Contents

0.1 Notes

See a post HERE, which listed various resources...

See Old Lecture Homepages of ETH HERE. There are some courses on algebraic geometry with solutions for homeworks.

See a post on "Best AG Textbook" HERE.

Great Notes on Algebraic Geometry and Commutative Algebra by Andreas Gathmann HERE.

See Horawa's Notes on Algebraic Geometry.

See HERE for a comprehensive guide to Hartshorne as well as solutions.

See Student Q& A in MITCourseware HERE.

See notes on algebraic geometry by Prof. Dr. Andreas Gathmann HERE.

See HERE for Aaron Landesman's notes.

See a course in 2012 by James McKernan HERE.

Book: Sheaf Theory through Examples by Daniel HERE.

See Math 216 Course Webpage HERE.

Nice online book by Mumford HERE.

Part I

Algebraic Geometry
Hartshorne

Chapter 1

Chapter 1.1.

1.1 Definition: Irreducible

For equivalent definitions, see Wiki HERE; Also see "dense" on Wiki HERE.

See Atiyah's [2] Exercise 19 from Chapter 1 for more information...

See Lemma 14. on Page 210 of [3] for equivalent characterisations of *irreducible*.

1.2 Example 1.1.3.

See Atiyah's [2] Exercise 19 from Chapter 1, which proves it must be dense.

For the sake of contradiction, assume a non-empty subset $A \subset X$ is reducible. Hence there exist two proper closed subset $A_1, A_2 \subset A$ such that $A = A_1 \cup A_2$. Then we have

$$X = (A^c \cup A_1) \cup (A^c \cup A_2),$$

which implies X is reducible.

×!

1.2.1

See a post HERE

✔

First approach applied some density argument. While the second approach, similarly, gave a decomposition of the whole space given the open set is reducible.

1.3 Example 1.1.4.

See Atiyah's [2] Exercise 20 from Chapter 1.

1.4 Definition.

"Induced topology". Definition of *quasi affine variety*, see HERE.

1.5 Prop 1.2 (d)

According to Hilbert's Nullstellensatz, I agree we'll get $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$. For the reverse inclusion, pick any $f \in A$ such that $f^r \in \mathfrak{a}$ where $r \in \mathbb{Z}_{>0}$. We wish to show that $f(P) = 0$ for any $P \in Z(\mathfrak{a})$. By definition, $f^r(P) = 0$ given $f^r \in \mathfrak{a}$. And this implies

$$f^r(P) = (f(P))^r = 0 \;\Rightarrow\; f(P) = 0$$

given the polynomial ring A is an integral domain. Therefore we get the inclusion

$$I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}.$$

See Theorem 6 on Page 183, Strong Nullstellensatz, [4].

1.6 Theorem 1.3A. Hilbert's Nullstellensatz

For the case where k isn't algebraically closed, see [14] Remarks in 5.6.

1.6.1

For the proof, see [9] Chapter 1 for details.

The followings are some comments for [9] Chapter 1 Theorem 1.7.

A post on surjective preimage for maximal ideal HERE.

A post on preimage for maximal ideal (not necessarily surj) HERE.

For completeness, a post on preimage of prime ideals HERE.

A post on image of prime ideals HERE, HERE, and HERE.

See Kemper's [9], Lemma 1.22 on Page 17, which completely described prime and maximal ideals in quotients.

1.7 Definition. Height

Here the definition *height* is specifically for prime ideal $\mathfrak{p} \lhd_{pr} R$ for some ring R. For a general definition, see a post HERE; see a webpage HERE; or see [9] Definition 6.10 on Page 68.

Nagata's example: Notherian ring with infinite Krull dimension, see HERE and a post HERE.

1.8 Proposition 1.7.

Difference between algebraic set and affine algebraic set? See HERE.

For an analogue in Projective, see Exercise 2.6 **??** in Chapter 1.2. And a post HERE.

1.9 Theorem 1.8A.

For transcendence degree, see HERE and a NOTE by Milne James.

1.10 Proposition 1.10.

Apart from the proof, we discussed *locally closed subset.* See HERE for its equivalent definitions.

1.11 Proposition 1.13.

See HERE.

1.12 Exercise 1.1.

1.12.1 (a)

By definition of affine coordinate ring we have

$$A(Y) = A/I(Y) = A/I(Z(f)) = A/\sqrt{\langle f \rangle}.$$

While f is irreducible in U.F.D. $k[x,y]$, the ideal it generated will be a prime ideal, which is radical. Therefore we can further simplify the expression as

$$A/\sqrt{\langle f \rangle} = A/\langle f \rangle = k[x,y]/\langle y - x^2 \rangle = k[x].$$

Hence we can conclude $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

1.12.2 (b)

1.13 Exericse 1.2.

My initial guess was incorrect, in which I thought $I(Y) = \langle z - xy \rangle$.

1.13.1 Solutions

See a post HERE, and HERE.
See a post HERE.
The correct one is $I(Y) = \langle z - x^3,\ z - y^2 \rangle$. Notice that

$$\dim Y = \dim A(Y) = \dim k[x,y,z]/\langle z - x^3,\ z - y^2 \rangle = \dim k[z] = \operatorname{tr.deg}_k k(z) = 1.$$

Therefore we proved that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

Question: How to compute $I(\cdot)$ precisely?

1.14 Exercise 1.4.

See a post HERE, HERE, and HERE.

The hint was to consider diagonal. Let the coordinate ring of $\mathbb{A}^2$ be $A(\mathbb{A}^2) = k[x, y]$ where k is algebraically closed. Then the diagonal

$$\Delta = \{(x, y) \in \mathbb{A}^2 \mid x = y\} = Z(x - y = 0)$$

is closed in Zariski topology of $\mathbb{A}^2$.

Wrong!

But it cannot be countably many union of closed subsets in product topology of copies of Zariski topology on $\mathbb{A}^1$, which are finitely many points.

Here we used the fact that k is infinite given it's algebraically closed.

1.14.1

See the first post! Closed subset of product topology on $\mathbb{A}^1 \times \mathbb{A}^1$ can be uncountable, for example, a line $\{(x_0, y) \in \mathbb{A}^1 \times A^1 \mid y \in k\}$ where $x_0 \in k$ is fixed would be closed. But it's uncountable.

The reason why Δ is closed in Zariski topology of $\mathbb{A}^2$ is correct.

For product topology, the reason why it's not closed is because Zariski topology on $\mathbb{A}^1$ is not Hausdorff, by a lemma we mentioned we know Δ isn't closed.

Verified!

1.15 Exercise 1.7.

1.15.1 (a)

(i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv): Zorn's Lemma.

(ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii): contrapositive, with Axiom of Dependent Choice.

(i) $\Leftrightarrow$ (iii): taking complement.

1.15.2 (b)

A similar statement is Exercise 17. (v) of [2], which states that $\operatorname{Spec} A$ is quasi-compact for a ring A.

For a given open cover of $\{U_i\}_{i \in I}$ of X, we can throw away open subset U_i such that $U_i \subset U_j$ for some $j \in I$. Here I is some index set. So we can assume in this open cover, we don't have $U_i \subset U_j$ for any $i, j \in I$. Then we have a ascending chain of open subsets

$$U_i \subseteq U_i \cup U_j \subseteq \cdots .$$

Noetherian condition imposed gives us finitely many index, and they form a finite subcover as expected.

A very simple proof HERE used maximality property.

A stronger statement HERE.

1.16 Exercise 1.10.

1.16.1 (a)

Clear, since any chain of Y is also a chain in X, and we use property of sup mentioned on Prop 2.8. of HERE.

1.16.2 (b)

1.16.3 (c)

1.16.4 (d)

1.16.5 (e)

See a post HERE, HERE,

See the notes by Vakil HERE.

See the notes from UofT HERE.

See a post HERE.

Example in topological space is actually easier to say than Nagata's example.

Take our topological space as $X = [0, 1] \subset \mathbb{R}$. And define the topology by declaring all closed subsets as

$$\{\emptyset, X, \{[1/n, 1] \subset \mathbb{R} \mid i \geq 2, i \in \mathbb{Z}\}\}.$$

Clearly we can check it satisfies infinite intersection and finite union axiom.

It's Noetherian for any descending chain of closed subsets must terminate.

It's of infinite Krull dimension. For any $n > 0$, we can construct a chain of such length. Hence the dimension, defined as supremum cannot be finite.

Another weird example could be found HERE, in which we give $\mathbb{N}$ the topology empty set, entire space, and $\{x \in \mathbb{N} \mid x \geq q\}$ for some $a \in \mathbb{N}$.

Chapter 2

2.1 Graded Ring

Say we wish to define variety in some projective space, i.e. some subsets of $\mathbb{P}^n$ that could be written as zero locus of some polynomials. If we don't add any restrictions on polynomials, for example some non-homogeneous polynomials. Then the zero locus doesn't make sense because it will depend on the representitive. See Gathmann's Notes Remark 6.5, notes on algebraic geometry, HERE.

2.2 Proposition 2.2.

There are two very technical claims need more details.

The first one is to prove $\varphi(Y) = Z(T') = Z(\alpha(T))$. Unwrap the notation precisely according to the definition

$$\begin{aligned} Z(\alpha(T)) &= \{ \ x \in \mathbb{A}^n \ \mid \ \alpha(g)(x) = 0 \ \forall \ g \in T \ \}, \\ \varphi(Y) &= \{ \ \varphi(y) \ \mid \ y \in Y \ \}. \end{aligned}$$

Notice that $y = [y_0, ..., y_n] \in Y \subset \overline{Y} = Z(T)$, therefore $g(y) = 0$ for any $g \in T$. More precisely, we have

$$\alpha(g)(\varphi(g)) = g(1, y_1/y_0, ..., y_n/y_n) = 0$$

given $g(y) = 0$ and $g \in T \subset S^h$, which proves $\varphi(Y) \subset Z(\alpha(T))$.

Conversely, let's start with an element $x = (x_1, ..., x_n) \in Z(\alpha(T))$. There's an element $y = [1, x_1, ..., x_n] \in Y$ such that $\varphi(y) = x$. Hence we've proved the equality.

And the second one is to check $\varphi^{-1}(W) = Z(\beta(T')) \cap U = Z(\beta(\alpha(T))) \cap U$.

2.3 Exercise 2.13.

*Let $V \subset \mathbb{P}^5$ be the Veronese (2-uple) embedding of $\mathbb{P}^2$. Prove that for any closed curve (a **curve** is a variety of dimension 1) $C \subset V$ there exists a hypersurface $H \subset \mathbb{P}^5$ such that $C = V \cap H$.*

2.3.1 References

A partial solution by REB HERE.
Another solution HERE.
And a post HERE.

Proof. The 2-uple embedding is defined as

$$\begin{aligned}\rho_2 : \mathbb{P}^2 &\to \mathbb{P}^5\\ [x:y:z] &\mapsto [x^2 : y^2 : z^2 : xy : yz : zx].\end{aligned}$$

By Exercise 2.12, we know that ρ_2 is a homeomorphism onto its image

$$\rho_2 : \mathbb{P}^2 \simeq V.$$

Therefore a curve $C \subset V$ is given by $C = \rho_2(Z(f))$ for some irreducible homogeneous polynomial $f \in S(\mathbb{P}^2)$ such that $\dim Z(f) = 1$.

Notice that $f^2 = g$ for some $g \in k[x^2 : y^2 : z^2 : xy : yz : zx]$. Therefore if we define $H = Z(g)$, then

$$C = Z(f) = Z(f^2) = Z(g) \cap \rho_2(\mathbb{P}^2) = H \cap V.$$

□

2.4 Exercise 2.12.

For all monomial of degree d in $n+1$ variables $x_0, ..., x_n$. There are

$$\binom{n+d}{n}$$

many monomials in total. Bars and balls argument: there are n bars and d balls. In $n+d$ many places, any choice of n bars will corresponds to a monomial, therefore N is the total number of monomials possible. While in we wish to consider them in projective space, we must define $N = \binom{n+d}{n} - 1$.

See a solution in lecture notes of Frank-Olaf Schreyer HERE.

A more detailed solution is given HERE.

2.5 Exercise 2.14.

2.5.1 Remarks

There's a prompt on Sándor's Notes, Lecture 13, Homework 2.79.

See Proposition 10. Chap 6 of [4] on Page 443.

2.5.2

Proof. First observation is that the map is well-defined. Because for P, Q in projective spaces we can find some nonzero a_i, b_j such that $a_i b_j \neq 0$ given they're from an integral domain.

□

2.6 Lemma 3.1.

See Sandor's Notes Lecture 4, Lemma 2.6.
Closedness can be checked locally. See a post HERE.

2.6.1 Closedness local criterion

This is HW 2.3. of Sandor's notes.

Let X be a topological space and $W \subset X$ a subset. Then W is closed if and only if for every $P \in X$ there is an open subset $U \subset X$ such that $P \in U$ and $W \cap U \subset U$ is a closed subset in U.

Proof. Assume W is closed, we can simply take $U = X$ for any P.

Conversely, we only need to verify that $X \setminus W$ is open. More precisely, we wish to prove that every point $P \in X \setminus W$ has an open neighborhood that contains in $X \setminus W$. This is ensured by Proposition 2.8. on Page 24 of [11].

Now start with an arbitrary point $q \in X \setminus W$, there exists open subset U_q of X such that

$$W \cap U_q \subset U_q$$

is closed in U_q. Then we can take $U_q \setminus W$ as the open neighborhood of q in $X \setminus W$ as expected. Hence we know $X \setminus W$ is open. □

2.7 Remark 3.1.1.

See Sandor's Notes Lecture 5, Corollary 2.10. Notice that Hartshorne defined variety to be irreducible. See a post explaining why the preimage is dense HERE.

See Lemma 14. on Page 210 of [3].

2.8 Definition: Ring of Regular Function

HERE is an explicit description on the ring structure of $\mathcal{O}_{P,Y}$.

2.9 Theorem 3.2.

See Sándor's Lecture Notes 09, STEP 03. See solution of problem 3 HERE. See a post HERE, HERE.

2.9.1 (c)

for each P, $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$, *and* $\dim \mathcal{O}_P = \dim Y$*;*

Proof. We begin with an injective homomorphism $\alpha : A(Y) \to \mathcal{O}(Y)$. And we define a map

$$A(Y)_{\mathfrak{m}_P} \to \mathcal{O}_{P,Y}$$
$$f/g \mapsto \langle V, \frac{\alpha(f)}{\alpha(g)} \rangle$$

where $\frac{\alpha(f)}{\alpha(g)} \in \mathcal{O}(V)$. Now we wish to give an explicit description of V. Since $\alpha(f) \in \mathcal{O}(Y)$, we know there exists an open subset $P \in V_1 \subset Y$ such that

$$\alpha(f) \mid_{V_1} = \frac{h_1}{h_2} \mid_{V_1}$$

where $h_1, h_2 \in A$ and $0 \notin h_2(V_1)$. Since $\alpha(g) \in \mathcal{O}(Y)$, we know there exists an open subset $P \in V_2 \subset Y$ such that

$$\alpha(g) \mid_{V_2} = \frac{h_3}{h_4} \mid_{V_2}$$

where $h_3, h_4 \in A$ and $0 \notin h_4(V_2)$. Here $g \notin \mathfrak{m}_P$ by definition of localisation, which gives us

$$g(P) \neq 0 \;\Rightarrow\; \alpha(g)(P) \neq 0 \;\Rightarrow\; \exists\, V_3 \subset Y,\ \alpha(g) \mid_{V_3} \neq 0.$$

Then we take $V = V_1 \cap V_2 \cap V_3$ will suffice to work. This is because for any point $P \in V$, we have

$$\frac{\alpha(f)}{\alpha(g)} = \frac{h_1 h_4}{h_2 h_3}$$

for $0 \notin h_2 h_3(V)$.

$$\begin{tikzcd}
A(Y) \arrow[r, "\alpha"] \arrow[dr] & \mathcal{O}(Y) \arrow[r, hook] & \mathcal{O}_{P,Y} \\
& A(Y)_{\mathfrak{m}_P} \arrow[ur, dashed] &
\end{tikzcd}$$

The induced map is given by universal property of localisation, for every elements in $A(Y) \setminus \mathfrak{m}_P$ will be mapped to a unit in $\mathcal{O}_{P,Y}$. And the map given by Lecture Notes 09 of Prof. Sándor satisfy the universal property (it makes the diagram commute and maps elements outside of $\mathfrak{m}_P$ to units).

□

2.10 Proposition 3.3.

See a post HERE.

2.11 Lemma 3.6.

See a post HERE.

Here are some details for proving $x_i \circ \psi$ being regular implies ψ is a morphism: Coordinate functions means

$$\psi(p) = (\psi_1(p), \psi_2(p), ..., \psi_n(p)) \in Y \subset \mathbb{A}^n$$

for any $p \in X$. Firstly, we check ψ is continuous. Take any closed subset $Z(f_1, ..., f_r) \subset Y$ for some polynomial $f_1, ..., f_r \in A = k[x_1, ..., x_n]$. We can compute the preimage as

$$\psi^{-1}(Z(f_1, ..., f_r)) = \{\ p \in X \ \mid\ \forall\ p \in X,\ f_i \circ \psi(p) = 0\ \}.$$

Notice that for any $p \in X$,

$$f_i \circ \psi(p) \ = \ f_i(\psi_i(p), ..., \psi_n(p))$$

is continuous since f_i is a polynomial and each $\psi_i := x_i \circ \psi$ is continuous by assumption that they're regular. Notice that the preimage of ψ is precisely intersection of $\psi_i^{-1}(\{0\})$ where $1 \leq i \leq n$. Hence the preimage is closed, and it follows that ψ is continuous as expected.

Secondly, fix an arbitrary open subset $V \subset Y$ with an arbitrary regular function $g : V \to k$, we wish to prove $g \circ \psi : \psi^{-1}(V) \to k$ is regular. For any $\psi(p) \in V$ with some $p \in X$, there exists a neighborhood $\psi(p) \in U \subset Y$ such that g equals to an expression of quotients of polynomial, i.e.

$$g = \frac{g_1}{g_2}$$

where $g_1, g_2 \in A$. Then for $p \in X$, take the open neighborhood of it as $\psi^{-1}(U)$, we can see

$$g \circ \psi(p) = \frac{g_1(\psi(p))}{g_2(\psi(p))}.$$

2.12 Exercise 3.5.

2.12.1 Hint

See a post HERE.

2.13 Exercise 3.6.

See a post HERE.

2.14 Exercise 3.7.

See this POST.

2.15 Exercise 3.10. Subvarieties

2.15.1 Locally Closed

See Wiki's notes HERE.
See a post HERE: image of a variety can be not even locally-closed.

