Definitions of PCC, ΔPCC and scalar estimate of ΔPCC, evaluation of PCO for use in the RingAnt project.

Convention for spherical coordinate system

Used convention for zenith angle θ_{zen} and azimuth angle φ_{az} is shown on fig 1. This convention corresponds to angles in ANTEX files.

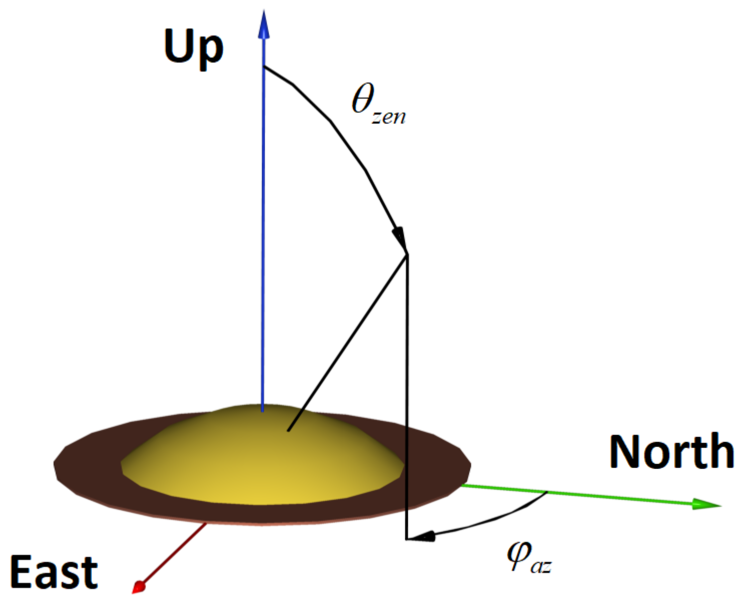

Fig 1. Convention for zenith and azimuth angles.

Phase center corrections definition

Given east, north and up components expressed in meters of antenna phase center offset (PCO) e_f, n_f, u_f for a GNSS frequency f the phase center corrections (PCC) for this GNSS frequency expressed in meters are defined as follows

$$\begin{aligned} PCC_f(\theta_{zen},\varphi_{az}) &= e_f \sin\varphi_{az}\sin\theta_{zen} + n_f\cos\varphi_{az}\sin\theta_{zen} + u_f\cos\theta_{zen} + PCV_f(\theta_{zen},\varphi_{az}) = \\ & PCO_f(\theta_{zen},\varphi_{az}) + PCV_f(\theta_{zen},\varphi_{az}) \end{aligned} \tag{1}$$

Where $PCV_f(\theta_{zen},\varphi_{az})$ are expressed in meters antenna phase center variations for GNSS frequency f. Note that PCV stored in ANTEX files have negative sign: $PCV_f(\theta_{zen},\varphi_{az}) = -PCV_{f,ANTEX}(\theta_{zen},\varphi_{az})$

For convenience of further presentation we will omit subscripts zen, az, f and will use θ,φ instead of $\theta_{zen},\varphi_{az}$.

Evaluation of PCO for a given PCC

If PCC for a GNSS frequency f is given as $PCC(\theta,\varphi)$, then the corresponding e,n,u are evaluated as the result of minimization of a functional

$$\int_0^{2\pi}\int_0^{\theta_0}\left(PCC(\theta,\varphi) - e\sin\varphi\sin\theta - n\cos\varphi\sin\theta - u\cos\theta - \rho\right)^2 w(\theta)\sin\theta\, d\theta\, d\varphi \tag{2}$$

by variables e, n, u and ρ_f, where $w(\theta)$ is a weight function, which may take into account typical de-weighting for low elevations in GNSS positioning, ρ is a constant. Next we will consider solutions for three weight functions $w(\theta)=1,\ w(\theta)=\cos(\theta),\ w(\theta)=1/\sin(\theta)$. Minimization of functional (2) corresponds to minimization of errors introduced by the antenna element when solving the GNSS positioning problem without taking into account its phase characteristics. Presumably uniform distribution of GNSS satellites across the sky in the process of solving a positioning problem, when the number of measurements for a certain solid angle is proportional to its value, is ensured by using differential element of the sphere surface $\sin\theta d\theta d\varphi$.

