Contents

1 Definition: Irreducible

For equivalent definitions, see Wiki HERE; Also see "dense" on Wiki HERE.

See Atiyah's [1] Exercise 19 from Chapter 1 for more information...

See Lemma 14. on Page 210 of [2] for equivalent characterisation of *irreducible*.

2 Example 1.1.3.

See Atiyah's [1] Exercise 19 from Chapter 1, which proves it must be dense.

For the sake of contradiction, assume a non-empty subset $A \subset X$ is reducible. Hence there exist two proper closed subset $A_1, A_2 \subset A$ such that $A = A_1 \cup A_2$. Then we have

$$X = (A^c \cup A_1) \cup (A^c \cup A_2),$$

which implies X is reducible. ✗!

2.1

See a post HERE ✔

First approach applied some density argument. While the second approach, similarly, gave a decomposition of the whole space given the open set is reducible.

3 Example 1.1.4.

See Atiyah's [1] Exercise 20 from Chapter 1.

4 Definition.

"Induced topology". Definition of *quasi affine variety*, see HERE.

5 Prop 1.2 (d)

According to Hilbert's Nullstellensatz, I agree we'll get $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$. For the reverse inclusion, pick any $f \in A$ such that $f^r \in \mathfrak{a}$ where $r \in \mathbb{Z}_{>0}$. We wish to show that $f(P) = 0$ for any $P \in Z(\mathfrak{a})$. By definition, $f^r(P) = 0$ given $f^r \in \mathfrak{a}$. And this implies

$$f^r(P) = (f(P))^r = 0 \ \Rightarrow \ f(P) = 0$$

given the polynomial ring A is an integral domain. Therefore we get the inclusion

$$I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}.$$

See Theorem 6 on Page 183, Strong Nullstellensatz, [3]. ✔

6 Theorem 1.3A. Hilbert's Nullstellensatz

For the case where k isn't algebraically closed, see [6] Remarks in 5.6.

6.1

For the proof, see [5] Chapter 1 for details.

The followings are some comments for [5] Chapter 1 Theorem 1.7.

A post on surjective preimage for maximal ideal HERE.

A post on preimage for maximal ideal (not necessarily surj) HERE.

For completeness, a post on preimage of prime ideals HERE.

A post on image of prime ideals HERE, HERE, and HERE.

See Kemper's [5], Lemma 1.22 on Page 17, which completely described prime and maximal ideals in quotients.

7 Definition. Height

Here the definition *height* is specifically for prime ideal $\mathfrak{p} \lhd_{pr} R$ for some ring R. For a general definition, see a post HERE; see a webpage HERE; or see [5] Definition 6.10 on Page 68.

8 Proposition 1.7.

Difference between algebraic set and affine algebraic set? See HERE.

For an analogue in Projective, see Exercise 2.6 **??** in Chapter 1.2. And a post HERE.

9 Theorem 1.8A.

For transcendence degree, see HERE and a NOTE by Milne James.

10 Proposition 1.10.

Apart from the proof, we discussed *locally closed subset.* See HERE for its equivalent definitions.

11 Proposition 1.13.

See HERE.

12 Exercise 1.1.

12.1 (a)

By definition of affine coordinate ring we have

$$A(Y) = A/I(Y) = A/I(Z(f)) = A/\sqrt{\langle f \rangle}.$$

While f is irreducible in U.F.D. $k[x, y]$, the ideal it generated will be a prime ideal, which is radical. Therefore we can further simplify the expression as

$$A/\sqrt{\langle f \rangle} = A/\langle f \rangle = k[x, y]/\langle y - x^2 \rangle = k[x].$$

Hence we can conclude $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

12.2 (b)

13 Proposition 2.2.

There are two very technical claims need more details.

The first one is to prove $\varphi(Y) = Z(T') = Z(\alpha(T))$. Unwrap the notation precisely according to the definition

$$\begin{aligned} Z(\alpha(T)) &= \{\ x \in \mathbb{A}^n \ \mid\ \alpha(g)(x) = 0\ \forall\ g \in T\ \}, \\ \varphi(Y) &= \{\ \varphi(y)\ \mid\ y \in Y\ \}. \end{aligned}$$

Notice that $y = [y_0, ..., y_n] \in Y \subset \overline{Y} = Z(T)$, therefore $g(y) = 0$ for any $g \in T$. More precisely, we have

$$\alpha(g)(\varphi(g)) = g(1, y_1/y_0, ..., y_n/y_n) = 0$$

given $g(y) = 0$ and $g \in T \subset S^h$, which proves $\varphi(Y) \subset Z(\alpha(T))$.

Conversely, let's start with an element $x = (x_1, ..., x_n) \in Z(\alpha(T))$. There's an element $y = [1, x_1, ..., x_n] \in Y$ such that $\varphi(y) = x$. Hence we've proved the equality.

