Contents

1 Definition: Irreducible

For equivalent definitions, see Wiki HERE; Also see "dense" on Wiki HERE.
See Atiyah's [1] Exercise 19 from Chapter 1 for more information...

2 Example 1.1.3.

See Atiyah's [1] Exercise 19 from Chapter 1, which proves it must be dense.

For the sake of contradiction, assume a non-empty subset $A \subset X$ is reducible. Hence there exist two proper closed subset $A_1, A_2 \subset A$ such that $A = A_1 \cup A_2$. Then we have

$$X = (A^c \cup A_1) \cup (A^c \cup A_2),$$

which implies X is reducible.

2.1

See a post HERE

First approach applied some density argument. While the second approach, similarly, gave a decomposition of the whole space given the open set is reducible.

3 Example 1.1.4.

See Atiyah's [1] Exercise 20 from Chapter 1.

4 Definition.

"Induced topology". Definition of *quasi affine variety*, see HERE.

5 Prop 1.2 (d)

According to Hilbert's Nullstellensatz, I agree we'll get $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$. For the reverse inclusion, pick any $f \in A$ such that $f^r \in \mathfrak{a}$ where $r \in \mathbb{Z}_{>0}$. We wish to show that $f(P) = 0$ for any $P \in Z(\mathfrak{a})$. By definition, $f^r(P) = 0$ given $f^r \in \mathfrak{a}$. And this implies

$$f^r(P) = (f(P))^r = 0 \;\Rightarrow\; f(P) = 0$$

given the polynomial ring A is an integral domain. Therefore we get the inclusion

$$I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}.$$

See Theorem 6 on Page 183, Strong Nullstellensatz, [2].

6 Theorem 1.3A. Hilbert's Nullstellensatz

For the case where k isn't algebraically closed, see [4] Remarks in 5.6.

6.1

For the proof, see [3] Chapter 1 for details.

The followings are some comments for [3] Chapter 1 Theorem 1.7.

A post on surjective preimage for maximal ideal HERE.

A post on preimage for maximal ideal (not necessarily surj) HERE.

For completeness, a post on preimage of prime ideals HERE.

A post on image of prime ideals HERE, HERE, and HERE.

See Kemper's [3], Lemma 1.22 on Page 17, which completely described prime and maximal ideals in quotients.

7 Definition. Height

Here the definition *height* is specifically for prime ideal $\mathfrak{p} \lhd_{pr} R$ for some ring R. For a general definition, see a post HERE; see a webpage HERE; or see [3] Definition 6.10 on Page 68.

8 Proposition 1.7.

Difference between algebraic set and affine algebraic set? See HERE.

For an analogue in Projective, see Exercise 2.6 **??** in Chapter 1.2. And a post HERE.

9 Theorem 1.8A.

For transcendence degree, see HERE and a NOTE by Milne James.

10 Proposition 1.10.

Apart from the proof, we discussed *locally closed subset.* See HERE for its equivalent definitions.

11 Proposition 1.13.

See HERE.

12 Exercise 1.1.

12.1 (a)

By definition of affine coordinate ring we have

$$A(Y) = A/I(Y) = A/I(Z(f)) = A/\sqrt{\langle f \rangle}.$$

While f is irreducible in U.F.D. $k[x, y]$, the ideal it generated will be a prime ideal, which is radical. Therefore we can further simplify the expression as

$$A/\sqrt{\langle f \rangle} = A/\langle f \rangle = k[x, y]/\langle y - x^2 \rangle = k[x].$$

Hence we can conclude $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

12.2 (b)

13 Proposition 2.2.

There are two very technical claims need more details.

