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1 Definition: Irreducible

For equivalent definitions, see Wiki [HERE](#); Also see "dense" on Wiki [HERE](#).
See Atiyah's [1] Exercise 19 from Chapter 1 for more information...

2 Example 1.1.3.

See Atiyah's [1] Exercise 19 from Chapter 1, which proves it must be dense.

For the sake of contradiction, assume a non-empty subset $A \subset X$ is reducible.
Hence there exist two proper closed subset $A_1, A_2 \subset A$ such that $A = A_1 \cup A_2$.
Then we have

$$X = (A^c \cup A_1) \cup (A^c \cup A_2),$$

which implies X is reducible.

✗!

2.1

See a post [HERE](#)

First approach applied some density argument. While the second approach, similarly, gave a decomposition of the whole space given the open set is reducible.

✓

3 Example 1.1.4.

See Atiyah's [1] Exercise 20 from Chapter 1.

4 Definition.

"Induced topology". Definition of *quasi affine variety*, see [HERE](#).

5 Prop 1.2 (d)

According to Hilbert's Nullstellensatz, I agree we'll get $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$. For the reverse inclusion, pick any $f \in A$ such that $f^r \in \mathfrak{a}$ where $r \in \mathbb{Z}_{>0}$. We wish to show that $f(P) = 0$ for any $P \in Z(\mathfrak{a})$. By definition, $f^r(P) = 0$ given $f^r \in \mathfrak{a}$. And this implies

$$f^r(P) = (f(P))^r = 0 \Rightarrow f(P) = 0$$

given the polynomial ring A is an integral domain. Therefore we get the inclusion

$$I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}.$$

See Theorem 6 on Page 183, Strong Nullstellensatz, [2].

✓

6 Theorem 1.3A. Hilbert's Nullstellensatz

For the case where k isn't algebraically closed, see [4] Remarks in 5.6.

6.1

For the proof, see [3] Chapter 1 for details.

The followings are some comments for [3] Chapter 1 Theorem 1.7.

A post on surjective preimage for maximal ideal [HERE](#).

A post on preimage for maximal ideal (not necessarily surj) [HERE](#).

For completeness, a post on preimage of prime ideals [HERE](#).

A post on image of prime ideals [HERE](#), [HERE](#), and [HERE](#).

See Kemper's [3], Lemma 1.22 on Page 17, which completely described prime and maximal ideals in quotients.

7 Definition. Height

Here the definition *height* is specifically for prime ideal $\mathfrak{p} \triangleleft_{pr} R$ for some ring R . For a general definition, see a post [HERE](#); see a webpage [HERE](#); or see [3] Definition 6.10 on Page 68.

8 Proposition 1.7.

Difference between algebraic set and affine algebraic set? See [HERE](#).

For an analogue in Projective, see Exercise 2.6 ?? in Chapter 1.2. And a post [HERE](#).

9 Theorem 1.8A.

For transcendence degree, see [HERE](#) and a NOTE by Milne James.

10 Proposition 1.10.

Apart from the proof, we discussed *locally closed subset*. See [HERE](#) for its equivalent definitions.

11 Proposition 1.13.

See [HERE](#).

12 Exercise 1.1.

12.1 (a)

By definition of affine coordinate ring we have

$$A(Y) = A/I(Y) = A/I(Z(f)) = A/\sqrt{\langle f \rangle}.$$

While f is irreducible in U.F.D. $k[x, y]$, the ideal it generated will be a prime ideal, which is radical. Therefore we can further simplify the expression as

$$A/\sqrt{\langle f \rangle} = A/\langle f \rangle = k[x, y]/\langle y - x^2 \rangle = k[x].$$

Hence we can conclude $A(Y)$ is isomorphic to a polynomial ring in one variable over k .

12.2 (b)

13 Proposition 2.2.

There are two very technical claims need more details.