Using least squares, differentiating (2) by unknowns PCO components and ρ, equating partial derivatives to zero we get normal equations:

$$\begin{pmatrix} \int_0^{2\pi}\int_0^{\theta_0} PCC(\theta,\varphi)\sin\varphi\sin\theta\, w\sin\theta d\theta d\varphi \\ \int_0^{2\pi}\int_0^{\theta_0} PCC(\theta,\varphi)\cos\varphi\sin\theta\, w\sin\theta d\theta d\varphi \\ \int_0^{2\pi}\int_0^{\theta_0} PCC(\theta,\varphi)\cos\theta\, w\sin\theta d\theta d\varphi \\ \int_0^{2\pi}\int_0^{\theta_0} PCC(\theta,\varphi)\, w\sin\theta d\theta d\varphi \end{pmatrix} = N\begin{pmatrix} e \\ n \\ u \\ \rho \end{pmatrix} \tag{3}$$

The matrix N of normal equations for three weight functions is as follows:

$$
\begin{gathered}
w(\theta)=\cos\theta: \quad N=\begin{pmatrix} \pi\sin^4\theta_0/4 & 0 & 0 & 0 \\ 0 & \pi\sin^4\theta_0/4 & 0 & 0 \\ 0 & 0 & \pi\left(1-\cos^4\theta_0\right)/2 & 2\pi\left(1-\cos^3\theta_0\right)/3 \\ 0 & 0 & 2\pi\left(1-\cos^3\theta_0\right)/3 & \pi\sin^2\theta_0 \end{pmatrix} \\
w(\theta)=1: \quad N=\begin{pmatrix} \dfrac{\pi\left(\cos\theta_0-1\right)^2\left(\cos\theta_0+2\right)}{3} & 0 & 0 & 0 \\ 0 & \dfrac{\pi\left(\cos\theta_0-1\right)^2\left(\cos\theta_0+2\right)}{3} & 0 & 0 \\ 0 & 0 & \dfrac{2\pi\left(1-\cos^3\theta_0\right)}{3} & \pi\sin^2\theta_0 \\ 0 & 0 & \pi\sin^2\theta_0 & 2\pi\left(1-\cos\theta_0\right) \end{pmatrix} \\
w(\theta)=1/\sin(\theta): \quad N=\begin{pmatrix} \dfrac{\pi\left(2\theta_0-\sin 2\theta_0\right)}{4} & 0 & 0 & 0 \\ 0 & \dfrac{\pi\left(2\theta_0-\sin 2\theta_0\right)}{4} & 0 & 0 \\ 0 & 0 & \dfrac{\pi\left(2\theta_0+\sin 2\theta_0\right)}{2} & 2\pi\sin\theta_0 \\ 0 & 0 & 2\pi\sin\theta_0 & 2\pi\theta_0 \end{pmatrix}
\end{gathered}
\tag{4}
$$

Integrals in the left part of (3) are computed numerically, considering that $PCC(\theta,\varphi)$ is given in a regular grid of points using linear interpolation between points.

Note that obtained estimates of PCO components e,n,u do not depend on additive constant of PCC, i.e. $PCC(\theta,\varphi)$ and $PCC(\theta,\varphi)+const$ have equal estimates of e,n,u. But estimates of ρ are different. To verify this one can evaluate the e,n,u for $PCC(\theta,\varphi)=const$; integrals in the left part of (3) in this case are evaluated analytically and multiplied by the inverse of N give zero e,n,u.

For given $PCC(\theta,\varphi)$ the PCO is affected by two factors: weight function $w(\theta)$ and elevation mask $\theta_{ele,0}=90-\theta_{zen,0}$. Fig 2 shows how these factors affect the vertical PCO component u (VPCO) and how they relate to VPCO taken from igs20.atx file for TPCCRG5C_NONE antenna in GPS L1 and GPS L2

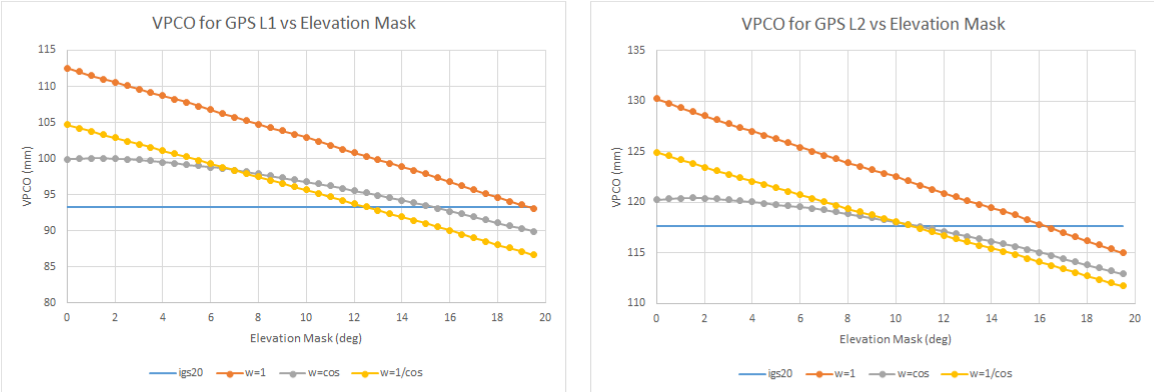

Fig 2. VPCO component computed for different weight functions versus elevations mask compared to VPCO value taken from igs20.atx for TPSCRG5C_NONE antenna

