Contents

1 Notes

See a post HERE, which listed various resources...

See Old Lecture Homepages of ETH HERE. There are some courses on algebraic geometry with solutions for homeworks.

See a post on "Best AG Textbook" HERE.

Great Notes on Algebraic Geometry and Commutative Algebra by Andreas Gathmann HERE.

See Horawa's Notes on Algebraic Geometry.

See HERE for a comprehensive guide to Hartshorne as well as solutions.

See Student Q& A in MITCourseware HERE.

See notes on algebraic geometry by Prof. Dr. Andreas Gathmann HERE.

See HERE for Aaron Landesman's notes.

See a course in 2012 by James McKernan HERE.

See a great book on Sheaf theory: Sheaf Theory through Examples by Daniel HERE.

Part I
Hartshorne

2 Definition: Irreducible

For equivalent definitions, see Wiki HERE; Also see ”dense” on Wiki HERE.

See Atiyah’s [1] Exercise 19 from Chapter 1 for more information...

See Lemma 14. on Page 210 of [2] for equivalent characterisation of *irreducible*.

3 Example 1.1.3.

See Atiyah’s [1] Exercise 19 from Chapter 1, which proves it must be dense.

For the sake of contradiction, assume a non-empty subset $A \subset X$ is reducible. Hence there exist two proper closed subset $A_1, A_2 \subset A$ such that $A = A_1 \cup A_2$. Then we have

$$X = (A^c \cup A_1) \cup (A^c \cup A_2),$$

which implies X is reducible. ✗!

3.1

See a post HERE ✔

First approach applied some density argument. While the second approach, similarly, gave a decomposition of the whole space given the open set is reducible.

4 Example 1.1.4.

See Atiyah’s [1] Exercise 20 from Chapter 1.

5 Definition.

”Induced topology”. Definition of *quasi affine variety*, see HERE.

6 Prop 1.2 (d)

According to Hilbert’s Nullstellensatz, I agree we’ll get $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$. For the reverse inclusion, pick any $f \in A$ such that $f^r \in \mathfrak{a}$ where $r \in \mathbb{Z}_{>0}$. We wish to show that $f(P) = 0$ for any $P \in Z(\mathfrak{a})$. By definition, $f^r(P) = 0$ given $f^r \in \mathfrak{a}$. And this implies

$$f^r(P) = (f(P))^r = 0 \;\Rightarrow\; f(P) = 0$$

given the polynomial ring A is an integral domain. Therefore we get the inclusion

$$I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}.$$

See Theorem 6 on Page 183, Strong Nullstellensatz, [3]. ✔

7 Theorem 1.3A. Hilbert's Nullstellensatz

For the case where k isn't algebraically closed, see [6] Remarks in 5.6.

7.1

For the proof, see [5] Chapter 1 for details.

The followings are some comments for [5] Chapter 1 Theorem 1.7.

A post on surjective preimage for maximal ideal HERE.

A post on preimage for maximal ideal (not necessarily surj) HERE.

For completeness, a post on preimage of prime ideals HERE.

A post on image of prime ideals HERE, HERE, and HERE.

See Kemper's [5], Lemma 1.22 on Page 17, which completely described prime and maximal ideals in quotients.

8 Definition. Height

Here the definition *height* is specifically for prime ideal $\mathfrak{p} \lhd_{pr} R$ for some ring R. For a general definition, see a post HERE; see a webpage HERE; or see [5] Definition 6.10 on Page 68.

Nagata's example: Notherian ring with infinite Krull dimension, see HERE and a post HERE.

9 Proposition 1.7.

Difference between algebraic set and affine algebraic set? See HERE.

For an analogue in Projective, see Exercise 2.6 **??** in Chapter 1.2. And a post HERE.

10 Theorem 1.8A.

For transcendence degree, see HERE and a NOTE by Milne James.

11 Proposition 1.10.

Apart from the proof, we discussed *locally closed subset.* See HERE for its equivalent definitions.

12 Proposition 1.13.

See HERE.

