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### 1 Definition: Irreducible

For equivalent definitions, see Wiki HERE; Also see "dense" on Wiki HERE. See Atiyah's [1] Exercise 19 from Chapter 1 for more information...

See Lemma 14. on Page 210 of [2] for equivalent characterisation of irreducible.

### 2 Example 1.1.3.

See Atiyah's [1] Exercise 19 from Chapter 1, which proves it must be dense. For the sake of contradiction, assume a non-empty subset  $A \subset X$  is reducible. Hence there exist two proper closed subset  $A_1, A_2 \subset A$  such that  $A = A_1 \cup A_2$ . Then we have

$$X = (A^c \cup A_1) \cup (A^c \cup A_2),$$

which implies X is reducible.

 $\times!$ 

#### 2.1

See a post HERE

First approach applied some density argument. While the second approach, similarly, gave a decomposition of the whole space given the open set is reducible.

### 3 Example 1.1.4.

See Atiyah's [1] Exercise 20 from Chapter 1.

#### 4 Definition.

"Induced topology". Definition of quasi affine variety, see HERE.

## 5 Prop 1.2 (d)

According to Hilbert's Nullstellensatz, I agree we'll get  $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$ . For the reverse inclusion, pick any  $f \in A$  such that  $f^r \in \mathfrak{a}$  where  $r \in \mathbb{Z}_{>0}$ . We wish to show that f(P) = 0 for any  $P \in Z(\mathfrak{a})$ . By definition,  $f^r(P) = 0$  given  $f^r \in \mathfrak{a}$ . And this implies

$$f^r(P) = (f(P))^r = 0 \implies f(P) = 0$$

given the polynomial ring A is an integral domain. Therefore we get the inclusion

$$I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}$$
.

See Theorem 6 on Page 183, Strong Nullstellensatz, [3].

#### 6 Theorem 1.3A. Hilbert's Nullstellensatz

For the case where k isn't algebraically closed, see [6] Remarks in 5.6.

#### 6.1

For the proof, see [5] Chapter 1 for details.

The followings are some comments for [5] Chapter 1 Theorem 1.7.

A post on surjective preimage for maximal ideal HERE.

A post on preimage for maximal ideal (not necessarily surj) HERE.

For completeness, a post on preimage of prime ideals HERE.

A post on image of prime ideals HERE, HERE, and HERE.

See Kemper's [5], Lemma 1.22 on Page 17, which completely described prime and maximal ideals in quotients.

### 7 Definition. Height

Here the definition *height* is specifically for prime ideal  $\mathfrak{p} \triangleleft_{pr} R$  for some ring R. For a general definition, see a post HERE; see a webpage HERE; or see [5] Definition 6.10 on Page 68.

### 8 Proposition 1.7.

Difference between algebraic set and affine algebraic set? See HERE. For an analogue in Projective, see Exercise 2.6 ?? in Chapter 1.2. And a post HERE.

#### 9 Theorem 1.8A.

For transcendence degree, see HERE and a NOTE by Milne James.

## 10 Proposition 1.10.

Apart from the proof, we discussed *locally closed subset*. See HERE for its equivalent definitions.

## 11 Proposition 1.13.

See HERE.

### 12 Exercise 1.1.

### 12.1 (a)

By definition of affine coordinate ring we have

$$A(Y) = A/I(Y) = A/I(Z(f)) = A/\sqrt{\langle f \rangle}.$$

While f is irreducible in U.F.D. k[x,y], the ideal it generated will be a prime ideal, which is radical. Therefore we can further simplify the expression as

$$A/\sqrt{\langle f \rangle} = A/\langle f \rangle = k[x,y]/\langle y - x^2 \rangle = k[x].$$

Hence we can conclude A(Y) is isomorphic to a polynomial ring in one variable over k.

### 12.2 (b)

### 13 Proposition 2.2.

There are two very technical claims need more details.

The first one is to prove  $\varphi(Y) = Z(T') = Z(\alpha(T))$ . Unwrap the notation precisely according to the definition

$$Z(\alpha(T)) = \{ x \in \mathbb{A}^n \mid \alpha(g)(x) = 0 \ \forall \ g \in T \},$$
  
$$\varphi(Y) = \{ \varphi(y) \mid y \in Y \}.$$

Notice that  $y = [y_0, ..., y_n] \in Y \subset \overline{Y} = Z(T)$ , therefore g(y) = 0 for any  $g \in T$ . More precisely, we have

$$\alpha(g)(\varphi(g)) = g(1, y_1/y_0, ..., y_n/y_n) = 0$$

given g(y) = 0 and  $g \in T \subset S^h$ , which proves  $\varphi(Y) \subset Z(\alpha(T))$ .

