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1 Notes

See a post [HERE](#), which listed various resources...

See Old Lecture Homepages of ETH [HERE](#). There are some courses on algebraic geometry with solutions for homeworks.

See a post on "Best AG Textbook" [HERE](#).

Great Notes on Algebraic Geometry and Commutative Algebra by Andreas Gathmann [HERE](#).

See Horawa's Notes on Algebraic Geometry.

See [HERE](#) for a comprehensive guide to Hartshorne as well as solutions.

See Student Q& A in MITCourseware [HERE](#).

See notes on algebraic geometry by Prof. Dr. Andreas Gathmann [HERE](#).

See [HERE](#) for Aaron Landesman's notes.

See a course in 2012 by James McKernan [HERE](#).

See a great book on Sheaf theory: Sheaf Theory through Examples by Daniel [HERE](#).

Part I

Hartshorne

2 Definition: Irreducible

For equivalent definitions, see Wiki [HERE](#); Also see "dense" on Wiki [HERE](#).
See Atiyah's [1] Exercise 19 from Chapter 1 for more information...

See Lemma 14. on Page 210 of [2] for equivalent characterisation of *irreducible*.

3 Example 1.1.3.

See Atiyah's [1] Exercise 19 from Chapter 1, which proves it must be dense.

For the sake of contradiction, assume a non-empty subset $A \subset X$ is reducible.
Hence there exist two proper closed subset $A_1, A_2 \subset A$ such that $A = A_1 \cup A_2$.
Then we have

$$X = (A^c \cup A_1) \cup (A^c \cup A_2),$$

which implies X is reducible.

✗!

3.1

See a post [HERE](#)

First approach applied some density argument. While the second approach, similarly, gave a decomposition of the whole space given the open set is reducible.

✓

4 Example 1.1.4.

See Atiyah's [1] Exercise 20 from Chapter 1.

5 Definition.

"Induced topology". Definition of *quasi affine variety*, see [HERE](#).

6 Prop 1.2 (d)

According to Hilbert's Nullstellensatz, I agree we'll get $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$. For the reverse inclusion, pick any $f \in A$ such that $f^r \in \mathfrak{a}$ where $r \in \mathbb{Z}_{>0}$. We wish to show that $f(P) = 0$ for any $P \in Z(\mathfrak{a})$. By definition, $f^r(P) = 0$ given $f^r \in \mathfrak{a}$. And this implies

$$f^r(P) = (f(P))^r = 0 \Rightarrow f(P) = 0$$

given the polynomial ring A is an integral domain. Therefore we get the inclusion

$$I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}.$$

See Theorem 6 on Page 183, Strong Nullstellensatz, [3].

✓

7 Theorem 1.3A. Hilbert's Nullstellensatz

For the case where k isn't algebraically closed, see [6] Remarks in 5.6.

7.1

For the proof, see [5] Chapter 1 for details.

The followings are some comments for [5] Chapter 1 Theorem 1.7.

A post on surjective preimage for maximal ideal [HERE](#).

A post on preimage for maximal ideal (not necessarily surj) [HERE](#).

For completeness, a post on preimage of prime ideals [HERE](#).

A post on image of prime ideals [HERE](#), [HERE](#), and [HERE](#).

See Kemper's [5], Lemma 1.22 on Page 17, which completely described prime and maximal ideals in quotients.

8 Definition. Height

Here the definition *height* is specifically for prime ideal $\mathfrak{p} \triangleleft_{pr} R$ for some ring R . For a general definition, see a post [HERE](#); see a webpage [HERE](#); or see [5] Definition 6.10 on Page 68.

Nagata's example: Notherian ring with infinite Krull dimension, see [HERE](#) and a post [HERE](#).

9 Proposition 1.7.

Difference between algebraic set and affine algebraic set? See [HERE](#).

For an analogue in Projective, see Exercise 2.6 ?? in Chapter 1.2. And a post [HERE](#).

10 Theorem 1.8A.

For transcendence degree, see [HERE](#) and a NOTE by Milne James.

11 Proposition 1.10.

Apart from the proof, we discussed *locally closed subset*. See [HERE](#) for its equivalent definitions.

12 Proposition 1.13.

See [HERE](#).

13 Exercise 1.1.

13.1 (a)

By definition of affine coordinate ring we have

$$A(Y) = A/I(Y) = A/I(Z(f)) = A/\sqrt{\langle f \rangle}.$$

While f is irreducible in U.F.D. $k[x, y]$, the ideal it generated will be a prime ideal, which is radical. Therefore we can further simplify the expression as

$$A/\sqrt{\langle f \rangle} = A/\langle f \rangle = k[x, y]/\langle y - x^2 \rangle = k[x].$$

Hence we can conclude $A(Y)$ is isomorphic to a polynomial ring in one variable over k .