13 Exercise 1.1.

13.1 (a)

By definition of affine coordinate ring we have

$$A(Y) = A/I(Y) = A/I(Z(f)) = A/\sqrt{\langle f \rangle}.$$

While f is irreducible in U.F.D. $k[x, y]$, the ideal it generated will be a prime ideal, which is radical. Therefore we can further simplify the expression as

$$A/\sqrt{\langle f \rangle} = A/\langle f \rangle = k[x, y]/\langle y - x^2 \rangle = k[x].$$

Hence we can conclude $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

13.2 (b)

14 Exericse 1.2.

My initial guess was incorrect, in which I thought $I(Y) = \langle z - xy \rangle$.

14.1

See a post HERE.
See a post HERE.
The correct one is $I(Y) = \langle z - x^3,\ z - y^2 \rangle$. Notice that

$$\dim Y = \dim A(Y) = \dim k[x, y, z]/\langle z - x^3,\ z - y^2 \rangle = \dim k[z] = \operatorname{tr.deg}_k k(z) = 1.$$

Therefore we proved that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

Question: How to compute $I(\cdot)$ precisely?

15 Exercise 1.4.

See a post HERE, HERE, and HERE.

The hint was to consider diagonal. Let the coordinate ring of $\mathbb{A}^2$ be $A(\mathbb{A}^2) = k[x, y]$ where k is algebraically closed. Then the diagonal

$$\Delta = \{(x, y) \in \mathbb{A}^2 \ \mid \ x = y\} = Z(x - y = 0)$$

is closed in Zariski topology of $\mathbb{A}^2$.

But it cannot be countably many union of closed subsets in product topology of copies of Zariski topology on $\mathbb{A}^1$, which are finitely many points.

Wrong!

Here we used the fact that k is infinite given it's algebraically closed.

15.1

See the first post! Closed subset of product topology on $\mathbb{A}^1 \times \mathbb{A}^1$ can be uncountable, for example, a line $\{(x_0, y) \in \mathbb{A}^1 \times A^1 \mid y \in k\}$ where $x_0 \in k$ is fixed would be closed. But it's uncountable.

The reason why Δ is closed in Zariski topology of $\mathbb{A}^2$ is correct.

For product topology, the reason why it's not closed is because Zariski topology on $\mathbb{A}^1$ is not Hausdorff, by a lemma we mentioned we know Δ isn't closed.

Verified!

16 Exercise 1.7.

16.1 (a)

(i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv): Zorn's Lemma.
(ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii): contrapositive, with Axiom of Dependent Choice.
(i) $\Leftrightarrow$ (iii): taking complement.

16.2 (b)

A similar statement is Exercise 17. (v) of [1], which states that $\operatorname{Spec} A$ is quasi-compact for a ring A.

For a given open cover of $\{U_i\}_{i \in I}$ of X, we can throw away open subset U_i such that $U_i \subset U_j$ for some $j \in I$. Here I is some index set. So we can assume in this open cover, we don't have $U_i \subset U_j$ for any $i, j \in I$. Then we have a ascending chain of open subsets

$$U_i \subseteq U_i \cup U_j \subseteq \cdots .$$

Noetherian condition imposed gives us finitely many index, and they form a finite subcover as expected.

16.2.1

A very simple proof HERE used maximality property.
A stronger statement HERE.

17 Exercise 1.10.

17.1 (a)

Clear, since any chain of Y is also a chain in X, and we use property of sup mentioned on Prop 2.8. of HERE.

17.2 (b)

17.3 (c)

17.4 (d)

17.5 (e)

See a post HERE, HERE,

See the notes by Vakil HERE.

See the notes from UofT HERE.

See a post HERE.

Example in topological space is actually easier to say than Nagata's example.

Take our topological space as $X = [0, 1] \subset \mathbb{R}$. And define the topology by declaring all closed subsets as

$$\{\emptyset, X, \{[1/n, 1] \subset \mathbb{R} \mid i \geq 2, i \in \mathbb{Z}\}\}.$$

Clearly we can check it satisfies infinite intersection and finite union axiom.

It's Noetherian for any descending chain of closed subsets must terminate.

It's of infinite Krull dimension. For any $n > 0$, we can construct a chain of such length. Hence the dimension, defined as supremum cannot be finite.

Another weird example could be found HERE, in which we give $\mathbb{N}$ the topology empty set, entire space, and $\{x \in \mathbb{N} \mid x \geq q\}$ for some $a \in \mathbb{N}$.

