84.4 Counterexample

Proof. For any $U \subset X$, where $\mathscr{F}$ is skycraper sheaf at $p \in X$ with value group A and $\mathscr{G}$ is a constant sheaf on topological space X with value group A. We claim that

$$\mathcal{H}om(\mathscr{F}, \mathscr{G})(U) = \mathrm{Mor}(\mathscr{F}\mid_U, \mathscr{G}\mid_U) = 0$$

as an Abelian group for arbitrary $U \subset X$.

It suffices to check the above statment is correct when $U = X$ for $\mathcal{H}om(\mathscr{F}, \mathscr{G})$ is a sheaf, i.e. we need to show the group

$$\mathcal{H}om(\mathscr{F}, \mathscr{G})(X) = \mathrm{Mor}(\mathscr{F}, \mathscr{G}) = 0$$

Pick any natural transformation $\alpha \in \mathrm{Mor}(\mathscr{F}, \mathscr{G})$. We wish to show the group homomorphism $\alpha(U)$ is 0 (i.e. sends everything to 0 in the codomain $\mathscr{G}(U)$). For an open subset U, it admits an open covering $\{U_i\}$ in which every U_i is connected. While $\mathscr{G}$ is a sheaf, to prove $\alpha(U) = 0$ it suffices to check $\alpha(U_i) = 0$. Fix i and denote $U_i = U_0$. When $p \notin U_0$, then $\mathscr{F}(U_0) = 0$ and the map $\alpha(U_0) = 0$.

Now assume $p \notin U_0$. We can still argue $\alpha(U_0)$ is 0 group homomorphism by restrict it to a smaller open subset that doesn't contain p, because the skycraper sheaf will be 0 group.

Assume p is closed an not open, which means it's not isolated. Then $V := U \setminus \{p\}$ is an open subset that doesn't contain p, hence $\mathscr{F}(V) = 0$ by definition.

?

$$\begin{tikzcd}
\mathscr{F}(U_0) \arrow[r, "\alpha(U)"] \arrow[d, "\mathrm{Res}_{U_0V}"'] & \mathscr{G}(U_0) = A \arrow[d] \\
\mathscr{F}(V) = 0 \arrow[r] & \mathscr{G}(V) = \oplus A
\end{tikzcd}$$

Notice that we assumed U_0 to connected, therefore $\mathscr{G}(U_0) = A$ is composed of single one copy of A. While $\mathscr{G}(V) = \oplus_{j \in J} A$ for some index set J. We claim that $\mathscr{G}(U_0) \to \mathscr{G}(V)$ is an injection:

- We claim that the map is in fact diagonal map by considering the following commutative diagram

$$\begin{tikzcd}
a \in \mathscr{G}(U_0) \arrow[r] \arrow[d, hook] & (a, a, ..., a) \in \mathscr{G}(V) \arrow[d, hook] \\
\amalg_{p \in U_0} \mathscr{G}_p = \amalg_{p \in U_0} A & \amalg_{p \in V} A \arrow[l, hook']
\end{tikzcd}$$

Therefore by commutativity of the diagram together with the injectivity of the above map we can conclude $\alpha(U_0) = 0$ as expected.

$\square$