And the second one is to check $\varphi^{-1}(W) = Z(\beta(T')) \cap U = Z(\beta(\alpha(T))) \cap U$.

14 Lemma 3.1.

See Sandor's Notes Lecture 4, Lemma 2.6.
Closedness can be checked locally. See a post HERE.

15 Remark 3.1.1.

See Sandor's Notes Lecture 5, Corollary 2.10. Notice that Hartshorne defined variety to be irreducible. See a post explaining why the preimage is dense HERE.
See Lemma 14. on Page 210 of [2].

16 Definition: Ring of Regular Function

HERE is an explicit description on the ring structure of $\mathcal{O}_{P,Y}$.

17 Theorem 3.2.

See Sándor's Lecture Notes 09, STEP 03. See solution of problem 3 HERE. See a post HERE, HERE.

17.1 (c)

for each P*,* $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$*, and* $\dim \mathcal{O}_P = \dim Y$*;*

Proof. We begin with an injective homomorphism $\alpha : A(Y) \to \mathcal{O}(Y)$. And we define a map

$$A(Y)_{\mathfrak{m}_P} \to \mathcal{O}_{P,Y}$$
$$f/g \mapsto \langle V, \frac{\alpha(f)}{\alpha(g)} \rangle$$

where $\frac{\alpha(f)}{\alpha(g)} \in \mathcal{O}(V)$. Now we wish to give an explicit description of V. Since $\alpha(f) \in \mathcal{O}(Y)$, we know there exists an open subset $P \in V_1 \subset Y$ such that

$$\alpha(f) \mid_{V_1} = \frac{h_1}{h_2} \mid_{V_1}$$

where $h_1, h_2 \in A$ and $0 \notin h_2(V_1)$. Since $\alpha(g) \in \mathcal{O}(Y)$, we know there exists an open subset $P \in V_2 \subset Y$ such that

$$\alpha(g) \mid_{V_2} = \frac{h_3}{h_4} \mid_{V_2}$$

where $h_3, h_4 \in A$ and $0 \notin h_4(V_2)$. Here $g \notin \mathfrak{m}_P$ by definition of localisation, which gives us

$$g(P) \neq 0 \;\Rightarrow\; \alpha(g)(P) \neq 0 \;\Rightarrow\; \exists\, V_3 \subset Y,\; \alpha(g) \mid_{V_3} \neq 0.$$

Then we take $V = V_1 \cap V_2 \cap V_3$ will suffice to work. This is because for any point $P \in V$, we have

$$\frac{\alpha(f)}{\alpha(g)} = \frac{h_1 h_4}{h_2 h_3}$$

for $0 \notin h_2 h_3(V)$.

$$\begin{tikzcd}
A(Y) \arrow[r, "\alpha"] \arrow[dr] & \mathcal{O}(Y) \arrow[r, hook] & \mathcal{O}_{P,Y} \\
& A(Y)_{\mathfrak{m}_P} \arrow[ur, dashed] &
\end{tikzcd}$$

The induced map is given by universal property of localisation, for every elements in $A(Y) \setminus \mathfrak{m}_P$ will be mapped to a unit in $\mathcal{O}_{P,Y}$. And the map given by Lecture Notes 09 of Prof. Sándor satisfy the universal property (it makes the diagram commute and maps elements outside of $\mathfrak{m}_P$ to units).

$\square$

18 Proposition 3.3.

See a post HERE.

19 Lemma 3.6.

See a post HERE.

Here are some details for proving $x_i \circ \psi$ being regular implies ψ is a morphism: Coordinate functions means

$$\psi(p) = (\psi_1(p), \psi_2(p), ..., \psi_n(p)) \in Y \subset \mathbb{A}^n$$

for any $p \in X$. Firstly, we check ψ is continuous. Take any closed subset $Z(f_1, ..., f_r) \subset Y$ for some polynomial $f_1, ..., f_r \in A = k[x_1, ..., x_n]$. We can compute the preimage as

$$\psi^{-1}(Z(f_1, ..., f_r)) = \{\ p \in X \ \mid \ \forall\ p \in X,\ f_i \circ \psi(p) = 0\ \}.$$

Notice that for any $p \in X$,

$$f_i \circ \psi(p) \ = \ f_i(\psi_i(p), ..., \psi_n(p))$$

is continuous since f_i is a polynomial and each $\psi_i := x_i \circ \psi$ is continuous by assumption that they're regular. Notice that the preimage of ψ is precisely intersection of $\psi_i^{-1}(\{0\})$ where $1 \leq i \leq n$. Hence the preimage is closed, and it follows that ψ is continuous as expected.

