), then X is in the form

$$X = \epsilon_X \rho_X \begin{bmatrix} \sinh \beta + \frac{(a-d)}{\sqrt{\Delta_A}} \cosh \beta & \frac{2b}{\sqrt{\Delta_A}} \cosh \beta \\ \frac{2c}{\sqrt{\Delta_A}} \cosh \beta & \sinh \beta - \frac{(a-d)}{\sqrt{\Delta_A}} \cosh \beta \end{bmatrix}.$$



Proof.

So, if n is odd, we have no solution for $X^n = A$, since

$$X^n = \epsilon_x^n \rho_x^n \begin{bmatrix} \sinh(n\beta) + \frac{(a-d)}{\sqrt{\Delta_A}} \cosh(n\beta) & \frac{2b}{\sqrt{\Delta_A}} \cosh(n\beta) \\ \frac{2c}{\sqrt{\Delta_A}} \cosh(n\beta) & \sinh(n\beta) - \frac{(a-d)}{\sqrt{\Delta_A}} \cosh(n\beta) \end{bmatrix}.$$

If n is even number, then

$$X^n = \rho_x^n \begin{bmatrix} \cosh(n\beta) + \frac{(a-d)}{\sqrt{\Delta_A}} \sinh(n\beta) & \frac{2b}{\sqrt{\Delta_A}} \sinh(n\beta) \\ \frac{2c}{\sqrt{\Delta_A}} \sinh(n\beta) & \cosh(n\beta) - \frac{(a-d)}{\sqrt{\Delta_A}} \sinh(n\beta) \end{bmatrix}.$$

Thus, we have another solution where $\beta = \frac{\theta}{n}$ and $\rho_x^n = \epsilon \rho$ for $\epsilon = 1$. At last, we can see that there is no solution, if X is in the form (11). \square

Now, let's give two examples for n is odd or even.

Example

Let's find $\sqrt[3]{A}$ for the matrix $A = \begin{bmatrix} -13 & -21 \\ 14 & 22 \end{bmatrix}$. A is a timelike hyperbolic matrix, since $\Delta_A = 49$ and $\det A = 8 > 0$. Then, the polar form of A is

$$A = \sqrt{8} \begin{bmatrix} \cosh \theta - 5 \sinh \theta & -6 \sinh \theta \\ 4 \sinh \theta & \cosh \theta + 5 \sinh \theta \end{bmatrix}$$

where $\theta = \ln 2\sqrt{2}$. Using the Theorem (6.7), we find $\sqrt[3]{A}$ as

$$\begin{aligned} \sqrt[3]{A} &= 8^{1/6} \begin{bmatrix} \cosh \frac{\ln 2\sqrt{2}}{3} - 5 \sinh \frac{\ln 2\sqrt{2}}{3} & -6 \sinh \frac{\ln 2\sqrt{2}}{3} \\ 4 \sinh \frac{\ln 2\sqrt{2}}{3} & \cosh \frac{\ln 2\sqrt{2}}{3} + 5 \sinh \frac{\ln 2\sqrt{2}}{3} \end{bmatrix} \\ &= \begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}. \end{aligned}$$

Example

Let's find $\sqrt[4]{A}$ for the matrix

$$A = \begin{bmatrix} 11 & 10 \\ 5 & 6 \end{bmatrix}.$$

A is a timelike hyperbolic matrix, since $\Delta_A = 225$ and $\det A = 16 > 0$.
Then, the polar form of A is

$$A = 4 \begin{bmatrix} \cosh \theta + \frac{1}{3} \sinh \theta & \frac{4}{3} \sinh \theta \\ \frac{2}{3} \sinh \theta & \cosh \theta - \frac{1}{3} \sinh \theta \end{bmatrix}$$

where $\theta = \ln 4$.