The first one is to prove $\varphi(Y) = Z(T') = Z(\alpha(T))$. Unwrap the notation precisely according to the definition

$$\begin{aligned} Z(\alpha(T)) &= \{ x \in \mathbb{A}^n \mid \alpha(g)(x) = 0 \ \forall \ g \in T \}, \\ \varphi(Y) &= \{ \varphi(y) \mid y \in Y \}. \end{aligned}$$

Notice that $y = [y_0, \dots, y_n] \in Y \subset \overline{Y} = Z(T)$, therefore $g(y) = 0$ for any $g \in T$. More precisely, we have

$$\alpha(g)(\varphi(y)) = g(1, y_1/y_0, \dots, y_n/y_0) = 0$$

given $g(y) = 0$ and $g \in T \subset S^h$, which proves $\varphi(Y) \subset Z(\alpha(T))$.

Conversely, let's start with an element $x = (x_1, \dots, x_n) \in Z(\alpha(T))$. There's an element $y = [1, x_1, \dots, x_n] \in Y$ such that $\varphi(y) = x$. Hence we've proved the equality.

And the second one is to check $\varphi^{-1}(W) = Z(\beta(T')) \cap U = Z(\beta(\alpha(T))) \cap U$.

14 Lemma 3.1.

See Sandor's Notes Lecture 4, Lemma 2.6.

Closedness can be checked locally. See a post [HERE](#).

15 Remark 3.1.1.

See Sandor's Notes Lecture 5, Corollary 2.10. Notice that Hartshorne defined variety to be irreducible. See a post explaining why the preimage is dense [HERE](#).

16 Lemma 3.6.

See a post [HERE](#).

Here are some details for proving $x_i \circ \psi$ being regular implies ψ is a morphism: Coordinate functions means

$$\psi(p) = (\psi_1(p), \psi_2(p), \dots, \psi_n(p)) \in Y \subset \mathbb{A}^n$$

for any $p \in X$. Firstly, we check ψ is continuous. Take any closed subset $Z(f_1, \dots, f_r) \subset Y$ for some polynomial $f_1, \dots, f_r \in A = k[x_1, \dots, x_n]$. We can compute the preimage as

$$\psi^{-1}(Z(f_1, \dots, f_r)) = \{ p \in X \mid \forall p \in X, f_i \circ \psi(p) = 0 \}.$$

Notice that for any $p \in X$,

$$f_i \circ \psi(p) = f_i(\psi_1(p), \dots, \psi_n(p))$$

is continuous since f_i is a polynomial and each $\psi_i := x_i \circ \psi$ is continuous by assumption that they're regular. Notice that the preimage of ψ is precisely intersection of $\psi_i^{-1}(\{0\})$ where $1 \leq i \leq n$. Hence the preimage is closed, and it follows that ψ is continuous as expected.

Secondly, fix an arbitrary open subset $V \subset Y$ with an arbitrary regular function $g : V \rightarrow k$, we wish to prove $g \circ \psi : \psi^{-1}(V) \rightarrow k$ is regular. For any $\psi(p) \in V$ with some $p \in X$, there exists a neighborhood $\psi(p) \in U \subset Y$ such that g equals to an expression of quotients of polynomial, i.e.

$$g = \frac{g_1}{g_2}$$

where $g_1, g_2 \in A$. Then for $p \in X$, take the open neighborhood of it as $\psi^{-1}(U)$, we can see

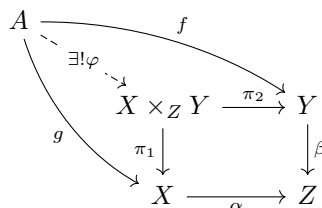
$$g \circ \psi(p) = \frac{g_1(\psi(p))}{g_2(\psi(p))}.$$

17 1.3.F. EXERCISE.

A post discussing this problem is [HERE](#).

18 1.3.N

A crutial step is to define the map such that the diagram commute. In order to prove it satisfies the universal property. We're given A be arbitrary with map $g : A \rightarrow X$ and $f : A \rightarrow Y$.



