Contents

1 Notes

See a post HERE, which listed various resources...

See Old Lecture Homepages of ETH HERE. There are some courses on algebraic geometry with solutions for homeworks.

See a post on "Best AG Textbook" HERE.

Great Notes on Algebraic Geometry and Commutative Algebra by Andreas Gathmann HERE.

See Horawa's Notes on Algebraic Geometry.

See HERE for a comprehensive guide to Hartshorne as well as solutions.

See Student Q& A in MITCourseware HERE.

See notes on algebraic geometry by Prof. Dr. Andreas Gathmann HERE.

See HERE for Aaron Landesman's notes.

See a course in 2012 by James McKernan HERE.

See a great book on Sheaf theory: Sheaf Theory through Examples by Daniel HERE.

See Math 216 Course Webpage HERE.

Part I

Hartshorne

2 Definition: Irreducible

For equivalent definitions, see Wiki HERE; Also see ”dense” on Wiki HERE.

See Atiyah’s [1] Exercise 19 from Chapter 1 for more information...

See Lemma 14. on Page 210 of [2] for equivalent characterisations of *irreducible*.

3 Example 1.1.3.

See Atiyah’s [1] Exercise 19 from Chapter 1, which proves it must be dense.

For the sake of contradiction, assume a non-empty subset $A \subset X$ is reducible. Hence there exist two proper closed subset $A_1, A_2 \subset A$ such that $A = A_1 \cup A_2$. Then we have

$$X = (A^c \cup A_1) \cup (A^c \cup A_2),$$

which implies X is reducible.

3.1

See a post HERE

First approach applied some density argument. While the second approach, similarly, gave a decomposition of the whole space given the open set is reducible.

4 Example 1.1.4.

See Atiyah’s [1] Exercise 20 from Chapter 1.

5 Definition.

”Induced topology”. Definition of *quasi affine variety*, see HERE.

6 Prop 1.2 (d)

According to Hilbert’s Nullstellensatz, I agree we’ll get $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$. For the reverse inclusion, pick any $f \in A$ such that $f^r \in \mathfrak{a}$ where $r \in \mathbb{Z}_{>0}$. We wish to show that $f(P) = 0$ for any $P \in Z(\mathfrak{a})$. By definition, $f^r(P) = 0$ given $f^r \in \mathfrak{a}$. And this implies

$$f^r(P) = (f(P))^r = 0 \Rightarrow f(P) = 0$$

given the polynomial ring A is an integral domain. Therefore we get the inclusion

$$I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}.$$

See Theorem 6 on Page 183, Strong Nullstellensatz, [3].

7 Theorem 1.3A. Hilbert's Nullstellensatz

For the case where k isn't algebraically closed, see [7] Remarks in 5.6.

7.1

For the proof, see [5] Chapter 1 for details.

The followings are some comments for [5] Chapter 1 Theorem 1.7.

A post on surjective preimage for maximal ideal HERE.

A post on preimage for maximal ideal (not necessarily surj) HERE.

For completeness, a post on preimage of prime ideals HERE.

A post on image of prime ideals HERE, HERE, and HERE.

See Kemper's [5], Lemma 1.22 on Page 17, which completely described prime and maximal ideals in quotients.

8 Definition. Height

Here the definition *height* is specifically for prime ideal $\mathfrak{p} \lhd_{pr} R$ for some ring R. For a general definition, see a post HERE; see a webpage HERE; or see [5] Definition 6.10 on Page 68.

Nagata's example: Notherian ring with infinite Krull dimension, see HERE and a post HERE.

9 Proposition 1.7.

Difference between algebraic set and affine algebraic set? See HERE.

For an analogue in Projective, see Exercise 2.6 **??** in Chapter 1.2. And a post HERE.

10 Theorem 1.8A.

For transcendence degree, see HERE and a NOTE by Milne James.

11 Proposition 1.10.

Apart from the proof, we discussed *locally closed subset*. See HERE for its equivalent definitions.

12 Proposition 1.13.

See HERE.

13 Exercise 1.1.

13.1 (a)

By definition of affine coordinate ring we have

$$A(Y) = A/I(Y) = A/I(Z(f)) = A/\sqrt{\langle f \rangle}.$$

While f is irreducible in U.F.D. $k[x,y]$, the ideal it generated will be a prime ideal, which is radical. Therefore we can further simplify the expression as

$$A/\sqrt{\langle f \rangle} = A/\langle f \rangle = k[x,y]/\langle y - x^2 \rangle = k[x].$$

Hence we can conclude $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

13.2 (b)

14 Exericse 1.2.

My initial guess was incorrect, in which I thought $I(Y) = \langle z - xy \rangle$.

14.1

See a post HERE.
See a post HERE.
The correct one is $I(Y) = \langle z - x^3,\ z - y^2 \rangle$. Notice that

$$\dim Y = \dim A(Y) = \dim k[x,y,z]/\langle z-x^3,\ z-y^2 \rangle = \dim k[z] = \operatorname{tr.deg}_k k(z) = 1.$$

Therefore we proved that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

Question: How to compute $I(\cdot)$ precisely?

15 Exercise 1.4.

See a post HERE, HERE, and HERE.

The hint was to consider diagonal. Let the coordinate ring of $\mathbb{A}^2$ be $A(\mathbb{A}^2) = k[x,y]$ where k is algebraically closed. Then the diagonal

$$\Delta = \{(x,y) \in \mathbb{A}^2 \ \mid\ x = y\} = Z(x - y = 0)$$

is closed in Zariski topology of $\mathbb{A}^2$.

But it cannot be countably many union of closed subsets in product topology of copies of Zariski topology on $\mathbb{A}^1$, which are finitely many points. Wrong!

Here we used the fact that k is infinite given it's algebraically closed.

15.1

See the first post! Closed subset of product topology on $\mathbb{A}^1 \times \mathbb{A}^1$ can be uncountable, for example, a line $\{(x_0, y) \in \mathbb{A}^1 \times A^1 \mid y \in k\}$ where $x_0 \in k$ is fixed would be closed. But it's uncountable.

The reason why Δ is closed in Zariski topology of $\mathbb{A}^2$ is correct.

For product topology, the reason why it's not closed is because Zariski topology on $\mathbb{A}^1$ is not Hausdorff, by a lemma we mentioned we know Δ isn't closed.

Verified!

16 Exercise 1.7.

16.1 (a)

(i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv): Zorn's Lemma.
(ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii): contrapositive, with Axiom of Dependent Choice.
(i) $\Leftrightarrow$ (iii): taking complement.

16.2 (b)

A similar statement is Exercise 17. (v) of [1], which states that $\operatorname{Spec} A$ is quasi-compact for a ring A.

For a given open cover of $\{U_i\}_{i \in I}$ of X, we can throw away open subset U_i such that $U_i \subset U_j$ for some $j \in I$. Here I is some index set. So we can assume in this open cover, we don't have $U_i \subset U_j$ for any $i, j \in I$. Then we have a ascending chain of open subsets

$$U_i \subseteq U_i \cup U_j \subseteq \cdots .$$

Noetherian condition imposed gives us finitely many index, and they form a finite subcover as expected.

16.2.1

A very simple proof HERE used maximality property.
A stronger statement HERE.

17 Exercise 1.10.

17.1 (a)

Clear, since any chain of Y is also a chain in X, and we use property of sup mentioned on Prop 2.8. of HERE.

17.2 (b)

17.3 (c)

17.4 (d)

17.5 (e)

See a post HERE, HERE,
See the notes by Vakil HERE.
See the notes from UofT HERE.
See a post HERE.
Example in topological space is actually easier to say than Nagata's example.

Take our topological space as $X = [0,1] \subset \mathbb{R}$. And define the topology by declaring all closed subsets as

$$\{\emptyset, X, \{[1/n, 1] \subset \mathbb{R} \mid i \geq 2, i \in \mathbb{Z}\}\}.$$

Clearly we can check it satisfies infinite intersection and finite union axiom.

It's Noetherian for any descending chain of closed subsets must terminate.

It's of infinite Krull dimension. For any $n > 0$, we can construct a chain of such length. Hence the dimension, defined as supremum cannot be finite.

Another weird example could be found HERE, in which we give $\mathbb{N}$ the topology empty set, entire space, and $\{x \in \mathbb{N} \mid x \geq q\}$ for some $a \in \mathbb{N}$.

18 Graded Ring

Say we wish to define variety in some projective space, i.e. some subsets of $\mathbb{P}^n$ that could be written as zero locus of some polynomials. If we don't add any restrictions on polynomials, for example some non-homogeneous polynomials. Then the zero locus doesn't make sense because it will depend on the representitive. See Gathmann's Notes Remark 6.5, notes on algebraic geometry, HERE.

19 Proposition 2.2.

There are two very technical claims need more details.

The first one is to prove $\varphi(Y) = Z(T') = Z(\alpha(T))$. Unwrap the notation precisely according to the definition

$$\begin{aligned} Z(\alpha(T)) &= \{\ x \in \mathbb{A}^n \mid \alpha(g)(x) = 0\ \forall\ g \in T\ \}, \\ \varphi(Y) &= \{\ \varphi(y) \mid y \in Y\ \}. \end{aligned}$$

Notice that $y = [y_0, ..., y_n] \in Y \subset \overline{Y} = Z(T)$, therefore $g(y) = 0$ for any $g \in T$. More precisely, we have

$$\alpha(g)(\varphi(g)) = g(1, y_1/y_0, ..., y_n/y_n) = 0$$

given $g(y) = 0$ and $g \in T \subset S^h$, which proves $\varphi(Y) \subset Z(\alpha(T))$.

Conversely, let's start with an element $x = (x_1, ..., x_n) \in Z(\alpha(T))$. There's an element $y = [1, x_1, ..., x_n] \in Y$ such that $\varphi(y) = x$. Hence we've proved the equality.

And the second one is to check $\varphi^{-1}(W) = Z(\beta(T')) \cap U = Z(\beta(\alpha(T))) \cap U$.

20 Exercise 2.13.

Let $V \subset \mathbb{P}^5$ be the Veronese (2-uple) embedding of $\mathbb{P}^2$. Prove that for any closed curve (a ***curve*** *is a variety of dimension 1) $C \subset V$ there exists a hypersurface $H \subset \mathbb{P}^5$ such that $C = V \cap H$.*

20.1 References

A partial solution by REB HERE.

Another solution HERE.

And a post HERE.