18 Graded Ring

Say we wish to define variety in some projective space, i.e. some subsets of $\mathbb{P}^n$ that could be written as zero locus of some polynomials. If we don't add any restrictions on polynomials, for example some non-homogeneous polynomials. Then the zero locus doesn't make sense because it will depend on the representitive. See Gathmann's Notes Remark 6.5, notes on algebraic geometry, HERE.

19 Proposition 2.2.

There are two very technical claims need more details.

The first one is to prove $\varphi(Y) = Z(T') = Z(\alpha(T))$. Unwrap the notation precisely according to the definition

$$\begin{aligned} Z(\alpha(T)) &= \{ \ x \in \mathbb{A}^n \ \mid \ \alpha(g)(x) = 0 \ \forall \ g \in T \ \}, \\ \varphi(Y) &= \{ \ \varphi(y) \ \mid \ y \in Y \ \}. \end{aligned}$$

Notice that $y = [y_0, ..., y_n] \in Y \subset \overline{Y} = Z(T)$, therefore $g(y) = 0$ for any $g \in T$. More precisely, we have

$$\alpha(g)(\varphi(g)) = g(1, y_1/y_0, ..., y_n/y_n) = 0$$

given $g(y) = 0$ and $g \in T \subset S^h$, which proves $\varphi(Y) \subset Z(\alpha(T))$.

Conversely, let's start with an element $x = (x_1, ..., x_n) \in Z(\alpha(T))$. There's an element $y = [1, x_1, ..., x_n] \in Y$ such that $\varphi(y) = x$. Hence we've proved the equality.

And the second one is to check $\varphi^{-1}(W) = Z(\beta(T')) \cap U = Z(\beta(\alpha(T))) \cap U$.

20 Exercise 2.12.

For all monomial of degree d in $n + 1$ variables $x_0, ..., x_n$. There are

$$\binom{n+d}{n}$$

many monomials in total. Bars and balls argument: there are n bars and d balls. In $n + d$ many places, any choice of n bars will corresponds to a monomial, therefore N is the total number of monomials possible. While in we wish to consider them in projective space, we must define $N = \binom{n+d}{n} - 1$.

See a solution in lecture notes of Frank-Olaf Schreyer HERE.

A more detailed solution is given HERE.

21 Exercise 2.14.

There's a prompt on Sándor's Notes, Lecture 13, Homework 2.79.

22 Lemma 3.1.

See Sandor's Notes Lecture 4, Lemma 2.6.
Closedness can be checked locally. See a post HERE.

23 Remark 3.1.1.

See Sandor's Notes Lecture 5, Corollary 2.10. Notice that Hartshorne defined variety to be irreducible. See a post explaining why the preimage is dense HERE.
See Lemma 14. on Page 210 of [2].

24 Definition: Ring of Regular Function

HERE is an explicit description on the ring structure of $\mathcal{O}_{P,Y}$.

25 Theorem 3.2.

See Sándor's Lecture Notes 09, STEP 03. See solution of problem 3 HERE. See a post HERE, HERE.

25.1 (c)

for each P, $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$, and $\dim \mathcal{O}_P = \dim Y$;

Proof. We begin with an injective homomorphism $\alpha : A(Y) \to \mathcal{O}(Y)$. And we define a map

$$\begin{aligned} A(Y)_{\mathfrak{m}_P} &\to \mathcal{O}_{P,Y} \\ f/g &\mapsto \langle V, \frac{\alpha(f)}{\alpha(g)} \rangle \end{aligned}$$

where $\frac{\alpha(f)}{\alpha(g)} \in \mathcal{O}(V)$. Now we wish to give an explicit description of V. Since $\alpha(f) \in \mathcal{O}(Y)$, we know there exists an open subset $P \in V_1 \subset Y$ such that

$$\alpha(f) \mid_{V_1} = \frac{h_1}{h_2} \mid_{V_1}$$

where $h_1, h_2 \in A$ and $0 \notin h_2(V_1)$. Since $\alpha(g) \in \mathcal{O}(Y)$, we know there exists an open subset $P \in V_2 \subset Y$ such that

$$\alpha(g) \mid_{V_2} = \frac{h_3}{h_4} \mid_{V_2}$$

where $h_3, h_4 \in A$ and $0 \notin h_4(V_2)$. Here $g \notin \mathfrak{m}_P$ by definition of localisation, which gives us

$$g(P) \neq 0 \;\Rightarrow\; \alpha(g)(P) \neq 0 \;\Rightarrow\; \exists\, V_3 \subset Y,\ \alpha(g) \mid_{V_3} \neq 0.$$