13.2 (b)

14 Exercise 1.2.

My initial guess was incorrect, in which I thought $I(Y) = \langle z - xy \rangle$.

14.1

See a post [HERE](#).

See a post [HERE](#).

The correct one is $I(Y) = \langle z - x^3, z - y^2 \rangle$. Notice that

$$\dim Y = \dim A(Y) = \dim k[x, y, z]/\langle z - x^3, z - y^2 \rangle = \dim k[z] = \text{tr. deg}_k k(z) = 1.$$

Therefore we proved that $A(Y)$ is isomorphic to a polynomial ring in one variable over k .

Question: How to compute $I(\cdot)$ precisely?

15 Exercise 1.4.

See a post [HERE](#), [HERE](#), and [HERE](#).

The hint was to consider diagonal. Let the coordinate ring of \mathbb{A}^2 be $A(\mathbb{A}^2) = k[x, y]$ where k is algebraically closed. Then the diagonal

$$\Delta = \{(x, y) \in \mathbb{A}^2 \mid x = y\} = Z(x - y = 0)$$

is closed in Zariski topology of \mathbb{A}^2 .

But it cannot be countably many union of closed subsets in product topology of copies of Zariski topology on \mathbb{A}^1 , which are finitely many points.

Here we used the fact that k is infinite given it's algebraically closed.

Wrong!

15.1

See the first post! Closed subset of product topology on $\mathbb{A}^1 \times \mathbb{A}^1$ can be uncountable, for example, a line $\{(x_0, y) \in \mathbb{A}^1 \times \mathbb{A}^1 \mid y \in k\}$ where $x_0 \in k$ is fixed would be closed. But it's uncountable.

The reason why Δ is closed in Zariski topology of \mathbb{A}^2 is correct.

For product topology, the reason why it's not closed is because Zariski topology on \mathbb{A}^1 is not Hausdorff, by a lemma we mentioned we know Δ isn't closed.

Verified!

16 Exercise 1.7.

16.1 (a)

(i) \Rightarrow (ii) and (iii) \Rightarrow (iv): Zorn's Lemma.

(ii) \Rightarrow (i) and (iv) \Rightarrow (iii): contrapositive, with Axiom of Dependent Choice.

(i) \Leftrightarrow (iii): taking complement.

16.2 (b)

A similar statement is Exercise 17. (v) of [1], which states that $\text{Spec } A$ is quasi-compact for a ring A .

For a given open cover of $\{U_i\}_{i \in I}$ of X , we can throw away open subset U_i such that $U_i \subset U_j$ for some $j \in I$. Here I is some index set. So we can assume in this open cover, we don't have $U_i \subset U_j$ for any $i, j \in I$. Then we have a ascending chain of open subsets

$$U_i \subseteq U_i \cup U_j \subseteq \dots$$

Noetherian condition imposed gives us finitely many index, and they form a finite subcover as expected.

16.2.1

A very simple proof HERE used maximality property.

A stronger statement HERE.

17 Exercise 1.10.

17.1 (a)

Clear, since any chain of Y is also a chain in X , and we use property of sup mentioned on Prop 2.8. of HERE.

17.2 (b)

17.3 (c)

17.4 (d)

17.5 (e)

See a post [HERE](#), [HERE](#),

See the notes by Vakil [HERE](#).

See the notes from UofT [HERE](#).

See a post [HERE](#).

Example in topological space is actually easier to say than Nagata's example.

Take our topological space as $X = [0, 1] \subset \mathbb{R}$. And define the topology by declaring all closed subsets as

$$\{\emptyset, X, \{[1/n, 1] \subset \mathbb{R} \mid i \geq 2, i \in \mathbb{Z}\}\}.$$

Clearly we can check it satisfies infinite intersection and finite union axiom.

It's Noetherian for any descending chain of closed subsets must terminate.

It's of infinite Krull dimension. For any $n > 0$, we can construct a chain of such length. Hence the dimension, defined as supremum cannot be finite.

Another weird example could be found [HERE](#), in which we give \mathbb{N} the topology empty set, entire space, and $\{x \in \mathbb{N} \mid x \geq q\}$ for some $a \in \mathbb{N}$.

18 Graded Ring

Say we wish to define variety in some projective space, i.e. some subsets of \mathbb{P}^n that could be written as zero locus of some polynomials. If we don't add any restrictions on polynomials, for example some non-homogeneous polynomials. Then the zero locus doesn't make sense because it will depend on the representative. See Gathmann's Notes Remark 6.5, notes on algebraic geometry, [HERE](#).