Example

Therefore, since $n = 4$ and $\epsilon_A = 1$, there are four roots $\sqrt[4]{A}$, and these are,

$$\sqrt[4]{A} = \pm \sqrt[4]{4} \begin{bmatrix} \cosh \frac{\ln 4}{4} + \frac{1}{3} \sinh \frac{\ln 4}{4} & \frac{4}{3} \sinh \frac{\ln 4}{4} \\ \frac{2}{3} \sinh \frac{\ln 4}{4} & \cosh \frac{\ln 4}{4} - \frac{1}{3} \sinh \frac{\ln 4}{4} \end{bmatrix}$$

$$A_{1,2} \simeq \pm \frac{1}{3} \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}.$$

$$\sqrt[4]{A} = \pm \sqrt[4]{4} \begin{bmatrix} \sinh \frac{\ln 4}{4} + \frac{1}{3} \cosh \frac{\ln 4}{4} & \frac{4}{3} \cosh \frac{\ln 4}{4} \\ \frac{2}{3} \cosh \frac{\ln 4}{4} & \sinh \frac{\ln 4}{4} - \frac{1}{3} \cosh \frac{\ln 4}{4} \end{bmatrix}$$

$$A_{3,4} = \pm \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a spacelike hyperbolic matrix whose polar representation is,

$$A = \rho \begin{bmatrix} \sinh \theta + \frac{(a-d)}{\sqrt{\Delta_A}} \cosh \theta & \frac{2b}{\sqrt{\Delta_A}} \cosh \theta \\ \frac{2c}{\sqrt{\Delta_A}} \cosh \theta & \sinh \theta - \left(\frac{a-d}{\sqrt{\Delta_A}} \right) \cosh \theta \end{bmatrix}.$$

i. If n is odd number, then $\sqrt[n]{A}$ is

$$\sqrt[n]{A} = \rho^{1/n} \begin{bmatrix} \sinh \frac{\theta}{n} + \frac{(a-d)}{\sqrt{\Delta_A}} \cosh \frac{\theta}{n} & \frac{2b}{\sqrt{\Delta_A}} \cosh \frac{\theta}{n} \\ \frac{2c}{\sqrt{\Delta_A}} \cosh \frac{\theta}{n} & \sinh \frac{\theta}{n} - \frac{(a-d)}{\sqrt{\Delta_A}} \cosh \frac{\theta}{n} \end{bmatrix}. \quad (25)$$

ii. If n is even number, then there is no n -th root of A .

Proof.

It can be proved similar to Theorem (6.7).



Example

Let's find $\sqrt[5]{A}$ for the matrix

$$A = \begin{bmatrix} -13 & 5 \\ -5 & 2 \end{bmatrix}.$$

A is a spacelike hyperbolic matrix, since $\Delta_A = 125$ and $\det A = -1 < 0$.

$$A = \begin{bmatrix} \sinh \theta - \frac{3\sqrt{5}}{5} \cosh \theta & \frac{2\sqrt{5}}{5} \cosh \theta \\ -\frac{2\sqrt{5}}{5} \cosh \theta & \sinh \theta + \frac{3\sqrt{5}}{5} \cosh \theta \end{bmatrix}$$

where $\theta = \ln \left(\frac{5\sqrt{5}-11}{2} \right)$. So, the only 5th root of A is

$$\begin{bmatrix} \sinh \frac{\theta}{5} - \frac{3\sqrt{5}}{5} \cosh \frac{\theta}{5} & \frac{2\sqrt{5}}{5} \cosh \frac{\theta}{5} \\ -\frac{2\sqrt{5}}{5} \cosh \frac{\theta}{5} & \sinh \frac{\theta}{5} + \frac{3\sqrt{5}}{5} \cosh \frac{\theta}{5} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}.$$

Example

There is no any root $\sqrt[n]{A}$ for the spacelike hyperbolic matrix

$A = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$, since a spacelike hyperbolic matrix has not an n th root if n is even.