Then we take $V = V_1 \cap V_2 \cap V_3$ will suffice to work. This is because for any point $P \in V$, we have

$$\frac{\alpha(f)}{\alpha(g)} = \frac{h_1 h_4}{h_2 h_3}$$

for $0 \notin h_2 h_3(V)$.

$$\begin{tikzcd}
A(Y) \arrow[r, "\alpha"] \arrow[dr] & \mathcal{O}(Y) \arrow[r, hook] & \mathcal{O}_{P,Y} \\
& A(Y)_{\mathfrak{m}_P} \arrow[ur, dashed] &
\end{tikzcd}$$

The induced map is given by universal property of localisation, for every elements in $A(Y) \setminus \mathfrak{m}_P$ will be mapped to a unit in $\mathcal{O}_{P,Y}$. And the map given by Lecture Notes 09 of Prof. Sándor satisfy the universal property (it makes the diagram commute and maps elements outside of $\mathfrak{m}_P$ to units).

$\square$

26 Proposition 3.3.

See a post HERE.

27 Lemma 3.6.

See a post HERE.

Here are some details for proving $x_i \circ \psi$ being regular implies ψ is a morphism: Coordinate functions means

$$\psi(p) = (\psi_1(p), \psi_2(p), ..., \psi_n(p)) \in Y \subset \mathbb{A}^n$$

for any $p \in X$. Firstly, we check ψ is continuous. Take any closed subset $Z(f_1, ..., f_r) \subset Y$ for some polynomial $f_1, ..., f_r \in A = k[x_1, ..., x_n]$. We can compute the preimage as

$$\psi^{-1}(Z(f_1, ..., f_r)) = \{\ p \in X \ \mid\ \forall\ p \in X,\ f_i \circ \psi(p) = 0\ \}.$$

Notice that for any $p \in X$,

$$f_i \circ \psi(p) \ = \ f_i(\psi_i(p), ..., \psi_n(p))$$

is continuous since f_i is a polynomial and each $\psi_i := x_i \circ \psi$ is continuous by assumption that they're regular. Notice that the preimage of ψ is precisely intersection of $\psi_i^{-1}(\{0\})$ where $1 \leq i \leq n$. Hence the preimage is closed, and it follows that ψ is continuous as expected.

Secondly, fix an arbitrary open subset $V \subset Y$ with an arbitrary regular function $g : V \to k$, we wish to prove $g \circ \psi : \psi^{-1}(V) \to k$ is regular. For any

$\psi(p) \in V$ with some $p \in X$, there exists a neighborhood $\psi(p) \in U \subset Y$ such that g equals to an expression of quotients of polynomial, i.e.

$$g = \frac{g_1}{g_2}$$

where $g_1, g_2 \in A$. Then for $p \in X$, take the open neighborhood of it as $\psi^{-1}(U)$, we can see

$$g \circ \psi(p) = \frac{g_1(\psi(p))}{g_2(\psi(p))}.$$

28 Exercise 3.6.

See a post HERE.

29 Definition: Dominant Rational Map

29.1

Well-definess for a rational map being *dominant*.

One thing important to keep in mind is both varieties X, Y are a priori irreducible. There's a completely point-set topological argument HERE. Here a technical detail is "The image of a dense subset under a surjective continuous function is again dense", which is from Wiki's entry. For details of this technicality, see HERE.

See a post on equivalent definition for dominant rational map HERE.

A good lecture note HERE.

Wiki's entry for Rational Map.

Very good note by Vakil HERE.

And a post HERE. However, it appears the way used in the proof implicitly required morphism is an open map? So I doubt...

29.2

Some posts share a technical details from general topology. The following statement is taken from Wiki...

The image of a dense subset under a surjective continuous function is again dense. More precisely, assume $f : X \to Y$ with E dense in X, then $f(E)$ is dense in $f(X)$.