Proof. The 2-uple embedding is defined as

$$\rho_2 : \mathbb{P}^2 \to \mathbb{P}^5$$
$$[x : y : z] \mapsto [x^2 : y^2 : z^2 : xy : yz : zx].$$

By Exercise 2.12, we know that ρ_2 is a homeomorphism onto its image

$$\rho_2 : \mathbb{P}^2 \simeq V.$$

Therefore a curve $C \subset V$ is given by $C = \rho_2(Z(f))$ for some irreducible homogeneous polynomial $f \in S(\mathbb{P}^2)$ such that $\dim Z(f) = 1$.

Notice that $f^2 = g$ for some $g \in k[x^2 : y^2 : z^2 : xy : yz : zx]$. Therefore if we define $H = Z(g)$, then

$$C = Z(f) = Z(f^2) = Z(g) \cap \rho_2(\mathbb{P}^2) = H \cap V.$$

□

21 Exercise 2.12.

For all monomial of degree d in $n + 1$ variables $x_0, ..., x_n$. There are

$$\binom{n+d}{n}$$

many monomials in total. Bars and balls argument: there are n bars and d balls. In $n + d$ many places, any choice of n bars will corresponds to a monomial, therefore N is the total number of monomials possible. While in we wish to consider them in projective space, we must define $N = \binom{n+d}{n} - 1$.

See a solution in lecture notes of Frank-Olaf Schreyer HERE.

A more detailed solution is given HERE.

22 Exercise 2.14.

There's a prompt on Sándor's Notes, Lecture 13, Homework 2.79.

23 Lemma 3.1.

See Sandor's Notes Lecture 4, Lemma 2.6.
Closedness can be checked locally. See a post HERE.

23.1 Closedness local criterion

This is HW 2.3. of Sandor's notes.

Let X be a topological space and $W \subset X$ a subset. Then W is closed if and only if for every $P \in X$ there is an open subset $U \subset X$ such that $P \in U$ and $W \cap U \subset U$ is a closed subset in U.

Proof. Assume W is closed, we can simply take $U = X$ for any P.

Conversely, we only need to verify that $X \setminus W$ is open. More precisely, we wish to prove that every point $P \in X \setminus W$ has an open neighborhood that contains in $X \setminus W$. This is ensured by Proposition 2.8. on Page 24 of [6].

Now start with an arbitrary point $q \in X \setminus W$, there exists open subset U_q of X such that

$$W \cap U_q \subset U_q$$

is closed in U_q. Then we can take $U_q \setminus W$ as the open neighborhood of q in $X \setminus W$ as expected. Hence we know $X \setminus W$ is open. □

24 Remark 3.1.1.

See Sandor's Notes Lecture 5, Corollary 2.10. Notice that Hartshorne defined variety to be irreducible. See a post explaining why the preimage is dense HERE.
See Lemma 14. on Page 210 of [2].

25 Definition: Ring of Regular Function

HERE is an explicit description on the ring structure of $\mathcal{O}_{P,Y}$.

26 Theorem 3.2.

See Sándor's Lecture Notes 09, STEP 03. See solution of problem 3 HERE. See a post HERE, HERE.

26.1 (c)

for each P, $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$, and $\dim \mathcal{O}_P = \dim Y$;

Proof. We begin with an injective homomorphism $\alpha : A(Y) \to \mathcal{O}(Y)$. And we define a map

$$\begin{aligned} A(Y)_{\mathfrak{m}_P} &\to \mathcal{O}_{P,Y} \\ f/g &\mapsto \langle V, \frac{\alpha(f)}{\alpha(g)} \rangle \end{aligned}$$

where $\frac{\alpha(f)}{\alpha(g)} \in \mathcal{O}(V)$. Now we wish to give an explicit description of V. Since $\alpha(f) \in \mathcal{O}(Y)$, we know there exists an open subset $P \in V_1 \subset Y$ such that

$$\alpha(f) \mid_{V_1} = \frac{h_1}{h_2} \mid_{V_1}$$

where $h_1, h_2 \in A$ and $0 \notin h_2(V_1)$. Since $\alpha(g) \in \mathcal{O}(Y)$, we know there exists an open subset $P \in V_2 \subset Y$ such that

$$\alpha(g) \mid_{V_2} = \frac{h_3}{h_4} \mid_{V_2}$$

where $h_3, h_4 \in A$ and $0 \notin h_4(V_2)$. Here $g \notin \mathfrak{m}_P$ by definition of localisation, which gives us

$$g(P) \neq 0 \;\Rightarrow\; \alpha(g)(P) \neq 0 \;\Rightarrow\; \exists\, V_3 \subset Y,\ \alpha(g) \mid_{V_3} \neq 0.$$

Then we take $V = V_1 \cap V_2 \cap V_3$ will suffice to work. This is because for any point $P \in V$, we have

$$\frac{\alpha(f)}{\alpha(g)} = \frac{h_1 h_4}{h_2 h_3}$$

for $0 \notin h_2 h_3(V)$.

$$\begin{tikzcd} A(Y) \arrow[r, "\alpha"] \arrow[dr] & \mathcal{O}(Y) \arrow[r, hook] & \mathcal{O}_{P,Y} \\ & A(Y)_{\mathfrak{m}_P} \arrow[ur, dashed] & \end{tikzcd}$$

The induced map is given by universal property of localisation, for every elements in $A(Y) \setminus \mathfrak{m}_P$ will be mapped to a unit in $\mathcal{O}_{P,Y}$. And the map given by Lecture Notes 09 of Prof. Sándor satisfy the universal property (it makes the diagram commute and maps elements outside of $\mathfrak{m}_P$ to units).

$\square$

27 Proposition 3.3.

See a post HERE.

28 Lemma 3.6.

See a post HERE.

Here are some details for proving $x_i \circ \psi$ being regular implies ψ is a morphism: Coordinate functions means

$$\psi(p) = (\psi_1(p), \psi_2(p), ..., \psi_n(p)) \in Y \subset \mathbb{A}^n$$

for any $p \in X$. Firstly, we check ψ is continuous. Take any closed subset $Z(f_1, ..., f_r) \subset Y$ for some polynomial $f_1, ..., f_r \in A = k[x_1, ..., x_n]$. We can compute the preimage as

$$\psi^{-1}(Z(f_1, ..., f_r)) = \{\ p \in X \ \mid \ \forall\ p \in X,\ f_i \circ \psi(p) = 0\ \}.$$

Notice that for any $p \in X$,

$$f_i \circ \psi(p) \ = \ f_i(\psi_i(p), ..., \psi_n(p))$$

is continuous since f_i is a polynomial and each $\psi_i := x_i \circ \psi$ is continuous by assumption that they're regular. Notice that the preimage of ψ is precisely intersection of $\psi_i^{-1}(\{0\})$ where $1 \leq i \leq n$. Hence the preimage is closed, and it follows that ψ is continuous as expected.

Secondly, fix an arbitrary open subset $V \subset Y$ with an arbitrary regular function $g : V \to k$, we wish to prove $g \circ \psi : \psi^{-1}(V) \to k$ is regular. For any $\psi(p) \in V$ with some $p \in X$, there exists a neighborhood $\psi(p) \in U \subset Y$ such that g equals to an expression of quotients of polynomial, i.e.

$$g = \frac{g_1}{g_2}$$

where $g_1, g_2 \in A$. Then for $p \in X$, take the open neighborhood of it as $\psi^{-1}(U)$, we can see

$$g \circ \psi(p) = \frac{g_1(\psi(p))}{g_2(\psi(p))}.$$

29 Exercise 3.5.

29.1 Hint

See a post HERE.

30 Exercise 3.6.

See a post HERE.

31 Exercise 3.10. Subvarieties

31.1 Locally Closed

See Wiki's notes HERE.
See a post HERE: image of a variety can be not even locally-closed.

32 Exercise 3.17.

Normal Varieties. *A vareity Y is* ***normal at a point*** *$P \in Y$ if $\mathscr{O}_p$ is an integrally closed ring. Y is* ***normal*** *if it is normal at every point.*

32.1 (a)

Show that every conic in $\mathbb{P}^2$ is normal.

Proof. According to Exercise 1.1.(c), we assume conic Y in $\mathbb{P}^2_{x,y,z}$ is defined by an irreducible homogeneous polynomial of degree 2.

And by Exercise 3.1.(c) we know every conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. To check it's normal, we need to show for any $P \in Y = \mathbb{P}^1$, the local ring $\mathscr{O}_P$ is an integrally closed ring.

Note that

$$\mathscr{O}_{P,\mathbb{P}^1} \simeq \mathscr{O}_{P,\mathbb{A}^1} = A(\mathbb{A}^1)_{\mathfrak{m}_P} = (k[x])_{\mathfrak{m}_P}.$$

While $k[x]$ is integrally closed, so we know its localisation is integrally closed hence every conic in $\mathbb{P}^2$ is normal. ???

Or we can use Exercise 3.18 (a). Notice that the homogeneous coordinate ring $S(\mathbb{P}^1)$ is clearly integrally closed, then it's projectively normal therefore normal.?

□

32.2 (b)

Show that the quadric surfaces Q_1, Q_2 in $\mathbb{P}^3$ given by equations $Q_1 : xy - zw$; $Q_2 : xy = z^2$ are normal (cf. (II. Ex. 6.4) for the latter.)

Proof. Denote $Q_1 = Z(xy - zw) \subset \mathbb{P}^3_{x,y,z,w}$. We have to compute the localisation of its homogeneous coordinate ring at some point $P \in Q_1$

$$\begin{aligned}\mathscr{O}_P =& S(Q_1)_{(\mathfrak{m}_P)}\\ =&(k[x,y,z,w]/I(Q_1))_{(\mathfrak{m}_P)}\\ =&(k[x,y,z,w]/\langle xy - zw\rangle)_{(\mathfrak{m}_P)}\end{aligned}$$

It suffices to check the quotient ring

$$k[x,y,z,w]/\langle xy - zw\rangle$$

is an integrally closed ring.???

Another approach is to use Jacobian matrix to check it's "smooth $\Rightarrow$ regular $\Rightarrow$ normal". Dimension of a surface is 2, then $n - r = 4 - 2 = 2 \geq 1$. Here 1 is the rank of Jacobian matrix at any point on the surface (for we choose only one polynomial $xy - zw$ as generator for $I(Q_1)$).

Similarly we can check for Q_2.

$\square$

32.3 (c)

Show that the cuspidal cubic $y^2 = x^3$ in $\mathbb{A}^2$ is not normal.