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a lightlike hyperbolic real matrix whose polar representation is

$$A = \text{tr}A \begin{bmatrix} \frac{a}{\text{tr}A} & \frac{b}{\text{tr}A} \\ \frac{c}{\text{tr}A} & \frac{d}{\text{tr}A} \end{bmatrix}.$$

Then, the n -th root of the matrix A is

$$\sqrt[n]{A} = (\text{tr}A)^{1/n} \begin{bmatrix} \frac{a}{\text{tr}A} & \frac{b}{\text{tr}A} \\ \frac{c}{\text{tr}A} & \frac{d}{\text{tr}A} \end{bmatrix}.$$

for $n \in \mathbb{Z}^+$.

n-th Roots of a Parabolic Matrix

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a \neq d$ be a timelike parabolic real matrix whose polar representation is

$$A = \frac{\text{tr}A}{2} \begin{bmatrix} \epsilon\theta + 1 & \frac{2b\epsilon\theta}{a-d} \\ \frac{2c\epsilon\theta}{a-d} & 1 - \epsilon\theta \end{bmatrix},$$

then the n-th roots of the matrix A are

$$\sqrt[n]{A} = \left(\frac{\text{tr}A}{2}\right)^{1/n} \begin{bmatrix} \frac{\epsilon\theta}{n} + 1 & \frac{2b\epsilon\theta}{n(a-d)} \\ \frac{2c\epsilon\theta}{n(a-d)} & 1 - \frac{\epsilon\theta}{n} \end{bmatrix} \quad (26)$$

where $\theta = \frac{a-d}{|a+d|}$ and $\epsilon = \text{sign}(\text{tr}A)$.

Proof.

Let X be a matrix satisfying the equality $X^n = A$. Because of that A is a timelike parabolic matrix, X must be a parabolic matrix according to Corollary (5.17). Then, the matrix X can be in the form

$$X = \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \frac{\text{tr}X}{2} \begin{bmatrix} \epsilon\beta + 1 & \frac{2y\epsilon\beta}{x-t} \\ \frac{2z\epsilon\beta}{x-t} & 1 - \epsilon\beta \end{bmatrix}.$$

where $\text{tr}X = x + t$, $\beta = \frac{x-t}{|x+t|}$ and $(x-t)^2 = -4yz$. According to (20), we have

$$X^n = \left(\frac{\text{tr}X}{2}\right)^n \begin{bmatrix} \epsilon n\beta + 1 & \frac{2y\epsilon n\beta}{x-t} \\ \frac{2z\epsilon n\beta}{x-t} & 1 - \epsilon n\beta \end{bmatrix}.$$



Proof.

Thus, using the equations $(X^n)_{11} = A_{11}$ and $(X^n)_{22} = A_{22}$, we find

$$\left(\frac{\text{tr}X}{2}\right)^n (\epsilon n\beta + 1) = \frac{\text{tr}A}{2} (\epsilon\theta + 1)$$

$$\left(\frac{\text{tr}X}{2}\right)^n (1 - \epsilon n\beta) = \frac{\text{tr}A}{2} (1 - \epsilon\theta).$$

Solving these two equations, we obtain $\frac{\text{tr}X}{2} = \left(\frac{\text{tr}A}{2}\right)^{1/n}$ and $\beta = \frac{\theta}{n}$.

Therefore, we have

$$X^n = \left(\frac{\text{tr}A}{2}\right) \begin{bmatrix} \epsilon\theta + 1 & \frac{2y\epsilon\theta}{x-t} \\ \frac{2z\epsilon\theta}{x-t} & 1 - \epsilon\theta \end{bmatrix}.$$



Proof.

Also, we can see that the equality $X^n = A$ satisfies if and only if

$$\frac{y}{x-t} = \frac{b}{a-d} \text{ and } \frac{z}{x-t} = \frac{c}{a-d} \text{ and } \frac{x-t}{|x+t|} = \frac{a-d}{n|a+d|}.$$

According to this, we can obtain $y = bk$, $z = ck$,

$$x = \left(\frac{a+d}{2}\right)^{1/n} + \left(\frac{a-d}{2}\right) k \text{ and } t = \left(\frac{a+d}{2}\right)^{1/n} - \left(\frac{a-d}{2}\right) k \text{ where}$$

$$k = \frac{1}{n} \left(\frac{a+d}{2}\right)^{\frac{1-n}{n}}. \text{ As a result, we find}$$

$$\sqrt[n]{A} = \left(\frac{\text{tr}A}{2}\right)^{1/n} \begin{bmatrix} \frac{\epsilon\theta}{n} + 1 & \frac{2b\epsilon\theta}{n(a-d)} \\ \frac{2c\epsilon\theta}{n(a-d)} & 1 - \frac{\epsilon\theta}{n} \end{bmatrix}.$$