2.16 Exercise 3.17.

Normal Varieties. *A vareity Y is* ***normal at a point*** *$P \in Y$ if $\mathscr{O}_p$ is an integrally closed ring. Y is* ***normal*** *if it is normal at every point.*

2.16.1 (a)

Show that every conic in $\mathbb{P}^2$ is normal.

Proof. According to Exercise 1.1.(c), we assume conic Y in $\mathbb{P}^2_{x,y,z}$ is defined by an irreducible homogeneous polynomial of degree 2.

And by Exercise 3.1.(c) we know every conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. To check it's normal, we need to show for any $P \in Y = \mathbb{P}^1$, the local ring $\mathscr{O}_P$ is an integrally closed ring.

Note that

$$\mathscr{O}_{P,\mathbb{P}^1} \simeq \mathscr{O}_{P,\mathbb{A}^1} = A(\mathbb{A}^1)_{\mathfrak{m}_P} = (k[x])_{\mathfrak{m}_P}.$$

While $k[x]$ is integrally closed, so we know its localisation is integrally closed hence every conic in $\mathbb{P}^2$ is normal. ???

Or we can use Exercise 3.18 (a). Notice that the homogeneous coordinate ring $S(\mathbb{P}^1)$ is clearly integrally closed, then it's projectively normal therefore normal.?

□

2.16.2 (b)

Show that the quadric surfaces Q_1, Q_2 in $\mathbb{P}^3$ given by equations $Q_1 : xy - zw$; $Q_2 : xy = z^2$ are normal (cf. (II. Ex. 6.4) for the latter.)

Proof. Denote $Q_1 = Z(xy - zw) \subset \mathbb{P}^3_{x,y,z,w}$. We have to compute the localisation of its homogeneous coordinate ring at some point $P \in Q_1$

$$\begin{aligned}\mathscr{O}_P =& S(Q_1)_{(\mathfrak{m}_P)}\\ =&(k[x,y,z,w]/I(Q_1))_{(\mathfrak{m}_P)}\\ =&(k[x,y,z,w]/\langle xy - zw\rangle)_{(\mathfrak{m}_P)}\end{aligned}$$

It suffices to check the quotient ring

$$k[x,y,z,w]/\langle xy - zw\rangle$$

is an integrally closed ring.???

Another approach is to use Jacobian matrix to check it's "smooth $\Rightarrow$ regular $\Rightarrow$ normal". Dimension of a surface is 2, then $n-r=4-2=2\geq 1$. Here 1 is the rank of Jacobian matrix at any point on the surface (for we choose only one polynomial $xy-zw$ as generator for $I(Q_1)$).

Similarly we can check for Q_2.

□

2.16.3 (c)

Show that the cuspidal cubic $y^2=x^3$ in $\mathbb{A}^2$ is not normal.

Proof. Let $P=(0,0)$, we wish to show that the local ring $\mathscr{O}_P$ is not integrally closed. Let $X=Z(y^2-x^3)$, and note that y^2-x^3 is irreducible in $k[x,y]$.

$$\mathscr{O}_P = A(X)_{\mathfrak{m}_P}=k[x,y]/I(X)_{\mathfrak{m}_P}=k[x,y]/(y^2-x^3)_{\mathfrak{m}_P}.$$

While integrally closed is a local property, it's equivalent to check $k[x,y]/(y^2-x^3)$ is an integrally closed domain. Since y^2-x^3 is irreducible in UFD, then this quotient ring an integral domain. Denote $R=k[x,y]/(y^2-x^3)$. Notice that $y/x\in\operatorname{Frac}(R)$ is an integral element since

$$(y/x)^2-x=0\in R.$$

On the other hand, the element $y/x\notin R$. Because otherwise we'll have $y/x=f$ for some polynomial $f\in R$, which is an integral domain. In integral domain, the difference between the degree of variables will give a contradiction. Hence $k[x,y]/(y^2-x^3)$ isn't integrally closed ring. □

See a post HERE, HERE.

Hint

According to Example 2 of Chapter 9 on Integral Extension on Page 65 of [12], we know

$$k[x,y]/\langle y^2-x^3\rangle\sim k[t^2,t^3]\subsetneq k[t].$$

Hence it's not an integrally closed domain.

2.16.4 (d)

If Y is affine, then Y is normal $\Leftrightarrow A(Y)$ is integrally closed.

Proof. For Y affine, we have Y being normal is equivalent to say $\mathscr{O}_{P,Y}=A(Y)_{\mathfrak{m}_P}$ is integrally closed for any point $P\in Y$. While being integrally closed is a local property, it's equivalent to say $A(Y)$ is integrally closed. □

2.16.5 (e)

Let Y be an affine variety. Show that there is a normal affine variety $\widetilde{Y}$, and a morphism $\pi : \widetilde{Y} \to Y$, with the property that whenever Z is a normal variety, and $\varphi : Z \to Y$ is a ***dominant*** *morphism (i.e., $\varphi(Z)$ is dense in Y), then there is a unique morphism $\theta : Z \to \widetilde{Y}$ such that $\varphi = \pi \circ \theta$. $\widetilde{Y}$ is called the* ***normalization*** *of Y. You will need (3.9A) above.*

Proof. Use Prof. Sándor's notes on Lecture 29 Theorem 4.16.

To complete the proof, we have to check the pre-variety defined in the proof is actually a variety. ? □

2.17 Exercise 3.18.

Projectively Normal Varieties. *A projective variety $Y \subset \mathbb{P}^n$ is* ***projectively normal*** *(with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.*

2.17.1 (a)

If Y is projectively normal, then Y is normal.

Proof. Since Y is projectively normal, then we know $S(Y)$ is integrally closed. Then we need to show

$$\mathscr{O}_P = S(Y)_{(\mathfrak{m}_P)}$$

is integrally closed. Clearly, $S(Y)_{\mathfrak{m}_P}$ is integrally closed. By definition we know $\mathscr{O}_P = S(Y)_{(\mathfrak{m}_P)}$ is a subring composed of degree 0 elements in $\mathscr{O}_P = S(Y)_{\mathfrak{m}_P}$. It is again integrally closed for any element $f/g \in \mathscr{O}_P = S(Y)_{\mathfrak{m}_P}$ that is integral over $\mathscr{O}_P = S(Y)_{(\mathfrak{m}_P)}$ satisfy the following equations

$$(f/g)^n + a_{n-1}(f/g)^{n-1} + a_0 = 0$$

for some $a_i \in \mathscr{O}_P = S(Y)_{(\mathfrak{m}_P)}$ where $0 \leq i \leq n-1$. Taking degree function on both sides, then it follows that f/g must be of degree 0, hence $\mathscr{O}_P = S(Y)_{(\mathfrak{m}_P)}$ is integrally closed.

□

2.17.2 (b)

2.18 Exercise 3.20.

Let Y be a variety of dimension ≥ 2, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

See a post HERE.

2.18.1 (a)

2.19 Definition: Dominant Rational Map

2.19.1

Well-definess for a rational map being *dominant*.

One thing important to keep in mind is both varieties X, Y are a priori irreducible. There's a completely point-set topological argument HERE. Here a technical detail is "The image of a dense subset under a surjective continuous function is again dense", which is from Wiki's entry. For details of this technicality, see HERE.

See a post on equivalent definition for dominant rational map HERE.

A good lecture note HERE.

Wiki's entry for Rational Map.

Very good note by Vakil HERE.

And a post HERE. However, it appears the way used in the proof implicitly required morphism is an open map? So I doubt...

2.19.2

Some posts share a technical details from general topology. The following statement is taken from Wiki...

The image of a dense subset under a surjective continuous function is again dense. More precisely, assume $f : X \to Y$ with E dense in X, then $f(E)$ is dense in $f(X)$.

Proof. By definition we have $E \subset f^{-1}(f(E)) \subset f^{-1}(\overline{f(E)})$, which is closed given f is continuous. It follows that

$$\overline{E} \subset f^{-1}(\overline{f(E)}) \;\Rightarrow\; f(\overline{E}) \subset \overline{f(E)}.$$

Conversely,

$$f(X) = \overline{f(E)} \cap f(X) \supset f(X) \cap f(X) = f(X) \;\Rightarrow\; f(X) \supset \overline{f(E)} \cap f(X).$$

Here $\overline{f(E)}$ denotes closure of $f(E)$ in Y, while it's intersection with $f(X)$ is the whole $f(X)$, then $f(E)$ is dense in $f(X)$. $\square$

2.19.3 A Pathological Example

Another equivalent statement required "surjectivity" and say $f(E)$ is dense in Y. It's curtial. Also we can only conclude $f(E)$ is dense merely in $f(X)$ instead of Y. Since we have the continuous inclusion map $\iota : \mathbb{R} \to \mathbb{C}$, then $\mathrm{id}(\mathbb{Q}) = \mathbb{Q}$ is just dense in $\mathbb{R}$ but not dense in $\mathbb{C}$.

2.19.4

Say we start with a dominant rational map $\varphi : X \to Y$ with two representatives

$$\langle U, f\rangle,\ \langle V, g\rangle.$$

By definintion of dominant, we know $f(U)$ is dense in Y. To check this definition is indepedent of the choice of the representative, we have to check $g(V)$ is dense in Y.

Notice that

$$Y = \overline{f(U)} = \overline{f(\overline{U\cap V}\cap U)} \subset \overline{f(\overline{U\cap V})} \subset \overline{\overline{f(U\cap V)}} = \overline{g(U\cap V)} \subset \overline{g(V)}.$$

for X is irreducible and both U, V are non-empty and open then $X = \overline{U\cap V}$. Here the third inclusion is given by the previous technical lemma.

2.19.5 Composing Dominant Rational Maps

See a post HERE.

$$X \overset{\phi}{\dashrightarrow} Y \overset{\psi}{\dashrightarrow} Z$$

$$(U, \phi_U) \qquad (V, \psi_V)$$

To prove the composition is a dominant rational map, we need to find a representative. We define

$$W := \phi_U^{-1}(V).$$

? slightly different than the post online

And we claim $(W,\ \psi_V \circ \phi_U)$ will be suitable for a representative for $\psi \circ \phi$. First of all, notice that W is non-empty. This is because $\phi_U(U) \cap V \neq \emptyset$ given $\phi_U(U)$ is dense in Y and V is assumed to be non-empty open subset. While Y is irreducible, by Lemma 14. of **??** on Page 210, which states that $\phi_U(U) \cap V$ is nontrivial. By definition this implies

$$\phi_U^{-1}(V) \neq \emptyset.$$

This is non-empty and open. Note X is irreducible, hence $W = \phi_U^{-1}(V)$ is dense in X. Hence $\psi_V \circ \phi_U(W)$ is dense in Z given both maps are continuous by being a morphism.

2.19.6

See a post HERE, HERE, and HERE.

2.20 Lemma 4.2.

See a post HERE.

2.21 Definition Blowing Up

On Page 29, Hartshorne defined blowing-up of a closed affine subvariety Y of $\mathbb{A}^n$ passing through O.

See Sándor's Notes, Lecture 22, *strict transform.*

See Daniel's notes HERE. Here the notation $(\cdot)^-$ stands for taking closure.

2.22 Exercise 4.1.

Define a function

$$h : U \cup V \to k \text{ by}$$
$$x \mapsto \begin{cases} f(x) & \text{when } x \in U \setminus V \\ g(x) & \text{when } x \in V \end{cases}.$$

Since $f = g$ on $U \cap V$, hence the function h is well-defined. For any point $p \in U \cup V$, if $p \in U$, then we apply assumption that f is regular. For $p \in V$, similarly apply assumption that g is regular. Hence h is regular on $U \cup V$.

Let f be a rational function on X. So we take all equivalence class $\{\langle U_i, f_i\rangle\}_{i\in I}$ that represents f. By the above lemma and the definition of regular function, there's a regular function g that's defined on $U := \bigcup_{i\in I} U_i$ that extends all f_i. Therefore we can take a representative of f as

$$\langle U, g\rangle.$$

Note $\langle U, g\rangle = \langle U_i, f_i\rangle$ by definition, hence it's indeed a representative of f.

Also U is the largest open set. Suppose it's not, then we have $\langle U_{i_0}, f_{i_0}\rangle$ represents f such that $U_{i_0} \supsetneq U$. And this will contradicts the construction of U, which must contain U_{i_0}.

2.23 Exercise 4.2.

We're given a rational map $\varphi : X \dashrightarrow Y$. Suppose we have two equivalent representatives

$$\langle U, \varphi_U\rangle, \ \langle V, \varphi_V\rangle.$$

This means two morphisms φ_U, φ_V agree on $U \cap V$.

It suffices to prove that we can define a morphism $\psi : U \cup V \to Y$ that extends both φ_U and φ_V. Similarly, we can apply argument of 2.22 to conclude the existence of a largest open set on which φ is represented by a morphism.

Both φ_U, φ_V are continuous function that agree on their intersection, then we can define

$$\psi : U \cup V \to Y \text{ by } \psi(x) = \begin{cases} \varphi_U(x) & \text{when } x \in U \setminus V \\ \varphi_V(x) & \text{when } x \in V \end{cases}.$$

Clearly it's a well-defined continuous function. For any open subset $W \subset Y$ with an arbitrary regular function $f : W \to k$. We have $f \circ \psi : \psi^{-1}(W) \to k$ is a regular function since it extends both

$$f \circ \varphi_U : \varphi_U^{-1}(W) \to k,\ f \circ \varphi_V : \varphi_V^{-1}(W) \to k.$$

While both two regular functions agree on their intersection, then we can conclude using 2.22 that $f \circ \psi$ is a regular function. And this proves that ψ is indeed a morphism on $U \cup V \to Y$.

2.24 Exercise 5.1.

2.24.1 (a)

According to this picture, (a) is a tacnode.

Denote $f_1 := x^2 - x^4 - y^4$, which is irreducible in UFD $k[x, y]$ hence it's prime. We can then compute the ideal defined by this affine variety

$$I(Z(\langle f_1 \rangle)) = \sqrt{\langle f_1 \rangle} = \langle f_1 \rangle.$$

The dimension of the affine variety $Z(\langle f_1 \rangle)$ is

$$\dim Z(\langle f_1 \rangle) = \dim k[x, y]/\sqrt{\langle f_1 \rangle} = \dim k[x, y] - \operatorname{height}\langle f_1 \rangle$$

By Krull's Hauptidealsatz we know the minimal prime ideal $\mathfrak{p}$ that contains $\langle f_1 \rangle$ has height exactly 1. While $\langle f_1 \rangle$ is a prime, we know it must be height of 1. Then by Theorem 4.7 in the notes, we know the Jacobian matrix at a singular point P cannot have rank $2 - 1 = 1$. While the matrix is 1×2, it follows that the matrix can only have dimension 0.

So we have to compute the Jacobian matrix of the above affine variety at some point $P \in \mathbb{A}^2$ on the affine variety. We choose f_1 itself as generators for the ideal of the affine variety and compute the Jacobian matrix

$$J(P) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(P) & \frac{\partial f_1}{\partial y}(P) \end{pmatrix} = \begin{pmatrix} 2x - 4x^3(P) & -4y^3(P) \end{pmatrix}.$$

Equivalently, we must havve $2x - 4x^3(P) = 0$ and $-4y^3(P) = 0$. Solving the equation, notice that P must lies on the tacnode, we'll get $P = (0, 0)$ is the only singular point.

2.24.2 (b)

According to this picture, (b) is a node.

Denote $f_2 = xy - x^6 - y^6$. Similarly, we choose f_2 itself as the generator for the affine variety it defined. Again, we have to compute the Jacobian matrix of $Z(f_2)$ at $P = (0, 0)$.

$$J(P) = \begin{pmatrix} \frac{\partial f_2}{\partial x}(P) & \frac{\partial f_2}{\partial y}(P) \end{pmatrix} = \begin{pmatrix} y - 6x^5(P) & x - 6y^5(P) \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix},$$

which as rank 0. Solving the equations $y - 6x^5(P) = 0$ and $x - 6y^5(P) = 0$ will implies that $P = (0, 0)$.

2.24.3 (c)

See this picture, then we know (c) is a cusp. Denote $f_3 = x^3 - y^2 - x^4 - y^4$. We just need to check the Jacobian matrix

$$J(P) = \begin{pmatrix} \frac{\partial f_2}{\partial x}(P) & \frac{\partial f_2}{\partial y}(P) \end{pmatrix} = \begin{pmatrix} 3x^2 - 4x^3(P) & -2y - 4y^3(P) \end{pmatrix}.$$

Solving the equations for points on cusp will forces $P = (0, 0)$.

2.24.4 (d)

See this picture, then we know (d) is the triple point. And we denote $f_4 = x^2y + xy^2 - x^4 - y^4$. Compute the Jacobian matrix gives us

$$J(P) = \begin{pmatrix} 2xy + y^2 - 4x^3(P) & x^2 + 2xy - 4y^3(P) \end{pmatrix}.$$

Solving the equations $2xy + y^2 - 4x^3(P) = 0$ and $x^2 + 2xy - 4y^3(P) = 0$ will give us $P = (0, 0)$.

2.24.5

See a post HERE.
See REB's solution HERE.
See a post on irreducibility of polynomial over $\mathbb{C}$ HERE.

2.25 Exercise 5.3.

Multiplicities. *Let $Y \subset \mathbb{A}^2$ be a curve defined by the equation $f(x, y) = 0$. Let $P = (a, b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0, 0)$. Then write f as a sum $f = f_0 + f_1 + \cdots + f_d$, where f_i is a homogeneous polynomial of degree i in x and y. Then we define the* ***multiplicity*** *of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note that $P \in Y \Leftrightarrow \mu_P(Y) > 0$.) The linear factors of f_r are called the* ***tangent directions*** *at P.*

Notice that $P = (0, 0)$ and

$$P \in Y \Leftrightarrow f(P) = 0 \Leftrightarrow f_0 = 0 \Leftrightarrow \mu_P(Y) > 0.$$

2.25.1 (a)

Proof. Notice that $\mu_P(Y) = 1$ is equivalent to say that $f_1 = ax + by \neq 0$ for some $a, b \in k$. Hence either a or b is non-zero. Now we compute the Jacobian matrix at P.

$$\begin{aligned} J(P) &= \begin{pmatrix} \frac{\partial f}{\partial x}(P) & \frac{\partial f}{\partial y}(P) \end{pmatrix} \\ &= \begin{pmatrix} 0 + \frac{\partial f_1}{\partial x}(P) + \frac{\partial f_2}{\partial x}(P) + \cdots & 0 + \frac{\partial f_1}{\partial y}(P) + \frac{\partial f_2}{\partial y}(P) + \cdots \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial f_1}{\partial x}(P) & \frac{\partial f_1}{\partial y}(P) \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix}. \end{aligned}$$

could we assume curve f is irreducible? ☛ See [8] Example 1.4.2. in Chapter 1 on Page 4

Verified HERE

This 2×1-matrix as dimension 1 exactly because either a or b is nonzero.