The first one is to prove $\varphi(Y) = Z(T') = Z(\alpha(T))$. Unwrap the notation precisely according to the definition

$$\begin{aligned} Z(\alpha(T)) &= \{\ x \in \mathbb{A}^n \ \mid\ \alpha(g)(x) = 0\ \forall\ g \in T\ \}, \\ \varphi(Y) &= \{\ \varphi(y)\ \mid\ y \in Y\ \}. \end{aligned}$$

Notice that $y = [y_0, ..., y_n] \in Y \subset \overline{Y} = Z(T)$, therefore $g(y) = 0$ for any $g \in T$. More precisely, we have

$$\alpha(g)(\varphi(g)) = g(1, y_1/y_0, ..., y_n/y_n) = 0$$

given $g(y) = 0$ and $g \in T \subset S^h$, which proves $\varphi(Y) \subset Z(\alpha(T))$.

Conversely, let's start with an element $x = (x_1, ..., x_n) \in Z(\alpha(T))$. There's an element $y = [1, x_1, ..., x_n] \in Y$ such that $\varphi(y) = x$. Hence we've proved the equality.

And the second one is to check $\varphi^{-1}(W) = Z(\beta(T')) \cap U = Z(\beta(\alpha(T))) \cap U$.

14 Lemma 3.1.

See Sandor's Notes Lecture 4, Lemma 2.6.
Closedness can be checked locally. See a post HERE.

15 Remark 3.1.1.

See Sandor's Notes Lecture 5, Corollary 2.10. Notice that Hartshorne defined variety to be irreducible. See a post explaining why the preimage is dense HERE.

16 Lemma 3.6.

See a post HERE.

Here are some details for proving $x_i \circ \psi$ being regular implies ψ is a morphism: Coordinate functions means

$$\psi(p) = (\psi_1(p), \psi_2(p), ..., \psi_n(p)) \in Y \subset \mathbb{A}^n$$

for any $p \in X$. Firstly, we check ψ is continuous. Take any closed subset $Z(f_1, ..., f_r) \subset Y$ for some polynomial $f_1, ..., f_r \in A = k[x_1, ..., x_n]$. We can compute the preimage as

$$\psi^{-1}(Z(f_1, ..., f_r)) = \{\ p \in X \ \mid \ \forall\ p \in X,\ f_i \circ \psi(p) = 0\ \}.$$

Notice that for any $p \in X$,

$$f_i \circ \psi(p) \ = \ f_i(\psi_i(p), ..., \psi_n(p))$$

is continuous since f_i is a polynomial and each $\psi_i := x_i \circ \psi$ is continuous by assumption that they're regular. Notice that the preimage of ψ is precisely intersection of $\psi_i^{-1}(\{0\})$ where $1 \leq i \leq n$. Hence the preimage is closed, and it follows that ψ is continuous as expected.

Secondly, fix an arbitrary open subset $V \subset Y$ with an arbitrary regular function $g : V \to k$, we wish to prove $g \circ \psi : \psi^{-1}(V) \to k$ is regular. For any $\psi(p) \in V$ with some $p \in X$, there exists a neighborhood $\psi(p) \in U \subset Y$ such that g equals to an expression of quotients of polynomial, i.e.

$$g = \frac{g_1}{g_2}$$

where $g_1, g_2 \in A$. Then for $p \in X$, take the open neighborhood of it as $\psi^{-1}(U)$, we can see

$$g \circ \psi(p) = \frac{g_1(\psi(p))}{g_2(\psi(p))}.$$

17 1.3.F. EXERCISE.

A post discussing this problem is HERE.

18 1.3.N

A crutial step is to define the map such that the diagram commute. In order to prove it satisfies the universal property. We're given A be arbitrary with map $g : A \to X$ and $f : A \to Y$.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists!\varphi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_Z Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\beta"] \\
& X \arrow[r, "\alpha"'] & Z
\end{tikzcd}$$

We can to define

$$\begin{aligned}
\varphi : A \to X \times_Z Y \quad \text{by} \\
a \mapsto (g(a), f(a)).
\end{aligned}$$

And we can verify this definition will make the diagram commute, and is unique.

19 1.3.O

It's indeed intersection. A post HERE.

A post HERE.

20 1.3.P.