Horizontal PCO (HPCO) is less affected by elevation mask and weight function (Fig 3)

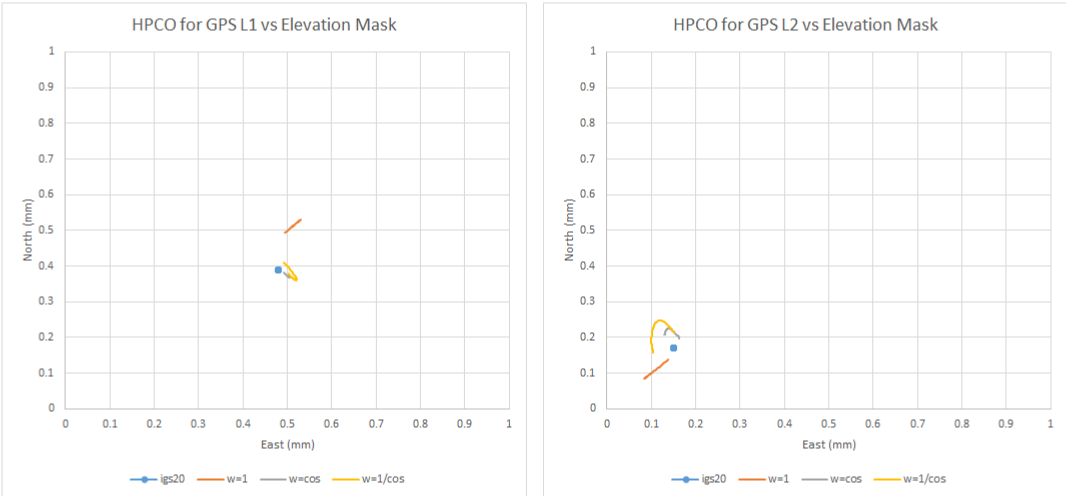

Fig 3. HPCO versus weight function and elevation mask for TPSCRG5C_NONE antenna calibration taken from igs20.atx. As elevation mask impacts the HPCO by less than tenth of millimeter, points on curves which correspond different elevation masks are not shown.

PCC difference (delta PCC) definition

Assuming that $PCC^{(1)}(\theta,\varphi)$ and $PCC^{(2)}(\theta,\varphi)$ are given in the form (1) and $PCV^{(1)}$ and $PCV^{(2)}$ are zero for zenith direction: $PCV^{(1)}(0,\varphi)=PCV^{(2)}(0,\varphi)=0$ define the $\Delta PCC(\theta,\varphi)$ as

$$\Delta PCC(\theta,\varphi)=PCC^{(1)}(\theta,\varphi)-PCC^{(2)}(\theta,\varphi)-r \tag{5}$$

where r is a constant whose value is determined further in the process of selecting a scalar estimate of the PCC difference.

Scalar estimate of the PCC difference

For a GNSS frequency f the scalar estimate σ of the PCC difference is as follows

$$\sigma = a\cdot\sqrt{\int_0^{2\pi}\int_0^{\pi/2}\Delta PCC(\theta,\varphi)^2\, w(\theta)\sin\theta\, d\theta d\varphi} = \\ a\cdot\sqrt{\int_0^{2\pi}\int_0^{\pi/2}\left(PCC^{(1)}(\theta,\varphi)-PCC^{(2)}(\theta,\varphi)-r\right)^2 w(\theta)\sin\theta\, d\theta d\varphi} \tag{6}$$

Where a and r are constants whose values are chosen such that if two PCC have equal PCV then the scalar estimate of their difference is equal to the distance between their PCO. It is assumed that PCO are determined by the process (2)-(3). Solving these problem (see Appendix) we obtain the following values for these parameters:

Weight function	a	r
$w(\theta)=1$	$\sqrt{\frac{3}{2\pi}}$	$\begin{cases} u^{(1)}-u^{(2)}, & \left(u^{(1)}-u^{(2)}\right)\rho\le 0 \\ 0, & \left(u^{(1)}-u^{(2)}\right)\rho>0 \end{cases}$
$w(\theta)=\cos\theta$	$\frac{2}{\sqrt{\pi}}$	$\begin{cases} \frac{4+\sqrt{7}}{6}\left(u^{(1)}-u^{(2)}\right), & \left(u^{(1)}-u^{(2)}\right)\rho\le 0 \\ \frac{4-\sqrt{7}}{6}\left(u^{(1)}-u^{(2)}\right), & \left(u^{(1)}-u^{(2)}\right)\rho>0 \end{cases}$
$w(\theta)=1/\sin\theta$	$\frac{2}{\pi}$	$\begin{cases} \frac{4+\sqrt{16-\pi^2}}{2\pi}\left(u^{(1)}-u^{(2)}\right), & \left(u^{(1)}-u^{(2)}\right)\rho\le 0 \\ \frac{4-\sqrt{16-\pi^2}}{2\pi}\left(u^{(1)}-u^{(2)}\right), & \left(u^{(1)}-u^{(2)}\right)\rho>0 \end{cases}$

where ρ is the solution of equations (3) for the

$$PCC(\theta,\varphi)=\left(e^{(1)}-e^{(2)}\right)\sin\varphi\sin\theta+\left(n^{(1)}-n^{(2)}\right)\cos\varphi\sin\theta+\left(u^{(1)}-u^{(2)}\right)\cos\theta+\Delta PCV(\theta,\varphi).$$

Fig. 4 shows the example of σ evaluation for ΔPCC of two successive calibrations of Harxon HXCCGX601A antenna performed on Topcon calibration facility. Scalar estimate was computed for different GNSS signals for three weight functions.