Proof. By definition we have $E \subset f^{-1}(f(E)) \subset f^{-1}(\overline{f(E)})$, which is closed given f is continuous. It follows that

$$\overline{E} \subset f^{-1}(\overline{f(E)}) \;\Rightarrow\; f(\overline{E}) \subset \overline{f(E)}.$$

Conversely,

$$f(X) = \overline{f(E)} \cap f(X) \supset f(X) \cap f(X) = f(X) \;\Rightarrow\; f(X) \supset \overline{f(E)} \cap f(X).$$

Here $\overline{f(E)}$ denotes closure of $f(E)$ in Y, while it's intersection with $f(X)$ is the whole $f(X)$, then $f(E)$ is dense in $f(X)$. $\square$

29.3 A Pathological Example

Another equivalent statement required "surjectivity" and say $f(E)$ is dense in Y. It's curtial. Also we can only conclude $f(E)$ is dense merely in $f(X)$ instead of Y. Since we have the continuous inclusion map $\iota : \mathbb{R} \to \mathbb{C}$, then $\mathrm{id}(\mathbb{Q}) = \mathbb{Q}$ is just dense in $\mathbb{R}$ but not dense in $\mathbb{C}$.

29.4

Say we start with a dominant rational map $\varphi : X \to Y$ with two representatives

$$\langle U, f\rangle,\ \langle V, g\rangle.$$

By definintion of dominant, we know $f(U)$ is dense in Y. To check this definition is indepedent of the choice of the representative, we have to check $g(V)$ is dense in Y.

Notice that

$$Y = \overline{f(U)} = \overline{f(\overline{U \cap V} \cap U)} \subset \overline{f(\overline{U \cap V})} \subset \overline{\overline{f(U \cap V)}} = \overline{g(U \cap V)} \subset \overline{g(V)}.$$

for X is irreducible and both U, V are non-empty and open then $X = \overline{U \cap V}$. Here the third inclusion is given by the previous technical lemma.

29.5 Composing Dominant Rational Maps

See a post HERE.

$$\begin{tikzcd} X \arrow[r, dashed, "\phi"] & Y \arrow[r, dashed, "\psi"] & Z \end{tikzcd}$$

$$(U, \phi_U) \qquad\qquad (V, \psi_V)$$

To prove the composition is a dominant rational map, we need to find a representative. We define

$$W := \phi_U^{-1}(V).$$

And we claim $(W,\ \psi_V \circ \phi_U)$ will be suitable for a representative for $\psi \circ \phi$. First of all, notice that W is non-empty. This is because $\phi_U(U) \cap V \neq \emptyset$ given $\phi_U(U)$ is dense in Y and V is assumed to be non-empty open subset. While Y is irreducible, by Lemma 14. of **??** on Page 210, which states that $\phi_U(U) \cap V$ is nontrivial. By definition this implies

? slightly different than the post online

$$\phi_U^{-1}(V) \neq \emptyset.$$

This is non-empty and open. Note X is irreducible, hence $W = \phi_U^{-1}(V)$ is dense in X. Hence $\psi_V \circ \phi_U(W)$ is dense in Z given both maps are continuous by being a morphism.

29.6

See a post HERE, HERE, and HERE.

30 Lemma 4.2.

See a post HERE.

31 Definition Blowing Up

On Page 29, Hartshorne defined blowing-up of a closed affine subvariety Y of $\mathbb{A}^n$ passing through O.

See Sándor's Notes, Lecture 22, *strict transform.*

See Daniel's notes HERE. Here the notation $(\cdot)^-$ stands for taking closure.

32 Exercise 4.1.

Define a function

$$h : U \cup V \to k \text{ by}$$
$$x \mapsto \begin{cases} f(x) & \text{when } x \in U \setminus V \\ g(x) & \text{when } x \in V \end{cases}.$$

Since $f = g$ on $U \cap V$, hence the function h is well-defined. For any point $p \in U \cup V$, if $p \in U$, then we apply assumption that f is regular. For $p \in V$, similarly apply assumption that g is regular. Hence h is regular on $U \cup V$.

Let f be a rational function on X. So we take all equivalence class $\{\langle U_i, f_i\rangle\}_{i\in I}$ that represents f. By the above lemma and the definition of regular function, there's a regular function g that's defined on $U := \bigcup_{i\in I} U_i$ that extends all f_i. Therefore we can take a representative of f as

$$\langle U, g\rangle.$$

Note $\langle U, g\rangle = \langle U_i, f_i\rangle$ by definition, hence it's indeed a representative of f.