Proof. Let $P = (0,0)$, we wish to show that the local ring $\mathscr{O}_P$ is not integrally closed. Let $X = Z(y^2 - x^3)$, and note that $y^2 - x^3$ is irreducible in $k[x,y]$.

$$\mathscr{O}_P = A(X)_{\mathfrak{m}_P} = k[x,y]/I(X)_{\mathfrak{m}_P} = k[x,y]/(y^2 - x^3)_{\mathfrak{m}_P}.$$

While integrally closed is a local property, it's equivalent to check $k[x,y]/(y^2 - x^3)$ is an integrally closed domain. Since $y^2 - x^3$ is irreducible in UFD, then this quotient ring an integral domain. Denote $R = k[x,y]/(y^2 - x^3)$. Notice that $y/x \in \operatorname{Frac}(R)$ is an integral element since

$$(y/x)^2 - x = 0 \in R.$$

On the other hand, the element $y/x \notin R$. Because otherwise we'll have $y/x = f$ for some polynomial $f \in R$, which is an integral domain. In integral domain, the difference between the degree of variables will give a contradiction. Hence $k[x,y]/(y^2 - x^3)$ isn't integrally closed ring. $\square$

See a post HERE, HERE.

32.3.1 Hint

According to Example 2 of Chapter 9 on Integral Extension on Page 65 of [**?**], we know

$$k[x,y]/\langle y^2 - x^3\rangle \sim k[t^2,t^3] \subsetneq k[t].$$

Hence it's not an integrally closed domain.

32.4 (d)

If Y is affine, then Y is normal $\Leftrightarrow$ $A(Y)$ is integrally closed.

Proof. For Y affine, we have Y being normal is equivalent to say $\mathscr{O}_{P,Y} = A(Y)_{\mathfrak{m}_P}$ is integrally closed for any point $P \in Y$. While being integrally closed is a local property, it's equivalent to say $A(Y)$ is integrally closed. $\square$

32.5 (e)

Let Y be an affine variety. Show that there is a normal affine variety $\widetilde{Y}$, and a morphism $\pi : \widetilde{Y} \to Y$, with the property that whenever Z is a normal variety, and $\varphi : Z \to Y$ is a ***dominant*** *morphism (i.e., $\varphi(Z)$ is dense in Y), then there is a unique morphism $\theta : Z \to \widetilde{Y}$ such that $\varphi = \pi \circ \theta$. $\widetilde{Y}$ is called the* ***normalization*** *of Y. You will need (3.9A) above.*

Proof. Use Prof. Sándor's notes on Lecture 29 Theorem 4.16.

To complete the proof, we have to check the pre-variety defined in the proof is actually a variety. ? □

33 Exercise 3.18.

Projectively Normal Varieties. *A projective variety $Y \subset \mathbb{P}^n$ is* ***projectively normal*** *(with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.*

33.1 (a)

If Y is projectively normal, then Y is normal.

Proof. Since Y is projectively normal, then we know $S(Y)$ is integrally closed. Then we need to show

$$\mathscr{O}_P = S(Y)_{(\mathfrak{m}_P)}$$

is integrally closed. Clearly, $S(Y)_{\mathfrak{m}_P}$ is integrally closed. By definition we know $\mathscr{O}_P = S(Y)_{(\mathfrak{m}_P)}$ is a subring composed of degree 0 elements in $\mathscr{O}_P = S(Y)_{\mathfrak{m}_P}$. It is again integrally closed for any element $f/g \in \mathscr{O}_P = S(Y)_{\mathfrak{m}_P}$ that is integral over $\mathscr{O}_P = S(Y)_{(\mathfrak{m}_P)}$ satisfy the following equations

$$(f/g)^n + a_{n-1}(f/g)^{n-1} + a_0 = 0$$

for some $a_i \in \mathscr{O}_P = S(Y)_{(\mathfrak{m}_P)}$ where $0 \leq i \leq n-1$. Taking degree function on both sides, then it follows that f/g must be of degree 0, hence $\mathscr{O}_P = S(Y)_{(\mathfrak{m}_P)}$ is integrally closed.

□

33.2 (b)

34 Exercise 3.20.

Let Y be a variety of dimension ≥ 2, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

See a post HERE.

34.1 (a)

35 Definition: Dominant Rational Map

35.1

Well-definess for a rational map being *dominant*.

One thing important to keep in mind is both varieties X, Y are a priori irreducible. There's a completely point-set topological argument HERE. Here a technical detail is "The image of a dense subset under a surjective continuous function is again dense", which is from Wiki's entry. For details of this technicality, see HERE.

See a post on equivalent definition for dominant rational map HERE.

A good lecture note HERE.

Wiki's entry for Rational Map.

Very good note by Vakil HERE.

And a post HERE. However, it appears the way used in the proof implicitly required morphism is an open map? So I doubt...

35.2

Some posts share a technical details from general topology. The following statement is taken from Wiki...

The image of a dense subset under a surjective continuous function is again dense. More precisely, assume $f : X \to Y$ with E dense in X, then $f(E)$ is dense in $f(X)$.

Proof. By definition we have $E \subset f^{-1}(f(E)) \subset f^{-1}(\overline{f(E)})$, which is closed given f is continuous. It follows that

$$\overline{E} \subset f^{-1}(\overline{f(E)}) \;\Rightarrow\; f(\overline{E}) \subset \overline{f(E)}.$$

Conversely,

$$f(X) = \overline{f(E)} \cap f(X) \supset f(X) \cap f(X) = f(X) \;\Rightarrow\; f(X) \supset \overline{f(E)} \cap f(X).$$

Here $\overline{f(E)}$ denotes closure of $f(E)$ in Y, while it's intersection with $f(X)$ is the whole $f(X)$, then $f(E)$ is dense in $f(X)$. □

35.3 A Pathological Example

Another equivalent statement required "surjectivity" and say $f(E)$ is dense in Y. It's curtial. Also we can only conclude $f(E)$ is dense merely in $f(X)$ instead of Y. Since we have the continuous inclusion map $\iota : \mathbb{R} \to \mathbb{C}$, then $\mathrm{id}(\mathbb{Q}) = \mathbb{Q}$ is just dense in $\mathbb{R}$ but not dense in $\mathbb{C}$.

35.4

Say we start with a dominant rational map $\varphi : X \to Y$ with two representatives

$$\langle U, f\rangle,\ \langle V, g\rangle.$$

By definintion of dominant, we know $f(U)$ is dense in Y. To check this definition is indepedent of the choice of the representative, we have to check $g(V)$ is dense in Y.

Notice that

$$Y = \overline{f(U)} = \overline{f(\overline{U \cap V} \cap U)} \subset \overline{f(\overline{U \cap V})} \subset \overline{\overline{f(U \cap V)}} = \overline{g(U \cap V)} \subset \overline{g(V)}.$$

for X is irreducible and both U, V are non-empty and open then $X = \overline{U \cap V}$. Here the third inclusion is given by the previous technical lemma.

35.5 Composing Dominant Rational Maps

See a post HERE.

$$X \overset{\phi}{\dashrightarrow} Y \overset{\psi}{\dashrightarrow} Z$$

$$(U, \phi_U) \qquad (V, \psi_V)$$

To prove the composition is a dominant rational map, we need to find a representative. We define

$$W := \phi_U^{-1}(V).$$

? slightly different than the post online

And we claim $(W,\ \psi_V \circ \phi_U)$ will be suitable for a representative for $\psi \circ \phi$. First of all, notice that W is non-empty. This is because $\phi_U(U) \cap V \neq \emptyset$ given $\phi_U(U)$ is dense in Y and V is assumed to be non-empty open subset. While Y is irreducible, by Lemma 14. of **??** on Page 210, which states that $\phi_U(U) \cap V$ is nontrivial. By definition this implies

$$\phi_U^{-1}(V) \neq \emptyset.$$

This is non-empty and open. Note X is irreducible, hence $W = \phi_U^{-1}(V)$ is dense in X. Hence $\psi_V \circ \phi_U(W)$ is dense in Z given both maps are continuous by being a morphism.

35.6

See a post HERE, HERE, and HERE.

36 Lemma 4.2.

See a post HERE.

37 Definition Blowing Up

On Page 29, Hartshorne defined blowing-up of a closed affine subvariety Y of $\mathbb{A}^n$ passing through O.

See Sándor's Notes, Lecture 22, *strict transform.*

See Daniel's notes HERE. Here the notation $(\cdot)^-$ stands for taking closure.

38 Exercise 4.1.

Define a function

$$h : U \cup V \to k \text{ by}$$
$$x \mapsto \begin{cases} f(x) & \text{when } x \in U \setminus V \\ g(x) & \text{when } x \in V \end{cases}.$$

Since $f = g$ on $U \cap V$, hence the function h is well-defined. For any point $p \in U \cup V$, if $p \in U$, then we apply assumption that f is regular. For $p \in V$, similarly apply assumption that g is regular. Hence h is regular on $U \cup V$.

Let f be a rational function on X. So we take all equivalence class $\{\langle U_i, f_i\rangle\}_{i\in I}$ that represents f. By the above lemma and the definition of regular function, there's a regular function g that's defined on $U := \bigcup_{i\in I} U_i$ that extends all f_i. Therefore we can take a representative of f as

$$\langle U, g\rangle.$$

Note $\langle U, g\rangle = \langle U_i, f_i\rangle$ by definition, hence it's indeed a representative of f.

Also U is the largest open set. Suppose it's not, then we have $\langle U_{i_0}, f_{i_0}\rangle$ represents f such that $U_{i_0} \supsetneq U$. And this will contradicts the construction of U, which must contain U_{i_0}.

39 Exercise 4.2.

We're given a rational map $\varphi : X \dashrightarrow Y$. Suppose we have two equivalent representatives

$$\langle U, \varphi_U\rangle, \ \langle V, \varphi_V\rangle.$$

This means two morphisms φ_U, φ_V agree on $U \cap V$.

It suffices to prove that we can define a morphism $\psi : U \cup V \to Y$ that extends both φ_U and φ_V. Similarly, we can apply argument of 38 to conclude the existence of a largest open set on which φ is represented by a morphism.

Both φ_U, φ_V are continuous function that agree on their intersection, then we can define

$$\psi : U \cup V \to Y \text{ by } \psi(x) = \begin{cases} \varphi_U(x) & \text{when } x \in U \setminus V \\ \varphi_V(x) & \text{when } x \in V \end{cases}.$$

Clearly it's a well-defined continuous function. For any open subset $W \subset Y$ with an arbitrary regular function $f : W \to k$. We have $f \circ \psi : \psi^{-1}(W) \to k$ is a regular function since it extends both

$$f \circ \varphi_U : \varphi_U^{-1}(W) \to k,\ f \circ \varphi_V : \varphi_V^{-1}(W) \to k.$$

While both two regular functions agree on their intersection, then we can conclude using 38 that $f \circ \psi$ is a regular function. And this proves that ψ is indeed a morphism on $U \cup V \to Y$.