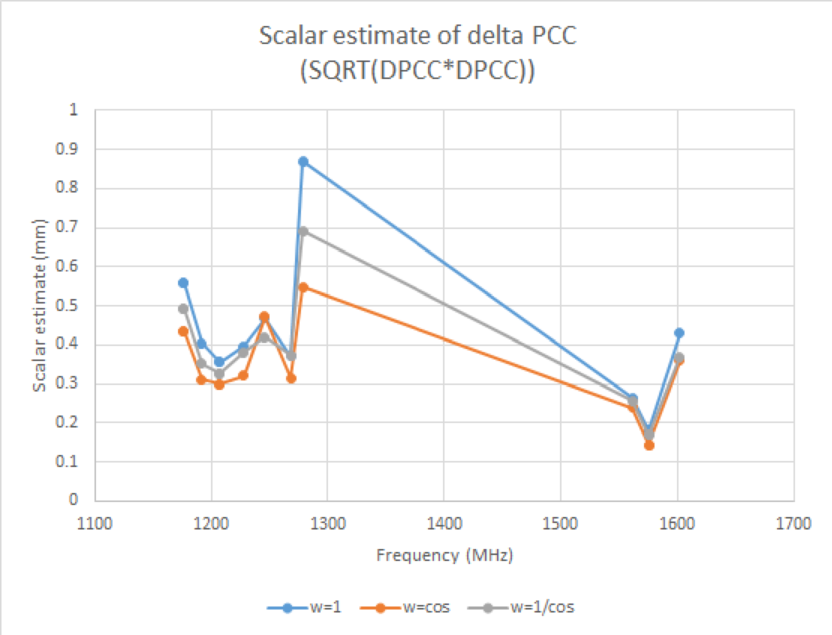

Fig 4. Scalar estimate (6) of ΔPCC versus frequency for two successive calibrations of Harxon HXCCGX601A antenna performed 09/24/2023 and 09/25/2023 on Topcon calibration facility.

Another possible option for introducing a scalar estimate is associated with forced zeroing of the term (A.21). This leads to a simpler expression

$$\sigma_{simple} = a \cdot \sqrt{\frac{e^2 + n^2 + u^2}{a^2} + \int_0^{2\pi} \int_0^{\pi/2} \Delta PCV(\theta, \varphi)^2 \, w(\theta) \sin\theta \, d\theta d\varphi} \tag{7}$$

which is a simple quadrature sum of scalar estimates of ΔPCO and ΔPCV. There is no need for selecting roots and signs, but here we assume that $\Delta PCO(\theta, \varphi)$ and $\Delta PCV(\theta, \varphi)$ are orthogonal in the sense of scalar product $\int_0^{2\pi} \int_0^{\pi/2} \Delta PCO(\theta, \varphi) \Delta PCV(\theta, \varphi) w(\theta) \sin\theta \, d\theta d\varphi$, which is not true in common case. Fig 5 shows the result of applying this to the same data. As it can be seen this scalar estimate is a little more optimistic.