If we can assume $\dim Y = 1$, then it follows that $n - r = 2 - 1$ equals to the rank of the Jacobian matrix, which proves that P is non-singular on Y. The converse direction is similar.

□

2.25.2 (b)

See solution HERE.

2.26 Exercise 5.6.

Blowing Up Curve Singularities.

2.26.1 (a)

Let Y be the cusp or node of (Ex. 5.1). Show that the curve $\widetilde{Y}$. obtained by blowing up Y at $O = (0,0)$ is nonsingular (cf. (4.9.1) and (Ex. 4.10)).

Proof. Let Y be the node curve, so $Y = I(xy - x^6 - y^6)$. Let x, y be coordinate of $\mathbb{A}^2$ and let u, v be coordinates for $\mathbb{P}^1$. For $\mathbb{A}^2_{x,y} \times \mathbb{P}^1_{u,v}$, we know the blowing-up of $\mathbb{A}^2$ at O is

$$\mathrm{Bl}_O \mathbb{A}^2 = Z(xv - yu) \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

And we denote the projection as $\varphi : \mathrm{Bl}_O \mathbb{A}^2 \to \mathbb{A}^2$. Now we're going to compute strict transform of Y

$$\widetilde{Y} = \overline{\varphi^{-1}(Y \setminus \{O\})}.$$

For any point $a_0 \in (x, y, u : v) \in \widetilde{Y}$, we know $\varphi(a_0) = (x, y) \in Y$. Hence we know at least

$$\widetilde{Y} \subset Z(xv - yu, xy - x^6 - y^6).$$

Now we try to restrict ourselves to one affine cover U_u of $\mathbb{P}^1$, i.e. let $u = 1$. Then

$$\begin{aligned} Z(xv - yu,\ xy - x^6 - y^6) \cap U_u &= Z(xv - y,\ xy - x^6 - y^6) \\ &= Z(x^2(v - x^4 - x^4 v^6)) \subset \mathbb{A}^3. \end{aligned}$$

Here $Z(x^2)$ is the exceptional set. And $\widetilde{Y} \cap U_u = Z(v - x^4 - x^4 v^6)$. Take $f_1 = v - x^4 - x^4 v^6$ and compute the Jacobian matrix at $P \in \mathbb{A}^2_{x,v}$

$$J(P) = \begin{pmatrix} \partial f_1/\partial x(P) & \partial f_1/\partial v(P) \end{pmatrix} = \begin{pmatrix} -4x^3 - 4v^6x^3(P) & 1 - 6x^4v^5(P) \end{pmatrix}.$$

Notice that there's no solution of P for equations $-4x^3 - 4v^6x^3(P) = 0$ and $1 - 6x^4v^5(P) = 0$. Therefore the matrix has rank exactly 1 becuase the coefficient k is a field. And by Krull's Hauptidealsatz, we know the dimension for the curve is 1. Then apply Theorem 4.7. from the notes we know $2 - 1 = 1$ is exactly the rank of the Jacobian matrix. Hence there's no singular points on $\widetilde{Y}$.

Similarly, we can check there's no singular points of $\widetilde{Y}$ on another affine cover U_v where $v = 1$. Hence we can conclude $\widetilde{Y}$ is non-singular. □

is it affine?

2.26.2 (b)

We define a ***node*** *(also called* ***ordinary double point****) to be a double point (i.e., a point of multiplicity 2) of a plane curve with distinct tangent directions (Ex. 5.3). If P is a node on a plane curve Y, show that $\varphi^{-1}(P)$ consists of two distinct nonsingular points on the blown-up curve $\widetilde{Y}$. We say that "blowing up P resolves the singularity at P".*

Proof. □

2.26.3 (c)

Let $P \in Y$ be the tacnode of (Ex. 5.1). If $\varphi : \widetilde{Y} \to Y$ is the blowing-up at P, show that $\varphi^{-1}(P)$ is a node. Using (b) we see that the tacnode can be resolved by two successive blowings-up.

2.26.4 (d)

Let Y be the plane curve $y^3 = x^5$, which has a "higher order cusp" at O. Show that O is a triple point; that blowing up O gives rise to a double point (what kind?) and that one further blowing up resolves the singularity.

2.27 Exercise 5.12.

Quadric Hypersurfaces*. Assume* char $k \neq 2$*, and let f be a homogeneous polynomial of degree 2 in $x_0, ..., x_n$.*

2.27.1 (a)

Show that after a suitable linear change of variables, f can be brought into the form $f = x_0^2 + \cdots + x_r^2$ for some $0 \leq r \leq n$.

Proof. Notice that it suffices to prove, after suitable linear transformation of variables, we can kill all terms $x_i x_j$ where $i \neq j$. Because then we know the polynomial will be $b_0 x_0^2 + \cdots + b_r x_r^2$, and we simply let $x_r \mapsto 1/\sqrt{b_r} x_r$ will yield the desired form. Denote our homogeneous polynomial f as

$$\begin{aligned} f =& a'_{00} x_0^2 + a'_{01} x_0 x_1 + \cdots + a'_{0n} x_0 x_n \\ &+ a'_{10} x_1 x_0 + \cdots + \\ &+ a'_{n0} x_n x_0 + \cdots + a'_{nn} x_n x_n \\ =& \sum_{0 \leq i \leq j \leq n} a_{ij} x_i x_j \\ =& a_{00} x_0^2 + a_{01} x_0 x_1 + a_{02} x_0 x_2 + \cdots + a_{nn} x_n x_n. \end{aligned}$$

Given char $k \neq 2$, we know $1/2 \neq 0$. We denote a symmetric $(n+1) \times (n+1)$-matrix with coefficients in k by

$$A = \begin{pmatrix} a_{00} & a_{01}/2 & a_{02}/2 & \cdots & a_{0n}/2 \\ a_{01}/2 & a_{11} & a_{12}/2 & \cdots & a_{1n}/2 \\ \vdots & \ddots & & & \\ a_{0n}/2 & a_{1n}/2 & a_{2n}/2 & \cdots & a_{nn}. \end{pmatrix}$$

. The matrix A is symmetric and except the diagonal, every coefficient has an extra coefficient $1/2$. Let a vector be $\mathbf{v} = [x_0 \ x_1 \ \cdots x_n]$. The reason we introduce this matrix is becuase the following identity

$$\begin{aligned} \mathbf{v} A \mathbf{v}^t =& a_{00} x_0 x_0 + 1/2 a_{01} x_0 x_1 + \cdots 1/2 a_{0n} x_0 x_n \\ &+ 1/2 a_{01} x_0 x_1 + a_{11} x_1 x_1 + 1/2 a_{12} x_1 x_2 + \cdots + 1/2 a_{2n} x_2 x_n \\ &+ \cdots \\ &+ 1/2 a_{0n} x_0 x_n + \cdots a_{nn} x_n x_n \\ =& a_{00} x_0 x_0 + a_{01} x_0 x_1 + \cdots + a_{nn} x_n x_n = f. \end{aligned}$$

While A is symmetric and over an algebraically closed field k, we can diagonalise it by some matrix B:

$$BAB^{-1} = \begin{pmatrix} a_{00} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}.$$

Here matrix B will provide information on linear change of variables. Also during linear change of variables, square terms can never be killed. So r depends
? on number of square terms in the original polynomial f. ∎

2.27.2 (b)

Show that f is irreducible if and only if $r \geq 2$.

Proof. Now we assume $f \in k[x_0, ..., x_r]$ for some $0 \leq r \leq n$. Suppose f is reducible. So we can find a factorisation $f = f_1 f_2$ for some non-unit polynomials $f_1, f_2 \in k[x_0, ..., x_r]$. This is equivalent to say that f_1, f_2 must be of homogeneous of degree 1 given the coordinate ring $k[x_0, ..., x_r]$ is an integral domain. Given that $r \geq 2$, we can express the factorisation without loss of generality as

$$f = f_1 f_2 = (x_0 + a_1 x_1 + \cdots + a_r x_r)(x_0 + b_1 x_1 + \cdots + b_n r x_r)$$

where all $a_i, b_i \in k \setminus \{0\}$ for $1 \leq i \leq r$. In order for the terms $x_i x_j$ where $i \neq j$ to be killed, we must have $a_i b_j + a_j b_i = 0$ for all $0 \leq i \leq j \leq r$. Here we assume $a_0 = b_0 = 1$. Hence we immediately have $a_j = -b_j$ for all $1 \leq j \leq r$ and the factorisation becomes

$$(x_0 + a_1 x_1 + \cdots + a_r x_r)(x_0 - a_1 x_1 - \cdots - a_r x_r).$$

And this means we can never kill the terms such as $x_1 x_2$ for it has coefficent $2a_1 a_2$. Then $f \neq f_1 f_2$, contradiction. It follows that f is irreducible.

$\square$

See a post HERE.

2.28 Exercise 6.1.

Recall that a curve is ***rational*** *if it is birationally equivalent to* $\mathbb{P}^1$ *(Ex 4.4). Let* Y *be a nonsingular rational curve which is not isomorphic to an open subset of* $\mathbb{P}^1$.

2.28.1 (a)

Show that Y *is isomorphic to an open subset of* $\mathbb{A}^1$.

Proof. Nonsingular rational curve Y is a nonsingular quasi-projective curve, which is isomorphic to an abstract nonsingular curve by Proposition 6.7.

According to Corollary 6.10., an abstract nonsingular curve is isomorphic to an open subset of a nonsingular projective curve Z. Therefore by Theorem 4.4. we have

$$\mathbb{P}^1 \sim_{\text{bir}} Y \sim_{\text{bir}} Z \;\Rightarrow\; K(\mathbb{P}^1) \simeq K(Z).$$

? This shows that it's isomorphic to an open subset of $\mathbb{P}^1$ therefore some open subset in $\mathbb{A}^1$.

$\square$

2.28.2 Hint

See a post HERE.
See REB's solution HERE.

2.29 Exercise 6.2.

2.29.1 (a)

Proof. We compute the Jacobian of $I(Y)$ as:

$$\begin{pmatrix} -3x^2+1(P) & 2y(P) \end{pmatrix}$$

This matrix evaluate at some point $P \in Y$ will have rank $n-r = 2-1 = 1$ given both functions cannot be 0 at the same time, therefore the curve is non-singular. Hence we know

$$\mathscr{O}_{P,Y}$$

is regular local ring for any $P \in Y$. While it's a local property(?), we know $A(Y)$ is regular local. While coordinate ring of this curve Y is Noetherian domain of dimension 1, hence Theorem 6.2A. implies it's an integrally closed domain. $\square$

2.30 Defintion: Presheaf

2.30.1 Two Pathological Examples

Here are two examples taken from Tennison's [18].

Let X be any topological space with more thatn one point, i.e. $X = \{0, 1\}$ or $X = \{0, 1\} \to \mathbb{R}$.

Define a presheaf $\mathscr{P}_1$ by

$$\begin{cases} \mathscr{P}_1(X) = \mathbb{Z} \\ \mathscr{P}_1(U) = 0 \text{ for open } U \subsetneq X \\ \text{All restrictions except } \rho_{XX} \text{ being constant maps.} \end{cases}.$$

Here 0 denotes the trivial Abelian group.

Pick $x_0 \in X$. Define a presheaf $\mathscr{P}_2$ by

$$\begin{cases} \mathscr{P}_2(U) = \mathbb{Z} \text{ for } U \text{ open in } X \text{ such that } x_0 \in U \\ \mathscr{P}_2(U) = 0 \text{ for } U \text{ open in } X \text{ such that } x_0 \notin U \\ \text{restrictions } \rho_{UV} = \begin{cases} \mathrm{id}_{\mathbb{Z}} & \text{if } x_0 \in V \subset U \\ 0 & \text{trivial map if not} \end{cases} \end{cases}.$$

Here the second appearance of 0 denotes the trivial map.

2.31 Example 1.0.3.

See some examples of presheaves that are not sheaves HERE; a post HERE.

In Wiki's page HERE, it introduced non-separated presheaf, i.e. presheaf that doesn't satisfy locality axiom for sheaf.

2.32 Proposition 1.1.

$\Rightarrow$: For this direction, notice that we have to identify $(\varphi^{-1})_P = (\varphi_P)^{-1}$.

Given $s_P = 0 \in \mathscr{F}_P$, we know $0 \in \mathscr{F}_P$ could be represented by $\langle W_1, 0\rangle$ and $s_P = \langle U, s\rangle$ for two open subsets $P \in W_1, U$.They're equal means there exist some open $W_P \subset U \cap W_1$ such that $s|_{W_P} = 0|_{W_P} = 0$. While P is arbitrary, therefore we have an open covering of U by $\{W_P\}_{P\in U}$. Then we can apply sheaf property to conclude $s = 0$ as desired.

2.33 Proposition-Definition 1.2. Sheafification

See *Sheafification* on The Stacks Project.

See solution of problem 3 HERE.

Of course, consult Ravi's Notes on Sheafification;

Or see Section 6.5 on Page 232 of [3].

Also, see a REU paper HERE by Daping Weng.

A short paper by Tom is HERE.

A post discussing two equivalent definitions of sheafification: one based on compatible germs, and the other one is based on continuous sections.

2.33.1 Isomorphism on stalk

This is for Lemma 007Z of Stacks Project. Similar contents could be found in Rising Sea 2.4.L.

$$\begin{tikzcd}
 & s \in \mathscr{F}(U) \arrow[r] \arrow[dl] \arrow[d] & \mathscr{F}^\sharp(U) \ni (s_u)_{u\in U} \arrow[d] \\
t \in \mathscr{F}(W) & (U, x) \in \mathscr{F}_x \arrow[r, dashed, "\exists!"] & \mathscr{F}^\sharp_x \ni (U, (s_u)_{u\in U})
\end{tikzcd}$$

According to Stacks Project, injectivity is proved.

We focus on proving the surjectivity of $\mathscr{F}_x \to \mathscr{F}^\sharp_x$. We constructed the induced map on stalks of presheaves based on universal property of $\mathscr{F}_x$ being a colimit. Therefore the map is unique. And we wish to get an explicit description of that. While for the diagram we know all but the induced map explicity, therefore the map defined such that making the diagram commute must be the induced map by uniqueness of universal property. It follows that we have

$$\begin{aligned}\mathscr{F}_x &\to \mathscr{F}^\sharp_x \\ (U, s) &\mapsto (U, (s_u)_{u\in U})\end{aligned}$$

where $x \in U \subset X$, $s \in \mathscr{F}(U)$, and $(s_u)_{u\in U} \in \prod_{x\in U} \mathscr{F}_x$ is a compatible germ.

We could also check this map is well-defined. But it must be for it makes the diagram commute, hence it's the unique induced map on stalk for a given $x \in X$.

do we care?

The facts that $x \in U$ and $(s_u)_{u\in U}$ is a compatible germ enable us, by definition of sheafification, find:

- a neighborhood $W \subset U$ such that $x \in W$;
- a section $t \in \mathscr{F}(W)$ such that $t_w = s_w$ for any $w \in W$.

And we claim $(W, t) \in \mathscr{F}_x$ will be the preimage of $(U, (s_u)_{u \in U})$. Now we're ready to compute the image of (W, t) as

$$\begin{tikzcd} t \in \mathscr{F}(W) \arrow[r] \arrow[d] & (t_u)_{u \in U} \in \mathscr{F}^\sharp(W) \arrow[d] \\ (W, t) \in \mathscr{F}_x \arrow[r, dashed] & (W, (t_u)_{u \in W}) \in \mathscr{F}^\sharp_x \end{tikzcd}$$

We claim that

$$(W, (t_u)_{u \in W}) = (U, (s_u)_{u \in U}) \in \mathscr{F}^\sharp_x$$

This is because there exists $W \subset W \cap U$ such that

$$\begin{aligned} (s_u)_{u \in U} \mid_W &= (s_u)_{u \in W} \\ &= (t_u)_{u \in W} \\ &= (t_u)_{u \in W} \mid_W . \end{aligned}$$

Here the first and third equality is given by restriction map. The second equality holds: by definition we claimed that $s_w = t_w$ for any $w \in W$. Therefore we've checked that (W, t) is the preimage for an arbitrary element $(U, (s_u)_{u \in U}) \in \mathscr{F}^\sharp_x$, and it follows that $\mathscr{F} \to \mathscr{F}^\sharp$ is surjective.

2.33.2 Hint

The above approach is based on Stacks Project. But Proposition 2.24. on Page 53 of Algebraic Geometry I: Schemes [7] gives another solution, which is **much** more efficient!

It identitifies, in the sense of colimi, that

$$\operatorname{colim} \mathscr{F}^\sharp = \mathscr{F}_x.$$

2.34 Definition: Image Sheaf

Caution: Given $\varphi : \mathscr{F} \to \mathscr{G}$ with $\mathscr{F}, \mathscr{G}$ being sheaves.

$$U \mapsto \operatorname{Im}(\varphi(U)),\ U \mapsto \operatorname{Coker}(\varphi(U))$$

are in general just presheaves.

See a post explaining an example of image presheaf that's not a sheaf.

See a very good solution for details.

See another example on Page 14 of a brilliant notes HERE by Yuri.

2.35 Definition: Inverse Image Sheaf

See Wiki for motivation of such definition.

See a POST that gives more details, as well as a counterexample.

From Prof. Sándor's Email: Let X by disjoint union of two copies of Y with a continuous map $f : X \to Y$. Assume Y is irreducible and let $\mathscr{G}$ be a constant sheaf on Y. We claim that $f^{-1}_{\text{pre}}\mathscr{G}$ is just a presheaf, but not a sheaf.

Any open subset $W_1, W_2 \in X$ will have intersection in Y. Then any section will agree on their intersections. Take two sections from $0 \amalg Y$ and $Y \amalg Y$, there won't be a global section such that restriction is either of them.

2.36 Definition: Restriction

See Rising Sea [19] Exercise 2.7.D. Section 2.137 and Example 2.2.8. The definition is very general, it's for both open and closed subsets. Usually, the definition given in Example 2.2.8. of [19] is easily to work with. And Exercise 2.7.D, states that we can avoid taking inverse image sheaf when the subset is *open*.