40 Exercise 5.1.

40.1 (a)

According to this picture, (a) is a tacnode.

Denote $f_1 := x^2 - x^4 - y^4$, which is irreducible in UFD $k[x, y]$ hence it's prime. We can then compute the ideal defined by this affine variety

$$I(Z(\langle f_1 \rangle)) = \sqrt{\langle f_1 \rangle} = \langle f_1 \rangle.$$

The dimension of the affine variety $Z(\langle f_1 \rangle)$ is

$$\dim Z(\langle f_1 \rangle) = \dim k[x, y]/\sqrt{\langle f_1 \rangle} = \dim k[x, y] - \text{height}\langle f_1 \rangle$$

By Krull's Hauptidealsatz we know the minimal prime ideal $\mathfrak{p}$ that contains $\langle f_1 \rangle$ has height exactly 1. While $\langle f_1 \rangle$ is a prime, we know it must be height of 1. Then by Theorem 4.7 in the notes, we know the Jacobian matrix at a singular point P cannot have rank $2 - 1 = 1$. While the matrix is 1×2, it follows that the matrix can only have dimension 0.

So we have to compute the Jacobian matrix of the above affine variety at some point $P \in \mathbb{A}^2$ on the affine variety. We choose f_1 itself as generators for the ideal of the affine variety and compute the Jacobian matrix

$$J(P) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(P) & \frac{\partial f_1}{\partial y}(P) \end{pmatrix} = \begin{pmatrix} 2x - 4x^3(P) & -4y^3(P) \end{pmatrix}.$$

Equivalently, we must havve $2x - 4x^3(P) = 0$ and $-4y^3(P) = 0$. Solving the equation, notice that P must lies on the tacnode, we'll get $P = (0, 0)$ is the only singular point.

40.2 (b)

According to this picture, (b) is a node.

Denote $f_2 = xy - x^6 - y^6$. Similarly, we choose f_2 itself as the generator for the affine variety it defined. Again, we have to compute the Jacobian matrix of $Z(f_2)$ at $P = (0, 0)$.

$$J(P) = \begin{pmatrix} \frac{\partial f_2}{\partial x}(P) & \frac{\partial f_2}{\partial y}(P) \end{pmatrix} = \begin{pmatrix} y - 6x^5(P) & x - 6y^5(P) \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix},$$

which as rank 0. Solving the equations $y - 6x^5(P) = 0$ and $x - 6y^5(P) = 0$ will implies that $P = (0, 0)$.

40.3 (c)

See this picture, then we know (c) is a cusp. Denote $f_3 = x^3 - y^2 - x^4 - y^4$. We just need to check the Jacobian matrix

$$J(P) = \begin{pmatrix} \frac{\partial f_2}{\partial x}(P) & \frac{\partial f_2}{\partial y}(P) \end{pmatrix} = \begin{pmatrix} 3x^2 - 4x^3(P) & -2y - 4y^3(P) \end{pmatrix}.$$

Solving the equations for points on cusp will forces $P = (0,0)$.

40.4 (d)

See this picture, then we know (d) is the triple point. And we denote $f_4 = x^2y + xy^2 - x^4 - y^4$. Compute the Jacobian matrix gives us

$$J(P) = \begin{pmatrix} 2xy + y^2 - 4x^3(P) & x^2 + 2xy - 4y^3(P) \end{pmatrix}.$$

Solving the equations $2xy + y^2 - 4x^3(P) = 0$ and $x^2 + 2xy - 4y^3(P) = 0$ will give us $P = (0,0)$.

40.5

See a post HERE.
See REB's solution HERE.
See a post on irreducibility of polynomial over $\mathbb{C}$ HERE.

41 Exercise 5.3.

Multiplicities. *Let $Y \subset \mathbb{A}^2$ be a curve defined by the equation $f(x,y) = 0$. Let $P = (a,b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0,0)$. Then write f as a sum $f = f_0 + f_1 + \cdots + f_d$, where f_i is a homogeneous polynomial of degree i in x and y. Then we define the* ***multiplicity*** *of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note that $P \in Y \Leftrightarrow \mu_P(Y) > 0$.) The linear factors of f_r are called the* ***tangent directions*** *at P.*

Notice that $P = (0,0)$ and

$$P \in Y \ \Leftrightarrow\ f(P) = 0 \ \Leftrightarrow\ f_0 = 0 \ \Leftrightarrow\ \mu_P(Y) > 0.$$

41.1 (a)

Proof. Notice that $\mu_P(Y) = 1$ is equivalent to say that $f_1 = ax + by \neq 0$ for some $a, b \in k$. Hence either a or b is non-zero. Now we compute the Jacobian matrix at P.

$$\begin{aligned}J(P) &= \begin{pmatrix}\frac{\partial f}{\partial x}(P) & \frac{\partial f}{\partial y}(P)\end{pmatrix}\\ &= \begin{pmatrix}0+\frac{\partial f_1}{\partial x}(P)+\frac{\partial f_2}{\partial x}(P)+\cdots & 0+\frac{\partial f_1}{\partial y}(P)+\frac{\partial f_2}{\partial y}(P)+\cdots\end{pmatrix}\\ &= \begin{pmatrix}\frac{\partial f_1}{\partial x}(P) & \frac{\partial f_1}{\partial y}(P)\end{pmatrix} = \begin{pmatrix}a & b\end{pmatrix}.\end{aligned}$$

This 2×1-matrix as dimension 1 exactly because either a or b is nonzero.

If we can assume $\dim Y = 1$, then it follows that $n - r = 2 - 1$ equals to the rank of the Jacobian matrix, which proves that P is non-singular on Y. The converse direction is similar.

could we assume curve f is irreducible? ☛ See [4] Example 1.4.2. in Chapter 1 on Page 4

□

41.2 (b)

See solution HERE.

Verified HERE

42 Exercise 5.6.

Blowing Up Curve Singularities.

42.1 (a)

Let Y be the cusp or node of (Ex. 5.1). Show that the curve $\widetilde{Y}$. obtained by blowing up Y at $O = (0,0)$ is nonsingular (cf. (4.9.1) and (Ex. 4.10)).

Proof. Let Y be the node curve, so $Y = I(xy - x^6 - y^6)$. Let x, y be coordinate of $\mathbb{A}^2$ and let u, v be coordinates for $\mathbb{P}^1$. For $\mathbb{A}^2_{x,y} \times \mathbb{P}^1_{u,v}$, we know the blowing-up of $\mathbb{A}^2$ at O is

$$\operatorname{Bl}_O \mathbb{A}^2 = Z(xv - yu) \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

And we denote the projection as $\varphi : \operatorname{Bl}_O \mathbb{A}^2 \to \mathbb{A}^2$. Now we're going to compute strict transform of Y

$$\widetilde{Y} = \overline{\varphi^{-1}(Y \setminus \{O\})}.$$

For any point $a_0 \in (x, y, u : v) \in \widetilde{Y}$, we know $\varphi(a_0) = (x, y) \in Y$. Hence we know at least

$$\widetilde{Y} \subset Z(xv - yu, xy - x^6 - y^6).$$

Now we try to restrict ourselves to one affine cover U_u of $\mathbb{P}^1$, i.e. let $u = 1$. Then

$$\begin{aligned}Z(xv - yu,\ xy - x^6 - y^6) \cap U_u &= Z(xv - y,\ xy - x^6 - y^6)\\ &= Z(x^2(v - x^4 - x^4v^6)) \subset \mathbb{A}^3.\end{aligned}$$

Here $Z(x^2)$ is the exceptional set. And $\widetilde{Y} \cap U_u = Z(v - x^4 - x^4 v^6)$. Take $f_1 = v - x^4 - x^4 v^6$ and compute the Jacobian matrix at $P \in \mathbb{A}^2_{x,v}$

$$J(P) = \begin{pmatrix} \partial f_1/\partial x(P) & \partial f_1/\partial v(P) \end{pmatrix} = \begin{pmatrix} -4x^3 - 4v^6x^3(P) & 1 - 6x^4v^5(P) \end{pmatrix}.$$

Notice that there's no solution of P for equations $-4x^3 - 4v^6x^3(P) = 0$ and $1 - 6x^4v^5(P) = 0$. Therefore the matrix has rank exactly 1 becuase the coefficient k is a field. And by Krull's Hauptidealsatz, we know the dimension for the curve is 1. Then apply Theorem 4.7. from the notes we know $2 - 1 = 1$ is exactly the rank of the Jacobian matrix. Hence there's no singular points on $\widetilde{Y}$.

Similarly, we can check there's no singular points of $\widetilde{Y}$ on another affine cover U_v where $v = 1$. Hence we can conclude $\widetilde{Y}$ is non-singular. □

is it affine?

42.2 (b)

We define a ***node*** *(also called* ***ordinary double point****) to be a double point (i.e., a point of multiplicity 2) of a plane curve with distinct tangent directions (Ex. 5.3). If P is a node on a plane curve Y, show that $\varphi^{-1}(P)$ consists of two distinct nonsingular points on the blown-up curve $\widetilde{Y}$. We say that "blowing up P resolves the singularity at P".*

Proof. □

42.3 (c)

Let $P \in Y$ be the tacnode of (Ex. 5.1). If $\varphi : \widetilde{Y} \to Y$ is the blowing-up at P, show that $\varphi^{-1}(P)$ is a node. Using (b) we see that the tacnode can be resolved by two successive blowings-up.

42.4 (d)

Let Y be the plane curve $y^3 = x^5$, which has a "higher order cusp" at O. Show that O is a triple point; that blowing up O gives rise to a double point (what kind?) and that one further blowing up resolves the singularity.

43 Exercise 5.12.

Quadric Hypersurfaces*. Assume* $\operatorname{char} k \neq 2$, *and let f be a homogeneous polynomial of degree 2 in $x_0, ..., x_n$.*

43.1 (a)

Show that after a suitable linear change of variables, f can be brought into the form $f = x_0^2 + \cdots + x_r^2$ for some $0 \leq r \leq n$.