Example

Let's find n -th roots of the parabolic matrix

$$A = \begin{bmatrix} 11 & -12 \\ 3 & -1 \end{bmatrix}.$$

The polar form of the matrix A is

$$A = 5 \begin{bmatrix} 1 + \theta & -2\theta \\ \theta/2 & 1 - \theta \end{bmatrix}$$

where $\theta = \frac{6}{5}$. Therefore, we get

$$\sqrt[n]{A} = 5^{1/n} \begin{bmatrix} 1 + \theta/n & -2\theta/n \\ \theta/2n & 1 - \theta/n \end{bmatrix}.$$

Theorem

If $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$ be a timelike parabolic real matrix, then $\sqrt[n]{A}$ is

$$\begin{bmatrix} 1 & 0 \\ c/an & 1 \end{bmatrix}, a^{1/n} \begin{bmatrix} 1 & b/an \\ 0 & 1 \end{bmatrix} \text{ or } a^{1/n} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

according to $b = 0, c = 0$ or $b = c = 0$, respectively. Moreover, if n is even and $b = c = 0$, then $\sqrt[n]{A}$ is

$$\begin{bmatrix} t & s \\ \frac{a^{2/n} - t^2}{s} & -t \end{bmatrix}$$

for $t, s \in \mathbb{R}, s \neq 0$.

Proof.

If A is timelike parabolic, then $bc = 0$. So, A can be one of the forms (13). So, according to Theorem 5.16, we find $\sqrt[n]{A}$ as

$$a^{1/n} \begin{bmatrix} 1 & 0 \\ c/an & 1 \end{bmatrix}, a^{1/n} \begin{bmatrix} 1 & b/an \\ 0 & 1 \end{bmatrix} \text{ or } a^{1/n} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for $b = 0$, $c = 0$ or $b = c = 0$, respectively.

Moreover, from the Theorem 5.7, we can find different roots for

$$A = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Assume that $X^n = A$ satisfies, where $X = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$.



Example

Let's find $\sqrt[6]{A}$ for

$$A = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}.$$

According to Theorem 6.18, we get

$$\sqrt[6]{A} = \begin{bmatrix} t & s \\ -\frac{1}{s}(t^2 - 2) & -t \end{bmatrix}$$

for $s, t \in \mathbb{R}$.

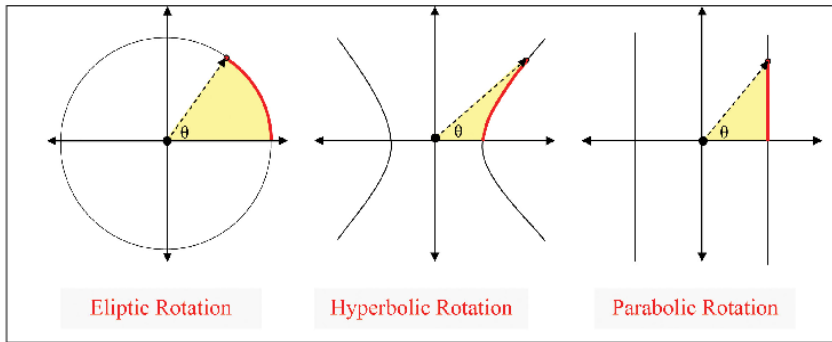
Some Conclusions

1. Type of a Power of A 2×2 matrix

Type and character of a power of a 2×2 matrix are summarized with the following table.