2.37 Exercise 1.1.

See "Rising Sea" by Ravi Exercise 2.2.E. on Page 74. It gave another potentially equivalent definition of constant sheaf, which could be easier to check it is indeed sheafification of constant pre-sheaf.

Let $\mathscr{F}$ denotes the constant pre-sheaf. Then we can compute, by using universal property of colimit that stalk at $p \in U$ is $\mathscr{F}_p = S$.

Now we have a concrete description of the compatible germs as

$$\begin{aligned}\mathscr{F}^\sharp(U) &= \{s : U \to \amalg_{p\in U}\mathscr{F}_p \ \mid \ ...\} \\ &= \{s : U \to \amalg_{p\in U} S \ \mid \\ &\quad \forall\ p \in U,\ \exists \text{ an open neighorbood } V \subset U \text{ containing } p \text{ and } \exists\ t \in \mathscr{F}(V). \\ &\quad \text{such that } s(q) = s_q = t_q\ \forall\ q \in V.\}\end{aligned}$$

But notice that for constant presheaf $\mathscr{F}$, all restriction maps and natural map to stalk is identity as

$$\begin{tikzcd} \mathscr{F}() = S \arrow[rr, "\text{id}"] \arrow[dr, "\text{id}"'] & & \mathscr{F}() = S \arrow[dl, "\text{id}"] \\ & \mathscr{F}_P = S & \end{tikzcd}$$

So we can simplify the expression as

$$\begin{aligned}\mathscr{F}^\sharp(U) = \{f : U \to \amalg_{p\in U} S \ \mid \\ &\forall\ p \in U,\ \exists \text{ an open neighorbood } V \subset U \text{ containing } p \text{ and } \exists\ t \in S. \\ &\text{such that } f(q) = t_q = \text{id}(t) = t\ \forall\ q \in V\}.\end{aligned}$$

We claim there's a bijection between two sets

$$\Phi : \mathscr{F}^\sharp(U) \to \underline{S}(U)$$

For a given $f \in \mathscr{F}^\sharp(U)$, we can define a map $g : U \to S$ by

$$g(p) := \mathrm{pr}_p \circ f(p)$$

where the projection map is $\mathrm{pr}_p : \amalg_{i \in I} S_i \to S_p$ defined by projection to p-coordinate. For any $p \in U$, by definition of sheafification we know there is an open neighborhood $p \in V \subset U$ and $t \in S$ such that

$$g(q) = \mathrm{pr}_q \circ f(q) = t$$

for any $q \in V$. And this is precisely saying g is locally constant. Different choices of f will result in different g, therefore it's injective.

Clearly, if we're given a locally constant map $g : U \to S$, we can form a tuple indexed by $p \in U$ as

$$\prod_{u \in U} (g(u)) \in \amalg_{u \in U} S.$$

And this corresponds to a function in $\mathscr{F}^\sharp(U)$ that satisfies the requirements precisely because g is locally constant. Therefore it's surjective.

What we've shown is that there's a bijection between

$$\Phi : \mathscr{F}^\sharp(U) \to \underline{S}(U)$$

where $\mathscr{F}$ is constant pre-sheaf. Therefore we conclude that constant sheaf is indeed the sheafification of constant pre-sheaf.

2.37.1 References

See a post HERE, HERE.

Basically, we need to prove $\underline{S}^\sharp_{\mathrm{pre}} \simeq \underline{S}$. And I did by exhibiting a bijection on when they both evaluate at an open subset U, i.e. I checked isomorphism between two functors by showing the natural transformation is natural isomorphism. I didn't do this on stalk for I'm afraid it could be more complicated.

But could we?

2.38 Exercise 1.2.

2.38.1 (a)

See 2.6.A. Exercise Section 2.135 of [19].

2.38.2 Hint

A categorical approach is to apply Lemma 002W of Tag 04AX. See HERE for an application.

2.39 Exercise 1.3.

See a post HERE for explicit information of induced map on stalks.
See the solution from a post HERE.
See HERE for a partial solution, as well as a counterexample.

2.39.1 (a)

Now assume φ is surjective. Fix an open subset $U \subset X$ and a section $s \in \mathscr{G}(U)$. Now we can pick any point $p \in U$, consider the stalk at it.

2.40 Exercise 1.8.

See Rotman's [15], Lemma 6.68. on Page 378.

2.41 Exercise 1.9. Direct Sum

In the context of the prompt, both sheaves on space X take value in category of Abelian group. Let $\{U_i\}_{i\in I}$ be a covering of arbitrary open subset $U \subset X$ where I is some index set. Assume $f = (s,t) \in \mathscr{F} \oplus \mathscr{G}(U)$ for some $s \in \mathscr{F}(U)$ and $t \in \mathscr{G}(U)$.

$$f|_{U_i} = \mathrm{Res}_{U,U_i}(f) = \mathrm{Res}_{U,U_i}((s,t)) = (\mathrm{Res}_{U,U_i}(s), \mathrm{Res}_{U,U_i}(t)) = 0 \in \mathscr{F}(U_i) \oplus \mathscr{G}(U_i).$$

Then we must have $s|_{U_i} = 0$ for all i, then $s|_U = 0$ for $\mathscr{F}$ is a sheaf. Then we can conclude that $f|_U = (s|_U, t|_U) = 0$.

For gluability, ...

2.41.1 Warning

Example 006Y on [16], which shows we cannot extend to taking infinitely many direct sum: it is a presheaf, but not sheaf in general.

2.41.2 Hint

See [15] Proposition 5.78. on Page 294.

2.42 Exercise 1.14.

Because for any point $q \in X$ such that $s_q = 0$, we can find a neighborhood U_q of q such that $\mathrm{Res}_{?U_q}\, s = 0$. Hence for any point $q' \in U_q$, we have $q' \notin \operatorname{Supp} s$, which proves it's closed for its complement is open.

2.42.1 Solution

See POST, POST, Tag 01AS.

2.43 Exercise 1.15.

See Rising Sea 2.3.C.

2.44 Exercise 1.18.

2.44.1 Solution

In fact, the sheaf on Y could be weaker: it suffices to assume $\mathcal{G}$ to be a presheaf.

See Proposition 2.27 Chapter 2 of [7], or see Tag 008C.

2.45 Exercise 1.19.

Extending a Sheaf by Zero. *Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subsets, and let $j : U \to X$ be its inclusion.*

2.45.1 (a)

Let $\mathscr{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathscr{F})_P$ of the direct image sheaf on X is $\mathscr{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ be zero outside Z. By abuse of notation we will sometimes write $\mathscr{F}$ instead of $i_*\mathscr{F}$, and say "consider $\mathscr{F}$ as a sheaf on X," when we mean "consider $i_*\mathscr{F}$."

Proof. For $P \in Z$, we have

$$\begin{aligned}
(i_*\mathscr{F})_P &= \operatorname*{Colim}_{P\in W\subset X,\ W \text{ open}} (i_*\mathscr{F})(W) \\
&= \operatorname*{Colim}_{P\in W\subset Z\subset X,\ W \text{ open}} (i_*\mathscr{F})(W) \\
&= \operatorname*{Colim}_{P\in W\subset Z\subset X,\ W \text{ open}} \mathscr{F}(i^{-1}(W)) \\
&= \operatorname*{Colim}_{P\in W\subset Z\subset X,\ W \text{ open}} \mathscr{F}(W) \\
&= \mathscr{F}_P
\end{aligned}$$

For $P \notin Z$, we have

$$\begin{aligned}
(i_*\mathscr{F})_P &= \operatorname*{Colim}_{P\in W\subset X,\ W \text{ open}} (i_*\mathscr{F})(W) \\
&= \operatorname*{Colim}_{P\in W\subset X,\ W \text{ open},\ W\cap Z=\emptyset} (i_*\mathscr{F})(W) \\
&= \operatorname*{Colim}_{P\in W\subset X,\ W \text{ open},\ W\cap Z=\emptyset} \mathscr{F}(i^{-1}(W)) \\
&= \operatorname*{Colim}_{P\in W\subset X,\ W \text{ open},\ W\cap Z=\emptyset} \mathscr{F}(\emptyset) \\
&= 0.
\end{aligned}$$

In summary, we know

$$(i_*\mathscr{F})_P = \begin{cases} \mathscr{F}_P & \text{when } P \in Z \\ 0 & \text{when } P \notin Z \end{cases}.$$

□

2.45.2 (b)

Now let $\mathscr{F}$ be a sheaf on U. Let $j_!(\mathscr{F})$ be the sheaf on X associate to the pre-sheaf $V \mapsto \mathscr{F}(V)$ if $V \subset U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathscr{F}))_P$ is equal to $\mathscr{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathscr{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathscr{F}$. We call $j_!\mathscr{F}$ the sheaf obtained by ***extending*** *$\mathscr{F}$* ***by zero*** *outside U.*

Proof. According to the definition, the sheaf $j_!\mathscr{F}$ is the sheafification of presheaf defined on $V \subset X$ by

$$(j_!\mathscr{F})^0(V) = \begin{cases} \mathscr{F}(V) & \text{when } V \subset U \\ 0 & \text{otherwise} \end{cases}.$$

Now we compute, for $P \in U$,

$$\begin{aligned}(j_!\mathscr{F})_P = (j_!\mathscr{F})^0_P &= \operatorname*{Colim}_{P\in W\subset X,\ W \text{ open in } X} (j_!\mathscr{F})^0(W) \\ &= \operatorname*{Colim}_{P\in W\subset U\subset X,\ W \text{ open in } X} (j_!\mathscr{F})^0(W) \\ &= \operatorname*{Colim}_{P\in W\subset U\subset X,\ W \text{ open in } X} \mathscr{F}(W) \\ &= \mathscr{F}_P.\end{aligned}$$

When $P \notin U$, we have

$$\begin{aligned}(j_!\mathscr{F})_P = (j_!\mathscr{F})^0_P &= \operatorname*{Colim}_{P\in W\subset X,\ W \text{ open in } X} (j_!\mathscr{F})^0(W) \\ &= \operatorname*{Colim}_{P\in W\subset U\subset X,\ W \text{ open in } X} (j_!\mathscr{F})^0(W) \\ &= \operatorname*{Colim}_{P\in W\subset U\subset X,\ W \text{ open in } X, W\not\subset U} \mathscr{F}(W) \\ &= \operatorname*{Colim}_{P\in W\subset U\subset X,\ W \text{ open in } X, W\not\subset U} \\ &= 0.\end{aligned}$$

In summary, we have

$$(j_!(\mathscr{F}))_P = \begin{cases} \mathscr{F}_P & \text{when } P \in U \\ 0 & \text{when } P \notin U \end{cases}.$$

Suppose we have another sheaf $\mathscr{G}$ on X such that $\mathscr{G}_P = \mathscr{F}_P$ if $P \in U$ and $\mathscr{G}_P = 0$ if $P \notin U$, then we necessarily have $j_!\mathscr{F} = \mathscr{G}$ by Proposition 1.1. Chap 2 [8]. Furthermore, we notice for any $P \in U$,

$$((j_!\mathscr{F})|_U)_P = \mathscr{F}_P \ \Rightarrow \ (j_!\mathscr{F})|_U = \mathscr{F}.$$

$\square$

2.45.3 (c)

Now let $\mathscr{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0.$$

Proof. We need to define map from $j_!(\mathscr{F}|_U) \to \mathscr{F}$, it suffices to define morphism of (pre-sheaves) from $\mathscr{G} := j_!(\mathscr{F}|_U)^0$ which denote the presheaf defined in part (b). For any $W \subset X$, we define ψ

$$\psi(W) : \mathscr{G}(W) \to \mathscr{F}(W) = \begin{cases} 0 & \text{when } W \not\subset U \\ \mathrm{id}_{\mathscr{F}(W)} & \text{when } W \subset R \end{cases}.$$

Restriction map from any $\mathscr{G}(W)$ where $W \not\subset U$ is 0 given $\mathscr{G}(W) = 0$. Then we can check for $W_1 \subset W_2 \subset X$, the following diagram is commutative

$$\begin{array}{ccc} \mathscr{G}(W_2) & \longrightarrow & \mathscr{F}(W_2) \\ \downarrow & & \downarrow \\ \mathscr{G}(W_1) & \longrightarrow & \mathscr{F}(W_1) \end{array}$$

in three cases respective: i) when $W_1, W_2 \not\subset U$, ii) when $W_2 \not\subset U$ and $W_1 \subset U$, iii) $W_1, W_2 \subset U$.

By universal property of sheafification, given $\mathscr{F}$ is a sheaf, there exists a morphism of sheaves

$$j_!(\mathscr{F}|_U) \to \mathscr{F}.$$

For any $W \subset X$, we define ϕ

$$\begin{aligned} \phi(W) : \mathscr{F}(W) &\to i_*(\mathscr{F}|_Z)(W) = \mathscr{F}|_Z(i^{-1}(W)) = \mathscr{F}(W \cap Z) \\ x &\mapsto x|_{W \cap Z} \end{aligned}$$

by restriction. Clearly it's compatible with the restriction maps hence ϕ is indeed a morphism of sheaves

$$\mathscr{F} \to i_*(\mathscr{F}|_Z).$$

With these two morphisms defined, it suffices to check exactness at each term. For $P \in X = U \amalg Z$, we firstly consider $x \in U$ where the following sequence is exact

$$0 \longrightarrow (j_!(\mathscr{F}|_U))_P = \mathscr{F}_P \longrightarrow \mathscr{F}_P \longrightarrow (i_*(\mathscr{F}|_Z))_P = 0 \longrightarrow 0$$

For $P \in Z$, we have exact sequence again as follows.

$$0 \longrightarrow (j_!(\mathscr{F}|_U))_P = 0 \longrightarrow \mathscr{F}_P \longrightarrow (i_*(\mathscr{F}|_Z))_P = \mathscr{F}_P \longrightarrow 0$$

Therefore we know $0 \to (j_!(\mathscr{F}|_U))_P \to \mathscr{F}_P \to (i_*(\mathscr{F}|_Z))_P \to 0$ for every $P \in X$, which proves

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0$$

is exact. □

2.45.4 Remarks

The sheaf $j_!\mathscr{F}$ is defined as sheafification of the presheaf that we denote as $\mathscr{G}$, i.e. $j_!\mathscr{F} = \mathscr{G}^\sharp$. We'll demonstrate that it's crutial to sheafify. We define the presheaf $\mathscr{G}$ on $U \subset X$ by

$$\mathscr{G}(V) = \begin{cases} \mathscr{F}(V) & \text{for } V \subset U \\ 0 & \text{for } V \not\subseteq U. \end{cases}$$

Let $X = \mathbb{R}$. Now we fix an open subset $W \subset X$ with $\{W_i\}_{i \in I}$ being an open cover for some index set I. We define

$$\mathscr{F}(U) := \{ \text{ continuous function from } U \to \mathbb{R} \ \}$$

- It satisfies Identity Axiom: if we're given an element $s \in \mathscr{G}(W)$ such that $s \mid_{W_i} = 0$, then $s = 0$ as a function.
- Let the open subset $W \subset X$ endowed with an open covering $\{W_1, W_2\}$ such that $W_1 \cap W_2 \not\subset U$. Then we can pick to different non-zero constant function $C_1 in \mathscr{G}(W_1), C_2 \in \mathscr{G}(W_2)$. They agree on the overlap for $C_1|_{W_1 \cap W_2} = 0 = C_2|_{W_1 \cap W_2}$. But there cannot exist a continuous function on...

I didn't find a counterexample according to the hint on Rising Sea

2.45.5 Comments

See this POST, and Tag 03S2.

2.45.6 A Counterexample

Let $X = \{u_1, u_2\}$ be a topological space endowed with discrete topology. The inclusion map is $j : \{u_1\} \to X$. Assume we have a sheaf $\mathscr{F}$ on $\{u_2\}$ such that $\mathscr{F}(\{u_2\}) \neq 0$. Show $j_!^{pre}\mathscr{F}$ is not a sheaf.

Proof. We denote $\mathscr{G} := j_!^{\text{pre}}\mathscr{F}$. In general, the pre-sheaf $\mathscr{G}$ satisfies the Identity Axiom. However, we can pick a non-zero section $s_1 \in \mathscr{G}(\{u_1\})$ and a section $s_2 \in \mathscr{G}(\{u_2\}) = 0$. Notice that

$$s_1|_{\{u_1\} \cap \{u_2\}} = s_1|_\emptyset = 0 = \cdots = s_2|_{\{u_1\} \cap \{u_2\}}.$$

Hence if we assume $\mathscr{G}$ is a sheaf then there must exist a global section $s_3 \in \mathscr{G}(X) = 0$ such that

$$0 = s_3|_{\{u_1\}} = s_1,$$

contradiction. □

2.45.7 References

See Rising Sea [19] Exercise 23.4.F.

2.46 Exercise 1.22.

2.46.1 Hint

See Stacks Project Glueing Sheaves, in which condition (2) is called *glueing data*.

According to the following Tag 00AL, there exists (not necessarily unique) a sheaf $\mathscr{F}$ on X such that ...

It remains to use (1) in the prompt to verify such a sheaf is unique.

See a post HERE.

2.47 Definition: Ringed Space

2.47.1 Local Homomorphism

The map $\mathscr{F}_P^\sharp$ is defined in Rising Sea's Exercise 2.2.I., PUSHFORWARD INDUCES MAPS OF STALKS.

The condition $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$ has equivalent definitions by Lemma 07BJ on Stacks Project.

Here is an example of ring homomorphism between local rings that's not a local homomorphism, see a post HERE.

2.47.2 Isomorphism of locally ringed space

See Lemma 01 HC Tag 01HA. This means that if we wish to prove an isomorphism of locally ringed space, we can avoid checking the induced map of local rings. It suffices to check it's isomorphism of ringed space.

2.48 Proposition 2.2.

2.48.1 (b)

Notice that the map ψ is clearly well-defined. For $a/f^n = b/f^m$ for some integers m, n, the maps is defined as

$$\begin{aligned}\psi : A_f &\to \mathcal{O}(D(f)) \\ a/f^n &\mapsto s : D(f) \to \prod_{\mathfrak{p}\in D(f)} A_{\mathfrak{p}}\end{aligned}$$

where $s(\mathfrak{p}) := a/f^n \in A_{\mathfrak{p}}$. Substituting b/f^m into a/f^n will not change the value of the map, therefore it's independent of the choice of representative.

2.49 Proposition 2.3.

See 7.3.2. Key Proposition [19].

2.50 Example 2.3.2.

2.50.1 References

See this NOTE.

2.51 Example 2.3.5.

For a more general treatment, see Section 01JA of [16].

2.52 Example 2.3.6.

Why it's not affine? See this POST and a detailed POST.

Some arguments heavily rely on *separated*.