Say we have $X \times Y$ and $X \times_Z Y$. By universal property of product and fibered product we can produce two unique map goes in between. Their composition must be identity, hence they're isomorphic. Notice it's important for Z being a final object.

$$\begin{tikzcd}
X \times Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\beta"] \\
X \arrow[r, "\alpha"'] & Z
\end{tikzcd}$$

When we trying to make a map from $X \times Y$ to $X \times_Z Y$, we have to make sure two maps

$$\beta \circ \pi_1 = \alpha \circ \pi_1$$

on $X \times Y$. While Z is the final object, hence they must agree.

There's a cleaner way to state it HERE. Crutial part is applying final property of object Z.

21 1.3.Q.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists!\varphi"] \arrow[drr, bend left, "g_1"] \arrow[ddr, dashed, "\exists!\psi"'] \arrow[dddr, bend right, "g_2"'] & & \\
& U \arrow[r, "f_1"] \arrow[d, "f_2"'] & V \arrow[d, "f_3"] \\
& W \arrow[r, "f_4"'] \arrow[d, "f_5"'] & X \arrow[d, "f_6"] \\
& Y \arrow[r, "f_7"'] & Z
\end{tikzcd}$$

Label the maps as indicated. To prove the universal property with respect to the "outside rectangle", we're given

$$f_6 f_3 g_1 = f_7 g_2$$

agree on A. While W is fibered product, apply universal property of fibered product with resepct to W we immediately get a unique map

$$\psi : A \to W$$

that makes the diagram involving A, W, X, Y, Z commute. In particularly, we know $f_4\psi = f_3 g_1$. Furthermore, recall that U is the fibered product. We're given the condition that $f_4\psi = f_3 g_1$, by universal property of U we know there exists a unique map

$$\varphi : A \to U$$

making the diagram involving A, U, V, W, X commute. And we claim that the diagram involving A, U, V, Y, Z commute. This is because

$$g_2 = f_5\psi = f_5 f_2\varphi = (f_5 f_2)\varphi \text{ and } g_1 = f_1\varphi.$$

And this proves that U is the fibered product for the diagram involving A, U, V, Y, Z.

A post is HERE.

22 1.3.R

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[drr, bend left, "\pi_1"] \arrow[ddr, bend right, "\pi_4"'] & & & \\
& X_1 \times_Z X_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_3"'] & X_2 \arrow[d, "f_3"] & \\
& X_1 \arrow[r, "f_2"'] & Y \arrow[dr, "f_1"] & \\
& & & Z
\end{tikzcd}$$

By the universal property of $X_1 \times_Z X_2$, we know there exists a unique map

$$\varphi : X_1 \times_Y X_2 \to X_1 \times_Z X_2$$

"Natural morphism", a convention discussed HERE.

23 Course Notes from Cornell

See HERE.

24 1.3.S. Magic Diagram

Didn't finish. Need to See HERE, HERE!!!

25 1.3.Y. (a)

YONEDA'S LEMMA Given what we have, define $g : A \to A'$ as

$$g := i_A(\mathrm{id}_A).$$

This is correct, see a post HERE.

26 1.4.C.

(a) See "A Term of Commutative Algebra", Example 7.3 on Page 52.

27 1.6.B.

Write out everything by definition, and we can finish the proof immediately by applying rank-nullity theorem for linear transformation...

28 Definition 2.2.6.

Comments on $\mathscr{F}(\emptyset)$. In category **Set**, the empty set is initial object and one element set is terminal. See Wiki's examples HERE.

29 References

[1] M.F. Atiyah and I.G. MacDonald. *Introduction To Commutative Algebra*. Addison-Wesley series in mathematics. Avalon Publishing, 1994.

[2] David Cox, John Little, and Donal OShea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. Springer Science & Business Media, 2013.

[3] Gregor Kemper. *A course in commutative algebra*, volume 1. Springer, 2011.

[4] Miles Reid. *Undergraduate commutative algebra*. Number 29. Cambridge University Press, 1995.