Proof. Notice that it suffices to prove, after suitable linear transformation of variables, we can kill all terms $x_i x_j$ where $i \neq j$. Because then we know the polynomial will be $b_0 x_0^2 + \cdots + b_r x_r^2$, and we simply let $x_r \mapsto 1/\sqrt{b_r} x_r$ will yield the desired form. Denote our homogeneous polynomial f as

$$\begin{aligned}
f =& a'_{00} x_0^2 + a'_{01} x_0 x_1 + \cdots + a'_{0n} x_0 x_n \\
&+ a'_{10} x_1 x_0 + \cdots + \\
&+ a'_{n0} x_n x_0 + \cdots + a'_{nn} x_n x_n \\
=& \sum_{0 \leq i \leq j \leq n} a_{ij} x_i x_j \\
=& a_{00} x_0^2 + a_{01} x_0 x_1 + a_{02} x_0 x_2 + \cdots + a_{nn} x_n x_n.
\end{aligned}$$

Given $\operatorname{char} k \neq 2$, we know $1/2 \neq 0$. We denote a symmetric $(n+1) \times (n+1)$-matrix with coefficients in k by

$$A = \begin{pmatrix} a_{00} & a_{01}/2 & a_{02}/2 & \cdots & a_{0n}/2 \\ a_{01}/2 & a_{11} & a_{12}/2 & \cdots & a_{1n}/2 \\ \vdots & \ddots & & & \\ a_{0n}/2 & a_{1n}/2 & a_{2n}/2 & \cdots & a_{nn}. \end{pmatrix}$$

. The matrix A is symmetric and except the diagonal, every coefficient has an extra coefficient $1/2$. Let a vector be $\mathbf{v} = [x_0 \ x_1 \ \cdots x_n]$. The reason we introduce this matrix is becuase the following identity

$$\begin{aligned}
\mathbf{v} A \mathbf{v}^t =& a_{00} x_0 x_0 + 1/2 a_{01} x_0 x_1 + \cdots 1/2 a_{0n} x_0 x_n \\
&+ 1/2 a_{01} x_0 x_1 + a_{11} x_1 x_1 + 1/2 a_{12} x_1 x_2 + \cdots + 1/2 a_{2n} x_2 x_n \\
&+ \cdots \\
&+ 1/2 a_{0n} x_0 x_n + \cdots a_{nn} x_n x_n \\
=& a_{00} x_0 x_0 + a_{01} x_0 x_1 + \cdots + a_{nn} x_n x_n = f.
\end{aligned}$$

While A is symmetric and over an algebraically closed field k, we can diagonalise it by some matrix B:

$$BAB^{-1} = \begin{pmatrix} a_{00} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}.$$

Here matrix B will provide information on linear change of variables. Also during linear change of variables, square terms can never be killed. So r depends on number of square terms in the original polynomial f. ∎ ?

43.2 (b)

Show that f is irreducible if and only if $r \geq 2$.

Proof. Now we assume $f \in k[x_0, ..., x_r]$ for some $0 \leq r \leq n$. Suppose f is reducible. So we can find a factorisation $f = f_1 f_2$ for some non-unit polynomials $f_1, f_2 \in k[x_0, ..., x_r]$. This is equivalent to say that f_1, f_2 must be of homogeneous of degree 1 given the coordinate ring $k[x_0, ..., x_r]$ is an integral domain. Given that $r \geq 2$, we can express the factorisation without loss of generality as

$$f = f_1 f_2 = (x_0 + a_1 x_1 + \cdots + a_r x_r)(x_0 + b_1 x_1 + \cdots + b_n r x_r)$$

where all $a_i, b_i \in k \setminus \{0\}$ for $1 \leq i \leq r$. In order for the terms $x_i x_j$ where $i \neq j$ to be killed, we must have $a_i b_j + a_j b_i = 0$ for all $0 \leq i \leq j \leq r$. Here we assume $a_0 = b_0 = 1$. Hence we immediately have $a_j = -b_j$ for all $1 \leq j \leq r$ and the factorisation becomes

$$(x_0 + a_1 x_1 + \cdots + a_r x_r)(x_0 - a_1 x_1 - \cdots - a_r x_r).$$

And this means we can never kill the terms such as $x_1 x_2$ for it has coefficent $2a_1 a_2$. Then $f \neq f_1 f_2$, contradiction. It follows that f is irreducible.

□

See a post HERE.

44 Exercise 6.1.

Recall that a curve is ***rational*** *if it is birationally equivalent to* $\mathbb{P}^1$ *(Ex 4.4). Let* Y *be a nonsingular rational curve which is not isomorphic to an open subset of* $\mathbb{P}^1$*.*

44.1 (a)

Show that Y *is isomorphic to an open subset of* $\mathbb{A}^1$*.*

Proof. Nonsingular rational curve Y is a nonsingular quasi-projective curve, which is isomorphic to an abstract nonsingular curve by Proposition 6.7.

According to Corollary 6.10., an abstract nonsingular curve is isomorphic to an open subset of a nonsingular projective curve Z. Therefore by Theorem 4.4. we have

$$\mathbb{P}^1 \sim_{\mathrm{bir}} Y \sim_{\mathrm{bir}} Z \;\Rightarrow\; K(\mathbb{P}^1) \simeq K(Z).$$

? This shows that it's isomorphic to an open subset of $\mathbb{P}^1$ therefore some open subset in $\mathbb{A}^1$.

□

44.2 Hint

See a post HERE.
See REB's solution HERE.

45 Exercise 6.2.

45.1 (a)

Proof. We compute the Jacobian of $I(Y)$ as:

$$\begin{pmatrix} -3x^2+1(P) & 2y(P) \end{pmatrix}$$

This matrix evaluate at some point $P \in Y$ will have rank $n-r = 2-1 = 1$ given both functions cannot be 0 at the same time, therefore the curve is non-singular. Hence we know

$$\mathscr{O}_{P,Y}$$

is regular local ring for any $P \in Y$. While it's a local property(?), we know $A(Y)$ is regular local. While coordinate ring of this curve Y is Noetherian domain of dimension 1, hence Theorem 6.2A. implies it's an integrally closed domain. $\square$

46 Defintion: Presheaf

46.1 Two Pathological Examples

Here are two examples taken from Tennison's [9].

Let X be any topological space with more thatn one point, i.e. $X = \{0,1\}$ or $X = \{0,1\} \to \mathbb{R}$.

Define a presheaf $\mathscr{P}_1$ by

$$\begin{cases} \mathscr{P}_1(X) = \mathbb{Z} \\ \mathscr{P}_1(U) = 0 \ \text{ for open } \ U \subsetneq X \\ \text{All restrictions except } \rho_{XX} \text{ being constant maps.} \end{cases}.$$

Here 0 denotes the trivial Abelian group.

Pick $x_0 \in X$. Define a presheaf $\mathscr{P}_2$ by

$$\begin{cases} \mathscr{P}_2(U) = \mathbb{Z} \ \text{ for } U \text{ open in } X \text{ such that } x_0 \in U \\ \mathscr{P}_2(U) = 0 \ \text{ for } U \text{ open in } X \text{ such that } x_0 \notin U \\ \text{restrictions } \ \rho_{UV} = \begin{cases} \mathrm{id}_{\mathbb{Z}} & \text{if } x_0 \in V \subset U \\ 0 & \text{trivial map if not} \end{cases} \end{cases}.$$

Here the second appearance of 0 denotes the trivial map.

47 Example 1.0.3.

See some examples of presheaves that are not sheaves HERE; a post HERE.

In Wiki's page HERE, it introduced non-separated presheaf, i.e. presheaf that doesn't satisfy locality axiom for sheaf.

48 Proposition-Definition 1.2. Sheafification

See *Sheafification* on The Stacks Project.
See solution of problem 3 HERE.
Of course, consult Ravi's Notes on Sheafification;
Or see Section 6.5 on Page 232 of [2].
Also, see a REU paper HERE by Daping Weng.
A short paper by Tom is HERE.

48.1 Isomorphism on stalk

This is for Lemma 007Z of Stacks Project. Similar contents could be found in Rising Sea 2.4.L.

$$
\begin{tikzcd}
 & s \in \mathscr{F}(U) \arrow[r] \arrow[d] \arrow[dl] & \mathscr{F}^{\sharp}(U) \ni (s_u)_{u\in U} \arrow[d] \\
t \in \mathscr{F}(W) & (U,x) \in \mathscr{F}_x \arrow[r, dashed, "\exists !"] & \mathscr{F}^{\sharp}_x \ni (U,(s_u)_{u\in U})
\end{tikzcd}
$$

According to Stacks Project, injectivity is proved.

We focus on proving the surjectivity of $\mathscr{F}_x \to \mathscr{F}^{\sharp}_x$. We constructed the induced map on stalks of presheaves based on universal property of $\mathscr{F}_x$ being a colimit. Therefore the map is unique. And we wish to get an explicit description of that. While for the diagram we know all but the induced map explicity, therefore the map defined such that making the diagram commute must be the induced map by uniqueness of universal property. It follows that we have

$$
\begin{aligned}
\mathscr{F}_x &\to \mathscr{F}^{\sharp}_x \\
(U,s) &\mapsto (U,(s_u)_{u\in U})
\end{aligned}
$$

where $x \in U \subset X$, $s \in \mathscr{F}(U)$, and $(s_u)_{u\in U} \in \prod_{x\in U} \mathscr{F}_x$ is a compatible germ.

We could also check this map is well-defined. But it must be for it makes the diagram commute, hence it's the unique induced map on stalk for a given $x \in X$. do we care?

The facts that $x \in U$ and $(s_u)_{u\in U}$ is a compatible germ enable us, by definition of sheafification, find:

- a neighborhood $W \subset U$ such that $x \in W$;
- a section $t \in \mathscr{F}(W)$ such that $t_w = s_w$ for any $w \in W$.

And we claim $(W,t) \in \mathscr{F}_x$ will be the preimage of $(U,(s_u)_{u\in U})$. Now we're ready to compute the image of (W,t) as

$$
\begin{tikzcd}
t \in \mathscr{F}(W) \arrow[r] \arrow[d] & (t_u)_{u\in U} \in \mathscr{F}^{\sharp}(W) \arrow[d] \\
(W,t) \in \mathscr{F}_x \arrow[r, dashed] & (W,(t_u)_{u\in W}) \in \mathscr{F}^{\sharp}_x
\end{tikzcd}
$$

We claim that

$$(W, (t_u)_{u \in W}) = (U, (s_u)_{u \in U}) \in \mathscr{F}_x^\sharp$$

This is because there exists $W \subset W \cap U$ such that

$$\begin{aligned}(s_u)_{u \in U} \mid_W =& (s_u)_{u \in W} \\ =& (t_u)_{u \in W} \\ =& (t_u)_{u \in W} \mid_W .\end{aligned}$$

Here the first and third equality is given by restriction map. The second equality holds: by definition we claimed that $s_w = t_w$ for any $w \in W$. Therefore we've checked that (W, t) is the preimage for an arbitrary element $(U, (s_u)_{u \in U}) \in \mathscr{F}_x^\sharp$, and it follows that $\mathscr{F} \to \mathscr{F}^\sharp$ is surjective.