2.53 Proj S: Homogeneous Spectrum

2.53.1 Homogeneous

See this POST for equivalent definitions of homogeneous prime ideal.

According to Tag 00JM, we have

To check that a homogeneous ideal $\mathfrak{p}$ *is prime it suffices to check that* $\star$ *if* $ab \in \mathfrak{p}$ *with* a, b *homogeneous then either* $a \in \mathfrak{p}$ *or* $b \in \mathfrak{p}$.

Say A, B are two elements in the graded ring S such that $AB \in \mathfrak{p}$, which is assumed to be homogeneous prime. We assume the assumption of $\star$ and we wish to show $A \in \mathfrak{p}$ or $B \in \mathfrak{p}$. First of all, we can decompose the elements

$$A = \sum_{1 \leq i \leq n} a_i, B = \sum_{1 \leq j \leq m} b_j$$

where $a_i \in S_i, b_j \in S_j$ and n, m are some integers. Suppose the assumption $\star$ holds for arbitrary pair of (a_i, b_j), it will implies their sum $A \in \mathfrak{p}$ or $B \in ....$

2.54 Proposition 2.5.

2.54.1 (a)

2.54.2 (b)

Notice that when defining homogeneous spectrum, we excluded those homenegeous prime ideal that contains S_+. This is because we wish to cover $\operatorname{Proj} S$ with $D_+(f)$. We need to prove the non-trivial inclusion

$$\operatorname{Proj} S \subset \bigcup_{f \in S_+ \cap S^h} D_+(f).$$

Pick an element from LHS, i.e. a homogeneous prime ideal $\mathfrak{p}$ such that $S_+ \not\subset \mathfrak{p}$. Hence there exists some $f \in S_+$ such that $f' \notin \mathfrak{p}$, then we can pick a homogeneous part $f := f' \cap S^h$ such that $f_0 \notin \mathfrak{p}$. This gives us

$$\mathfrak{p} \in D_+(f_0).$$

And this proves the desired inclusion, with the converse inclusion apparently holds, we can conclude

$$\{D_+(f)\}_{f \in S_+ \cap S^h}$$

is an open covering of homogeneous spectrum $\operatorname{Proj} S$.

2.54.3 References

For more details, see [13] Chap 2 Lemma 3.36. on Page 52.

2.55 Definition: scheme over...

2.55.1 Counterexample

Give examples where two S-schemes are isomorphic but not isomorphic as S-schemes.

2.56 Proposition 2.6.

2.56.1 References

See this NOTE Theorem 13.13. for details.

2.57 Exercise 2.1.

One technical detail is regarding localising twice: See [1] Proposition 11.16.

Proof. The homeomorphism of topological spaces between $\operatorname{Spec} A_f$ and $\mathbf{D}(f)$ is immediate. Denote $f : \operatorname{Spec} A_f \to \mathbf{D}(f)$, we need to show the induced morphism between sheaves

$$f^\sharp : \mathcal{O}_{\mathbf{D}(f)} \to f_* \mathcal{O}_{\operatorname{Spec} A_f}$$

is an isomorphism. And the induced map $f^\sharp_{\mathfrak{p}}$ is a local ring homomorphism: ? □

2.57.1 References

See [13] Chap 2 Lemma 3.7. and Proposition 3.9. See Tag 01I3.

See a counterexample in POST regarding Open subschemes of affine schemes are affine?

See a POST "The distinguished open sets are affine subschemes".

2.58 Exercise 2.2.

Assume X admits affine covering $\{V_i = \operatorname{Spec} A_i\}_{i \in I}$ for some index set I and ring A_i. Notice that $V_i \cap U$ is open, hence we can cover it by distinguished open sets. By previous exercise Section 2.57, we know distinguished open subset is an affine scheme. Therefore the locally ringed space $(U, \mathcal{O}_X|_U)$ is a scheme because it's covered by affine schemes.

2.58.1 References

See [13] Chap 2 Lemma 3.9.

2.59 Exercise 2.3.

Reduced Schemes*. A scheme $(X, \mathcal{O}_X)$ is* ***reduced*** *if for every open set $U \subset X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.*

2.59.1 (a)

Proof. $\Leftarrow$: We can express the stalk as a filtered colimit

$$\mathcal{O}_{X,P} = \operatorname{colim}_{P \in U} \mathcal{O}_X(U).$$

?

where each $\mathcal{O}_X(U)$ is assumed to be reduced. And I claim colimit of reduced ring is again reduced : Suppose it has a nilpotent element $a \in \mathcal{O}_{X,P}$, then there exists some $(U, a) \in \mathcal{O}_X(U)$ such that a is the image of (U, a) under colimit map φ which is a ring homomorphism. Then we have $(U, a)^n$ for some integer n will be mapped to 0 in the stalk $\mathcal{O}_{X,P}$. By [1] Corollary 7.5. (c), there exists some $\mathcal{O}_X(V)$ for $V \subset U$ such that the image of $(U, a)^n$ will be mapped to 0 under a transition map ρ_{UV}, hence $\rho_{UV}((U, a))^n = 0 \in \mathcal{O}_X(V)$, which forces $\rho_{UV}((U, a)) = 0$ given ring $\mathcal{O}_X(V)$ is reduced. By commutativity, we have a, being an image of $\rho_{UV}((U, a))$, equals to 0.

$\Rightarrow$: See Lemma 01J1 of Tag 01IZ. Pick a section $f \in \mathcal{O}_X(U)$ such that $f^n = 0$ for some integer n. Consider the ring homomorphism $O_X(U) \to O_{X,P}$, which will map f to a nilpotent in $O_{X,P}$ that we assumed to be reduced. While $P \in U$ is arbitrary, then f will be mapped to 0 under injective map

$$O_X(U) \to \prod_{P \in U} O_{X,P}.$$

This forces $f = 0$ as $\mathcal{O}_X$ is a sheaf. $\square$

2.59.2 Failed Attempt of (b)

The reduced scheme $(X, (\mathcal{O}_X)_{\text{red}})$ is indeed a scheme. For every point $P \in X$, there exists an open subset U such that $(U, \mathcal{O}_X|_U) \simeq (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

To define a morphism from $X_{\text{red}} \to X$, we take identity $\mathrm{id} : \mathrm{sp}(X_{\text{red}}) \to \mathrm{sp}(X)$ as the homeomorphism between the underlying topological spaces. It remains to define a morphism of sheaves from

$$f^\sharp : \mathcal{O}_X \to \mathrm{id}_*(\mathcal{O}_X)_{\text{red}} =: \mathcal{G}$$

such that $f^\sharp_P : \mathcal{O}_{X,P} \to \mathcal{G}_P$ is a local ring homomorphism.

Fix an open subset $U \subset X$, we define $f^\sharp(U)$ as

$$f^\sharp(U) : \mathcal{O}_X(U) \to \mathcal{G}(U) = \mathrm{id}_*(\mathcal{O}_X)_{\text{red}}(U) = \mathcal{O}_{\text{red}}(U)$$

2.59.3 References

See Tag 01IZ. See POST, and POST discussing colimit of reduced ring. See a solution HERE.

On reduced presheaf isn't a sheaf HERE. Discussion on Ex 2.3. (b), POST, POST. For (b), a standard treatment is [13] Chapter 2 2.4 on Page 59.

2.60 Exercise 2.4.

"Morphism into an affine scheme"

Isn't this a direct application of Proposition 2.3. [8]? We'll use Prop 2.3. in the proof. See Section 2.63 for morphism from an affine scheme with global section of a field.

2.61 references

See Theorem I-40 on Page 30 of [5]. See [7] Proposition 3.4. Chap 3 on Page 69. Rising Sea [19] 7.3.4.

2.62 Exercise 2.5.

Corollary of previous Exercise 2.4 Section 2.60. See Corollary 3.6. Chap 3 of [7].

2.63 Exercise 2.7.

The following proof is based on Lecture 53 of Sándor's Notes on Algebraic Geometry [10].

Proof. To specify a morphism between locally ringed spaces, we have to give a continuous map of topological spaces and morhism of sheave of rings. A continuous map is given by choosing a point x of X for the spectrum $\operatorname{Spec} K$ is a single point. Here we denote this map as $f : \operatorname{Spec} K \to X$.

As definition specified, the morphism of sheaves $\mathscr{O}_X \to f_*\mathscr{O}_{\operatorname{Spec} K}$ induces a local ring homomorphism on stalks at $x \in X$. Notice that $0 \in \operatorname{Spec} K$ is the pre-image of x, then

$$\mathscr{O}_{X,x} \to f_*\mathscr{O}_{\operatorname{Spec} K,0} = K$$

is a local ring homomorphism. As K is a field, the pre-image of maximal ideal 0 must be exactly the maximal ideal of $\mathscr{O}_{X,x}$. This morphism cannot be zero morphism, so it must be surjective. The kernel of this morphism is exactly $\Updownarrow_x \triangleleft_{\max} \mathscr{O}_{X,x}$, which implies the isomorphism of

$$\mathscr{O}_{X,x}/\Updownarrow_x = \kappa(x) \simeq k.$$

$\square$

2.63.1 First Attempt

We wish to prove a morphism on sheaves of rings

$$f^\sharp : \mathscr{O}_X \to f_*\mathscr{O}_{\operatorname{Spec} K}$$

will corresponds bijectively to an inclusion map $k(x) \to K$.

Use Tag 008K to simp
the construction of the
morphism?

Give an open subset $U \subset X$, we can define $f^\sharp(U)$ as

$$\mathscr{O}_X(U) \to f_*\mathscr{O}_{\operatorname{Spec} K}(U) =$$

2.63.2 Remarks

For converse direction, see Proposition 3.8. Chap 3 of [7]. Notice that this problem is NOT completely analoguous to Section 2.60, because here the global section of the affine scheme is a field. There is a slightly more general statement regarding when global section is a local ring, see comments of Prop 3.8. Chap 3 of [7].

2.64 Exercise 2.9.

If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

Proof. Let $Z \subset X$ be a non-empty irreducible closed subsets. If we assume the existence of a generic point, then it must be unique. Otherwise we take the closure of two generic points will contradict irreducibility of the Z.

It remains to check the existence. Take an open subset $U := X \setminus (X \setminus Z)^- \subset Z$, and we can find an affine open $\operatorname{Spec} R \subset\subset U$ for affine opens form a basis for the scheme. Here R is a ring. And the generic point $\xi \in \operatorname{Spec} R$ corresponds to the minimal prime of R. The existence of minimal prime implies the existence of generic point ξ. Furthermore, the point ξ will be the generic point for Z because $\operatorname{Spec} R$ is dense in Z given Z is irreducible.

$$\mathrm{Cl}_Z(\xi) = \mathrm{Cl}_Z(\mathrm{Cl}_U(\xi)) = \mathrm{Cl}_Z(U) = Z.$$

$\square$

2.64.1 Solution

See this POST "Every irreducible closed has a generic point". Another way to describe the generic point ξ is to define it as the nilradical of R. Notice that the nilradical here is a prime because "nonempty affine scheme $X = \operatorname{Spec}(A)$ is irreducible if and only if the nilradical of A is a prime ideal" by this NOTE Lemma 3 on Page 2.

For existence of minimal prime, see "A Term..." [1] 3.17 and Exercise 3.16 on Page 21.

See Proposition 3.23 Chap 3 on Page 78 [7].

2.65 Exercise 2.12.

2.65.1 Remarks

Notice that in order to be defined, the diagram of morphism $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ admits a diagram as in Tag 01JA.

See Rising Sea Exercise 4.4.A. Section 2.162, in fact the inverse condition given as (1) and identity are redundent.

2.66 Exercise 2.13.

2.66.1 (a)

$\Leftarrow$: In particular, the space X is quasi-compact. For any ascending chain of open subsets, they form an open covering of X, hence we can reduce to finite many of them. Take the largest one open set U_0, the chain will terminate there.

$\Rightarrow$: Fix an open subset U with an opne covering $\{U_i\}_{i\in I}$ where I is an index set. It must reduce to finite sub-cover otherwise we can form an (strictly) ascending chain of open subsets in U, hence in X, which contradicts the fact that X is Noetherian.

2.66.2 (b)

For space $\mathrm{sp}(X)$ is quasi-compact see Atiyah's Exercise in Chapter 1.

Example of an affine scheme X whose underlying space $\mathrm{sp}(X)$ is not Noetherian:

$$\operatorname{Spec} k[x_1, x_2, x_3, ...]$$

because we have infinitely descending chain of closed subsets, which corresponds to infinitely ascending ideals.

?

2.66.3 (d)

See a standard example HERE.

2.66.4 Verification

See this POST, POST.

For (b), see this POST.

See standard counterexample for part (d) HERE, HERE.

2.67 Exercise 2.16.

2.67.1 (a)

Proof.

$$\begin{tikzcd}
f \in \mathscr{O}_X(X) \arrow[r, mapsto] \arrow[dr, mapsto] & \overline{f} \in \Gamma(U, \mathscr{O}_{X|U}) = B \arrow[d, mapsto] \\
& f_{\mathfrak{p}} = \overline{f}_{\mathfrak{p}} \in \mathfrak{m}_{\mathfrak{p}} \triangleleft_{\text{prime}} \mathscr{O}_{X,\mathfrak{p}}
\end{tikzcd}$$

$$\begin{aligned}
U \cap X_f &= \{x \in U \mid f_x \notin \mathfrak{m}_x\} \\
&= \{x \in U \mid f_x \in \mathscr{O}_{X,x}^{\times}\} \\
&= \{\mathfrak{p} \in U \mid f_{\mathfrak{p}} \in \mathscr{O}_{X,\mathfrak{p}}^{\times}\} \\
&= \{\mathfrak{p} \in U \mid f_{\mathfrak{p}} \in (B_{\mathfrak{p}})^{\times}\} \\
&= \{\mathfrak{p} \in U \mid f_{\mathfrak{p}} \in \mathfrak{m}_{\mathfrak{p}} := \mathfrak{p}B_{\mathfrak{p}}\}^C \\
&= \{\mathfrak{p} \in \operatorname{Spec} B = U \mid (\overline{f}) \subset \mathfrak{p}\}^C \quad \text{by commutativity of the diagram} \\
&= \mathbf{D}_U(\overline{f}) = \mathbf{D}(\overline{f}).
\end{aligned}$$

Therefore X_f is an open subset because all affine open form an open basis and intersection with each is open inside each basis element. □

In fact, the notion of X_f generalises this equality $(\operatorname{Spec} B)_g = \operatorname{Spec} B_g$ in which left hand side is regarded in the definition of this exercise and right hand side is regarded in the usual sense.

2.67.2 (b)

Proof. Restriction map gives us the following commutative diagram for each $1 \le i \le n$.

$$\begin{tikzcd}
& f, a \in \Gamma(X, \mathcal{O}_X) = A \arrow[dl] \arrow[dr] & \\
\Gamma(U_i, \mathcal{O}_X|_{U_i}) = B_i \arrow[dr] & & \Gamma(X_f, \mathcal{O}_X) \ni 0 \arrow[dl] \\
& \Gamma(U_i \cap X_f, \mathcal{O}_X|_{U_i}) \ni 0 &
\end{tikzcd}$$

We can compute by part (a) that for $1 \le i \le n$,

$$\begin{aligned}
\Gamma(U_i \cap X_f, \mathcal{O}_X|_{U_i}) &= \Gamma(D_{U_i}(\overline{f}), \mathcal{O}_X|_{U_i}) \\
&= \Gamma(D_{U_i}(\operatorname{Res}_i(f)), \mathcal{O}_X|_{U_i}) \\
&= \{1, \operatorname{Res}_i(f), \operatorname{Res}_i(f)^2, ...\}^{-1} B_i.
\end{aligned}$$

While a will restrict to 0 in X_f, hence it will be mapped to $0 \in \{1, \mathrm{Res}_i(f), \mathrm{Res}_i(f)^2, ...\}^{-1}B_i$.

Given X is quasi-compact, let $U_1, ..., U_n$ be affine opens of X where n is some integer with global sections $B_1, ..., B_n$. We have the following diagram

$$\begin{tikzcd}
& f, a \in \Gamma(X, \mathcal{O}_X) = A \arrow[dl, "\mathrm{Res}_1"'] \arrow[d, "..."] \arrow[dr, "\mathrm{Res}_n"] & \\
\Gamma(U_1, \mathcal{O}_X|_{U_1}) = B_1 \arrow[d] & \cdots \arrow[d] & \Gamma(U_n, \mathcal{O}_X|_{U_n}) = B_n \arrow[d] \\
B_{1\mathrm{Res}_1(f)} = \{1, \mathrm{Res}_1(f), \mathrm{Res}_1(f)^2, ...\}^{-1}B_1 & \cdots & B_{n\mathrm{Res}_n(f)}
\end{tikzcd}$$

We claim lower map from $B_i \to \{1, \mathrm{Res}_i(f), \mathrm{Res}_i(f)^2, ...\}^{-1}B_i$ is localisation. One possible interpretation is to recall construction of spectrum of a ring on Page 70 of [8]. Another interpretation is that, by (a) and Section 2.57 we know the open subsets $U_i \cap X_f = D_{U_i}(\overline{f})$ is an affine scheme. Then apply Proposition 2.3 Chap 2 of Hartshrone [8] proves the localisation map is indeed the map between them.

Now, we can apply definition of localisation:

$$\begin{aligned}
\mathrm{Res}_i(a)/1 = 0 \in \{1, \mathrm{Res}_i(f), \mathrm{Res}_i(f)^2, ...\}^{-1}B_i \;\Leftrightarrow \\
\exists n_i \in \mathbb{Z},\; \mathrm{Res}_i(f)^n \, \mathrm{Res}_i(a) = \mathrm{Res}_i(f^n a) = 0 \in B_i.
\end{aligned}$$

Now we pick the largest index $N = \max n_{i1 \le i \le n}$. Notice that $f^N a$ will be mapped to 0 in each $B_i = \Gamma(U_i, \mathcal{O}_X|_{U_i})$, while $\{U_i\}_{1 \le i \le n}$ is an open covering of the whole space, by sheaf axiom we know $f^N a = 0 \in \Gamma(X, \mathcal{O}_X)$.

□

2.67.3 (c)

Statement of the prompt is misleading. I combined part (c) and (d).