Type and Character of Power of a 2×2 Matrix						
A	Elliptic	Parabolic	Hyperbolic ($\text{tr} A \neq 0$)	Spacelike Hyperbolic ($\text{tr} A = 0$)	Spacelike Hyperbolic ($\text{tr} A = 0$), ($\det A \neq 0$)	Lightlike Hyperbolic
A^n	Elliptic	Parabolic	Hyperbolic	Parabolic if n is even; Hyperbolic if n is odd	Spacelike Hyperbolic	Lightlike Hyperbolic

2. De Moivre's Formula for 2D Rotation Matrices



The polar representations of the matrices

$$R_E = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad R_L = \begin{bmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{bmatrix}, \quad R_G = \begin{bmatrix} \theta + 1 & -\theta \\ \theta & 1 - \theta \end{bmatrix}$$

are themselves, since $\Delta_E = -\sin^2 \theta$, $\Delta_L = \sinh^2 \theta$, where R_E, R_L and, R_G are rotations matrices in the Euclidean, Lorentzian and Galilean plane, respectively.

So, de Moivre's formula for these matrices as follows :

$$R_E^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}, \quad R_L^n = \begin{bmatrix} \cosh n\theta & \sinh n\theta \\ \sinh n\theta & \cosh n\theta \end{bmatrix}, \quad R_G^n = \begin{bmatrix} n\theta + 1 & -n\theta \\ n\theta & 1 - n\theta \end{bmatrix}.$$

for $n \in \mathbb{Z}$.

Number of n-th roots of a 2x2 real matrix

As it can be seen from the above theorems,
A 2×2 real matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

may have

**one,
two,
four,
n or
infinitely many**

n th roots, or **it may have not a root**. We can summarize all results with the following corollary.

Corollary

A matrix B is said to be an n -th root of a matrix A if $B^n = A$, where $n \geq 2$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the number of n th roots of a A is as follows

- An elliptic matrix has exactly n n th roots.
- A timelike hyperbolic matrix has exactly four n th roots if n is even.
- A timelike hyperbolic matrix has only one n th root if n is odd.
- A spacelike hyperbolic matrix has not an n th root if n is even.
- A spacelike hyperbolic matrix has only one n th root if n is odd.
- A lightlike hyperbolic matrix has only one n th root if n is odd.
- A lightlike hyperbolic matrix has two n th roots if n is even.
- A timelike parabolic matrix has only one n th root if $a \neq d$.
- A nonzero lightlike parabolic matrix has not an n th root.
- A non-scalar timelike parabolic matrix has only one n th root if $a = d$.
- A scalar matrix has infinitely many n th roots.
- Each lightlike parabolic matrix is a root of zero matrix.