19 Proposition 2.2.

There are two very technical claims need more details.

The first one is to prove $\varphi(Y) = Z(T') = Z(\alpha(T))$. Unwrap the notation precisely according to the definition

$$\begin{aligned} Z(\alpha(T)) &= \{ x \in \mathbb{A}^n \mid \alpha(g)(x) = 0 \ \forall \ g \in T \}, \\ \varphi(Y) &= \{ \varphi(y) \mid y \in Y \}. \end{aligned}$$

Notice that $y = [y_0, \dots, y_n] \in Y \subset \bar{Y} = Z(T)$, therefore $g(y) = 0$ for any $g \in T$. More precisely, we have

$$\alpha(g)(\varphi(y)) = g(1, y_1/y_0, \dots, y_n/y_0) = 0$$

given $g(y) = 0$ and $g \in T \subset S^h$, which proves $\varphi(Y) \subset Z(\alpha(T))$.

Conversely, let's start with an element $x = (x_1, \dots, x_n) \in Z(\alpha(T))$. There's an element $y = [1, x_1, \dots, x_n] \in Y$ such that $\varphi(y) = x$. Hence we've proved the equality.

And the second one is to check $\varphi^{-1}(W) = Z(\beta(T')) \cap U = Z(\beta(\alpha(T))) \cap U$.

20 Exercise 2.12.

For all monomial of degree d in $n + 1$ variables x_0, \dots, x_n . There are

$$\binom{n+d}{n}$$

many monomials in total. Bars and balls argument: there are n bars and d balls. In $n + d$ many places, any choice of n bars will corresponds to a monomial, therefore N is the total number of monomials possible. While in we wish to consider them in projective space, we must define $N = \binom{n+d}{n} - 1$.

See a solution in lecture notes of Frank-Olaf Schreyer [HERE](#).

A more detailed solution is given [HERE](#).

21 Exercise 2.14.

There's a prompt on Sándor's Notes, Lecture 13, Homework 2.79.

22 Lemma 3.1.

See Sandor's Notes Lecture 4, Lemma 2.6.

Closedness can be checked locally. See a post [HERE](#).

23 Remark 3.1.1.

See Sandor's Notes Lecture 5, Corollary 2.10. Notice that Hartshorne defined variety to be irreducible. See a post explaining why the preimage is dense [HERE](#).

See Lemma 14. on Page 210 of [2].

24 Definition: Ring of Regular Function

[HERE](#) is an explicit description on the ring structure of $\mathcal{O}_{P,Y}$.

25 Theorem 3.2.

See Sándor's Lecture Notes 09, STEP 03. See solution of problem 3 [HERE](#).
See a post [HERE](#), [HERE](#).

25.1 (c)

for each P , $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$, and $\dim \mathcal{O}_P = \dim Y$;

Proof. We begin with an injective homomorphism $\alpha : A(Y) \rightarrow \mathcal{O}(Y)$. And we define a map

$$\begin{aligned} A(Y)_{\mathfrak{m}_P} &\rightarrow \mathcal{O}_{P,Y} \\ f/g &\mapsto \langle V, \frac{\alpha(f)}{\alpha(g)} \rangle \end{aligned}$$

where $\frac{\alpha(f)}{\alpha(g)} \in \mathcal{O}(V)$. Now we wish to give an explicit description of V . Since $\alpha(f) \in \mathcal{O}(Y)$, we know there exists an open subset $P \in V_1 \subset Y$ such that

$$\alpha(f) \mid_{V_1} = \frac{h_1}{h_2} \mid_{V_1}$$

where $h_1, h_2 \in A$ and $0 \notin h_2(V_1)$. Since $\alpha(g) \in \mathcal{O}(Y)$, we know there exists an open subset $P \in V_2 \subset Y$ such that

$$\alpha(g) \mid_{V_2} = \frac{h_3}{h_4} \mid_{V_2}$$

where $h_3, h_4 \in A$ and $0 \notin h_4(V_2)$. Here $g \notin \mathfrak{m}_P$ by definition of localisation, which gives us

$$g(P) \neq 0 \Rightarrow \alpha(g)(P) \neq 0 \Rightarrow \exists V_3 \subset Y, \alpha(g) \mid_{V_3} \neq 0.$$

Then we take $V = V_1 \cap V_2 \cap V_3$ will suffice to work. This is because for any point $P \in V$, we have

$$\frac{\alpha(f)}{\alpha(g)} = \frac{h_1 h_4}{h_2 h_3}$$

for $0 \notin h_2 h_3(V)$.

$$\begin{array}{ccccc} A(Y) & \xrightarrow{\alpha} & \mathcal{O}(Y) & \hookrightarrow & \mathcal{O}_{P,Y} \\ & \searrow & & \nearrow & \\ & & A(Y)_{\mathfrak{m}_P} & & \end{array}$$

The induced map is given by universal property of localisation, for every elements in $A(Y) \setminus \mathfrak{m}_P$ will be mapped to a unit in $\mathcal{O}_{P,Y}$. And the map given by Lecture Notes 09 of Prof. Sándor satisfy the universal property (it makes the diagram commute and maps elements outside of \mathfrak{m}_P to units). \square

26 Proposition 3.3.

See a post [HERE](#).