48.2 Hint

The above approach is based on Stacks Project. But Proposition 2.24. on Page 53 of Algebraic Geometry I: Schemes [**?**] gives another solution, which is **much** more efficient!

It identitifies, in the sense of colimi, that

$$\operatorname{colim} \mathscr{F}^\sharp = \mathscr{F}_x.$$

49 Definition: Inverse Image Sheaf

See Wiki for motivation of such definition.

See a POST that gives more details, as well as a counterexample.

From Prof. Sándor's Email: Let X by disjoint union of two copies of Y with a continuous map $f : X \to Y$. Assume Y is irreducible and let $\mathscr{G}$ be a constant sheaf on Y. We claim that $f_{\text{pre}}^{-1}\mathscr{G}$ is just a presheaf, but not a sheaf.

Any open subset $W_1, W_2 \in X$ will have intersection in Y. Then any section will agree on their intersections. Take two sections from $0 \amalg Y$ and $Y \amalg Y$, there won't be a global section such that restriction is either of them.

50 Exercise 1.1.

See "Rising Sea" by Ravi Exercise 2.2.E. on Page 74. It gave another potentially equivalent definition of constant sheaf, which could be easier to check it is indeed sheafification of constant pre-sheaf.

Let $\mathscr{F}$ denotes the constant pre-sheaf. Then we can compute, by using universal property of colimit that stalk at $p \in U$ is $\mathscr{F}_p = S$.

Now we have a concrete description of the compatible germs as

$$\begin{aligned}\mathscr{F}^\sharp(U) &= \{s : U \to \amalg_{p \in U} \mathscr{F}_p \ \mid \ ...\} \\ &= \{s : U \to \amalg_{p \in U} S \ \mid \\ &\quad \forall\ p \in U,\ \exists \text{ an open neighorbood } V \subset U \text{ containing } p \text{ and } \exists\ t \in \mathscr{F}(V). \\ &\quad \text{such that } s(q) = s_q = t_q\ \forall\ q \in V.\}\end{aligned}$$

But notice that for constant presheaf $\mathscr{F}$, all restriction maps and natural map to stalk is identity as

$$\begin{tikzcd}
\mathscr{F}() = S \arrow[rr, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & & \mathscr{F}() = S \arrow[dl, "\mathrm{id}"] \\
& \mathscr{F}_P = S &
\end{tikzcd}$$

So we can simplify the expression as

$$\begin{aligned}
\mathscr{F}^\sharp(U) = \{ f : U \to \amalg_{p\in U} S \ \ | \\
&\forall\, p \in U,\ \exists \text{ an open neighorbood } V \subset U \text{ containing } p \text{ and } \exists\, t \in S. \\
&\text{such that } f(q) = t_q = \mathrm{id}(t) = t \ \forall\, q \in V \}.
\end{aligned}$$

We claim there's a bijection between two sets

$$\Phi : \mathscr{F}^\sharp(U) \to \underline{S}(U)$$

For a given $f \in \mathscr{F}^\sharp(U)$, we can define a map $g : U \to S$ by

$$g(p) := \mathrm{pr}_p \circ f(p)$$

where the projection map is $\mathrm{pr}_p : \amalg_{i\in I} S_i \to S_p$ defined by projection to p-coordinate. For any $p \in U$, by definition of sheafification we know there is an open neighborhood $p \in V \subset U$ and $t \in S$ such that

$$g(q) = \mathrm{pr}_q \circ f(q) = t$$

for any $q \in V$. And this is precisely saying g is locally constant. Different choices of f will result in different g, therefore it's injective.

Clearly, if we're given a locally constant map $g : U \to S$, we can form a tuple indexed by $p \in U$ as

$$\prod_{u\in U} (g(u)) \in \amalg_{u\in U} S.$$

And this corresponds to a function in $\mathscr{F}^\sharp(U)$ that satisfies the requirements precisely because g is locally constant. Therefore it's surjective.

What we've shown is that there's a bijection between

$$\Phi : \mathscr{F}^\sharp(U) \to \underline{S}(U)$$

where $\mathscr{F}$ is constant pre-sheaf. Therefore we conclude that constant sheaf is indeed the sheafification of constant pre-sheaf.

50.1 References

See a post HERE, HERE.

Basically, we need to prove $\underline{S}^\sharp_{\mathrm{pre}} \simeq \underline{S}$. And I did by exhibiting a bijection on when they both evaluate at an open subset U, i.e. I checked isomorphism between two functors by showing the natural transformation is natural isomorphism. I didn't do this on stalk for I'm afraid it could be more complicated.

But could we?

51 Exercise 1.3.

See a post HERE for explicit information of induced map on stalks.
See the solution from a post HERE.
See HERE for a partial solution, as well as a counterexample.

51.1 (a)

Now assume φ is surjective. Fix an open subset $U \subset X$ and a section $s \in \mathscr{G}(U)$. Now we can pick any point $p \in U$, consider the stalk at it.

52 Exercise 1.8.

See Rotman's [8], Lemma 6.68. on Page 378.

53 Exercise 1.15.

See Rising Sea 2.3.C.

54 Exercise 1.22.

54.1 Hint

See Stacks Project Glueing Sheaves, in which condition (2) is called *glueing data*.

According to the following Lemma 00AL, there exists (not necessarily unique) a sheaf $\mathscr{F}$ on X such that ...

It remains to use (1) in the prompt to verify such a sheaf is unique.

See a post HERE.

55 Definition: Ringed Space

55.1 Local Homomorphism

The map $\mathscr{F}^\sharp_P$ is defined in Rising Sea's Exercise 2.2.I., PUSHFORWARD INDUCES MAPS OF STALKS.

The condition $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$ has equivalent definitions by Lemma 07BJ on Stacks Project.

Here is an example of ring homomorphism between local rings that's not a local homomorphism, see a post HERE.

Part II
The Rising Sea

56 1.3.F. EXERCISE.

A post discussing this problem is HERE.

57 1.3.N

A crutial step is to define the map such that the diagram commute. In order to prove it satisfies the universal property. We're given A be arbitrary with map $g : A \to X$ and $f : A \to Y$.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists! \varphi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_Z Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\beta"] \\
& X \arrow[r, "\alpha"'] & Z
\end{tikzcd}$$

We can to define

$$\begin{aligned} \varphi : A \to X \times_Z Y \quad \text{by} \\ a \mapsto (g(a), f(a)). \end{aligned}$$

And we can verify this definition will make the diagram commute, and is unique.

58 1.3.O

It's indeed intersection. A post HERE.
A post HERE.

59 1.3.P.

Say we have $X \times Y$ and $X \times_Z Y$. By universal property of product and fibered product we can produce two unique map goes in between. Their composition must be identity, hence they're isomorphic. Notice it's important for Z being a final object.

$$\begin{tikzcd}
X \times Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\beta"] \\
X \arrow[r, "\alpha"'] & Z
\end{tikzcd}$$

When we trying to make a map from $X \times Y$ to $X \times_Z Y$, we have to make sure two maps

$$\beta \circ \pi_1 = \alpha \circ \pi_1$$

on $X \times Y$. While Z is the final object, hence they must agree.

There's a cleaner way to state it HERE. Crutial part is applying final property of object Z.

60 1.3.Q.

$$\begin{tikzcd}
A \arrow[drr, bend left, "g_1"] \arrow[ddr, bend right, "g_2"'] \arrow[dr, dashed, "\exists!\varphi"] \arrow[dr, dashed, "\exists!\psi"'] & & \\
& U \arrow[r, "f_1"] \arrow[d, "f_2"'] & V \arrow[d, "f_3"] \\
& W \arrow[r, "f_4"'] \arrow[d, "f_5"'] & X \arrow[d, "f_6"] \\
& Y \arrow[r, "f_7"'] & Z
\end{tikzcd}$$

Label the maps as indicated. To prove the universal property with respect to the "outside rectangle", we're given

$$f_6 f_3 g_1 = f_7 g_2$$

agree on A. While W is fibered product, apply universal property of fibered product with resepct to W we immediately get a unique map

$$\psi : A \to W$$

that makes the diagram involving A, W, X, Y, Z commute. In particularly, we know $f_4 \psi = f_3 g_1$. Furthermore, recall that U is the fibered product. We're given the condition that $f_4 \psi = f_3 g_1$, by universal property of U we know there exists a unique map

$$\varphi : A \to U$$

making the diagram involving A, U, V, W, X commute. And we claim that the diagram involving A, U, V, Y, Z commute. This is because

$$g_2 = f_5 \psi = f_5 f_2 \varphi = (f_5 f_2)\varphi \quad \text{and} \quad g_1 = f_1 \varphi.$$

And this proves that U is the fibered product for the diagram involving A, U, V, Y, Z.

A post is HERE.

61 1.3.R

$$
\begin{tikzcd}
X_1 \times_Y X_2 \arrow[drr, bend left, "\pi_1"] \arrow[ddr, bend right, "\pi_4"'] & & & \\
& X_1 \times_Z X_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_3"'] & X_2 \arrow[d, "f_3"] & \\
& X_1 \arrow[r, "f_2"'] & Y \arrow[dr, "f_1"] & \\
& & & Z
\end{tikzcd}
$$

By the universal property of $X_1 \times_Z X_2$, we know there exists a unique map

$$\varphi : X_1 \times_Y X_2 \to X_1 \times_Z X_2$$

"Natural morphism", a convention discussed HERE.

62 Course Notes from Cornell

See HERE.

63 1.3.S. Magic Diagram

Didn't finish. Need to See HERE, HERE!!!

64 1.3.Y. (a)

Yoneda's Lemma Given what we have, define $g : A \to A'$ as

$$g := i_A(\mathrm{id}_A).$$

This is correct, see a post HERE.

65 1.4.C.

(a) See "A Term of Commutative Algebra", Example 7.3 on Page 52.

66 1.6.B.

Write out everything by definition, and we can finish the proof immediately by applying rank-nullity theorem for linear transformation...

67 2.2.6. Definition: Sheaf.

Comments on $\mathscr{F}(\emptyset)$. In category **Set**, the empty set is initial object and one element set is terminal. See Wiki's examples HERE.

67.1 Example

68 2.2.B. Presheaves that are not SHEAVES.

68.1 (a)

For (a): see Wiki's counterexample HERE, which gave an explanation for presheaves on $\mathbb{R}$ instead of $\mathbb{C}$. See a post HERE.

68.2 (b)

See a post HERE, HERE, and HERE.