Secondly, fix an arbitrary open subset $V \subset Y$ with an arbitrary regular function $g : V \to k$, we wish to prove $g \circ \psi : \psi^{-1}(V) \to k$ is regular. For any

$\psi(p) \in V$ with some $p \in X$, there exists a neighborhood $\psi(p) \in U \subset Y$ such that g equals to an expression of quotients of polynomial, i.e.

$$g = \frac{g_1}{g_2}$$

where $g_1, g_2 \in A$. Then for $p \in X$, take the open neighborhood of it as $\psi^{-1}(U)$, we can see

$$g \circ \psi(p) = \frac{g_1(\psi(p))}{g_2(\psi(p))}.$$

20 Example 1.0.3.

See some examples of presheaves that are not sheaves HERE; a post HERE.

In Wiki's page HERE, it introduced non-separated presheaf, i.e. presheaf that doesn't satisfy locality axiom for sheaf.

21 Proposition-Definition 1.2.

See *Sheafification* on The Stacks Project.
See solution of problem 3 HERE.
Of course, consult Ravi's Notes on Sheafification;
or see Section 6.5 on Page 232 of [2].
Also, see a REU paper HERE by Daping Weng.
A short paper by Tom is HERE.

22 Definition: Inverse Image Sheaf

See Wiki for motivation of such definition.
See a POST that gives more details, as well as a counterexample.

23 1.3.F. EXERCISE.

A post discussing this problem is HERE.

24 1.3.N

A crutial step is to define the map such that the diagram commute. In order to prove it satisfies the universal property. We're given A be arbitrary with map $g : A \to X$ and $f : A \to Y$.

$$\begin{tikzcd} A \arrow[dr, dashed, "\exists!\varphi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\ & X \times_Z Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\beta"] \\ & X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

We can to define

$$\varphi : A \to X \times_Z Y \quad \text{by}$$
$$a \mapsto (g(a), f(a)).$$

And we can verify this definition will make the diagram commute, and is unique.

25 1.3.O

It's indeed intersection. A post HERE.

A post HERE.

26 1.3.P.

Say we have $X \times Y$ and $X \times_Z Y$. By universal property of product and fibered product we can produce two unique map goes in between. Their composition must be identity, hence they're isomorphic. Notice it's important for Z being a final object.

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

When we trying to make a map from $X \times Y$ to $X \times_Z Y$, we have to make sure two maps

$$\beta \circ \pi_1 = \alpha \circ \pi_1$$

on $X \times Y$. While Z is the final object, hence they must agree.

There's a cleaner way to state it HERE. Crutial part is applying final property of object Z.

27 1.3.Q.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists!\varphi"] \arrow[drr, bend left, "g_1"] \arrow[ddr, dashed, "\exists!\psi"'] \arrow[dddr, bend right, "g_2"'] & & \\
& U \arrow[r, "f_1"] \arrow[d, "f_2"'] & V \arrow[d, "f_3"] \\
& W \arrow[r, "f_4"'] \arrow[d, "f_5"'] & X \arrow[d, "f_6"] \\
& Y \arrow[r, "f_7"'] & Z
\end{tikzcd}$$

Label the maps as indicated. To prove the universal property with respect to the "outside rectangle", we're given

$$f_6 f_3 g_1 = f_7 g_2$$

agree on A. While W is fibered product, apply universal property of fibered product with resepct to W we immediately get a unique map

$$\psi : A \to W$$

that makes the diagram involving A, W, X, Y, Z commute. In particularly, we know $f_4\psi = f_3 g_1$. Furthermore, recall that U is the fibered product. We're given the condition that $f_4\psi = f_3 g_1$, by universal property of U we know there exists a unique map

$$\varphi : A \to U$$

making the diagram involving A, U, V, W, X commute. And we claim that the diagram involving A, U, V, Y, Z commute. This is because

$$g_2 = f_5\psi = f_5 f_2 \varphi = (f_5 f_2)\varphi \quad \text{and} \quad g_1 = f_1\varphi.$$

And this proves that U is the fibered product for the diagram involving A, U, V, Y, Z.

A post is HERE.

28 1.3.R

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[drr, bend left, "\pi_1"] \arrow[ddr, bend right, "\pi_4"'] & & & \\
& X_1 \times_Z X_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_3"'] & X_2 \arrow[d, "f_3"] & \\
& X_1 \arrow[r, "f_2"'] & Y \arrow[dr, "f_1"] & \\
& & & Z
\end{tikzcd}$$

By the universal property of $X_1 \times_Z X_2$, we know there exists a unique map

$$\varphi : X_1 \times_Y X_2 \to X_1 \times_Z X_2$$

"Natural morphism", a convention discussed HERE.

29 Course Notes from Cornell

See HERE.

30 1.3.S. Magic Diagram

Didn't finish. Need to See HERE, HERE!!!

