Appendix B: Equilibria and Stability Analysis

B.1 Equilibria Derivations

Plasmid-free only equilibrium (F^*, P^*) where $P^* = 0$

$$\frac{dF}{dt} = r(1-s)\left(1-\frac{F}{K}\right)F - \mu F = 0$$

Factor F:

$$F\left[r(1-s)\left(1-\frac{F}{K}\right)-\mu\right] = r(1-s)\left(1-\frac{F^*}{K}\right)-\mu = 0$$

Solve for F^*:

$$1-\frac{F^*}{K} = \frac{\mu}{r(1-s)}$$

$$F^* = K\left(1-\frac{\mu}{r(1-s)}\right)$$

Thus, plasmid-free equilibrium:

$$(F^*, P^*) = \left(K\left(1-\frac{\mu}{r(1-s)}\right), 0\right)$$

And existence condition:

$$\mu < r(1-s)$$

Plasmid-bearing only equilibrium (F^*, P^*) where $F^* = 0$

$$\frac{dF}{dt} = r(1-c)\left(1-\frac{P}{K}\right)P - \mu P - \delta P = 0$$

Factor P:

$$P\left[r(1-c)\left(1-\frac{P}{K}\right)-\mu+\delta\right] = r(1-c)\left(1-\frac{P^*}{K}\right)-\mu+\delta = 0$$

Solve for P^*:

$$1-\frac{P^*}{K} = \frac{\mu+\delta}{r(1-c)}$$

$$P^* = K\left(1-\frac{\mu+\delta}{r(1-c)}\right)$$

Thus, plasmid-bearing equilibrium:

$$(F^*, P^*) = \left(0, \left(1-\frac{\mu+\delta}{r(1-c)}\right)\right)$$

And existence condition:

$$\mu + \delta < r(1-c)$$

B.2 Jacobian Matrix

The Jacobian is the matrix of the first derivatives:

$$J(F,P) = \begin{pmatrix} \frac{\partial \dot{\mathrm{F}}}{\partial F} & \frac{\partial \dot{\mathrm{F}}}{\partial P} \\ \frac{\partial \dot{\mathrm{P}}}{\partial F} & \frac{\partial \dot{\mathrm{P}}}{\partial P} \end{pmatrix}$$

The $\dot{\mathrm{F}}$ equation:

$$\dot{\mathrm{F}} = r(1-s)F - \frac{r(1-s)}{K}F^2 - \frac{r(1-s)}{K}FP - \mu F + \delta P - \beta FP$$

With respect to F:

$$\frac{\partial \dot{\mathrm{F}}}{\partial F} = r(1-s) - 2\frac{r(1-s)}{K}F - \frac{r(1-s)}{K}P - \mu - \beta P$$

$$\frac{\partial \dot{\mathrm{F}}}{\partial F} = r(1-s)\left(1 - \frac{2F+P}{K}\right) - \mu - \beta P$$

With respect to P:

$$\frac{\partial \dot{\mathrm{F}}}{\partial P} = -\frac{r(1-s)}{K}F + \delta - \beta F$$

The $\dot{\mathrm{P}}$ equation:

$$\dot{\mathrm{F}} = r(1-c)P - \frac{r(1-c)}{K}P^2 - \frac{r(1-c)}{K}FP - (\mu+\delta)P + \beta FP$$

With respect to F:

$$\frac{\partial \dot{\mathrm{P}}}{\partial F} = -\frac{r(1-c)}{K}P + \delta P$$

With respect to P:

$$\frac{\partial \dot{\mathrm{P}}}{\partial P} = r(1-c) - 2\frac{r(1-c)}{K}P - \frac{r(1-c)}{K}F - (\mu+\delta) + \beta F$$

$$\frac{\partial \dot{\mathrm{P}}}{\partial P} = r(1-c)\left(1 - \frac{2P+F}{K}\right) - (\mu+\delta) + \beta F$$

Thus, the full Jacobian:

$$J(F,P) = \begin{pmatrix} r(1-s)\left(1 - \frac{2F+P}{K}\right) - \mu - \beta P & -\frac{r(1-s)}{K}F + \delta - \beta F \\ -\frac{r(1-c)}{K}P + \delta P & r(1-c)\left(1 - \frac{2P+F}{K}\right) - (\mu+\delta) + \beta F \end{pmatrix}$$

B.3 Invasion Eigenvalue

To test whether plasmids can invade a plasmid-free population, evaluate the Jacobian at the plasmid-free equilibrium $(F^*, P^*) = (K(1 - \mu/r(1 - s))),0)$:

The lower-right entry (effect on P) is the invasion eigenvalue:

$$\lambda_{invade} = \partial\dot{P}/\partial P|_{(F^*,0)} = r(1-c)\left(1-\frac{F^*}{K}\right) - (\mu+\delta) + \beta F^*$$

Substitute $F^* = K(1 - \mu/r(1 - s))$:

$$\lambda_{invade} = r(1-c)\frac{\mu}{r(1-s)} - (\mu+\delta) + \beta K\left(1-\frac{\mu}{r(1-s)}\right)$$

Simplified:

$$\lambda_{invade} = \frac{r(1-c)\mu}{r(1-s)} - \mu - \delta + \beta K\left(1-\frac{\mu}{r(1-s)}\right)$$

$$\lambda_{invade} = \beta K\left(1-\frac{\mu}{r(1-s)}\right) - \delta - \mu\frac{c}{1-s}$$

Plasmids can invade if $\lambda_{invade} > 0$.

B.4 Critical Values for Plasmid Invasion

Set $\lambda_{invade} = 0$ to solve for the critical transfer rate β_{crit}:

$$\beta_{crit} = \frac{\delta + \mu\frac{c}{1-s}}{\left(1-\frac{\mu}{r(1-s)}\right)}$$

When $s = 0$:

$$\beta_{crit} = \frac{\delta + \mu c}{1-\frac{\mu}{r}}$$

This is the minimum conjugation rate required for plasmids to successfully invade a plasmid-free population.

Set $\lambda_{invade} = 0$ to solve for the critical selective pressure s_{crit}:

$$s_{crit} = \frac{\mu\left(\frac{\beta K}{r} + c\right)}{\beta K - \delta}$$

This is minimum selective pressure needed for plasmids to provide a significant advantage over plasmid-free cells and invade the population.