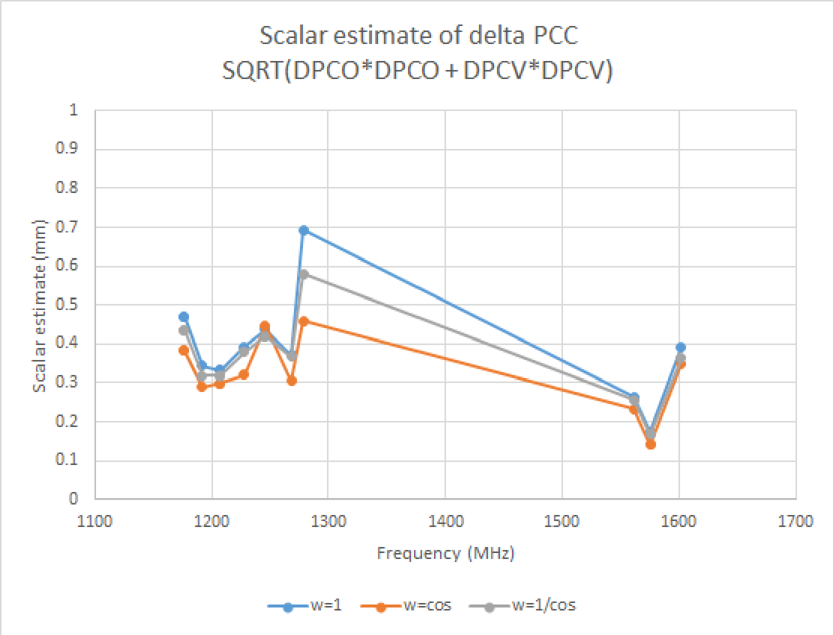

Fig 5. Simple scalar estimate (7) of ΔPCC versus frequency for two successive calibrations of Harxon HXCCGX601A antenna performed 09/24/2023 and 09/25/2023 on Topcon calibration facility.

The worst-case GALLILEO E6 $\Delta PCC(\theta,\varphi)$ for this data is shown on Fig 6 to visually compare $\Delta PCC(\theta,\varphi)$ with scalar estimates.

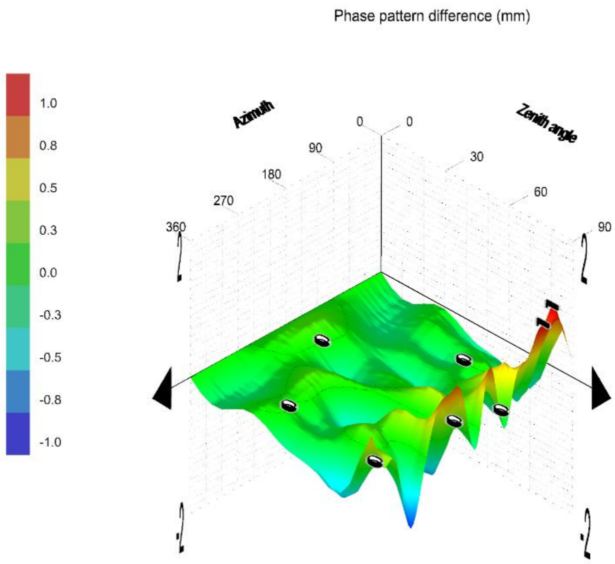

Fig 6. GALLILEO E6 $\Delta PCC(\theta,\varphi)$ for two successive calibrations of Harxon HXCCGX601A antenna performed 09/24/2023 and 09/25/2023 on Topcon calibration facility.

Conclusion

If the presented options for calculating estimate of PCO, ΔPCC and the scalar estimate of ΔPCC do not raise serious objections, then it would be good to choose one of the three options for weighting functions (I like $w(\theta)=\cos\theta$ best as it reflects the features of solving GNSS positioning problems) and one of the two presented methods for calculating the scalar estimate σ of ΔPCC (here I have no particular preference, the first one is more complicated but seems more correct, the second one is simpler and more optimistic).

Appendix. Evaluating scalar measure of delta PCC

If delta PCV = 0, then according to definition (1) and (6)

$$\sigma = a\cdot\sqrt{\int_0^{2\pi}\int_0^{\pi/2} \Delta PCC_f(\theta,\varphi)^2\, w(\theta)\sin\theta\, d\theta d\varphi} =$$
$$a\cdot\sqrt{\int_0^{2\pi}\int_0^{\pi/2} (e\sin\varphi\sin\theta + n\cos\varphi\sin\theta + u\cos\theta - r)^2\, w(\theta)\sin\theta d\theta d\varphi} =$$
$$a\sqrt{\int_0^{2\pi}\int_0^{\pi/2} \left(e^2\sin^2\varphi\sin^2\theta + n^2\cos^2\varphi\sin^2\theta + u^2\cos^2\theta - 2ur\cos\theta + r^2\right) w(\theta)\sin\theta d\theta d\varphi} \tag{A.1}$$

Where $e = e_f^{(1)} - e_f^{(2)}$, $n = n_f^{(1)} - n_f^{(2)}$, $u = u_f^{(1)} - u_f^{(2)}$ - PCO components differences.

Our goal is to find a and r such that

$$\sigma = \sqrt{e^2 + n^2 + u^2} \tag{A.2}$$

First find r such that

$$\int_0^{2\pi}\int_0^{\pi/2} \left(e^2\sin^2\varphi\sin^2\theta + n^2\cos^2\varphi\sin^2\theta + u^2\cos^2\theta - 2ur\cos\theta + r^2\right) w(\theta)\sin(\theta) d\theta d\varphi = const\cdot\left(e^2 + n^2 + u^2\right) \tag{A.3}$$