We can to define

$$\begin{aligned} \varphi : A &\rightarrow X \times_Z Y \text{ by} \\ a &\mapsto (g(a), f(a)). \end{aligned}$$

And we can verify this definition will make the diagram commute, and is unique.

19 1.3.O

It's indeed intersection. A post [HERE](#).

A post [HERE](#).

20 1.3.P.

Say we have $X \times Y$ and $X \times_Z Y$. By universal property of product and fibered product we can produce two unique map goes in between. Their composition must be identity, hence they're isomorphic. Notice it's important for Z being a final object.

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_2} & Y \\ \pi_1 \downarrow & & \downarrow \beta \\ X & \xrightarrow{\alpha} & Z \end{array}$$

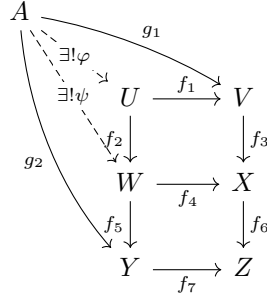
When we trying to make a map from $X \times Y$ to $X \times_Z Y$, we have to make sure two maps

$$\beta \circ \pi_1 = \alpha \circ \pi_1$$

on $X \times Y$. While Z is the final object, hence they must agree.

There's a cleaner way to state it [HERE](#). Crutial part is applying final property of object Z .

21 1.3.Q.



Label the maps as indicated. To prove the universal property with respect to the "outside rectangle", we're given

$$f_6 f_3 g_1 = f_7 g_2$$

agree on A . While W is fibered product, apply universal property of fibered product with respect to W we immediately get a unique map

$$\psi : A \rightarrow W$$

that makes the diagram involving A, W, X, Y, Z commute. In particular, we know $f_4 \psi = f_3 g_1$. Furthermore, recall that U is the fibered product. We're given the condition that $f_4 \psi = f_3 g_1$, by universal property of U we know there exists a unique map

$$\varphi : A \rightarrow U$$

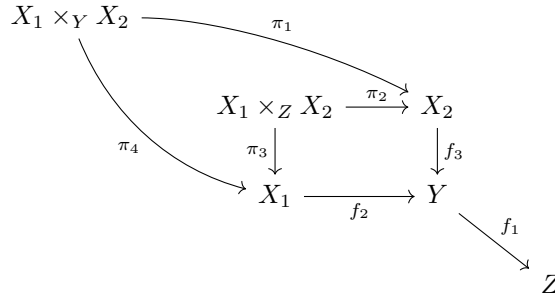
making the diagram involving A, U, V, W, X commute. And we claim that the diagram involving A, U, V, Y, Z commute. This is because

$$g_2 = f_5 \psi = f_5 f_2 \varphi = (f_5 f_2) \varphi \quad \text{and} \quad g_1 = f_1 \varphi.$$

And this proves that U is the fibered product for the diagram involving A, U, V, Y, Z .

A post is [HERE](#).

22 1.3.R



By the universal property of $X_1 \times_Z X_2$, we know there exists a unique map

$$\varphi : X_1 \times_Y X_2 \rightarrow X_1 \times_Z X_2$$

"Natural morphism", a convention discussed [HERE](#).

23 Course Notes from Cornell

See [HERE](#).

24 1.3.S. Magic Diagram

Didn't finish. Need to See [HERE](#), [HERE!!!](#)

25 1.3.Y. (a)

YONEDA'S LEMMA Given what we have, define $g : A \rightarrow A'$ as

$$g := i_A(\text{id}_A).$$

This is correct, see a post [HERE](#).

26 1.4.C.

(a) See "A Term of Commutative Algebra", Example 7.3 on Page 52.

27 1.6.B.

Write out everything by definition, and we can finish the proof immediately by applying rank-nullity theorem for linear transformation...

28 2.2.6. Definition: Sheaf.

Comments on $\mathcal{F}(\emptyset)$. In category **Set**, the empty set is initial object and one element set is terminal. See Wiki's examples [HERE](#).