Conversely, let's start with an element  $x=(x_1,...,x_n)\in Z(\alpha(T))$ . There's an element  $y=[1,x_1,...,x_n]\in Y$  such that  $\varphi(y)=x$ . Hence we've proved the equality.

And the second one is to check  $\varphi^{-1}(W) = Z(\beta(T')) \cap U = Z(\beta(\alpha(T))) \cap U$ .

#### 14 Lemma 3.1.

See Sandor's Notes Lecture 4, Lemma 2.6. Closedness can be checked locally. See a post HERE.

### 15 Remark 3.1.1.

See Sandor's Notes Lecture 5, Corollary 2.10. Notice that Hartshorne defined variety to be irreducible. See a post explaining why the preimage is dense HERE. See Lemma 14. on Page 210 of [2].

### 16 Definition: Ring of Regular Function

HERE is an explicit description on the ring structure of  $\mathcal{O}_{P,Y}$ .

### 17 Theorem 3.2.

See Sándor's Lecture Notes 09, STEP 03. See solution of problem 3 HERE. See a post HERE, HERE.

### 17.1 (c)

for each P,  $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$ , and  $\dim \mathcal{O}_P = \dim Y$ ;

*Proof.* We begin with an injective homomorphism  $\alpha:A(Y)\to\mathcal{O}(Y)$ . And we define a map

$$A(Y)_{\mathfrak{m}_P} \to \mathcal{O}_{P,Y}$$
  
 $f/g \mapsto \langle V, \frac{\alpha(f)}{\alpha(g)} \rangle$ 

where  $\frac{\alpha(f)}{\alpha(g)} \in \mathcal{O}(V)$ . Now we wish to give an explicit description of V. Since  $\alpha(f) \in \mathcal{O}(Y)$ , we know there exists an open subset  $P \in V_1 \subset Y$  such that

$$\alpha(f) \mid_{V_1} = \frac{h_1}{h_2} \mid_{V_1}$$

where  $h_1, h_2 \in A$  and  $0 \notin h_2(V_1)$ . Since  $\alpha(g) \in \mathcal{O}(Y)$ , we know there exists an open subset  $P \in V_2 \subset Y$  such that

$$\alpha(g)\mid_{V_2}=\frac{h_3}{h_4}\mid_{V_2}$$

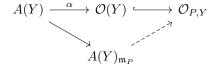
where  $h_3, h_4 \in A$  and  $0 \notin h_4(V_2)$ . Here  $g \notin \mathfrak{m}_P$  by definition of localisation, which gives us

$$g(P) \neq 0 \implies \alpha(g)(P) \neq 0 \implies \exists V_3 \subset Y, \ \alpha(g) \mid_{V_3} \neq 0.$$

Then we take  $V = V_1 \cap V_2 \cap V_3$  will suffice to work. This is because for any point  $P \in V$ , we have

$$\frac{\alpha(f)}{\alpha(g)} = \frac{h_1 h_4}{h_2 h_3}$$

for  $0 \notin h_2h_3(V)$ .



The induced map is given by universal property of localisation, for every elements in  $A(Y) \setminus \mathfrak{m}_P$  will be mapped to a unit in  $\mathcal{O}_{P,Y}$ . And the map given by Lecture Notes 09 of Prof. Sándor satisfy the universal property (it makes the diagram commute and maps elements outside of  $\mathfrak{m}_P$  to units).

### 18 Proposition 3.3.

See a post HERE.

### 19 Lemma 3.6.

See a post HERE.

Here are some details for proving  $x_i \circ \psi$  being regular implies  $\psi$  is a morphism: Coordinate functions means

$$\psi(p) = (\psi_1(p), \psi_2(p), ..., \psi_n(p)) \in Y \subset \mathbb{A}^n$$

for any  $p \in X$ . Firstly, we check  $\psi$  is continuous. Take any closed subset  $Z(f_1,...,f_r) \subset Y$  for some polynomial  $f_1,...,f_r \in A = k[x_1,...,x_n]$ . We can compute the preimage as

$$\psi^{-1}(Z(f_1,...,f_r)) = \{ p \in X \mid \forall p \in X, f_i \circ \psi(p) = 0 \}.$$

Notice that for any  $p \in X$ ,

$$f_i \circ \psi(p) = f_i(\psi_i(p), ..., \psi_n(p))$$

is continuous since  $f_i$  is a polynomial and each  $\psi_i := x_i \circ \psi$  is continuous by assumption that they're regular. Notice that the preimage of  $\psi$  is precisely intersection of  $\psi_i^{-1}(\{0\})$  where  $1 \le i \le n$ . Hence the preimage is closed, and it follows that  $\psi$  is continuous as expected.