Proof.

$$\begin{tikzcd}
f \in A = \Gamma(X, \mathcal{O}_X) \arrow[rr, "\mathrm{Res}"] \arrow[dr] & & \Gamma(X_f, \mathcal{O}_{X_f}) \\
& A_f \arrow[ur, dashed] &
\end{tikzcd}$$

Firstly, we try to construct a map from A_f to $\Gamma(X_f, \mathcal{O}_{X_f})$ by universal property of localisation. It suffices to check f will be mapped to a unit under restriction map Res.

$$\begin{tikzcd}
f \in \mathcal{O}_X(U_x) \ni g \arrow[d, shift right] \arrow[d, mapsto, shift left] \\
f_x \in \mathcal{O}_{X,x} \ni 1/f_x
\end{tikzcd}$$

Now let's consider for $x \in X_f = \{x \in X \mid f(x) \neq 0 \in \kappa(x)\}$. Section $f \in A$ admits a representative in $\mathcal{O}_{X,x}$, i.e. there exist $U'_x \subset X$ contains x such that $f \in \mathcal{O}_X(U'_x)$. Similarly, for $1/f_x$, it will corresponds a section $g \in \mathcal{O}_X(U''_x)$ on some open neighborhood U''_x of x such that $g_x = 1/f_x$ given f_x is a unit in the stalk $\mathcal{O}_{X,x}$. Denote $U_x = U'_x \cap U''_x$. Locally on $\mathcal{O}(U_x)$, we have

$$f|_{U_x} g|_{U_x} = 1$$

because in ring homomorphism $\mathcal{O}_X(U_x) \to \mathcal{O}_{X,x}$, the preimage of 1 must be 1 and $f_x g_x = 1$ by our construction.

For different $x \in X_f$, we have an open cover $\{U_x\}_{x \in X_f}$. Since $f \in \Gamma(X, \mathcal{O}_X)$, so it satisfy all sheaf axiom.

$$g|_{U_x}|_{U_x \cap U_y} = (1/f|_{U_x})|_{U_x \cap U_y} = (1/f|_{U_y})|_{U_x \cap U_y} = g|_{U_y}|_{U_x \cap U_y}$$

implies there exist $g \in \Gamma(X_f, \mathcal{O}_{X_f})$ such that $\operatorname{Res} fg = 1 \in \Gamma(X_f, \mathcal{O}_{X_f})$ by sheaf axiom(used twice, one for existence of g, one for identifying $\operatorname{Res} fg$). Hence we have the map by universal property of localisation.

It remains to check surjectivity of this map.

□

2.67.4 Hint

Probably the most elegant and official proof is given as Proposition 3.12 in Chap 2 of [13]. It didn't involve the check according to the hint of part (a) and construct the map in part (c) immediately. See this POST. See this Webpage which leads to this POST. See POST for Hartshorne ex II 2.16 part b. See POST for details.

2.68 Exercise 2.17.

2.68.1 (a)

On Page 73, Definition of ringed space, we have an equivalent definition of a morphism $(f, f^\sharp)$ being an isomorphism. One proof I didn't write down(I'm afraid it will evolve many glue arguments) was to prove the ring isomorphism $f^\sharp(V) : \mathcal{O}_Y(V) \to f_*\mathcal{O}_X(V)$. A more convenient approach was to apply Prop 1.1. directly given that we observe the isomorphism on the level of stalks.

Proof. For topological space, the inverse is clear to define. It remains to check

$$f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$$

is an isomorphism.

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "f^\sharp(V)"] \arrow[d] & \mathcal{O}_X(f^{-1}(V)) = f_*\mathcal{O}_X \arrow[d] \\
\mathcal{O}_Y(V\cap U_i) \arrow[r, "\simeq"'] \arrow[d] & \mathcal{O}_X(f^{-1}(V\cap U_i)) \arrow[d] \\
(\mathcal{O}_Y)_P \arrow[r, "\simeq"'] & (f_*\mathcal{O}_X)_P = \mathcal{O}_{X,f^{-1}(P)}
\end{tikzcd}$$

Here V denotes an open subset of Y. We'll apply Proposition 1.1. Chapter 2 [8]. For an arbitrary $P \in V$, we can find some $U_i \cap V$ contains P for $\{U_i\}$ covers Y.

Notice that we have, on one hand $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$, and on the other hand $\mathcal{O}_{U_i} \to (f|_{f^{-1}(U_i)})_*\mathcal{O}_{f^{-1}(U_i)}$. They'll induced the same stalks as we observe that for $P \in U_i$,

$$(\mathcal{O}_Y)_P = (\mathcal{O}_{U_i})_P,$$
$$(f_*\mathcal{O}_X)_P = ((f|_{f^{-1}(U_i)})_*\mathcal{O}_{f^{-1}(U_i)})_P.$$

Given $f^{-1}(U_i) \to U_i$ is an isomorphism, therefore

$$\mathcal{O}_{U_i} \to (f|_{f^{-1}(U_i)})_*\mathcal{O}_{f^{-1}(U_i)}$$

is an isomorphism of sheaves. Therefore we have

$$(\mathcal{O}_Y)_P \simeq (f_*\mathcal{O}_X)_P$$

for $P \in U_i$. While $\{U_i\}$ is an open covering, then the statement could be generalised for any $P \in Y$. This proves

$$f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$$

is an isomorphism of sheaves by Proposition 1.1. Chap 2 [8]. □

2.68.2 (b)

$\Rightarrow$: When X is affine, we can take finitely many such f_i (which generate the global section $\Gamma(X, \mathcal{O}_X)$) for affine scheme is quasi-compact. And we know distinguished open is affine.

$\Leftarrow$: Assume that each X_{f_i} is affine. By Chap 2 Ex 2.16. of Hartshorne [8] Section 2.67, $\Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}}) = A_{f_i}$ for some ring A_i where $1 \le i \le r$.

We need to show that $X \to \operatorname{Spec} A$ is an isomorphism. Given the localisation map $A \to A_{f_i} = \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}})$, Ex 2.4 induces morphism $h_i : X_{f_i} \to \operatorname{Spec} A$ for each $1 \le i \le r$. Gluei morphisms h_i. Since $\{\mathbf{D}(f_i)\}_{1\le i\le r}$ is an open covering of ?? Spec A given $f_1, ..., f_r$ generate the unit ideal. By part (a), it remains to check...

2.69 Comments

See this POST. See this POST regarding part (a). In fact, after reading this post I realised the topological part of the proof could take some words.

?

2.70 Exercise 2.18.

2.70.1 (a)

Notice that $\sqrt{\langle f \rangle} = \sqrt{\langle f^n \rangle} = 0 \subset \mathfrak{p}$ for any prime ideal $\mathfrak{p} \triangleleft A$. For the dirction "empty $\Rightarrow$ nilpotent"...

need to finish

2.70.2 (b)

Assume $\varphi : A \to B$ is injective. In order to check $f^\sharp$ is injective, it suffices to check on $\mathbf{D}(g) \subset X = \operatorname{Spec} A$, the map $f^\sharp(\mathbf{D}(g)) : \mathcal{O}_X(\mathbf{D}(g)) \to f_*\mathcal{O}_Y(\mathbf{D}(g))$ is injective.

$$\begin{aligned} & f^\sharp(\mathbf{D}(g)) : \mathcal{O}_X(\mathbf{D}(g)) \to f_*\mathcal{O}_Y(\mathbf{D}(g)) = \mathcal{O}_Y(f^{-1}(\mathbf{D}(g))) = \mathcal{O}_Y(\mathbf{D}(\varphi(g))) \\ \Rightarrow \ & f^\sharp(\mathbf{D}(g)) : A_g \to B_{\varphi(g)} \end{aligned}$$

Is this map the localisation induced by $\varphi : A \to B$? If so, then we can conclude injectivity of $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$.

$$\begin{tikzcd} g \in A = \mathcal{O}_X(X) \arrow[r, "\varphi = f^\sharp(X)"] \arrow[d] & B = f_*\mathcal{O}_Y(X) \ni \varphi(g) \arrow[d] \\ A_g \arrow[r, bend left, "f^\sharp(\mathbf{D}(g))"] \arrow[r, dashed, bend right, "\exists!"'] & B_{\varphi(g)} \end{tikzcd}$$

Here we wish to identify $f^\sharp$ with the localization. Notice that we can check by universal property of localisation that g will be mapped to a unit in $B_{\varphi(g)}$, therefore there exists a unique map such that the above diagram commutes. While $f^\sharp$ being a natural transformation we know $f^\sharp(\mathbf{D}(g))$ will make this diagram commute, then two maps is the same. And we can conclude the injectivity of $f^\sharp$.

What we've done is to check the injectivity of the morphism of sheaves on base, which will induced an injective morphism of sheaf by Exercise 2.7. Chap 2 on Page 40 [13].

Conversely, if we assume $f^\sharp$ is injective. We can localise at every point of X which give rise to an injective morphism by Prop 3.9. on Page 40 of Atiyah [2]. Or we can just take the global section, which recovers to ring map $\varphi : A \to B$ more quickly.

2.70.3 Solution

For example, see Tag 00DY. See a post HERE, HERE, and HERE.

2.71 Exercise 2.19.

Firstly prove (ii) $\Leftrightarrow$ (iii), then prove (iii) $\Rightarrow$ (i).
For (i) $\Rightarrow$ (iii),

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \Gamma(U, \mathcal{O}|_{\dots}) \arrow[d] & \\
\Gamma(V, \mathcal{O}|_{\dots}) \arrow[r] \arrow[drr] & 1 = \Gamma(\emptyset, \mathcal{O}|\dots) & \\
& & ?
\end{tikzcd}$$

?

2.71.1 Hint

Because Spec is a contravariant functor.

2.72 Example 3.0.1.

See Page 2 of this NOTE.

2.73 Caution 3.1.1.

It's finitely many of Noetherian topological spaces (in the affine case), which is again Noetherian. See POST.

2.74 Proposition 3.2.

2.75 Theorem 3.3.

2.75.1 Step 3: Glueing Morphisms

For a general statement for glueing morphisms between ringed spaces, see Rising Sea 7.2.A. [19].

Proof. To construct f, we need to specify the topological map and the corresponding sheaf morphism. For topological map, it's clearly well-defined and it satisfies our requirements. It remains to:

1. construct the morphism $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$.
2. check morphism $f = (f, f^\flat)$ is unique.
3. check $f|_{U_i} = f_i$.

For each $f_i : U_i \to Y$, we have $f_i^\flat : \mathcal{O}_Y \to f_{i,*}\mathcal{O}_{U_i}$ and $f_i^\sharp : f_i^{-1}\mathcal{O}_Y \to \mathcal{O}_{U_i}$.

For an open subset $V \subset Y$, we define

$$\begin{aligned} f^\sharp(V) \ : \ \mathcal{O}_Y(V) &\to \mathcal{O}_X(f^{-1}(V)) \\ \varphi &\mapsto s \end{aligned}$$

finish this...

where $s|_{U_i \cap f^{-1}(V)} = f_i^\sharp(U_i)\varphi$.

Wouldn't it be easier to construct the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$? To construct $f^\flat$, it's equivalent to construct $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. In prompty, we're given $f_i : U_i \to Y$ with $f_i^\sharp : f_i^{-1}\mathcal{O}_Y \to \mathcal{O}_{U_i}$. Hence we can define $f^\sharp$ on all open subsets $U \subset X$ as

$$\begin{aligned} f^\sharp(U) \ : \ f^{-1}\mathcal{O}_Y(U) &\to \mathcal{O}_X(U) \\ s &\mapsto S \end{aligned}$$

where $S|_{U_i} = f_i^\sharp(U_i) : (\mathcal{O}_Y(U_i) \to \mathcal{O}_{U_i}(U_i))(s)$. Recall that we're given $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, in particularly we know

$$f_i^\sharp : f_i^{-1}\mathcal{O}_Y \to \mathcal{O}_{U_i \cap U_j} \ = \ f_j^\sharp : f_j^{-1}\mathcal{O}_Y \to \mathcal{O}_{U_i \cap U_j}.$$

Therefore compatibility is satisfied:

$$\begin{aligned} S|_{U_i}|_{U_i \cap U_j} &= \ldots \\ &= \ldots \\ &= S|_{U_j}|_{U_i \cap U_j} \end{aligned}$$

And by sheaf axiom of $\mathcal{O}_X$ there exists a unique S as above.

???

`check the later two conditions after translating the requirements correctly by adjoint?`

□

2.75.2 Solutions

See POST, POST.

2.76 Definition: Base Extension

If $S'' \to S' \to S$ are two morphisms, then $(X \times_S S') \times_{S'} S'' \cong X \times_S S''$.

Proof. By universal property of $X \times_S S''$ we have a unique map $(X \times_S S') \times_{S'} S'' \to X \times_S S''$. Furthermore, universal property of $X \times_S S'$ give rise to $X \times_S S'' \to X \times_S S'$. And this enable us to use universal property of $(X \times_S S') \times_{S'} S''$ to produce a unique map backwards, which proves the isomorphism. □

2.77 Definition: Stable under Base Extension

2.78 Example 3.3.1.

Recall Example 3.0.1 Section 2.72, we know $\operatorname{Spec} k[x,y,t]/(ty-x^2)$ is an integral scheme given the global section ring is an integral domain. For $k[x,y,t]/(ty-x^2)$ is an integral domain, see this POST.

2.79 Exercise 3.1.

We assume Prop 5.3.1. of Rising Sea [19]. We know for a specific choice of open affine covers of Y, each open affine has $\star$ property. First condition of affine communication lemma enable us to conclude all small distinguished open subsets

Proof. One direction is clear.

For another direction, assume the morphism f is locally of finite type and we wish to prove that for every open affine $V = \operatorname{Spec} B \subset Y$, preimage $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$ where A_j is f.g. B-algebra.

Given f is locally of finite type. This means that scheme Y admits an open covering of open affine subsets $\{V_i\}_{i\in I}$(here I is an index set) such that:

- for any i, preimage $f^{-1}(V_i)$ can be covered by $\{U_{ij} = \operatorname{Spec} A_{ij}\}_{j\in J_i}$ where A_{ij} is f.g. B_i-algebra.

We'll check two conditions for Affine Communication Lemma. Assume $\operatorname{Spec} C \to Y$ admits such property, i.e. the preimage $f^{-1}(\operatorname{Spec} C)$ can be covered by $\{U_j = \operatorname{Spec} D_j\}_{j\in J}$ where D_j is f.g. C-algebra. Fix $c \in C$, we have

$$\begin{aligned} f^{-1}(\operatorname{Spec} C_c) &= f^{-1}(\operatorname{Spec} C_c \cap \operatorname{Spec} C) \\ &= \bigcup_{j\in J} \left(f^{-1}(\operatorname{Spec} C_c) \cap U_j\right). \end{aligned}$$

Now we further assume $f^{-1}(\operatorname{Spec} C_c)$ could be covered by affine opens $\{V_l = \operatorname{Spec} E_l\}_{l\in L}$. According to 5.3.1. Proposition of Rising Sea [19], we can cover each $\operatorname{Spec} E_l \cap \operatorname{Spec} D_j$ by distinguished open subsets on both sides, i.e.

$$\operatorname{Spec} E_l \cap \operatorname{Spec} D_j = \bigcup_{k\in K_{lj}} \operatorname{Spec}\{1,?,?^2...\}^{-1} D_j.$$

Therefore we can cover the preimage

$$\begin{aligned} f^{-1}(\operatorname{Spec} C) &= \bigcup_{j\in J} \bigcup_{l\in L} \left(f^{-1}(\operatorname{Spec} E_l) \cap U_j\right) \\ &= \bigcup_{j\in J} \bigcup_{l\in L} \bigcup_{k\in K_{lj}} \operatorname{Spec}\{1,h,h^2...\}^{-1} D_j \end{aligned}$$

where $f \in D_j$. Recall that D_j is f.g. C-algebra

□

2.79.1 Solutions

See Tag 01T0 for a slightly different treatment. Or Proposition and Definition 10.5. [7].

2.80 Exercise 3.2.

Proof. We need to check two conditions of Affine Communication Lemma. For the second condition, use the fact that 3.6.H. (a) [19].

□

2.80.1 Comments

See 8.3.1. [19] for another equivalent definition that's "geometrically correct". See Tag 01K2.

2.81 Exercise 3.6.

Proof. Given $(X, \mathcal{O})$ an integral scheme, then there exists a unique generic point ξ by Exercise 2.9 in Section 2.64. Now we compute the local ring at ξ. Pick an affine open $\operatorname{Spec} A \subset X$ which contains ξ, then we know the closure of ξ in $\operatorname{Spec} A$ must be whole $\operatorname{Spec} A$. While X is integral, section A is an integral domain, hence ξ corresponds to the zero ideal in A.

$$\begin{aligned}\mathcal{O}_\xi &= \operatorname*{Colim}_{U \ni \xi,\ U \text{ open in } X} \mathcal{O}(U) \\ &= \operatorname*{Colim}_{U \ni \xi,\ U \text{ open in } \operatorname{Spec} A} \mathcal{O}|_{\operatorname{Spec} A}(U) \\ &= A_0 = \operatorname{Frac} A,\end{aligned}$$

for which we define as the *function field* $K(X)$.

□

2.81.1 Hint

See this POST.

2.82 Exercise 3.7.

A morphism $f : X \to Y$, with Y irreducible, is ***generically finite*** *if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y.*

Proof. As a dominant morphism, f will map generic points of X to Y. So we'll use $\xi := f^{-1}(\eta)$ to denote a generic point of X. Let $\{V_i = \operatorname{Spec} B_i \subset Y\}_{i\in I}$ be the affine cover of Y such that f is of finite type. Hence $f^{-1}(V_i)$ could be covered by finitely many $\{U_{ij} = \operatorname{Spec} A_{ij}\}_{j\in J_i}$ such that A_{ij} is f.g. B_i-algebra. According to Exercise 3.6. Chap 2 [8], we know the function fields are

$$K(X) = (O_X|_{U_{ij}})_\xi = \operatorname{Frac} A_{ij},\ K(Y) = (O_Y|_{V_i})_\eta = \operatorname{Frac} B_i.$$

We are given A_{ij} is f.g. B_i-algebra.

□

2.83 Exercise 3.11.

2.83.1 (a)

Closed immersions are stable under base extension:...

Okay, so if we start by assuming X, X', Y are all affine schemes, how do we glue back? Following the order of construction of fibre product, we first generalise that either Y or X' is a scheme, then assume X is a scheme.

But in general case, how do we glue back, or even compute the image of the $Y \times_X X'$?

Proof. We assume X, X', Y are all affine schemes with $X = \operatorname{Spec} A$, $X' = \operatorname{Spec} B$, and $Y = \operatorname{Spec} C$. Then by construction of Theorem 3.3, we know

$$Y \times_X X' = \operatorname{Spec}(B \otimes_A C).$$

$$\begin{array}{ccc} Y \times_X X' = \operatorname{Spec} C \otimes_A B & \xrightarrow{p_1} & X' = \operatorname{Spec} B \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle g} \\ Y = \operatorname{Spec} C & \xrightarrow[f]{} & X = \operatorname{Spec} A \end{array}$$

□

2.83.2 Remarks

See this POST.