Also U is the largest open set. Suppose it's not, then we have $\langle U_{i_0}, f_{i_0}\rangle$ represents f such that $U_{i_0} \supsetneq U$. And this will contradicts the construction of U, which must contain U_{i_0}.

33 Exercise 4.2.

We're given a rational map $\varphi : X \dashrightarrow Y$. Suppose we have two equivalent representatives

$$\langle U, \varphi_U\rangle,\ \langle V, \varphi_V\rangle.$$

This means two morphisms φ_U, φ_V agree on $U \cap V$.

It suffices to prove that we can define a morphism $\psi : U \cup V \to Y$ that extends both φ_U and φ_V. Similarly, we can apply argument of 32 to conclude the existence of a largest open set on which φ is represented by a morphism.

Both φ_U, φ_V are continuous function that agree on their intersection, then we can define

$$\psi : U \cup V \to Y \text{ by } \psi(x) = \begin{cases} \varphi_U(x) & \text{when } x \in U \setminus V \\ \varphi_V(x) & \text{when } x \in V \end{cases}.$$

Clearly it's a well-defined continuous function. For any open subset $W \subset Y$ with an arbitrary regular function $f : W \to k$. We have $f \circ \psi : \psi^{-1}(W) \to k$ is a regular function since it extends both

$$f \circ \varphi_U : \varphi_U^{-1}(W) \to k,\ f \circ \varphi_V : \varphi_V^{-1}(W) \to k.$$

While both two regular functions agree on their intersection, then we can conclude using 32 that $f \circ \psi$ is a regular function. And this proves that ψ is indeed a morphism on $U \cup V \to Y$.

34 Defintion: Presheaf

34.1 Two Pathological Examples

Here are two examples taken from Tennison's [8].

Let X be any topological space with more thatn one point, i.e. $X = \{0, 1\}$ or $X = \{0, 1\} \to \mathbb{R}$.

Define a presheaf $\mathscr{P}_1$ by

$$\begin{cases} \mathscr{P}_1(X) = \mathbb{Z} \\ \mathscr{P}_1(U) = 0 \quad \text{for open} \quad U \subsetneq X \\ \text{All restrictions except } \rho_{XX} \text{ being constant maps.} \end{cases}.$$

Here 0 denotes the trivial Abelian group.

Pick $x_0 \in X$. Define a presheaf $\mathscr{P}_2$ by

$$\begin{cases} \mathscr{P}_2(U) = \mathbb{Z} \quad \text{for } U \text{ open in } X \text{ such that } x_0 \in U \\ \mathscr{P}_2(U) = 0 \quad \text{for } U \text{ open in } X \text{ such that } x_0 \notin U \\ \text{restrictions} \quad \rho_{UV} = \begin{cases} \mathrm{id}_{\mathbb{Z}} & \text{if } x_0 \in V \subset U \\ 0 & \text{trivial map if not} \end{cases} \end{cases}.$$

Here the second appearance of 0 denotes the trivial map.

35 Example 1.0.3.

See some examples of presheaves that are not sheaves HERE; a post HERE.

In Wiki's page HERE, it introduced non-separated presheaf, i.e. presheaf that doesn't satisfy locality axiom for sheaf.

36 Proposition-Definition 1.2.

See *Sheafification* on The Stacks Project.

See solution of problem 3 HERE.

Of course, consult Ravi's Notes on Sheafification;

or see Section 6.5 on Page 232 of [2].

Also, see a REU paper HERE by Daping Weng.

A short paper by Tom is HERE.

37 Definition: Inverse Image Sheaf

See Wiki for motivation of such definition.

See a POST that gives more details, as well as a counterexample.

From Prof. Sándor's Email: Let X by disjoint union of two copies of Y with a continuous map $f : X \to Y$. Assume Y is irreducible and let $\mathscr{G}$ be a constant sheaf on Y. We claim that $f^{-1}_{\text{pre}}\mathscr{G}$ is just a presheaf, but not a sheaf.

Any open subset $W_1, W_2 \in X$ will have intersection in Y. Then any section will agree on their intersections. Take two sections from $0 \amalg Y$ and $Y \amalg Y$, there won't be a global section such that restriction is either of them.