27 Lemma 3.6.

See a post [HERE](#).

Here are some details for proving $x_i \circ \psi$ being regular implies ψ is a morphism: Coordinate functions means

$$\psi(p) = (\psi_1(p), \psi_2(p), \dots, \psi_n(p)) \in Y \subset \mathbb{A}^n$$

for any $p \in X$. Firstly, we check ψ is continuous. Take any closed subset $Z(f_1, \dots, f_r) \subset Y$ for some polynomial $f_1, \dots, f_r \in A = k[x_1, \dots, x_n]$. We can compute the preimage as

$$\psi^{-1}(Z(f_1, \dots, f_r)) = \{ p \in X \mid \forall p \in X, f_i \circ \psi(p) = 0 \}.$$

Notice that for any $p \in X$,

$$f_i \circ \psi(p) = f_i(\psi_1(p), \dots, \psi_n(p))$$

is continuous since f_i is a polynomial and each $\psi_i := x_i \circ \psi$ is continuous by assumption that they're regular. Notice that the preimage of ψ is precisely intersection of $\psi_i^{-1}(\{0\})$ where $1 \leq i \leq n$. Hence the preimage is closed, and it follows that ψ is continuous as expected.

Secondly, fix an arbitrary open subset $V \subset Y$ with an arbitrary regular function $g : V \rightarrow k$, we wish to prove $g \circ \psi : \psi^{-1}(V) \rightarrow k$ is regular. For any

$\psi(p) \in V$ with some $p \in X$, there exists a neighborhood $\psi(p) \in U \subset Y$ such that g equals to an expression of quotients of polynomial, i.e.

$$g = \frac{g_1}{g_2}$$

where $g_1, g_2 \in A$. Then for $p \in X$, take the open neighborhood of it as $\psi^{-1}(U)$, we can see

$$g \circ \psi(p) = \frac{g_1(\psi(p))}{g_2(\psi(p))}.$$

28 Exercise 3.6.

See a post [HERE](#).

29 Definition: Dominant Rational Map

29.1

Well-define for a rational map being *dominant*.

One thing important to keep in mind is both varieties X, Y are a priori irreducible. There's a completely point-set topological argument [HERE](#). Here a technical detail is "The image of a dense subset under a surjective continuous function is again dense", which is from Wiki's entry. For details of this technicality, see [HERE](#).

See a post on equivalent definition for dominant rational map [HERE](#).

A good lecture note [HERE](#).

Wiki's entry for Rational Map.

Very good note by Vakil [HERE](#).

And a post [HERE](#). However, it appears the way used in the proof implicitly required morphism is an open map? So I doubt...

29.2

Some posts share a technical details from general topology. The following statement is taken from Wiki...

The image of a dense subset under a surjective continuous function is again dense. More precisely, assume $f : X \rightarrow Y$ with E dense in X , then $f(E)$ is dense in $f(X)$.

Proof. By definition we have $E \subset f^{-1}(f(E)) \subset f^{-1}(\overline{f(E)})$, which is closed given f is continuous. It follows that

$$\overline{E} \subset f^{-1}(\overline{f(E)}) \Rightarrow f(\overline{E}) \subset \overline{f(E)}.$$

Conversely,

$$f(X) = \overline{f(E)} \cap f(X) \supset f(X) \cap f(X) = f(X) \Rightarrow f(X) \supset \overline{f(E)} \cap f(X).$$

Here $\overline{f(E)}$ denotes closure of $f(E)$ in Y , while it's intersection with $f(X)$ is the whole $f(X)$, then $f(E)$ is dense in $f(X)$. \square

29.3 A Pathological Example

Another equivalent statement required "surjectivity" and say $f(E)$ is dense in Y . It's curtail. Also we can only conclude $f(E)$ is dense merely in $f(X)$ instead of Y . Since we have the continuous inclusion map $\iota : \mathbb{R} \rightarrow \mathbb{C}$, then $\text{id}(\mathbb{Q}) = \mathbb{Q}$ is just dense in \mathbb{R} but not dense in \mathbb{C} .