For different weight functions $w(\theta)$ we get

For $w(\theta) = 1$

$$\int_0^{2\pi}\int_0^{\pi/2} e^2\sin^2\varphi\sin^2\theta w(\theta)\sin(\theta) d\theta d\varphi = \frac{2\pi}{3}e^2 \tag{A.4}$$

$$\int_0^{2\pi}\int_0^{\pi/2} n^2\cos^2\varphi\sin^2\theta w(\theta)\sin(\theta) d\theta d\varphi = \frac{2\pi}{3}n^2 \tag{A.5}$$

$$\int_0^{2\pi}\int_0^{\pi/2} \left(u^2\cos^2\theta - 2ur\cos\theta + r^2\right) w(\theta)\sin(\theta) d\theta d\varphi = \frac{2\pi}{3}\left(3r^2 - 3ru + u^2\right) \tag{A.6}$$

And we get two possible values for r

$$r_1 = u;\quad r_2 = 0 \tag{A.7}$$

For $w(\theta)=\cos\theta$

$$\int_0^{2\pi}\int_0^{\pi/2} e^2\sin^2\varphi\sin^2\theta\cos\theta\sin\theta d\theta d\varphi=\frac{\pi}{4}e^2 \quad (A.8)$$

$$\int_0^{2\pi}\int_0^{\pi/2} n^2\cos^2\varphi\sin^2\theta\cos\theta\sin\theta d\theta d\varphi=\frac{\pi}{4}n^2 \quad (A.9)$$

$$\int_0^{2\pi}\int_0^{\pi/2}\left(u^2\cos^2\theta-2ur\cos\theta+r^2\right)\cos\theta\sin\theta d\theta d\varphi=\pi\left(r^2-\frac{4}{3}ru+\frac{1}{2}u^2\right) \quad (A.10)$$

Equating coefficients before e^2, n^2 and u^2 we obtain the following equations

$$\frac{1}{4}=\frac{r^2}{u^2}-\frac{4}{3}\frac{r}{u}+\frac{1}{2}\ \ or\ \ x^2-\frac{4}{3}x+\frac{1}{4}=0,\ \ x=\frac{r}{u}$$

$$D=\frac{16}{9}-1=\frac{7}{9},\ \sqrt{D}=\frac{\sqrt{7}}{3},x_1=\frac{\frac{4}{3}+\frac{\sqrt{7}}{3}}{2}=\frac{4+\sqrt{7}}{6},x_2=\frac{\frac{4}{3}-\frac{\sqrt{7}}{3}}{2}=\frac{4-\sqrt{7}}{6} \quad (A.12)$$

$$r_1=\frac{4+\sqrt{7}}{6}u,\ \ r_2=\frac{4-\sqrt{7}}{6}u$$

For $w(\theta)=1/\sin(\theta)$

$$\int_0^{2\pi}\int_0^{\pi/2} e^2\sin^2\varphi\sin^2\theta d\theta d\varphi=\frac{\pi^2}{4}e^2 \quad (A.13)$$

$$\int_0^{2\pi}\int_0^{\pi/2} n^2\cos^2\varphi\sin^2\theta d\theta d\varphi=\frac{\pi^2}{4}n^2 \quad (A.14)$$

$$\int_0^{2\pi}\int_0^{\pi/2}\left(u^2\cos^2\theta-2ur\cos\theta+r^2\right)d\theta d\varphi=\pi^2\left(r^2-\frac{4}{\pi}ru+\frac{1}{2}u^2\right) \quad (A.15)$$

Equating coefficients before e^2, n^2 and u^2 we obtain the following equations

$$\frac{1}{4}=\frac{r^2}{u^2}-\frac{4}{\pi}\frac{r}{u}+\frac{1}{2},\ \ x^2-\frac{4}{\pi}x+\frac{1}{4}=0,\ \ x=\frac{r}{u}$$

$$D=\frac{16}{\pi^2}-1=\frac{16-\pi^2}{\pi^2},\ \ \sqrt{D}=\frac{\sqrt{16-\pi^2}}{\pi}$$

$$x_1=\frac{\frac{4}{\pi}+\frac{\sqrt{16-\pi^2}}{\pi}}{2}=\frac{4+\sqrt{16-\pi^2}}{2\pi},\ x_1=\frac{4-\sqrt{16-\pi^2}}{2\pi} \quad (A.16)$$

$$r_1=\frac{4+\sqrt{16-\pi^2}}{2\pi}u,\ \ r_2=\frac{4-\sqrt{16-\pi^2}}{2\pi}u$$

Next find the value of a substituting found r into (A.1) and (A.2). Note that both r_1 and r_2 values of r give the same result for a

For $w(\theta)=1$

$$a\cdot\sqrt{\int_0^{2\pi}\int_0^{\pi/2}\left(e\sin\varphi\sin\theta+n\cos\varphi\sin\theta+u\cos\theta\right)^2\sin\theta d\theta d\varphi}=a\sqrt{\frac{2\pi}{3}\left(e^2+n^2+u^2\right)}=\sqrt{e^2+n^2+u^2}$$

$$a=\sqrt{\frac{3}{2\pi}}$$

(A.17)