38 Exercise 1.3.

See a post HERE for explicit information of induced map on stalks.
See the solution from a post HERE.
See HERE for a partial solution, as well as a counterexample.

38.1 (a)

Now assume φ is surjective. Fix an open subset $U \subset X$ and a section $s \in \mathscr{G}(U)$. Now we can pick any point $p \in U$, consider the stalk at it.

39 Exercise 1.8.

See Rotman's [7], Lemma 6.68. on Page 378.

Part II
The Rising Sea

40 1.3.F. EXERCISE.

A post discussing this problem is HERE.

41 1.3.N

A crutial step is to define the map such that the diagram commute. In order to prove it satisfies the universal property. We're given A be arbitrary with map $g : A \to X$ and $f : A \to Y$.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists ! \varphi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_Z Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\beta"] \\
& X \arrow[r, "\alpha"'] & Z
\end{tikzcd}$$

We can to define

$$\begin{aligned}
\varphi : A \to X \times_Z Y \quad &\text{by} \\
a \mapsto (g(a), f(a)).&
\end{aligned}$$

And we can verify this definition will make the diagram commute, and is unique.

42 1.3.O

It's indeed intersection. A post HERE.

A post HERE.

43 1.3.P.

Say we have $X \times Y$ and $X \times_Z Y$. By universal property of product and fibered product we can produce two unique map goes in between. Their composition must be identity, hence they're isomorphic. Notice it's important for Z being a final object.

$$\begin{tikzcd}
X \times Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\beta"] \\
X \arrow[r, "\alpha"'] & Z
\end{tikzcd}$$

When we trying to make a map from $X \times Y$ to $X \times_Z Y$, we have to make sure two maps

$$\beta \circ \pi_1 = \alpha \circ \pi_1$$

on $X \times Y$. While Z is the final object, hence they must agree.

There's a cleaner way to state it HERE. Crutial part is applying final property of object Z.

44 1.3.Q.

$$\begin{tikzcd}
A \arrow[drr, bend left, "g_1"] \arrow[dr, dashed, "\exists!\varphi"] \arrow[ddr, dashed, "\exists!\psi"'] \arrow[dddr, bend right, "g_2"'] & & \\
& U \arrow[r, "f_1"] \arrow[d, "f_2"'] & V \arrow[d, "f_3"] \\
& W \arrow[r, "f_4"'] \arrow[d, "f_5"'] & X \arrow[d, "f_6"] \\
& Y \arrow[r, "f_7"'] & Z
\end{tikzcd}$$

Label the maps as indicated. To prove the universal property with respect to the "outside rectangle", we're given

$$f_6 f_3 g_1 = f_7 g_2$$

agree on A. While W is fibered product, apply universal property of fibered product with resepct to W we immediately get a unique map

$$\psi : A \to W$$

that makes the diagram involving A, W, X, Y, Z commute. In particularly, we know $f_4\psi = f_3 g_1$. Furthermore, recall that U is the fibered product. We're given the condition that $f_4\psi = f_3 g_1$, by universal property of U we know there exists a unique map

$$\varphi : A \to U$$

making the diagram involving A, U, V, W, X commute. And we claim that the diagram involving A, U, V, Y, Z commute. This is because

$$g_2 = f_5\psi = f_5 f_2 \varphi = (f_5 f_2)\varphi \text{ and } g_1 = f_1\varphi.$$

And this proves that U is the fibered product for the diagram involving A, U, V, Y, Z.

A post is HERE.

45 1.3.R

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[drr, bend left, "\pi_1"] \arrow[ddr, bend right, "\pi_4"'] & & & \\
& X_1 \times_Z X_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_3"'] & X_2 \arrow[d, "f_3"] & \\
& X_1 \arrow[r, "f_2"'] & Y \arrow[dr, "f_1"] & \\
& & & Z
\end{tikzcd}$$

By the universal property of $X_1 \times_Z X_2$, we know there exists a unique map

$$\varphi : X_1 \times_Y X_2 \to X_1 \times_Z X_2$$

"Natural morphism", a convention discussed HERE.

46 Course Notes from Cornell

See HERE.

47 1.3.S. Magic Diagram

Didn't finish. Need to See HERE, HERE!!!