29.4

Say we start with a dominant rational map $\varphi : X \rightarrow Y$ with two representatives

$$\langle U, f \rangle, \langle V, g \rangle.$$

By definition of dominant, we know $f(U)$ is dense in Y . To check this definition is independent of the choice of the representative, we have to check $g(V)$ is dense in Y .

Notice that

$$Y = \overline{f(U)} = \overline{f(\overline{U \cap V} \cap U)} \subset \overline{f(\overline{U \cap V})} \subset \overline{f(\overline{U \cap V})} = \overline{g(\overline{U \cap V})} \subset \overline{g(V)}.$$

for X is irreducible and both U, V are non-empty and open then $X = \overline{U \cap V}$. Here the third inclusion is given by the previous technical lemma.

29.5 Composing Dominant Rational Maps

See a post [HERE](#).

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$$

$$(U, \phi_U) \quad (V, \psi_V)$$

To prove the composition is a dominant rational map, we need to find a representative. We define

$$W := \phi_U^{-1}(V).$$

And we claim $(W, \psi_V \circ \phi_U)$ will be suitable for a representative for $\psi \circ \phi$. First of all, notice that W is non-empty. This is because $\phi_U(U) \cap V \neq \emptyset$ given $\phi_U(U)$ is dense in Y and V is assumed to be non-empty open subset. While Y is irreducible, by Lemma 14. of ?? on Page 210, which states that $\phi_U(U) \cap V$ is nontrivial. By definition this implies

$$\phi_U^{-1}(V) \neq \emptyset.$$

This is non-empty and open. Note X is irreducible, hence $W = \phi_U^{-1}(V)$ is dense in X . Hence $\psi_V \circ \phi_U(W)$ is dense in Z given both maps are continuous by being a morphism.

? slightly different than the post online

29.6

See a post [HERE](#), [HERE](#), and [HERE](#).

30 Lemma 4.2.

See a post [HERE](#).

31 Definition Blowing Up

On Page 29, Hartshorne defined blowing-up of a closed affine subvariety Y of \mathbb{A}^n passing through O .

See Sándor's Notes, Lecture 22, *strict transform*.

See Daniel's notes [HERE](#). Here the notation $(\cdot)^-$ stands for taking closure.

32 Exercise 4.1.

Define a function

$$h : U \cup V \rightarrow k \text{ by}$$

$$x \mapsto \begin{cases} f(x) & \text{when } x \in U \setminus V \\ g(x) & \text{when } x \in V \end{cases}.$$

Since $f = g$ on $U \cap V$, hence the function h is well-defined. For any point $p \in U \cup V$, if $p \in U$, then we apply assumption that f is regular. For $p \in V$, similarly apply assumption that g is regular. Hence h is regular on $U \cup V$.

Let f be a rational function on X . So we take all equivalence class $\{\langle U_i, f_i \rangle\}_{i \in I}$ that represents f . By the above lemma and the definition of regular function, there's a regular function g that's defined on $U := \bigcup_{i \in I} U_i$ that extends all f_i . Therefore we can take a representative of f as

$$\langle U, g \rangle.$$

Note $\langle U, g \rangle = \langle U_i, f_i \rangle$ by definition, hence it's indeed a representative of f .

Also U is the largest open set. Suppose it's not, then we have $\langle U_{i_0}, f_{i_0} \rangle$ represents f such that $U_{i_0} \supsetneq U$. And this will contradicts the construction of U , which must contain U_{i_0} .

33 Exercise 4.2.

We're given a rational map $\varphi : X \dashrightarrow Y$. Suppose we have two equivalent representatives

$$\langle U, \varphi_U \rangle, \langle V, \varphi_V \rangle.$$

This means two morphisms φ_U, φ_V agree on $U \cap V$.

It suffices to prove that we can define a morphism $\psi : U \cup V \rightarrow Y$ that extends both φ_U and φ_V . Similarly, we can apply argument of 32 to conclude the existence of a largest open set on which φ is represented by a morphism.

Both φ_U, φ_V are continuous function that agree on their intersection, then we can define

$$\psi : U \cup V \rightarrow Y \text{ by } \psi(x) = \begin{cases} \varphi_U(x) & \text{when } x \in U \setminus V \\ \varphi_V(x) & \text{when } x \in V \end{cases}.$$

Clearly it's a well-defined continuous function. For any open subset $W \subset Y$ with an arbitrary regular function $f : W \rightarrow k$. We have $f \circ \psi : \psi^{-1}(W) \rightarrow k$ is a regular function since it extends both

$$f \circ \varphi_U : \varphi_U^{-1}(W) \rightarrow k, \quad f \circ \varphi_V : \varphi_V^{-1}(W) \rightarrow k.$$

While both two regular functions agree on their intersection, then we can conclude using 32 that $f \circ \psi$ is a regular function. And this proves that ψ is indeed a morphism on $U \cup V \rightarrow Y$.