Proof. This problem is based on some knowledge from complex analysis. Here are some facts:

- When does a complex function have a square root?
- Theorem 6.2 on Page 100 of [**?**]. This means $f(x) = x$ as a function on $\mathbb{C}$ does not admit a square root for it will vanish.

However, we can cover $\mathbb{C}$ by two slit regions $U_1 = \mathbb{C} - (-\infty, 0], U_2 = \mathbb{C} - [0, \infty)$. And on each U_i, $f\mid_{U_i}$ admits a square root and satisfy gluability axiom.

The solution given in HERE is saying on annulus, $f(x) = x$ cannot have a square root. And then we can follow argument of HERE.

$\square$

69 2.2.7.

See Daping's Notes Definition 2.5 on Page 4 HERE; also a post HERE; also a post HERE.

See a post HERE and HERE regarding gluing sheaves.

70 2.2.C.

Since $\cup U_i$ is colimit for $\{U_i\}$, then $\mathscr{F}(\cup U_i)$ will be limit because of the contravariance.

?

See an expository post HERE.

See another post, with more detailed explanations HERE. "In fancy language, it's stack"...

71 2.2.D.

(b) Motivating example for definition of sheaf.

72 2.2.E.

See comments for Exercise 1.1. of Chapter 2 [4].

73 2.2.F.

Almost by definition.

74 2.2.G.

74.1 (a)

75 2.2.9.

See a post HERE.

76 2.2.10.

It's different from a post HERE, and Wiki's page on Constant pre-Sheaf. Why???

Clearly it's a contravariant functor

$$\mathscr{F} := \underline{S}_{\text{pre}} : \mathbf{Top}(X) \to \mathbf{Set}$$

Let $X = \{a, b\}$ with discrete topology. Pick two sections

$$s_1 \in \mathscr{F}(\{a\}) = S,\ \ s_2 \in \mathscr{F}(\{b\}) = S$$

such that $s_1 \neq s_2$ given S has at least two distinct elements. Clearly we have

$$s_1 \mid_{\{a\}\cap\{b\}} = s_1 \mid_{\emptyset} = e = \cdots = s_2 \mid_{\{a\}\cap\{b\}} .$$

If it's a sheaf then there exists a global section $s \in \mathscr{F}(\{a, b\}) = S$ such that

$$s_1 = s \mid_{\{a\}} = s \mid_{\{b\}} = s_2,$$

contradiction. It follows that constant presheaf defined this way is not necessarily a sheaf.

77 2.2.E.

We have to deceptively identical pre-sheaves $\mathscr{F}_1$ defined as locally closed, and $\mathscr{F}_2$ defined by giving S discrete topology...

We wish to prove they, as pre-sheaves, are isomorphic. Equivalent, we need to exhibit a natural transformation that admits an inverse. And it suffices to prove by element inclusions:

- Let $f : U \to S$ be a map that's locally constant. Now we take $g(u) = f(u)$ as a map $g : U \to S$ with S endowed with a discrete topology. We claim that g is continuous. It suffices to check for each $s \in S$, the fiber $g^{-1}(s)$ is open. For any point $a \in g^{-1}(s) = f^{-1}(s)$, there exists an open neighborhood $V_a \subset f^{-1}(s)$ such that
$$f(V_a) = \{s\}$$
given f is locally constant. While $V_a \subset g^{-1}(s)$, therefore we know $g^{-1}(s)$ is open and g is continuous.

- Conversely, we assume $g : U \to S$ with S given a discrete topology is continuous. We claim $f = g$ is locally constant. For any point $p \in U$, there is an open neighborhood
$$g^{-1}(f(p)) \ni p$$
such that f is constant because $f(g^{-1}(f(p))) = \{p\}$.

Now we try to check constant sheaf $\mathscr{F}$ is indeed a sheaf. We're going to prove identity axiom and gluability axiom using the "better description", which is much easier to check:

- If we have two functions, which equal whenever we restric to any open subset from an open cover, then they must be equal. For functions are precisely defined this way.

- Define the global section for any choice, and it's going to be well-defined for they're compatible.

Therefore $\mathscr{F} = \underline{S}$ is indeed a sheaf.

78 2.2.F.

Same argument as 71.

79 2.2.G.

79.1 (a)

It's clearly a pre-sheaf.

Fix an open subset $U \subset X$ with an open cover $\{U_i\}_{i \in I}$ for some index set I. Denote the presheaf as $\mathscr{F}$.

Pick two continuous maps $s_1, s_2 : Y \to X$ that satisfying the requirements, i.e. $s_1, s_2 \in \mathscr{F}(U)$.

Both functions will agree on U since

$$\mathrm{Res}_{U,U_i} \ \ s_1 = \mathrm{Res}_{U,U_i} \ \ s_1$$

for arbitrary U_i, whose union is U. So we must have $s_1 = s_2$.

Again with this open cover $\{U_i\}_{i \in I}$ and $a_i \in \mathscr{F}(U_i)$ for $i \in I$. Equivalently, we know $a_i : U_i \to Y$ is a continuous map satisfying $\mu \circ a_i = \mathrm{Id}\,|_{U_i}$. Now let's define a map

$$\begin{aligned} f : U &\to Y \\ u &\mapsto a_i(u) \ \ \text{when} \ \ u \in U_i. \end{aligned}$$

It's well-defined by our assumption. Also it's continuous since preimage of an open set in $V \subset Y$ is a union of open subsets given by continuity of each a_i. Similarly we can check $\mu \circ f = \mathrm{Id}\,|_U$ as expected. Unverified ?

79.2 (b)

80 2.2.11. Espace Étalé

See a post discussion accent letter in LaTeX HERE.

See an exercise in [4] Chapter 2, Exercise 1.13.

See the discussion after Lemma 7. on Page 229 of [2].

For *section*, see Wiki's explanation for *section* in context of fiber bundle; and *section* in terms of category theory.

See a detailed post HERE.

81 2.2.H.

Clearly it's again a contravariant functor, therefore $\pi_* \mathscr{F}$ must be a pre-sheaf. When $\mathscr{F}$ is a sheaf, I checked identity axiom (lots of things to write down).

82 2.3.A.

I'm planning to use universal property to define the induced map ϕ_P.

One crutial step is to verify the diagram below is commutative

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[rr, "\rho_{UV}"] \arrow[d, "\phi(U)"'] & & \mathscr{F}(V) \arrow[d, "\phi(V)"] \\
\mathscr{G}(U) \arrow[rr, "\tau_{UV}"'] \arrow[dr] & & \mathscr{G}(V) \arrow[dl] \\
& \mathscr{G}_P &
\end{tikzcd}$$

And this is because the square diagram in the upper half commute given ϕ is a natural transformation; the lower half is by definition of $\mathscr{G}_P$. Then by universal property of colimit induces a map

$$\phi_P : \mathscr{F}_P \to \mathscr{G}_P$$

which makes the diagram commute.

See a post defined the map HERE.

83 2.3.B.

To define a functor $\pi_* : \mathbf{Set}_X \to \mathbf{Set}_Y$. Firstly, we have to define for any $\mathscr{F} \in \mathbf{Set}_X$,

$$\pi_*(\mathscr{F})(U) = \mathscr{F}(\pi^{-1}(U))$$

for any $U \in \mathfrak{Top}(X)$ as in **??**.

Secondly, for any natural transformation $\phi : \mathscr{F} \to \mathscr{G}$, we define $\pi_*(\phi)$ by specifying

$$\pi_*(\phi)(U) \mapsto \mathscr{F}(\pi^{-1}(U)) \to \mathscr{G}(\pi^{-1}(U)).$$

? Is this correct

84 2.3.C.

This is Exercise 1.15. from Chapter II of [4] on Page 67.

Proof. Clearly $\mathrm{Hom}(\mathscr{F}, \mathscr{G})(U)$ takes value in the set of all natural transformations from $\mathscr{F}|_U$ to $\mathscr{G}|_U$. Namely, we have

$$U \mapsto \mathrm{Mor}(\mathscr{F}|_U, \mathscr{G}|_U).$$

The restriction map induced by $V \subset U$ is given by consider a natural morphism $\alpha \in \mathrm{Mor}(\mathscr{F}|_U, \mathscr{G}|_U)$ as a natural morphism from

$$\mathscr{F}|_V \to \mathscr{G}|_V .$$

So set-theoretically restriction map is identity map, with exception that it regard an element as a presheaf on a smaller open subset. ?

Hence $\mathrm{Hom}(\mathscr{F}, \mathscr{G})(\cdot)$ is a presheaf.

Fix an open subset $U \subset X$, with an open covering $\{U_i\}_{i\in I}$ for some index set I. Pick two natural transformations $\alpha, \beta \in \mathrm{Mor}(\mathscr{F}|_U, \mathscr{G}|_U)$. Assume for any $i \in I$,

$$\mathrm{Res}_{U,U_i}\,\alpha = \mathrm{Res}_{U,U_i}\,\beta.$$

More precisely, this means for any open subset $V_i \subset U_i$ where $i \in I$ is arbitrary, we have

$$\alpha\,|_{U_i}\,(V_i) = \beta\,|_{U_i}\,(V_i).$$

However, note $\{U_i\}_{i\in I}$ is an open cover for the whole space U we're considering. It follows that for any open subset $W \subset U$, we can denote $W_i = W \cap U_i$ and express W as a union of W_i where $i \in I$.

$$\begin{aligned}
\alpha(W) =& \alpha\left(\bigcup_{i\in I} W_i\right) \in \mathrm{Obj}(\mathbf{Set}) \\
=& \bigcup_{i\in I} \alpha(W_i) \\
=& \bigcup_{i\in I} \alpha\,|_{U_i}\,(W_i) \\
=& \bigcup_{i\in I} \beta\,|_{U_i}\,(W_i) \\
=& \cdots \\
=& \beta(W).
\end{aligned}$$

Here the third equality holds by the definition of restriction map. While $W \subset U$ is arbitrary, it follows that $\alpha = \beta \in \mathrm{Mor}(\mathscr{F}|_U, \mathscr{G}|_U)$ as expected.