2.84 Proposition 4.1.

2.85 Definition: Morphism of sheaves of modules

For a non-example... Notice that there are morphism of two modules that's not module morphism (i.e. does not respect action of the ring, but it's indeed a morphism between two Abelian groups). See HERE, in which they considered conjugation between to $\mathbb{C}$-module(over itself)

$$\bar{\cdot} : \mathbb{C} \to \mathbb{C}$$

and noticed that $\overline{i \cdot 1} \neq i \cdot \overline{1}$.

2.86 Definition: Tensor Product of Sheaves

See this POST.

2.87 Definition: Quasi-coherent Sheaves

See 6.1.2. Theorem of Rising Sea.

2.88 Example 5.2.4.

Consider a closed immersion $\iota : \operatorname{Spec} \mathbb{F}_p = Y \to \operatorname{Spec} \mathbb{Z} = X$. I'm confused by the notation $\mathscr{O}_{X|Y}$. But in fact we have

$$\mathscr{O}_{X|Y} = \iota^{-1}(\mathscr{O}_{\operatorname{Spec} \mathbb{Z}}) = (\mathscr{O}_{\operatorname{Spec} \mathbb{Z}})_{\mathfrak{p}} = \mathbb{Z}_{(p)}.$$

Then $\mathscr{O}_{X|Y}(Y)$ is not $\mathscr{O}_Y$-module. Because $\mathscr{O}_Y(Y) = \mathbb{F}_p$, while $\mathscr{O}_{X|Y}(Y) = ... = \mathbb{Z}_{(p)}$. However, $\mathbb{Z}_{(p)}$ doesn't admit $\mathbb{F}_p$-module structure. Because for any non-zero element $x \in \mathbb{Z}_{(p)}$, we have

$$0 = px = x + ... + x \in \mathbb{Z}_{(p)},$$

which is impossible.

2.89 Definition Universal

For details when defining universal δ functors, see Def F.201 (2) on Page 806 of Algebraic Geometry II: Cohomology of Schemes.

2.90 Proposition 2.1.A.

See Theorem 3.38. of Section 3.2. on Page 123 of [15].

2.91 Lemma 2.4.

2.91.1 Comments

See Tag 09SV and Tag 00A7.

Part II

The Rising Sea

2.92 1.3.F. EXERCISE.

A post discussing this problem is HERE.

2.93 1.3.N

A crutial step is to define the map such that the diagram commute. In order to prove it satisfies the universal property. We're given A be arbitrary with map $g : A \to X$ and $f : A \to Y$.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\exists!\varphi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_Z Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\beta"] \\
& X \arrow[r, "\alpha"'] & Z
\end{tikzcd}$$

We can to define

$$\begin{aligned} \varphi : A \to X \times_Z Y \quad \text{by} \\ a \mapsto (g(a), f(a)). \end{aligned}$$

And we can verify this definition will make the diagram commute, and is unique.

2.94 1.3.O

It's indeed intersection. A post HERE.

A post HERE.

2.95 1.3.P.

Say we have $X \times Y$ and $X \times_Z Y$. By universal property of product and fibered product we can produce two unique map goes in between. Their composition must be identity, hence they're isomorphic. Notice it's important for Z being a final object.

$$\begin{tikzcd}
X \times Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\beta"] \\
X \arrow[r, "\alpha"'] & Z
\end{tikzcd}$$

When we trying to make a map from $X \times Y$ to $X \times_Z Y$, we have to make sure two maps

$$\beta \circ \pi_1 = \alpha \circ \pi_1$$

on $X \times Y$. While Z is the final object, hence they must agree.

There's a cleaner way to state it HERE. Crutial part is applying final property of object Z.

2.96 1.3.Q.

$$\begin{tikzcd}
A \arrow[drr, bend left, "g_1"] \arrow[dr, dashed, "\exists!\varphi"] \arrow[ddr, dashed, "\exists!\psi"'] \arrow[dddr, bend right, "g_2"'] & & \\
& U \arrow[r, "f_1"] \arrow[d, "f_2"'] & V \arrow[d, "f_3"] \\
& W \arrow[r, "f_4"'] \arrow[d, "f_5"'] & X \arrow[d, "f_6"] \\
& Y \arrow[r, "f_7"'] & Z
\end{tikzcd}$$

Label the maps as indicated. To prove the universal property with respect to the "outside rectangle", we're given

$$f_6 f_3 g_1 = f_7 g_2$$

agree on A. While W is fibered product, apply universal property of fibered product with resepct to W we immediately get a unique map

$$\psi : A \to W$$

that makes the diagram involving A, W, X, Y, Z commute. In particularly, we know $f_4\psi = f_3 g_1$. Furthermore, recall that U is the fibered product. We're given the condition that $f_4\psi = f_3 g_1$, by universal property of U we know there exists a unique map

$$\varphi : A \to U$$

making the diagram involving A, U, V, W, X commute. And we claim that the diagram involving A, U, V, Y, Z commute. This is because

$$g_2 = f_5\psi = f_5 f_2 \varphi = (f_5 f_2)\varphi \quad \text{and} \quad g_1 = f_1\varphi.$$

And this proves that U is the fibered product for the diagram involving A, U, V, Y, Z.

A post is HERE.

2.97 1.3.R

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[drr, bend left, "\pi_1"] \arrow[ddr, bend right, "\pi_4"'] & & & \\
& X_1 \times_Z X_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_3"'] & X_2 \arrow[d, "f_3"] & \\
& X_1 \arrow[r, "f_2"'] & Y \arrow[dr, "f_1"] & \\
& & & Z
\end{tikzcd}$$

By the universal property of $X_1 \times_Z X_2$, we know there exists a unique map

$$\varphi : X_1 \times_Y X_2 \to X_1 \times_Z X_2$$

"Natural morphism", a convention discussed HERE.

2.98 Course Notes from Cornell

See HERE.

2.99 1.3.S. Magic Diagram

Didn't finish. Need to See HERE, HERE!!!

2.100 1.3.Y. (a)

YONEDA'S LEMMA Given what we have, define $g : A \to A'$ as

$$g := i_A(\mathrm{id}_A).$$

This is correct, see a post HERE.

2.101 1.4.C.

(a) See "A Term of Commutative Algebra", Example 7.3 on Page 52.

2.102 1.6.B.

Write out everything by definition, and we can finish the proof immediately by applying rank-nullity theorem for linear transformation...

2.103 2.2.6. Definition: Sheaf.

Comments on $\mathscr{F}(\emptyset)$. In category **Set**, the empty set is initial object and one element set is terminal. See Wiki's examples HERE.

2.103.1 Example

2.104 2.2.B. Presheaves that are not SHEAVES.

2.104.1 (a)

For (a): see Wiki's counterexample HERE, which gave an explanation for presheaves on $\mathbb{R}$ instead of $\mathbb{C}$. See a post HERE.

2.104.2 (b)

See a post HERE, HERE, and HERE.

Proof. This problem is based on some knowledge from complex analysis. Here are some facts:

- When does a complex function have a square root?
- Theorem 6.2 on Page 100 of [17]. This means $f(x) = x$ as a function on $\mathbb{C}$ does not admit a square root for it will vanish.

However, we can cover $\mathbb{C}$ by two slit regions $U_1 = \mathbb{C} - (-\infty, 0], U_2 = \mathbb{C} - [0, \infty)$. And on each U_i, $f\mid_{U_i}$ admits a square root and satisfy gluability axiom.

The solution given in HERE is saying on annulus, $f(x) = x$ cannot have a square root. And then we can follow argument of HERE.

$\square$

2.105 2.2.7.

See Daping's Notes Definition 2.5 on Page 4 HERE; also a post HERE; also a post HERE.

See a post HERE and HERE regarding gluing sheaves.

2.106 2.2.C.

?

Since $\cup U_i$ is colimit for $\{U_i\}$, then $\mathscr{F}(\cup U_i)$ will be limit because of the contravariance.

See an expository post HERE.

See another post, with more detailed explanations HERE. "In fancy language, it's stack"...

2.107 2.2.D.

(b) Motivating example for definition of sheaf.

2.108 2.2.E.

See comments for Exercise 1.1. of Chapter 2 [8].

2.109 2.2.F.

Almost by definition.

2.110 2.2.G.

2.110.1 (a)

2.111 2.2.9.

See a post HERE.

2.112 2.2.10.

2.112.1 References

See this regarding constant presheaf. It's different from a post HERE, and Wiki's page on Constant pre-Sheaf.

Why???

2.112.2

Clearly it's a contravariant functor

$$\mathscr{F} := \underline{S}_{\text{pre}} : \mathbf{Top}(X) \to \mathbf{Set}$$

Let $X = \{a, b\}$ with discrete topology. Pick two sections

$$s_1 \in \mathscr{F}(\{a\}) = S, \ s_2 \in \mathscr{F}(\{b\}) = S$$

such that $s_1 \neq s_2$ given S has at least two distinct elements. Clearly we have

$$s_1 \mid_{\{a\}\cap\{b\}} = s_1 \mid_{\emptyset} = e = \cdots = s_2 \mid_{\{a\}\cap\{b\}} .$$

If it's a sheaf then there exists a global section $s \in \mathscr{F}(\{a, b\}) = S$ such that

$$s_1 = s \mid_{\{a\}} = s \mid_{\{b\}} = s_2,$$

contradiction. It follows that constant presheaf defined this way is not necessarily a sheaf.

2.113 2.2.E.

We have to deceptively identical pre-sheaves $\mathscr{F}_1$ defined as locally closed, and $\mathscr{F}_2$ defined by giving S discrete topology. . .

We wish to prove they, as pre-sheaves, are isomorphic. Equivalent, we need to exhibit a natural transformation that admits an inverse. And it suffices to prove by element inclusions:

- Let $f : U \to S$ be a map that's locally constant. Now we take $g(u) = f(u)$ as a map $g : U \to S$ with S endowed with a discrete topology. We claim that g is continuous. It suffices to check for each $s \in S$, the fiber $g^{-1}(s)$ is open. For any point $a \in g^{-1}(s) = f^{-1}(s)$, there exists an open neighborhood $V_a \subset f^{-1}(s)$ such that
$$f(V_a) = \{s\}$$
given f is locally constant. While $V_a \subset g^{-1}(s)$, therefore we know $g^{-1}(s)$ is open and g is continuous.

- Conversely, we assume $g : U \to S$ with S given a discrete topology is continuous. We claim $f = g$ is locally constant. For any point $p \in U$, there is an open neighborhood
$$g^{-1}(f(p)) \ni p$$
such that f is constant because $f(g^{-1}(f(p))) = \{p\}$.

Now we try to check constant sheaf $\mathscr{F}$ is indeed a sheaf. We're going to prove identity axiom and gluability axiom using the "better description", which is much easier to check:

- If we have two functions, which equal whenever we restric to any open subset from an open cover, then they must be equal. For functions are precisely defined this way.

- Define the global section for any choice, and it's going to be well-defined for they're compatible.

Therefore $\mathscr{F} = \underline{S}$ is indeed a sheaf.

2.114 2.2.F.

Same argument as 2.107.

2.115 2.2.G.

2.115.1 (a)

It's clearly a pre-sheaf.

Fix an open subset $U \subset X$ with an open cover $\{U_i\}_{i\in I}$ for some index set I. Denote the presheaf as $\mathscr{F}$.

Pick two continuous maps $s_1, s_2 : Y \to X$ that satisfying the requirements, i.e. $s_1, s_2 \in \mathscr{F}(U)$.

Both functions will agree on U since
$$\mathrm{Res}_{U,U_i}\ s_1 = \mathrm{Res}_{U,U_i}\ s_1$$

for arbitrary U_i, whose union is U. So we must have $s_1 = s_2$.

Again with this open cover $\{U_i\}_{i\in I}$ and $a_i \in \mathscr{F}(U_i)$ for $i \in I$. Equivalently, we know $a_i : U_i \to Y$ is a continuous map satisfying $\mu \circ a_i = \mathrm{Id}\,|_{U_i}$. Now let's define a map

$$\begin{aligned} f : U &\to Y \\ u &\mapsto a_i(u) \quad \text{when} \quad u \in U_i. \end{aligned}$$

It's well-defined by our assumption. Also it's continuous since preimage of an open set in $V \subset Y$ is a union of open subsets given by continuity of each a_i. Similarly we can check $\mu \circ f = \mathrm{Id}\,|_U$ as expected. Unverified ?

2.115.2 (b)

2.116 2.2.11. Espace Étalé

See a post discussion accent letter in LaTeX HERE.

See an exercise in [8] Chapter 2, Exercise 1.13.

See the discussion after Lemma 7. on Page 229 of [3].

For *section*, see Wiki's explanation for *section* in context of fiber bundle; and *section* in terms of category theory.

See a detailed post HERE.

2.117 2.2.H.

Clearly it's again a contravariant functor, therefore $\pi_*\mathscr{F}$ must be a pre-sheaf. When $\mathscr{F}$ is a sheaf, I checked identity axiom (lots of things to write down).

2.118 2.3.A.

I'm planning to use universal property to define the induced map ϕ_P.

One crutial step is to verify the diagram below is commutative

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[rr, "\rho_{UV}"] \arrow[d, "\phi(U)"'] & & \mathscr{F}(V) \arrow[d, "\phi(V)"] \\
\mathscr{G}(U) \arrow[rr, "\tau_{UV}"'] \arrow[dr] & & \mathscr{G}(V) \arrow[dl] \\
& \mathscr{G}_P &
\end{tikzcd}$$

And this is because the square diagram in the upper half commute given ϕ is a natural transformation; the lower half is by definition of $\mathscr{G}_P$. Then by universal property of colimit induces a map

$$\phi_P : \mathscr{F}_P \to \mathscr{G}_P$$

which makes the diagram commute.

See a post defined the map HERE.

2.119 2.3.B.

To define a functor $\pi_* : \mathbf{Set}_X \to \mathbf{Set}_Y$. Firstly, we have to define for any $\mathscr{F} \in \mathbf{Set}_X$,

$$\pi_*(\mathscr{F})(U) = \mathscr{F}(\pi^{-1}(U))$$

for any $U \in \mathfrak{Top}(X)$ as in **??**.

Secondly, for any natural transformation $\phi : \mathscr{F} \to \mathscr{G}$, we define $\pi_*(\phi)$ by specifying

$$\pi_*(\phi)(U) \mapsto \mathscr{F}(\pi^{-1}(U)) \to \mathscr{G}(\pi^{-1}(U)).$$

? Is this correct

2.120 2.3.C.

This is Exercise 1.15. from Chapter II of [8] on Page 67.

Proof. Clearly $\mathrm{Hom}(\mathscr{F},\mathscr{G})(U)$ takes value in the set of all natural transformations from $\mathscr{F}|_U$ to $\mathscr{G}|_U$. Namely, we have

$$U \mapsto \mathrm{Mor}(\mathscr{F}|_U, \mathscr{G}|_U).$$

The restriction map induced by $V \subset U$ is given by consider a natural morphism $\alpha \in \mathrm{Mor}(\mathscr{F}|_U, \mathscr{G}|_U)$ as a natural morphism from

$$\mathscr{F}|_V \to \mathscr{G}|_V .$$

?

So set-theoretically restriction map is identity map, with exception that it regard an element as a presheaf on a smaller open subset.

Hence $\mathrm{Hom}(\mathscr{F},\mathscr{G})(\cdot)$ is a presheaf.

Fix an open subset $U \subset X$, with an open covering $\{U_i\}_{i\in I}$ for some index set I. Pick two natural transformations $\alpha, \beta \in \mathrm{Mor}(\mathscr{F}|_U, \mathscr{G}|_U)$. Assume for any $i \in I$,

$$\mathrm{Res}_{U,U_i} \alpha = \mathrm{Res}_{U,U_i} \beta.$$

More precisely, this means for any open subset $V_i \subset U_i$ where $i \in I$ is arbitrary, we have

$$\alpha|_{U_i}(V_i) = \beta|_{U_i}(V_i).$$

However, note $\{U_i\}_{i\in I}$ is an open cover for the whole space U we're considering. It follows that for any open subset $W \subset U$, we can denote $W_i = W \cap U_i$ and express W as a union of W_i where $i \in I$.

$$\begin{aligned}
\alpha(W) =& \alpha\left(\bigcup_{i\in I} W_i\right) \in \mathrm{Obj}(\mathbf{Set}) \\
=& \bigcup_{i\in I} \alpha(W_i) \\
=& \bigcup_{i\in I} \alpha \mid_{U_i} (W_i) \\
=& \bigcup_{i\in I} \beta \mid_{U_i} (W_i) \\
=& \cdots \\
=& \beta(W).
\end{aligned}$$

Here the third equality holds by the definition of restriction map. While $W \subset U$ is arbitrary, it follows that $\alpha = \beta \in \mathrm{Mor}(\mathscr{F} \mid_U, \mathscr{G} \mid_U)$ as expected.

It remains to check gluability. Again $\{U_i\}_{i\in I}$ is an open cover of U. Pick natural transformations $\alpha_i \in \mathrm{Mor}(\mathscr{F} \mid_{U_i}, \mathscr{G} \mid_{U_i})$. They're compatiable in the sense that for any open subset $W_{ij} \subset U_i \cap U_j$, we know

$$\mathrm{Res}_{U_i, U_i\cap U_j} \alpha_i = \mathrm{Res}_{U_j, U_i\cap U_j} \alpha_j \;\Rightarrow\; \alpha_i(W_{ij}) = \alpha_j(W_{ij}).$$

Now we try to define a natrual transformation $\alpha \in \mathrm{Mor}(\mathscr{F} \mid_U, \mathscr{G} \mid_U)$ such that $\alpha \mid_{U_i} = \alpha_i$. For any open subset $Y \subset U$, denotes $Y_i := Y \cap U_i$.

$$\begin{aligned}
\alpha(Y) : \mathscr{F} \mid_U (Y) \to \mathscr{G} \mid_U (Y) \\
\mathscr{F} \mid_U (Y_i) \ni x \mapsto \alpha_i(x).
\end{aligned}$$

This is map in sets, it's well-defined for $\{Y_i\}_{i\in I}$ is an open covering for Y and each α_i is compatiable. By construction we know $\alpha \mid_{U_i} = \alpha_i$. Hence we've checked gluability.

Need to check, but I think it's basically unwrapping a long long defintion

□

2.120.1 Verification

See a post HERE.