34 Exercise 5.1.

34.1 (a)

According to this picture, (a) is a tacnode.

Denote $f_1 := x^2 - x^4 - y^4$, which is irreducible in UFD $k[x, y]$ hence it's prime. We can then compute the ideal defined by this affine variety

$$I(Z(\langle f_1 \rangle)) = \sqrt{\langle f_1 \rangle} = \langle f_1 \rangle.$$

To locate all singular points, we'll use Theorem 4.6 on the notes. So we have to compute the Jacobian matrix of the above affine variety at some point $P \in \mathbb{A}^2$ on the affine variety. We choose f_1 itself as generators for the ideal of the affine variety and compute the Jacobian matrix

$$J(P) = \left(\frac{\partial f_1}{\partial x}(P) \quad \frac{\partial f_1}{\partial y}(P) \right) = (2x - 4x^3(P) \quad -4y^3(P)).$$

The rank of the matrix is 0. By Krull's Hauptidealsatz, since $f_1 \in k[x, y]$ is neither a zero-divisor nor a unit, we know every minimal prime containing $\langle f_1 \rangle$ has height 1. But $\langle f_1 \rangle$ is already a prime ideal, so it has height exactly 1. We're ready to compute the dimension of the affine variety

$$\dim(Z(f_1)) = \dim A(Z(f_1)) = \dim(k[x, y]) - \text{height}(\langle f_1 \rangle) = 2 - 1 = 1.$$

Therefore we must have the rank of the matrix as $n-d = 2-1 = 1$, contradiction. It follows that P is a singular point.

34.2 (b)

According to this picture, (b) is a node.

Denote $f_2 = xy - x^6 - y^6$. Similarly, we choose f_2 itself as the generator for the affine variety it defined. Again, we have to compute the Jacobian matrix of $Z(f_2)$ at $P = (0, 0)$.

$$J(P) = \left(\frac{\partial f_2}{\partial x}(P) \quad \frac{\partial f_2}{\partial y}(P) \right) = (y - 6x^5(P) \quad x - 6y^5(P)) = (0 \quad 0),$$

which as rank 0. We can compute the dimension of the affine variety as 1, it follows that P is a singular point given the rank of matrix isn't $2 - 1 = 1$.

34.3 (c)

We just need to check the Jacobian matrix

$$J(P) = \left(\frac{\partial f_2}{\partial x}(P) \quad \frac{\partial f_2}{\partial y}(P) \right) =$$

34.4

See a post [HERE](#).

See REB's solution [HERE](#).

See a post on irreducibility of polynomial over \mathbb{C} [HERE](#).

35 Definition: Presheaf

35.1 Two Pathological Examples

Here are two examples taken from Tennison's [8].

Let X be any topological space with more than one point, i.e. $X = \{0, 1\}$ or $X = \{0, 1\} \rightarrow \mathbb{R}$.

Define a presheaf \mathcal{P}_1 by

$$\begin{cases} \mathcal{P}_1(X) = \mathbb{Z} \\ \mathcal{P}_1(U) = 0 \text{ for open } U \subsetneq X \\ \text{All restrictions except } \rho_{XX} \text{ being constant maps.} \end{cases}.$$

Here 0 denotes the trivial Abelian group.

Pick $x_0 \in X$. Define a presheaf \mathcal{P}_2 by

$$\begin{cases} \mathcal{P}_2(U) = \mathbb{Z} \text{ for } U \text{ open in } X \text{ such that } x_0 \in U \\ \mathcal{P}_2(U) = 0 \text{ for } U \text{ open in } X \text{ such that } x_0 \notin U \\ \text{restrictions } \rho_{UV} = \begin{cases} \text{id}_{\mathbb{Z}} & \text{if } x_0 \in V \subset U \\ 0 & \text{trivial map if not} \end{cases} \end{cases}.$$

Here the second appearance of 0 denotes the trivial map.

36 Example 1.0.3.

See some examples of presheaves that are not sheaves [HERE](#); a post [HERE](#).

In Wiki's page [HERE](#), it introduced non-separated presheaf, i.e. presheaf that doesn't satisfy locality axiom for sheaf.

37 Proposition-Definition 1.2.

See *Sheafification* on The Stacks Project.

See solution of problem 3 [HERE](#).

Of course, consult Ravi's Notes on Sheafification; or see Section 6.5 on Page 232 of [2].

Also, see a REU paper [HERE](#) by Daping Weng.

A short paper by Tom is [HERE](#).

38 Definition: Inverse Image Sheaf

See Wiki for motivation of such definition.