29 2.2.B.

For (a): see Wiki's counterexample [HERE](#), which gave an explanation for presheaves on \mathbb{R} instead of \mathbb{C} . See a post [HERE](#).

30 2.2.7.

See Daping's Notes Definition 2.5 on Page 4 [HERE](#); also a post [HERE](#); also a post [HERE](#).

31 2.2.10.

It's different from a post [HERE](#), and Wiki's page on Constant pre-Sheaf.

Why???

32 2.2.G.

It's clearly a pre-sheaf.

Fix an open subset $U \subset X$ with an open cover $\{U_i\}_{i \in I}$ for some index set I . Denote the presheaf as \mathcal{F} .

Pick two continuous maps $s_1, s_2 : Y \rightarrow X$ that satisfying the requirements, i.e. $s_1, s_2 \in \mathcal{F}(U)$.

Both functions will agree on U since

$$\text{Res}_{U, U_i} s_1 = \text{Res}_{U, U_i} s_2$$

for arbitrary U_i , whose union is U . So we must have $s_1 = s_2$.

Again with this open cover $\{U_i\}_{i \in I}$ and $a_i \in \mathcal{F}(U_i)$ for $i \in I$. Equivalently, we know $a_i : U_i \rightarrow Y$ is a continuous map satisfying $\mu \circ a_i = \text{Id}|_{U_i}$. Now let's define a map

$$\begin{aligned} f : U &\rightarrow Y \\ u &\mapsto a_i(u) \text{ when } u \in U_i. \end{aligned}$$

It's well-defined by our assumption. Also it's continuous since preimage of an open set in $V \subset Y$ is a union of open subsets given by continuity of each a_i . Similarly we can check $\mu \circ f = \text{Id}|_U$ as expected.

Unverified ?

33 2.3.A.

I'm planning to use universal property to define the induced map ϕ_P .
One crucial step is to verify the diagram below is commutative

$$\begin{array}{ccc}
 \mathcal{F}(U) & \xrightarrow{\rho_{UV}} & \mathcal{F}(V) \\
 \phi(U) \downarrow & & \downarrow \phi(V) \\
 \mathcal{G}(U) & \xrightarrow{\tau_{UV}} & \mathcal{G}(V) \\
 & \searrow & \swarrow \\
 & \mathcal{G}_P &
 \end{array}$$

And this is because the square diagram in the upper half commute given ϕ is a natural transformation; the lower half is by definition of \mathcal{G}_P . Then by universal property of colimit induces a map

$$\phi_P : \mathcal{F}_P \rightarrow \mathcal{G}_P$$

which makes the diagram commute.

See a post defined the map [HERE](#).

34 2.3.B.

To define a functor $\pi_* : \mathbf{Set}_X \rightarrow \mathbf{Set}_Y$. Firstly, we have to define for any $\mathcal{F} \in \mathbf{Set}_X$,

$$\pi_*(\mathcal{F})(U) = \mathcal{F}(\pi^{-1}(U))$$

for any $U \in \mathbf{Top}(Y)$ as in ??.

Secondly, for any natural transformation $\phi : \mathcal{F} \rightarrow \mathcal{G}$, we define $\pi_*(\phi)$ by specifying

$$\pi_*(\phi)(U) \mapsto \mathcal{F}(\pi^{-1}(U)) \rightarrow \mathcal{G}(\pi^{-1}(U)).$$

? Is this correct

35 References

- [1] M.F. Atiyah and I.G. MacDonald. *Introduction To Commutative Algebra*. Addison-Wesley series in mathematics. Avalon Publishing, 1994.
- [2] David Cox, John Little, and Donal OShea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. Springer Science & Business Media, 2013.
- [3] Gregor Kemper. *A course in commutative algebra*, volume 1. Springer, 2011.
- [4] Miles Reid. *Undergraduate commutative algebra*. Number 29. Cambridge University Press, 1995.