For $w(\theta)=\cos(\theta)$

$$a\cdot\sqrt{\int_0^{2\pi}\int_0^{\pi/2}\left(e\sin\varphi\sin\theta+n\cos\varphi\sin\theta+u\cos\theta-\frac{4-\sqrt{7}}{6}u\right)^2\cos\theta\sin\theta d\theta d\varphi}=$$

$$a\sqrt{\frac{\pi}{4}\left(e^2+n^2+u^2\right)}=\sqrt{e^2+n^2+u^2} \qquad \text{(A.18)}$$

$$a=\frac{2}{\sqrt{\pi}}$$

For $w(\theta)=1/\cos(\theta)$

$$a\cdot\sqrt{\int_0^{2\pi}\int_0^{\pi/2}\left(e\sin\varphi\sin\theta+n\cos\varphi\sin\theta+u\cos\theta-\frac{4+\sqrt{16-\pi^2}}{2\pi}u\right)^2 d\theta d\varphi}=$$

$$a\sqrt{\frac{\pi^2}{4}\left(e^2+n^2+u^2\right)}=\sqrt{e^2+n^2+u^2} \qquad \text{(A.19)}$$

$$a=\frac{2}{\pi}$$

So, the parameter a is defined for all three weight functions. But for the parameter r there are still two values in each case. As we just found out in case of a PCC difference with equal PCV both r_1 and r_2 values give the same σ. The same is true in case when PCV are different but VPCO are the same, because r is included in the expressions for σ as a multiplier of VPCO difference. But in common case when both VPCO and PCV are different the scalar estimates for r_1 and r_2 may be different too.

To select a right value for r consider the following expression

$$\sigma=a\cdot\sqrt{\int_0^{2\pi}\int_0^{\pi/2}\left(\Delta PCO(\theta,\varphi)+\Delta PCV(\theta,\varphi)\right)^2 w(\theta)\sin\theta\, d\theta d\varphi}=$$

$$a\cdot\sqrt{\int_0^{2\pi}\int_0^{\pi/2}\left(\Delta PCO(\theta,\varphi)^2+2\Delta PCO(\theta,\varphi)\Delta PCV(\theta,\varphi)+\Delta PCV(\theta,\varphi)^2\right)w(\theta)\sin\theta\, d\theta d\varphi}=$$

$$a\cdot\sqrt{\frac{e^2+n^2+u^2}{a^2}+2\int_0^{2\pi}\int_0^{\pi/2}\Delta PCO(\theta,\varphi)\Delta PCV(\theta,\varphi)w(\theta)\sin\theta\, d\theta d\varphi+\int_0^{2\pi}\int_0^{\pi/2}\Delta PCV(\theta,\varphi)^2 w(\theta)\sin\theta\, d\theta d\varphi}$$

(A.20)

where $\Delta PCO(\theta,\varphi) = e\sin\varphi\sin\theta + n\cos\varphi\sin\theta + u\cos\theta - r$,
$\Delta PCV(\theta,\varphi) = PCV^{(1)}(\theta,\varphi) - PCV^{(2)}(\theta,\varphi)$.

Integral of $\Delta PCO(\theta,\varphi)\Delta PCV(\theta,\varphi)w(\theta)\sin\theta$ can be evaluated as follows

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi/2} \Delta PCO(\theta,\varphi)\Delta PCV(\theta,\varphi)w(\theta)\sin\theta\, d\theta d\varphi = \\
&\int_0^{2\pi}\int_0^{\pi/2} (e\sin\varphi\sin\theta + n\cos\varphi\sin\theta + u\cos\theta - r)\Delta PCV(\theta,\varphi)w(\theta)\sin\theta\, d\theta d\varphi = \\
&\int_0^{2\pi}\int_0^{\pi/2} e\sin\varphi\sin\theta\ \Delta PCV(\theta,\varphi)w(\theta)\sin\theta\, d\theta d\varphi + \int_0^{2\pi}\int_0^{\pi/2} n\cos\varphi\sin\theta\ \Delta PCV(\theta,\varphi)w(\theta)\sin\theta\, d\theta d\varphi + \\
&\int_0^{2\pi}\int_0^{\pi/2} u\cos\theta\ \Delta PCV(\theta,\varphi)w(\theta)\sin\theta\, d\theta d\varphi - \int_0^{2\pi}\int_0^{\pi/2} r\ \Delta PCV(\theta,\varphi)w(\theta)\sin\theta\, d\theta d\varphi
\end{aligned} \tag{A.21}$$

It follows from the expression (3) that first two terms are zero. The third and the fourth terms can be evaluated using expressions (3) and (4)