See a POST that gives more details, as well as a counterexample.

From Prof. Sándor's Email: Let X be disjoint union of two copies of Y with a continuous map $f : X \rightarrow Y$. Assume Y is irreducible and let \mathcal{G} be a constant sheaf on Y . We claim that $f_{\text{pre}}^{-1}\mathcal{G}$ is just a presheaf, but not a sheaf.

Any open subset $W_1, W_2 \in X$ will have intersection in Y . Then any section will agree on their intersections. Take two sections from $0 \amalg Y$ and $Y \amalg Y$, there won't be a global section such that restriction is either of them.

39 Exercise 1.3.

See a post [HERE](#) for explicit information of induced map on stalks.

See the solution from a post [HERE](#).

See [HERE](#) for a partial solution, as well as a counterexample.

39.1 (a)

Now assume φ is surjective. Fix an open subset $U \subset X$ and a section $s \in \mathcal{G}(U)$. Now we can pick any point $p \in U$, consider the stalk at it.

40 Exercise 1.8.

See Rotman's [7], Lemma 6.68. on Page 378.

Part II

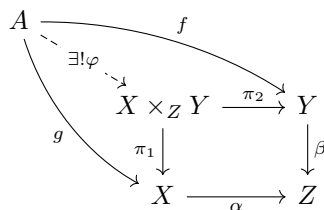
The Rising Sea

41 1.3.F. EXERCISE.

A post discussing this problem is [HERE](#).

42 1.3.N

A crutial step is to define the map such that the diagram commute. In order to prove it satisfies the universal property. We're given A be arbitrary with map $g : A \rightarrow X$ and $f : A \rightarrow Y$.



We can to define

$$\begin{aligned} \varphi : A &\rightarrow X \times_Z Y \text{ by} \\ a &\mapsto (g(a), f(a)). \end{aligned}$$

And we can verify this definition will make the diagram commute, and is unique.

43 1.3.O

It's indeed intersection. A post [HERE](#).

A post [HERE](#).

44 1.3.P.

Say we have $X \times Y$ and $X \times_Z Y$. By universal property of product and fibered product we can produce two unique map goes in between. Their composition must be identity, hence they're isomorphic. Notice it's important for Z being a final object.

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_2} & Y \\ \pi_1 \downarrow & & \downarrow \beta \\ X & \xrightarrow{\alpha} & Z \end{array}$$

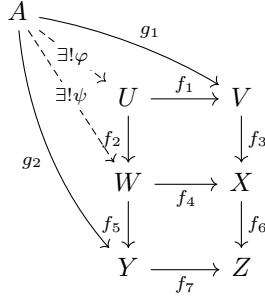
When we trying to make a map from $X \times Y$ to $X \times_Z Y$, we have to make sure two maps

$$\beta \circ \pi_1 = \alpha \circ \pi_1$$

on $X \times Y$. While Z is the final object, hence they must agree.

There's a cleaner way to state it [HERE](#). Crutial part is applying final property of object Z .

45 1.3.Q.



Label the maps as indicated. To prove the universal property with respect to the "outside rectangle", we're given

$$f_6 f_3 g_1 = f_7 g_2$$

agree on A . While W is fibered product, apply universal property of fibered product with respect to W we immediately get a unique map

$$\psi : A \rightarrow W$$

that makes the diagram involving A, W, X, Y, Z commute. In particular, we know $f_4 \psi = f_3 g_1$. Furthermore, recall that U is the fibered product. We're given the condition that $f_4 \psi = f_3 g_1$, by universal property of U we know there exists a unique map

$$\varphi : A \rightarrow U$$

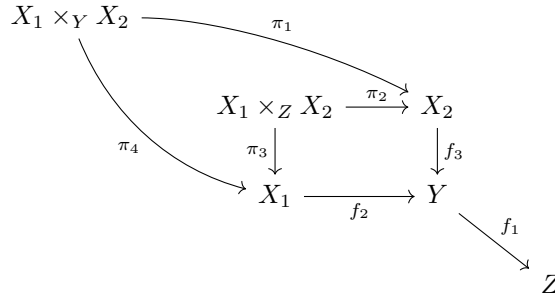
making the diagram involving A, U, V, W, X commute. And we claim that the diagram involving A, U, V, Y, Z commute. This is because

$$g_2 = f_5 \psi = f_5 f_2 \varphi = (f_5 f_2) \varphi \quad \text{and} \quad g_1 = f_1 \varphi.$$

And this proves that U is the fibered product for the diagram involving A, U, V, Y, Z .

A post is [HERE](#).