Secondly, fix an arbitrary open subset  $V \subset Y$  with an arbitrary regular function  $g: V \to k$ , we wish to prove  $g \circ \psi : \psi^{-1}(V) \to k$  is regular. For any

 $\psi(p) \in V$  with some  $p \in X$ , there exists a neighborhood  $\psi(p) \in U \subset Y$  such that g equals to an expression of quotients of polynomial, i.e.

$$g = \frac{g_1}{g_2}$$

where  $g_1, g_2 \in A$ . Then for  $p \in X$ , take the open neighborhood of it as  $\psi^{-1}(U)$ , we can see

$$g \circ \psi(p) = \frac{g_1(\psi(p))}{g_2(\psi(p))}.$$

### 20 Example 1.0.3.

See some examples of presheaves that are not sheaves HERE; a post HERE. In Wiki's page HERE, it introduced non-separated presheaf, i.e. presheaf that doesn't satisfy locality axiom for sheaf.

### 21 Proposition-Definition 1.2.

See Sheafification on The Stacks Project. See solution of problem 3 HERE. Of course, consult Ravi's Notes on Sheafification; or see Section 6.5 on Page 232 of [2]. Also, see a REU paper HERE by Daping Weng. A short paper by Tom is HERE.

## 22 Definition: Inverse Image Sheaf

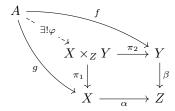
See Wiki for motivation of such definition. See a POST that gives more details, as well as a counterexample.

#### 23 1.3.F. EXERCISE.

A post discussing this problem is HERE.

### 24 1.3.N

A crutial step is to define the map such that the diagram commute. In order to prove it satisfies the universal property. We're given A be arbitrary with map  $g:A\to X$  and  $f:A\to Y$ .



We can to define

$$\varphi: A \to X \times_Z Y$$
 by  $a \mapsto (g(a), f(a)).$ 

And we can verify this definition will make the diagram commute, and is unique.

### 25 1.3.O

It's indeed intersection. A post HERE. A post HERE.

### 26 1.3.P.

Say we have  $X \times Y$  and  $X \times_Z Y$ . By universal property of product and fibered product we can produce two unique map goes in between. Their composition must be identity, hence they're isomorphic. Notice it's important for Z being a final object.

$$\begin{array}{ccc} X\times Y & \xrightarrow{\pi_2} Y \\ \downarrow^{\pi_1} & & \downarrow^{\beta} \\ X & \xrightarrow{\alpha} Z \end{array}$$

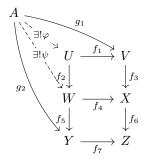
When we trying to make a map from  $X \times Y$  to  $X \times_Z Y$ , we have to make sure two maps

$$\beta \circ \pi_1 = \alpha \circ \pi_1$$

on  $X \times Y$ . While Z is the final object, hence they must agree.

There's a cleaner way to state it HERE. Crutial part is applying final property of object Z.

### 27 1.3.Q.



Label the maps as indicated. To prove the universal property with respect to the "outside rectangle", we're given

$$f_6 f_3 g_1 = f_7 g_2$$

agree on A. While W is fibered product, apply universal property of fibered product with resepct to W we immediately get a unique map

$$\psi:A\to W$$

that makes the diagram involving A, W, X, Y, Z commute. In particularly, we know  $f_4\psi = f_3g_1$ . Furthermore, recall that U is the fibered product. We're given the condition that  $f_4\psi = f_3g_1$ , by universal property of U we know there exists a unique map

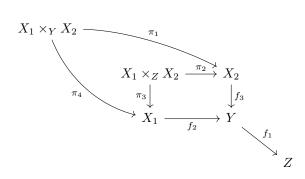
$$\varphi:A\to U$$

making the diagram involving A, U, V, W, X commute. And we claim that the diagram involving A, U, V, Y, Z commute. This is because

$$g_2 = f_5 \psi = f_5 f_2 \varphi = (f_5 f_2) \varphi$$
 and  $g_1 = f_1 \varphi$ .

And this proves that U is the fibered product for the diagram involving A, U, V, Y, Z. A post is HERE.