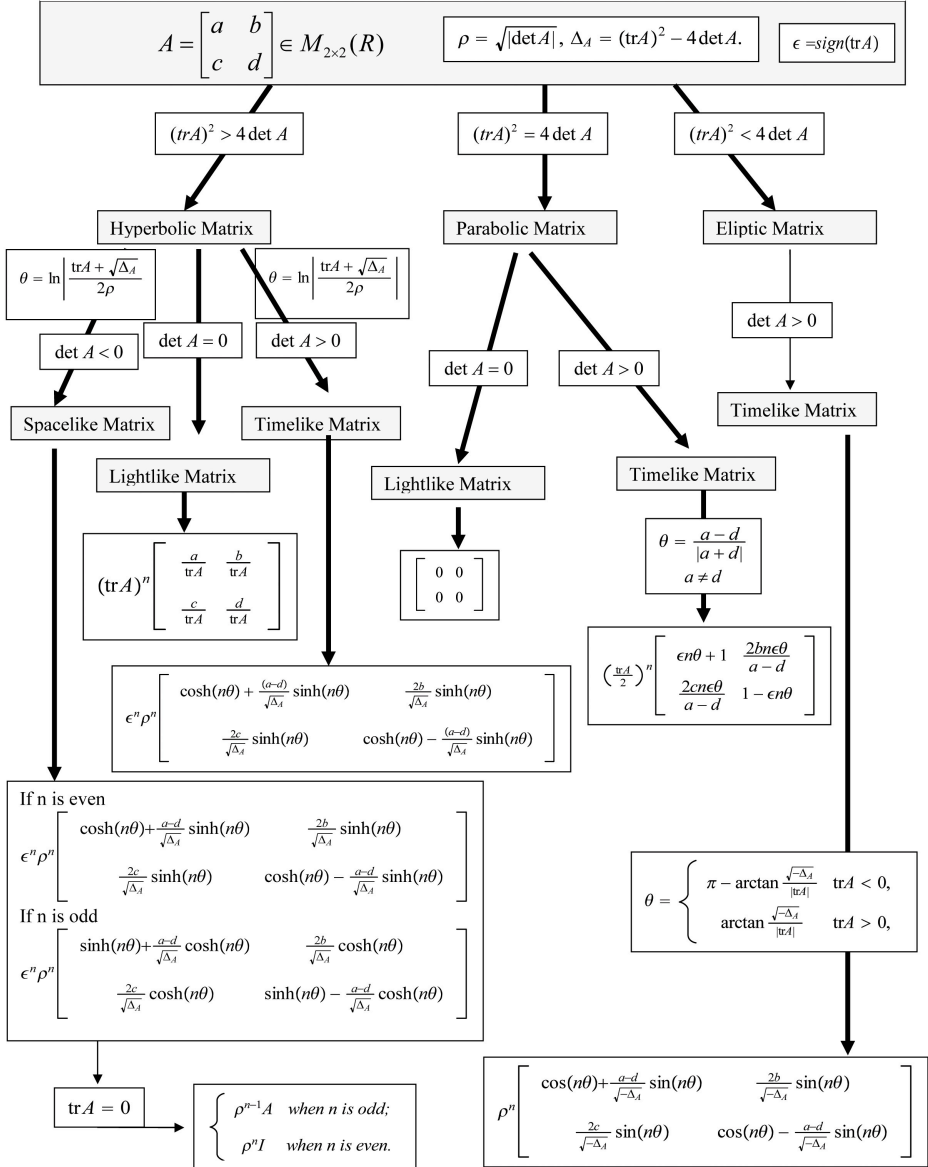
Comparing The Known Methods

- **Basic Algebraic Method** : Solving the system of higher degree equations can be difficult and messy.
- **Diagonalization** : It is a commonly used method. But, it works only for diagonalizable matrices.
- **Schur Decomposition Method** : It works only for triangularizable matrices. A disadvantage of this method is that if A has nonreal eigenvalues, the method necessitates complex arithmetic if the root which is computed should be real.
- **Cayley Hamilton Method** : Even if it is the best method for finding square roots, it is difficult to apply for finding the roots of degree greater than 2. For higher degrees, the computations can become long and messy.
- **Newton Method** : It does not give results in rootless matrices. It is long and tedious for $n > 2$. If it is not known that matrix has not a root, it will be a complete waste of time.

- **Using Complex, dual and hyperbolic numbers** : It can only be used for some three specific matrix types.
- **Abel-Mobius Method** : It is the most difficult, tedious and complicated one in the methods.
- **De Moivre's Formula (Hybrid Numbers)** : This method can be used for all 2×2 matrices. It is suitable for the computer algorithm, and after type and character of the matrix is determined, the result can be directly calculated by substituting the n and θ values in the appropriate formula. No complicated function and process information is required. The basic matrix, trigonometry and hyperbolic function knowledge is sufficient to obtain the result. In addition to all of these, if the number n is greater than 2, operations are not confused and difficult. That is, in the case $n > 2$, it is an easy and fast alternative method that can be used to find n -th root of any 2×2 matrix. But, this method can be used only to find the real roots of a matrix.

Comparing The Known Methods (good to bad)

No	Name of Method	Disadvantages
1	Diagonalization	It is valid only for diagonalizable matrices
2	Schur Decomp. Method	It is valid only for triangularizable matrices
3	Cayley Hamilton Method	It is difficult to apply in roots of degree greater than 2
4	Newton Method	It does not give results in rootless matrices It is long and tedious for $n > 2$.
5	Using \mathbb{C}, \mathbb{D} and \mathbb{D}	It can only be used for some three specific matrix types.
6	Basic Algebraic Method	Solving the higher degree equation can be difficult
7	Abel-Mobius Method	To apply this method is very difficult, long and tedious.

De MOIVRE'S FORMULAS FOR 2×2 REAL MATRICES



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THANKS