46 1.3.R



By the universal property of $X_1 \times_Z X_2$, we know there exists a unique map

$$\varphi : X_1 \times_Y X_2 \rightarrow X_1 \times_Z X_2$$

"Natural morphism", a convention discussed [HERE](#).

47 Course Notes from Cornell

See [HERE](#).

48 1.3.S. Magic Diagram

Didn't finish. Need to See [HERE](#), [HERE!!!](#)

49 1.3.Y. (a)

YONEDA'S LEMMA Given what we have, define $g : A \rightarrow A'$ as

$$g := i_A(\text{id}_A).$$

This is correct, see a post [HERE](#).

50 1.4.C.

(a) See "A Term of Commutative Algebra", Example 7.3 on Page 52.

51 1.6.B.

Write out everything by definition, and we can finish the proof immediately by applying rank-nullity theorem for linear transformation...

52 2.2.6. Definition: Sheaf.

Comments on $\mathcal{F}(\emptyset)$. In category **Set**, the empty set is initial object and one element set is terminal. See Wiki's examples [HERE](#).

53 2.2.B.

For (a): see Wiki's counterexample [HERE](#), which gave an explanation for presheaves on \mathbb{R} instead of \mathbb{C} . See a post [HERE](#).

54 2.2.7.

See Daping's Notes Definition 2.5 on Page 4 [HERE](#); also a post [HERE](#); also a post [HERE](#).

55 2.2.10.

It's different from a post [HERE](#), and Wiki's page on Constant pre-Sheaf.

Why???

56 2.2.G.

It's clearly a pre-sheaf.

Fix an open subset $U \subset X$ with an open cover $\{U_i\}_{i \in I}$ for some index set I . Denote the presheaf as \mathcal{F} .

Pick two continuous maps $s_1, s_2 : Y \rightarrow X$ that satisfying the requirements, i.e. $s_1, s_2 \in \mathcal{F}(U)$.

Both functions will agree on U since

$$\text{Res}_{U, U_i} s_1 = \text{Res}_{U, U_i} s_2$$

for arbitrary U_i , whose union is U . So we must have $s_1 = s_2$.

Again with this open cover $\{U_i\}_{i \in I}$ and $a_i \in \mathcal{F}(U_i)$ for $i \in I$. Equivalently, we know $a_i : U_i \rightarrow Y$ is a continuous map satisfying $\mu \circ a_i = \text{Id}|_{U_i}$. Now let's define a map

$$\begin{aligned} f : U &\rightarrow Y \\ u &\mapsto a_i(u) \text{ when } u \in U_i. \end{aligned}$$

It's well-defined by our assumption. Also it's continuous since preimage of an open set in $V \subset Y$ is a union of open subsets given by continuity of each a_i . Similarly we can check $\mu \circ f = \text{Id}|_U$ as expected.

Unverified ?

57 2.2.11. Espace Étale

See a post discussion accent letter in LaTeX [HERE](#).

See an exercise in [4] Chapter 2, Exercise 1.13.

See the discussion after Lemma 7. on Page 229 of [2].

For *section*, see Wiki's explanation for *section* in context of fiber bundle; and *section* in terms of category theory.

58 2.3.A.

I'm planning to use universal property to define the induced map ϕ_P .

One crucial step is to verify the diagram below is commutative

$$\begin{array}{ccc}
 \mathcal{F}(U) & \xrightarrow{\rho_{UV}} & \mathcal{F}(V) \\
 \phi(U) \downarrow & & \downarrow \phi(V) \\
 \mathcal{G}(U) & \xrightarrow{\tau_{UV}} & \mathcal{G}(V) \\
 & \searrow & \swarrow \\
 & \mathcal{G}_P &
 \end{array}$$

And this is because the square diagram in the upper half commute given ϕ is a natural transformation; the lower half is by definition of \mathcal{G}_P . Then by universal property of colimit induces a map

$$\phi_P : \mathcal{F}_P \rightarrow \mathcal{G}_P$$

which makes the diagram commute.

See a post defined the map [HERE](#).

59 2.3.B.

To define a functor $\pi_* : \mathbf{Set}_X \rightarrow \mathbf{Set}_Y$. Firstly, we have to define for any $\mathcal{F} \in \mathbf{Set}_X$,

$$\pi_*(\mathcal{F})(U) = \mathcal{F}(\pi^{-1}(U))$$

for any $U \in \mathbf{Top}(X)$ as in ??.

Secondly, for any natural transformation $\phi : \mathcal{F} \rightarrow \mathcal{G}$, we define $\pi_*(\phi)$ by specifying

$$\pi_*(\phi)(U) \mapsto \mathcal{F}(\pi^{-1}(U)) \rightarrow \mathcal{G}(\pi^{-1}(U)).$$

? Is this correct

60 References

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