#### 28 1.3.R



By the universal property of  $X_1 \times_Z X_2$ , we know there exists a unique map

$$\varphi: X_1 \times_Y X_2 \to X_1 \times_Z X_2$$

"Natural morphism", a convention discussed HERE.

### 29 Course Notes from Cornell

See HERE.

### 30 1.3.S. Magic Diagram

Didn't finish. Need to See HERE, HERE!!!

## 31 1.3.Y. (a)

Yoneda's Lemma Given what we have, define  $g:A\to A'$  as

$$g := i_A(\mathrm{id}_A).$$

This is correct, see a post HERE.

### 32 1.4.C.

(a) See "A Term of Commutative Algebra", Example 7.3 on Page 52.

### 33 1.6.B.

Write out everything by definition, and we can finish the proof immediately by applying rank-nullity theorem for linear transformation...

### 34 2.2.6. Definition: Sheaf.

Comments on  $\mathscr{F}(\emptyset)$ . In category **Set**, the empty set is initial object and one element set is terminal. See Wiki's examples HERE.

#### 35 2.2.B.

For (a): see Wiki's counterexample HERE, which gave an explanation for presheaves on  $\mathbb{R}$  instead of  $\mathbb{C}$ . See a post HERE.

### 36 2.2.7.

See Daping's Notes Definition 2.5 on Page 4 HERE; also a post HERE; also a post HERE.

### 37 2.2.10.

It's different from a post HERE, and Wiki's page on Constant pre-Sheaf.

Why???

#### 38 2.2.G.

It's clearly a pre-sheaf.

Fix an open subset  $U \subset X$  with an open cover  $\{U_i\}_{i \in I}$  for some index set I. Denote the presheaf as  $\mathscr{F}$ .

Pick two continuous maps  $s_1, s_2: Y \to X$  that satisfying the requirements, i.e.  $s_1, s_2 \in \mathscr{F}(U)$ .

Both functions will agree on U since

$$\operatorname{Res}_{U,U_i} s_1 = \operatorname{Res}_{U,U_i} s_1$$

for arbitrary  $U_i$ , whose union is U. So we must have  $s_1 = s_2$ .

Again with this open cover  $\{U_i\}_{i\in I}$  and  $a_i\in \mathscr{F}(U_i)$  for  $i\in I$ . Equivalently, we know  $a_i:U_i\to Y$  is a continuous map satisfying  $\mu\circ a_i=\mathrm{Id}\mid_{U_i}$ . Now let's define a map

$$f: U \to Y$$
  
 $u \mapsto a_i(u)$  when  $u \in U_i$ .

It's well-defined by our assumption. Also it's continuous since preimage of an open set in  $V \subset Y$  is a union of open subsets given by continuity of each  $a_i$ . Similarly we can check  $\mu \circ f = \operatorname{Id}|_U$  as expected.

Unverified?

# 39 2.2.11. Espace Étalé

See a post discussion accent letter in LaTeX HERE.

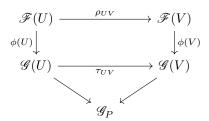
See an exercise in [4] Chapter 2, Exercise 1.13.

See the discussion after Lemma 7. on Page 229 of [2].

For *section*, see Wiki's explanation for *section* in context of fiber bundle; and *section* in terms of category theory.

#### 40 2.3.A.

I'm planning to use universal property to define the induced map  $\phi_P$ . One crutial step is to verify the diagram below is commutative



And this is because the square diagram in the upper half commute given  $\phi$  is a natural transformation; the lower half is by definition of  $\mathscr{G}_P$ . Then by universal property of colimit induces a map

$$\phi_P: \mathscr{F}_P \to \mathscr{G}_P$$

which makes the diagram commute.

See a post defined the map HERE.

#### 41 2.3.B.

To define a functor  $\pi_* : \mathbf{Set}_X \to \mathbf{Set}_Y$ . Firstly, we have to define for any  $\mathscr{F} \in \mathbf{Set}_X$ ,

$$\pi_*(\mathscr{F})(U) = \mathscr{F}(\pi^{-1}(U))$$

for any  $U \in \mathfrak{Top}(X)$  as in ??.

Secondly, for any natural transformation  $\phi: \mathscr{F} \to \mathscr{G}$ , we define  $\pi_*(\phi)$  by specifying

$$\pi_*(\phi)(U) \mapsto \mathscr{F}(\pi^{-1}(U)) \to \mathscr{G}(\pi^{-1}(U)).$$

? Is this correct

### 42 References

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