31 1.3.Y. (a)

YONEDA'S LEMMA Given what we have, define $g : A \to A'$ as

$$g := i_A(\mathrm{id}_A).$$

This is correct, see a post HERE.

32 1.4.C.

(a) See "A Term of Commutative Algebra", Example 7.3 on Page 52.

33 1.6.B.

Write out everything by definition, and we can finish the proof immediately by applying rank-nullity theorem for linear transformation...

34 2.2.6. Definition: Sheaf.

Comments on $\mathscr{F}(\emptyset)$. In category **Set**, the empty set is initial object and one element set is terminal. See Wiki's examples HERE.

35 2.2.B.

For (a): see Wiki's counterexample HERE, which gave an explanation for presheaves on $\mathbb{R}$ instead of $\mathbb{C}$. See a post HERE.

36 2.2.7.

See Daping's Notes Definition 2.5 on Page 4 HERE; also a post HERE; also a post HERE.

37 2.2.10.

It's different from a post HERE, and Wiki's page on Constant pre-Sheaf. Why???

38 2.2.G.

It's clearly a pre-sheaf.

Fix an open subset $U \subset X$ with an open cover $\{U_i\}_{i \in I}$ for some index set I. Denote the presheaf as $\mathscr{F}$.

Pick two continuous maps $s_1, s_2 : Y \to X$ that satisfying the requirements, i.e. $s_1, s_2 \in \mathscr{F}(U)$.

Both functions will agree on U since

$$\mathrm{Res}_{U,U_i} \; s_1 = \mathrm{Res}_{U,U_i} \; s_1$$

for arbitrary U_i, whose union is U. So we must have $s_1 = s_2$.

Again with this open cover $\{U_i\}_{i \in I}$ and $a_i \in \mathscr{F}(U_i)$ for $i \in I$. Equivalently, we know $a_i : U_i \to Y$ is a continuous map satisfying $\mu \circ a_i = \mathrm{Id}\,|_{U_i}$. Now let's define a map

$$\begin{aligned} f : U &\to Y \\ u &\mapsto a_i(u) \;\text{ when }\; u \in U_i. \end{aligned}$$

It's well-defined by our assumption. Also it's continuous since preimage of an open set in $V \subset Y$ is a union of open subsets given by continuity of each a_i. Unverified ?
Similarly we can check $\mu \circ f = \mathrm{Id}\,|_U$ as expected.

39 2.2.11. Espace Étalé

See a post discussion accent letter in LaTeX HERE.

See an exercise in [4] Chapter 2, Exercise 1.13.

See the discussion after Lemma 7. on Page 229 of [2].

For *section*, see Wiki's explanation for *section* in context of fiber bundle; and *section* in terms of category theory.

40 2.3.A.

I'm planning to use universal property to define the induced map ϕ_P.

One crutial step is to verify the diagram below is commutative

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[rr, "\rho_{UV}"] \arrow[d, "\phi(U)"'] & & \mathscr{F}(V) \arrow[d, "\phi(V)"] \\
\mathscr{G}(U) \arrow[rr, "\tau_{UV}"'] \arrow[dr] & & \mathscr{G}(V) \arrow[dl] \\
& \mathscr{G}_P &
\end{tikzcd}$$

And this is because the square diagram in the upper half commute given ϕ is a natural transformation; the lower half is by definition of $\mathscr{G}_P$. Then by universal property of colimit induces a map

$$\phi_P : \mathscr{F}_P \to \mathscr{G}_P$$

which makes the diagram commute.

See a post defined the map HERE.

41 2.3.B.

To define a functor $\pi_* : \mathbf{Set}_X \to \mathbf{Set}_Y$. Firstly, we have to define for any $\mathscr{F} \in \mathbf{Set}_X$,

$$\pi_*(\mathscr{F})(U) = \mathscr{F}(\pi^{-1}(U))$$

for any $U \in \mathfrak{Top}(X)$ as in **??**.

Secondly, for any natural transformation $\phi : \mathscr{F} \to \mathscr{G}$, we define $\pi_*(\phi)$ by specifying

$$\pi_*(\phi)(U) \mapsto \mathscr{F}(\pi^{-1}(U)) \to \mathscr{G}(\pi^{-1}(U)).$$

? Is this correct

42 References

[1] M.F. Atiyah and I.G. MacDonald. *Introduction To Commutative Algebra*. Addison-Wesley series in mathematics. Avalon Publishing, 1994.

[2] Siegfried Bosch et al. *Algebraic geometry and commutative algebra*. Springer, 2013.

[3] David Cox, John Little, and Donal OShea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. Springer Science & Business Media, 2013.

[4] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[5] Gregor Kemper. *A course in commutative algebra*, volume 1. Springer, 2011.

[6] Miles Reid. *Undergraduate commutative algebra*. Number 29. Cambridge University Press, 1995.