It remains to check gluability. Again $\{U_i\}_{i\in I}$ is an open cover of U. Pick natural transformations $\alpha_i \in \mathrm{Mor}(\mathscr{F}|_{U_i}, \mathscr{G}|_{U_i})$. They're compatiable in the sense that for any open subset $W_{ij} \subset U_i \cap U_j$, we know

$$\mathrm{Res}_{U_i,U_i\cap U_j}\,\alpha_i = \mathrm{Res}_{U_j,U_i\cap U_j}\,\alpha_j \;\Rightarrow\; \alpha_i(W_{ij}) = \alpha_j(W_{ij}).$$

Now we try to define a natrual transformation $\alpha \in \mathrm{Mor}(\mathscr{F}|_U, \mathscr{G}|_U)$ such that $\alpha\,|_{U_i} = \alpha_i$. For any open subset $Y \subset U$, denotes $Y_i := Y \cap U_i$.

$$\begin{aligned}
\alpha(Y) : \mathscr{F}\,|_U\,(Y) \to \mathscr{G}\,|_U\,(Y) \\
\mathscr{F}\,|_U\,(Y_i) \ni x \mapsto \alpha_i(x).
\end{aligned}$$

This is map in sets, it's well-defined for $\{Y_i\}_{i\in I}$ is an open covering for Y and each α_i is compatiable. By construction we know $\alpha\,|_{U_i} = \alpha_i$. Hence we've checked gluability.

Need to check, but I think it's basically unwrapping a long long defintion

$\square$

84.1 Verification

See a post HERE.

Also see a lemma from Stacks Project HERE. This lemma basically proves gluability and uniqueness, based on the fact that Sheaf Hom is already a presheaf. In the proof of the above lemma, we defined the natural transformation in the way such that the following diagram commute

$$\begin{tikzcd} s \in \mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \mathscr{F}(V) \arrow[r, "\varphi_V"'] & \mathscr{G}(V) \end{tikzcd}$$

And this relies on the fact that $\mathscr{G}$ is a sheaf!, by looking at every $U \cap U_i$, which covers U.

84.2 Warning

Sheaf Hom does not commute with taking stalks. But there exists at least one map from

$$\begin{aligned} &\mathrm{Hom}(\mathscr{F}, \mathscr{G})_p \to \mathrm{Hom}(\mathscr{F}_p, \mathscr{G}_p) \\ &\{(\alpha, U) \ \mid \ p \in U, \ \alpha \in \mathrm{Mor}(\mathscr{F}\mid_U, \mathscr{G}\mid_U)\}/\sim_1 \mapsto \ldots \end{aligned}$$

84.3 Warning: References

See a post HERE, which contains a link to the detailed version of the counterexample HERE. According to post HERE, The direction was correct.

See Stefan's notes on Page 18 for a concrete example HERE. One comment on "Hom functor preserve limit" HERE.

84.4 Counterexample

Proof. For any $U \subset X$, where $\mathscr{F}$ is skycraper sheaf at $p \in X$ with value group A and $\mathscr{G}$ is a constant sheaf on topological space X with value group A. We claim that

$$\mathcal{H}om(\mathscr{F}, \mathscr{G})(U) = \mathrm{Mor}(\mathscr{F}\mid_U, \mathscr{G}\mid_U) = 0$$

as an Abelian group for arbitrary $U \subset X$.

It suffices to check the above statment is correct when $U = X$ for $\mathcal{H}om(\mathscr{F}, \mathscr{G})$ is a sheaf, i.e. we need to show the group

$$\mathcal{H}om(\mathscr{F}, \mathscr{G})(X) = \mathrm{Mor}(\mathscr{F}, \mathscr{G}) = 0$$

Pick any natural transformation $\alpha \in \mathrm{Mor}(\mathscr{F}, \mathscr{G})$. We wish to show the group homomorphism $\alpha(U)$ is 0 (i.e. sends everything to 0 in the codomain $\mathscr{G}(U)$). For an open subset U, it admits an open covering $\{U_i\}$ in which every U_i is connected. While $\mathscr{G}$ is a sheaf, to prove $\alpha(U) = 0$ it suffices to check $\alpha(U_i) = 0$. Fix i and denote $U_i = U_0$. When $p \notin U_0$, then $\mathscr{F}(U_0) = 0$ and the map $\alpha(U_0) = 0$.

Now assume $p \notin U_0$. We can still argue $\alpha(U_0)$ is 0 group homomorphism by restrict it to a smaller open subset that doesn't contain p, because the skycraper sheaf will be 0 group.

Assume p is closed an not open, which means it's not isolated. Then $V := U \setminus \{p\}$ is an open subset that doesn't contain p, hence $\mathscr{F}(V) = 0$ by definition. ?

$$\begin{tikzcd}
\mathscr{F}(U_0) \arrow[r, "\alpha(U)"] \arrow[d, "\mathrm{Res}_{U_0 V}"'] & \mathscr{G}(U_0) = A \arrow[d] \\
\mathscr{F}(V) = 0 \arrow[r] & \mathscr{G}(V)
\end{tikzcd}$$

We claim that $\mathscr{G}(U_0) \to \mathscr{G}(V)$ is an injection:

- ?

Therefore by commutativity of the diagram we know $\alpha(U_0) = 0$ as expected. □

84.5 Abelian group structure

The Abelian group structure on $\mathcal{H}om(\mathscr{F}, \mathscr{G})(U)$ is given by defining

$$\alpha + \beta(U) : \mathscr{F}(U) \to \mathscr{G}(U) \ x \mapsto \alpha(U)x + \beta(U)x$$

for two natrual transformations $\alpha, \beta \in \mathcal{H}om(\mathscr{F}, \mathscr{G})(U)$. Notice that $\alpha + \beta$ is indeed a natural transformation, because it's compatible with restriction maps, which we could check by definition... Element $0 \in \mathcal{H}om(\mathscr{F}, \mathscr{G})(U)$ is just a natural transformation sends everything to $0 \in \mathscr{G}(U)$.

85 3.1.A.

?

86 3.2.A.

86.1 Example 5

A complete description of Spec($\mathbb{R}[x]$) is given by a post HERE.

To find all prime ideals in P.I.D. $\mathbb{R}[x]$ is generated by one single element, i.e. a polynomial $f \in \mathbb{R}[x]$. While $\mathbb{R}[x]$ is U.F.D., a prime element is equivalent to an irreducible element. Therefore we need to find all irreducible polynomials in $\mathbb{R}[x]$. Notice that field extension

$$[\mathbb{C} : \mathbb{R}] = 2 \;\Rightarrow\; \deg f \leq 2$$

for any irreducible polynomial f.

In case of $\deg f = 2$, it's precisely the case where we require f to be an irreducible quadratic.

See a post HERE.

87 3.2.B.

For an irreducible polynomial $x^2 + ax + b \in \mathbb{R}[x]$, it cannot admit a real root. Therefore in algebraic closure of R, i.e. in $\mathbb{C}$, we can find two roots α_1, α_2 of the polynomial.

88 3.2.C.

?

89 3.2.D.

I didn't figure out the Euclid's proof for this.

For Euclid's proof, see a post HERE.

Basic ideal is consider

$$f = \prod_{i \in I} f_i + 1$$

where $\langle f_i \rangle = \mathfrak{p}_i$ for a *finite* index set I. Here $\mathfrak{p}_i \lhd k[x]$ is a prime ideal. Each f_i must have degree $\deg f_i \geq 1$ because otherwise $\mathfrak{p}_i$ will contain a unit. While in integral domain $k[x]$, we know degree f is again large or equal to 1, hence not a unit. Clearly it's nonzero. As a non-zero non-unit element in U.F.D. $k[x]$, it could be written as a product of irreducible elements, say g_1. It is will also generate a prime ideal, which means

$$\langle g_1 \rangle = \mathfrak{p}_i = \langle f_i \rangle \ \Rightarrow \ g_1 = u f_i$$

for some $i \in I$ and unit u. Then $f \equiv 1 \pmod{g_i}$, which means g_i will divide unit 1, contradiction.

See a post HERE, HERE.

90 3.2.L. Exercise

Proof. Localisation commute with quotient, therefore we plan to prove the isomophism by constructing a ring homomorphism from $\psi : \mathbb{C}[x, y]_x \to \mathbb{C}[x]_x$ and compute the kernel. The map ψ is defined as

$$\begin{aligned} \psi : \mathbb{C}[x, y]_x &\to \mathbb{C}[x]_x \\ f(x, y)/x^i &\mapsto f(x, 0)/x^i \end{aligned}$$

where $f(x, y) \in \mathbb{C}[x, y]$ and i is some integer.

Clearly we see $(xy)_x \subset \operatorname{Ker} \psi$. Conversely, let's pick an element $f(x, y)/x^i$ such that $f(x, 0)/x^i = 0 \in \mathbb{C}[x]_x$, this implies there exists $j \in \mathbb{N}$ such that

$$x^j f(x, 0) = 0 \in \mathbb{C}[x] \ \Rightarrow \ f(x, 0) = 0 \in \mathbb{C}[x].$$

stopped,...

The conclusion follows immediately if we realise

$$(y)_x = (xy)_x \subset \mathbb{C}[x, y]_x.$$

One way to interpret this is as follow: both $(x)_x$ and $(xy)_x$ are image of localisation map of a principal ideas $(x), (xy) \lhd \mathbb{C}[x, y]$. And two principal ideals are the same if they differ by a unit, say x. Or we can perform a double inclusion argument. $\square$

90.1 Hint

See a post HERE.

Some details are not clear, so see the following.

Proof. Two facts to recall is that we have two exact sequences

$$0 \longrightarrow (xy)_x \subset \mathbb{C}[x, y]_x \longrightarrow \mathbb{C}[x, y]_x \longrightarrow (\mathbb{C}[x, y]/(xy))_x \longrightarrow 0$$

$$0 \longrightarrow (y) \subset \mathbb{C}[x, y] \longrightarrow \mathbb{C}[x, y] \longrightarrow \mathbb{C}[x] \longrightarrow 0$$

Here the second rows will induce, by exactness of localisation

$$0 \longrightarrow (y)_x = (xy)_x \subset \mathbb{C}[x, y]_x \longrightarrow \mathbb{C}[x, y]_x \longrightarrow \mathbb{C}[x]_x \longrightarrow 0$$

And this concludes the desired isomorphism. One crutial observation I missed was $(xy)_x = (y)_x$.

$\square$

Part III
Algebraic Geometry and Arithmetic Curves Qing Liu

91 Theorem 2.4.

Notice that

$$N_2' = N'/N_1 \subset N/N_1 = N_2.$$

92 Lemma 2.9.

$$\begin{tikzcd}
\mathcal{F}(V_x) \arrow[rr, "\mathrm{Res}"] \arrow[dr] & & \mathcal{F}(U_x) \arrow[dl] \\
& s_x \in \mathcal{F}_x &
\end{tikzcd}$$

Assume $s_x = 0$, while in Category of Abelian groups the homomorphism is exactly group homomorphism. So there must exist some U_x such that $s\mid_{U_x} = 0$ as a pre-image of s_x.

Part IV
Gathmann

93 Exercise 2.40.

See a post HERE.

94 References

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