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi/2} u\cos\theta\,\Delta PCV(\theta,\varphi)w(\theta)\sin\theta d\theta d\varphi = \\
&\int_0^{2\pi}\int_0^{\pi/2} u\cos\theta\left(PCC - e\sin\varphi\sin\theta - n\cos\varphi\sin\theta - u\cos\theta\right)w(\theta)\sin\theta d\theta d\varphi = \\
&u\int_0^{2\pi}\int_0^{\pi/2} PCC\cos\theta\, w(\theta)\sin\theta d\theta d\varphi - u\int_0^{2\pi}\int_0^{\pi/2} \cos\theta\, u\cos\theta\, w(\theta)\sin\theta d\theta d\varphi = \\
&u\left(N_{3,3}u + N_{3,4}\rho\right) - uN_{3,3}u = uN_{3,4}\rho
\end{aligned} \tag{A.22}$$

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi/2} r\ \Delta PCV(\theta,\varphi)w(\theta)\sin\theta\, d\theta d\varphi = \\
&\int_0^{2\pi}\int_0^{\pi/2} r\left(PCC - e\sin\varphi\sin\theta - n\cos\varphi\sin\theta - u\cos\theta\right)w(\theta)\sin\theta d\theta d\varphi = \\
&r\int_0^{2\pi}\int_0^{\pi/2} PCC\, w(\theta)\sin\theta d\theta d\varphi - r\int_0^{2\pi}\int_0^{\pi/2} u\cos\theta\, w(\theta)\sin\theta d\theta d\varphi = \\
&r\left(N_{4,3}u + N_{4,4}\rho\right) - rN_{4,3}u = rN_{4,4}\rho
\end{aligned} \tag{A.23}$$

where $N_{3,3}, N_{3,4}, N_{4,3}, N_{4,4}$ are elements of matrix N from (3), ρ is the solution of (3) for $PCC = e\sin\varphi\sin\theta + n\cos\varphi\sin\theta + u\cos\theta + \Delta PCV(\theta,\varphi)$.

So, we obtain

$$\int_0^{2\pi}\int_0^{\pi/2} \Delta PCO(\theta,\varphi)\Delta PCV(\theta,\varphi)w(\theta)\sin\theta\, d\theta d\varphi = \left(uN_{3,4} - rN_{4,4}\right)\rho \tag{A.24}$$

and

$$
\sigma = a \cdot \sqrt{\int_0^{2\pi}\int_0^{\pi/2}\left(\Delta PCO(\theta,\varphi)+\Delta PCV(\theta,\varphi)\right)^2 w(\theta)\sin\theta\, d\theta d\varphi} =
$$

$$
a \cdot \sqrt{\frac{e^2+n^2+u^2}{a^2} + 2\left(uN_{3,4} - rN_{4,4}\right)\rho + \int_0^{2\pi}\int_0^{\pi/2} \Delta PCV(\theta,\varphi)^2 w(\theta)\sin\theta\, d\theta d\varphi} \tag{A.25}
$$

$$
\left(uN_{3,4} - rN_{4,4}\right)\rho = \begin{cases}
(u\pi - r2\pi)\rho = \left(u\pi - \begin{Bmatrix} u \\ 0 \end{Bmatrix} 2\pi\right)\rho = \mp\pi u\rho, \quad w(\theta) = 1 \\
\left(u\dfrac{2\pi}{3} - r\pi\right)\rho = u\left(\dfrac{2\pi}{3} - \dfrac{\pi 4 \pm \pi\sqrt{7}}{6}\right)\rho = \mp\pi\dfrac{\sqrt{7}}{6}u\rho, \quad w(\theta) = \cos\theta \\
\left(u2\pi - r\pi^2\right)\rho = u\left(2\pi - \pi^2\dfrac{4 \pm \sqrt{16-\pi^2}}{2\pi}\right)\rho = \mp\pi\dfrac{\sqrt{16-\pi^2}}{2}u\rho, \quad w(\theta) = 1/\sin\theta
\end{cases} \tag{A.26}
$$

Upper sign in these expressions is chosen if the solution r_1 of (A.7), (A.12) and (A.16) is selected and lower sign if r_2.

The choice of the root of the equation (A.7), (A.12) and (A.16) as well as the sign in (A.26) can be made from the following considerations: In case of zero $\Delta PCV(\theta,\varphi)$ expressions (A.26) are zero too (because ρ is zero) and as it was stated above the scalar estimate of the ΔPCC is equal to the distance between PCO. Now if a non-zero ΔPCV appears and ΔPCO does not change, then our scalar estimate must increase too. It will always happen if expressions (A.26) are positive. And this is the criterion for choosing their sign and, accordingly, the root of (A.7), (A.12) and (A.16). So, if $u\rho$ is positive, then we have to choose lower sign in (A.26) and the root r_2, otherwise upper sign and the root r_1 is chosen.

It should be noted that even if the expression (A.26) is negative this does not always imply a decrease in the scalar estimate when a non-zero ΔPCV appears, because the last term in (A.25) may grow faster than $2\left(uN_{3,4} - rN_{4,4}\right)\rho$. Using the declared criterion we obtain a pessimistic scalar estimate.

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