48 1.3.Y. (a)

Yoneda's Lemma Given what we have, define $g : A \to A'$ as

$$g := i_A(\mathrm{id}_A).$$

This is correct, see a post HERE.

49 1.4.C.

(a) See "A Term of Commutative Algebra", Example 7.3 on Page 52.

50 1.6.B.

Write out everything by definition, and we can finish the proof immediately by applying rank-nullity theorem for linear transformation...

51 2.2.6. Definition: Sheaf.

Comments on $\mathscr{F}(\emptyset)$. In category **Set**, the empty set is initial object and one element set is terminal. See Wiki's examples HERE.

52 2.2.B.

For (a): see Wiki's counterexample HERE, which gave an explanation for presheaves on $\mathbb{R}$ instead of $\mathbb{C}$. See a post HERE.

53 2.2.7.

See Daping's Notes Definition 2.5 on Page 4 HERE; also a post HERE; also a post HERE.

54 2.2.10.

It's different from a post HERE, and Wiki's page on Constant pre-Sheaf. Why???

55 2.2.G.

It's clearly a pre-sheaf.

Fix an open subset $U \subset X$ with an open cover $\{U_i\}_{i \in I}$ for some index set I. Denote the presheaf as $\mathscr{F}$.

Pick two continuous maps $s_1, s_2 : Y \to X$ that satisfying the requirements, i.e. $s_1, s_2 \in \mathscr{F}(U)$.

Both functions will agree on U since

$$\mathrm{Res}_{U,U_i}\ s_1 = \mathrm{Res}_{U,U_i}\ s_1$$

for arbitrary U_i, whose union is U. So we must have $s_1 = s_2$.

Again with this open cover $\{U_i\}_{i \in I}$ and $a_i \in \mathscr{F}(U_i)$ for $i \in I$. Equivalently, we know $a_i : U_i \to Y$ is a continuous map satisfying $\mu \circ a_i = \mathrm{Id}\,|_{U_i}$. Now let's define a map

$$\begin{aligned} f : U &\to Y \\ u &\mapsto a_i(u) \quad \text{when} \quad u \in U_i. \end{aligned}$$

It's well-defined by our assumption. Also it's continuous since preimage of an open set in $V \subset Y$ is a union of open subsets given by continuity of each a_i. Unverified ?

Similarly we can check $\mu \circ f = \mathrm{Id}\,|_U$ as expected.

56 2.2.11. Espace Étalé

See a post discussion accent letter in LaTeX HERE.

See an exercise in [4] Chapter 2, Exercise 1.13.

See the discussion after Lemma 7. on Page 229 of [2].

For *section*, see Wiki's explanation for *section* in context of fiber bundle; and *section* in terms of category theory.

57 2.3.A.

I'm planning to use universal property to define the induced map ϕ_P.

One crutial step is to verify the diagram below is commutative

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[rr, "\rho_{UV}"] \arrow[d, "\phi(U)"'] & & \mathscr{F}(V) \arrow[d, "\phi(V)"] \\
\mathscr{G}(U) \arrow[rr, "\tau_{UV}"'] \arrow[dr] & & \mathscr{G}(V) \arrow[dl] \\
& \mathscr{G}_P &
\end{tikzcd}$$

And this is because the square diagram in the upper half commute given ϕ is a natural transformation; the lower half is by definition of $\mathscr{G}_P$. Then by universal property of colimit induces a map

$$\phi_P : \mathscr{F}_P \to \mathscr{G}_P$$

which makes the diagram commute.

See a post defined the map HERE.

58 2.3.B.

To define a functor $\pi_* : \mathbf{Set}_X \to \mathbf{Set}_Y$. Firstly, we have to define for any $\mathscr{F} \in \mathbf{Set}_X$,

$$\pi_*(\mathscr{F})(U) = \mathscr{F}(\pi^{-1}(U))$$

for any $U \in \mathfrak{Top}(X)$ as in **??**.

Secondly, for any natural transformation $\phi : \mathscr{F} \to \mathscr{G}$, we define $\pi_*(\phi)$ by specifying

$$\pi_*(\phi)(U) \mapsto \mathscr{F}(\pi^{-1}(U)) \to \mathscr{G}(\pi^{-1}(U)).$$

? Is this correct

59 References

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