Also see a lemma from Stacks Project HERE. This lemma basically proves gluability and uniqueness, based on the fact that Sheaf Hom is already a presheaf. In the proof of the above lemma, we defined the natural transformation in the way such that the following diagram commute

$$\begin{array}{ccc}
s \in \mathscr{F}(U) & \xrightarrow{\varphi_U} & \mathscr{G}(U) \\
\downarrow & & \downarrow \\
\mathscr{F}(V) & \xrightarrow[\varphi_V]{} & \mathscr{G}(V)
\end{array}$$

And this relies on the fact that $\mathscr{G}$ is a sheaf!, by looking at every $U \cap U_i$, which covers U.

2.120.2 Warning

Sheaf Hom does not commute with taking stalks. But there exists at least one map from

$$\begin{aligned}&\mathrm{Hom}(\mathscr{F},\mathscr{G})_p \to \mathrm{Hom}(\mathscr{F}_p,\mathscr{G}_p)\\&\{(\alpha,U)\ \mid\ p\in U,\ \alpha\in \mathrm{Mor}(\mathscr{F}\mid_U,\mathscr{G}\mid_U)\}/\sim_1\mapsto \ldots\end{aligned}$$

2.120.3 Warning: References

☛

See a post HERE, which contains a link to the detailed version of the counterexample HERE. According to post HERE, The direction was correct.

See Stefan's notes on Page 18 for a concrete example HERE. One comment on "Hom functor preserve limit" HERE.

Also, the first argument $\mathscr{F}$ doesn't have to be a sheaf, a pre-sheaf is enough. See a post HERE.

2.120.4 Counterexample

I'll add some details to the first example mentioned in the post linked above.

Proof. For any $U\subset X$, where $\mathscr{F}$ is skycraper sheaf at $p\in X$ with value group A and $\mathscr{G}$ is a constant sheaf on topological space X with value group A. We claim that

$$\mathcal{H}om(\mathscr{F},\mathscr{G})(U)=\mathrm{Mor}(\mathscr{F}\mid_U,\mathscr{G}\mid_U)=0$$

as an Abelian group for arbitrary $U\subset X$.

It suffices to check the above statment is correct when $U=X$ for $\mathcal{H}om(\mathscr{F},\mathscr{G})$ is a sheaf, i.e. we need to show the group

$$\mathcal{H}om(\mathscr{F},\mathscr{G})(X)=\mathrm{Mor}(\mathscr{F},\mathscr{G})=0$$

Pick any natural transformation $\alpha\in\mathrm{Mor}(\mathscr{F},\mathscr{G})$. We wish to show the group homomorphism $\alpha(U)$ is 0 (i.e. sends everything to 0 in the codomain $\mathscr{G}(U)$). For an open subset U, it admits an open covering $\{U_i\}$ in which every U_i is connected. While $\mathscr{G}$ is a sheaf, to prove $\alpha(U)=0$ it suffices to check $\alpha(U_i)=0$. Fix i and denote $U_i=U_0$. When $p\notin U_0$, then $\mathscr{F}(U_0)=0$ and the map $\alpha(U_0)=0$.

Now assume $p\notin U_0$. We can still argue $\alpha(U_0)$ is 0 group homomorphism by restrict it to a smaller open subset that doesn't contain p, because the skycraper sheaf will be 0 group.

?

Assume p is closed an not open, which means it's not isolated. Then $V:=U\setminus\{p\}$ is an open subset that doesn't contain p, hence $\mathscr{F}(V)=0$ by definition.

$$\begin{tikzcd}
\mathscr{F}(U_0) \arrow[r, "\alpha(U)"] \arrow[d, "\mathrm{Res}_{U_0V}"'] & \mathscr{G}(U_0)=A \arrow[d] \\
\mathscr{F}(V)=0 \arrow[r] & \mathscr{G}(V)=\oplus A
\end{tikzcd}$$

Notice that we assumed U_0 to connected, therefore $\mathscr{G}(U_0) = A$ is composed of single one copy of A. While $\mathscr{G}(V) = \oplus_{j\in J} A$ for some index set J. We claim that $\mathscr{G}(U_0) \to \mathscr{G}(V)$ is an injection:

- We claim that the map is in fact diagonal map by considering the following commutative diagram

$$\begin{tikzcd} a \in \mathscr{G}(U_0) \arrow[r] \arrow[d, hook] & (a, a, ..., a)_{j\in J} \in \mathscr{G}(V) \arrow[d, hook] \\ \amalg_{q\in U_0} \mathscr{G}_q = \amalg_{q\in U_0} A & \amalg_{q\in V} A \arrow[l, hook'] \end{tikzcd}$$

  Because the map from $\mathscr{G}(U_0) \to \amalg_{q\in U_0} \mathscr{G}_q = \amalg_{q\in U_0} A$ is the diagonal map, therefore the map we considering must factor through it. Hence it could only be the diagonal map, with index set J.

Therefore by commutativity of the diagram together with the injectivity of the above map we can conclude $\alpha(U_0) = 0$ as expected.

$\square$

2.120.5 Abelian group structure

The Abelian group structure on $\mathcal{H}om(\mathscr{F}, \mathscr{G})(U)$ is given by defining

$$\alpha + \beta(U) : \mathscr{F}(U) \to \mathscr{G}(U) \ x \mapsto \alpha(U)x + \beta(U)x$$

for two natrual transformations $\alpha, \beta \in \mathcal{H}om(\mathscr{F}, \mathscr{G})(U)$. Notice that $\alpha + \beta$ is indeed a natural transformation, because it's compatible with restriction maps, which we could check by definition... Element $0 \in \mathcal{H}om(\mathscr{F}, \mathscr{G})(U)$ is just a natural transformation sends everything to $0 \in \mathscr{G}(U)$.

2.121 2.3.D.

2.121.1 (a)

Since the stalk doesn't commute with sheaf hom, I guess it could be more complicated to compute the stalk for sheaf hom $\mathcal{H}om$. So I'll check two functors are isomorphic in terms of *natural transformation.*

Let $\mathscr{C} := \underline{\{p\}}$ denotes the constant sheaf with value set $\{p\}$. Firstly, by definition we know for any open subset $U \subset X$

$$\mathcal{H}om(\mathscr{C}, \mathscr{F})(U) = \operatorname{Mor}(\mathscr{C}|_U, \mathscr{F}|_U).$$

Pick any element $\alpha \in \operatorname{Mor}(\mathscr{C}|_U, \mathscr{F}|_U)$, which is a natural transformation.

Endow $\{p\}$ with discrete topology. For any open subset $V_1 \subset U$, we have set map $\alpha(V_1)$

$$\alpha(V_1) : \ \mathscr{C}|_U(V_1) = \mathscr{C}(V_1) \longrightarrow \mathscr{F}|_U(V_1) = \mathscr{F}(V_1).$$

The domain only contains one element: the set is all continuous map from some open subset in V_1 to $\{p\}$, hence there's only such function and we denote it as f. It follows that the map is completely determined by where f goes, $\alpha(V_1)$ could be identified with an element in $\mathscr{F}(V_1)$. In particular, for any $\alpha \in \mathrm{Mor}(\mathscr{C}|_U, \mathscr{F}|_U)$, $\alpha \in \mathscr{F}(U)$, and

$$\mathcal{H}om(\mathscr{C}, \mathscr{F})(U) = \mathrm{Mor}(\mathscr{C}|_U, \mathscr{F}|_U) = \mathscr{F}(U)$$

Need to verify

for any $U \subset X$, notice that it's in fact set-theorically bijection, hence it's a natural isomorphism and therefore two functors are isomorphic as expected.

2.121.2 References

A potentially related post HERE.

2.122 2.3.E.

- Diagram Chase: ?
- Universal Property: According to Definition 1.6.4., Kernel was defined using universal property. Therefore we can use universal property of $\mathrm{Ker}(\phi(U))$ to deduce the unique existence of such map for $\mathrm{Ker}(\phi(V)) \to \mathscr{G}(U) = 0$.

2.123 2.3.G. 2.3.H.

Notice that we're talking about pre-sheaves here. Hence these properties are satisfied by definition...

2.124 2.3.J.

Question: wouldn't it be better if $\underline{\mathbb{Z}}$ is replaced by $\underline{\mathbb{C}}$?

2.124.1 References

See a post HERE. In fact, the quotient sheaf could be not separated, see a post HERE.

See a detailed lecture note HERE.

And a solution HERE and HERE.

Here's another example where image presheaf might not be a sheaf.

2.125 2.4.A.

2.125.1 Hint

See G ortz's book [7], page 52, Prop 2.23. (1); or Lemma 2.9. on Page 35 of [13].

2.126 2.4.2. Compatible Germs

See a post regarding compatible germs and Espace Étalé.

2.126.1 Two Equivalent Definitions

The following two definitions are equivalent:

- *Element* $(s_p)_{p\in U} \in \prod_{p\in U} \mathscr{F}_p$ *consists of compatible germs ...*
- *There is an open covering* $\{U_i\}$ *of* U, *...*

Proof. For any $p \in U$, pick some representative of $s_p \in \mathscr{F}_p$ as follow:

$$(U_p, \tilde{s}_p)$$

where open subset $U_p \subset U$ contains p with $\tilde{s}_p \in \mathscr{F}(U_p)$. Then we take

$$\{U_p\}_{p\in U}$$

as an open covering of U and take corresponding sections $\tilde{s}_p \in \mathscr{F}(U_p)$ for $p \in U$. We can check conditions for the second statement is satisfied: Clearly $p \in U_p$ and s_p is the germ of $\tilde{s}_p$ at p for the first condition required this is true for any $q \in U_p$.

Conversely, pick any $p \in U$. We can find some U_i that contains p with a section $f_i \in \mathscr{F}(U_i)$ such that s_p is the germ of f_i at p. So we take

$$(f_i \in \mathscr{F}(U_i),\ U_p) \in \mathscr{F}_p$$

as representative of f_i. Notice that germ of f_i at any $q \in U_i$ is s_q as expected. □

2.126.2 Section induces a choice of compatible germs

Any sections s of $\mathscr{F}$ over U gives a choice of compatible germs for U.

Proof. Let $s \in \mathscr{F}(U)$ as given and pick any open covering $\{U_i\}_{i\in I}$ of U where I is an index set. And define $f_i := s|_{U_i} \in \mathscr{F}(U_i)$. Now we may assume $p \in U_i$ for some U_i. The following diagram is commute for any $p \in U_i$, with element being the image under corresponding maps.

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] & \mathscr{F}(U_i) \arrow[r] & \mathscr{F}_p \\
s \arrow[r, mapsto] \arrow[rr, bend right] & f_i := s|_{U_i} & s_p
\end{tikzcd}$$

Assume $p \in U_i$ for a fixed $i \in I$, then

$$s_p = (\mathscr{F}(U) \to \mathscr{F}_p)(s) = (\mathscr{F}(U_p) \to \mathscr{F}_p)(f_i)$$

by commutativity of the above diagram. Hence s_p is germ of f_i at p as expected.

□

2.127 2.4.B.

$$\mathscr{F}(U) \to \prod_{p \in U} \mathscr{F}_p$$

In the last subsection of Section 2.126, we checked that any section will induce something called compatible germs on the left of the above map. Question is that what are the images of the above map? The following lemma will give us complete description of the image of the map above.

2.127.1

Any choice of compatible germs is the image of a section.

Proof. I'll take the second definitions of a choice of compatible germs: there is an open cover $\{U_i\}_{i \in I}$ of U where I is understood as an index set, and there are sections $f_i \in \mathscr{F}(U_i)$ such that s_p is the germ of f_i at any $p \in U_i$.

Let's consider two open subsets $U_i, U_j \subset U$ where $i, j \in I$ with $f_i \in \mathscr{F}(U_i), f_j \in \mathscr{F}(U_j)$. Assume $U_i \cap U_j \neq \emptyset$. Then for any $p \in U_i \cap U_j$, s_p is the germ of f_i and f_j at p, i.e.

$$[U_i, f_i] = s_p = [U_j, f_j] \in \mathscr{F}_p.$$

Hence there exists open subset $W_{ij,p} \subset U_i \cap U_j$ containing p such that $f_i|_{W_{ij,p}} = f_j|_{W_{ij}}$. Notice that $p \in U_i \cap U_j$ is arbitrary, therefore for any p there exists some $W_{ij,p}$ (depending on p). And all such $\{W_{ij,p}\}_{p \in U_i \cap U_j}$ will be an open covering for $U_i \cap U_j$, by Identity Axiom we know

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

While i, j were arbitrary, Gluability Axiom implies there exists $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$. And we can check that f is indeed the preimage of the given choice of compatible germs: pick some U_i containing p,

$$f_p = (\mathscr{F}(U_i) \to \mathscr{F}_p)(f|_{U_i})$$

Hence f_p is the germ of f_i at p.

□

2.127.2 Hint

See this post.

2.128 2.4.C.

According to the hint, we have the following diagram commute for according to construction of 2.3.A.

$$
\begin{tikzcd}
\mathscr{F}(U) \arrow[r, shift left, "\phi_1(U)"] \arrow[r, shift right, "\phi_2(U)"'] \arrow[d, "\pi"'] & \mathscr{G}(U) \arrow[d, hook, "\iota"] \\
\prod_{p\in U} \mathscr{F}_p \arrow[r, "\Phi"'] & \prod_{p\in U} \mathscr{G}_p
\end{tikzcd}
$$

Hence we have

$$\iota \circ \phi_1(U) = \Phi \circ \pi = \iota \circ \phi_2(U) \;\Rightarrow\; \phi_1(U) = \phi_2(U)$$

for ι is injective.

2.129 2.4.D.

See Hartshorne [8] Chapter II, Proposition 1.1.

2.129.1 Warning

It could happen that two non-isomorphic sheaves have isomorphic stalks. See Sheaf Hom Section 2.120 regarding skycraper sheaf and constant sheaf.

See a post HERE.

2.130 2.4.E.

2.130.1 (a)

Pick a non-separated presheaf: See Stacks Project Definition 007A ?

Here is an example of gluable unseparated presheaf.

See a post discussing two examples where section isn't determined by stalks.

2.130.2 (b)

For Section 2.128, it's important to assume $\mathscr{G}$ is a sheaf.

See the answer by Andreas Blass HERE. For two pre-sheaves $\mathscr{F}, \mathscr{G}$, we might encounter a counterexample as follows:

Let $X = \{p, q\}$ be a discrete topological space with two points. Define two pre-sheaves $\mathscr{F}, \mathscr{G}$ on X with values in set $\{0, 1\}$. We specify two pre-sheaves as follow:

$U \subset X$	$\{p,q\}$	$\{p\}$	$\{q\}$	$\emptyset$
$\mathscr{F}(U)$	$\{0,1\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\mathscr{G}(U)$	$\{0,1\}$	$\{1\}$	$\{1\}$	$\{1\}$

where the restriction maps for $\mathscr{F}$ and $\mathscr{G}$ are the only possible ones. Now we wish to define two different morphism of sheaves $\alpha, \beta : \mathscr{F} \to \mathscr{G}$ as natural transformations. For all open subsets except $U = \{p,q\}$, we have

$$\alpha(U) = \beta(U) : \{0\} \to \{1\}.$$

The place we intentionally make them different is when $U = \{p,q\}$:

$$\alpha(U) : \{0,1\} \to \{0,1\} \text{ by } 0 \mapsto 0,\ 1 \mapsto 1;$$
$$\beta(U) : \{0,1\} \to \{0,1\} \text{ by } 0 \mapsto 1,\ 1 \mapsto 0.$$

Clearly as morphisms $\alpha \neq \beta$. But we have

$$\alpha_p \times \alpha_q : \prod_{x \in \{p,q\}} \mathscr{F}_x \to \prod_{x \in \{p,q\}} \mathscr{G}_x$$
$$\{0\} \times \{0\} \mapsto \{1\} \times \{1\}.$$

?

But notice that there's only **one** choice of induced morphism between $\mathscr{F}_x \to \mathscr{G}_x$ for any $x \in X = \{p,q\}$. This means, even if we're given another different maps β, it will induce the same map on stalks.

2.130.3 (c)

Take a pre-sheaf $\mathscr{F}$ (on space X) that is not a sheaf. For example, constant pre-sheaf. We know the stalk of its sheafification at any $p \in X$ is

$$\mathscr{F}_p = \mathscr{F}_p^\sharp.$$

2.131 2.4.O.

A post HERE, HERE.

Another related post explaining isomorphic sheaves will give rise to injective and surjective maps on sections, this does not contradict the fact that surjectivity of morphism of sheaves cannot have analogus statement on sections.

See Richard on YouTube, Scheme Lecture 03 last example. Let $U = \mathbb{C} \setminus \{0\}$ with classical topology.

$$\mathscr{O}_X(U) \to \mathscr{O}_X^*(U)$$

is not surjective for $x \in \mathscr{O}_X^*(U)$ doesn't admits a holomorphic log for $f(0) = 0$ while exponential function can never be 0. Because we cannot find $logx$ as a holomorphic function on $\mathbb{C}$.

See a very detailed solution HERE.

2.131.1 Complex analysis background

See Theorem 6.1., 6.2. Chap 3 Page 98 [17].
See this great POST!
See a note HERE. A post HERE.

2.132 2.5.1.

2.132.1 Remarks

We can also define, for general open subset by taking limit: see comments of 009H by [16]; and [7] Proposition 2.20. on Page 51 with discussion above it.

2.133 2.5.B.

Proof. ... □

2.133.1 Another Proof

?

Proof. A potentially wrong approach is to recognise that, in the definition of $\mathscr{F}(U)$: we set

$$\mathscr{F}(B) := \{(f_p \in F_p)_{p \in B} \ \mid \ \star\}.$$

The condition $\star$ is exactly the definition of compatible germs. While $F(B)$ could be identified with it's image in an injective map

$$F(B) \to \prod_{p \in B} F_p.$$

We proved in Section 2.127 that the image is precisely the compatible germs, therefore we must have they're isomorphic. □

2.133.2 Remarks

In fact, for sheaf on base F, the map

$$F(U) \to \prod_{p \in U} F_p$$

is injective. The proof is essentially the same as for sheaf on X.

2.134 2.5.D.

See [13] Chap 2 Exercise 2.7. on Page 44. This exercise will be used for Rising Sea [19] 4.3.B.

2.135 2.6.A.

I don't know what "natural" means. Is there a natural transformation?

I'll just check the isomorphism. And assume kernel sheaf has its corresponding the universal property.

Firstly, we have $(\mathrm{Ker}(\mathcal{F} \to \mathcal{G}))_p \to \mathrm{Ker}(\mathcal{F}_p \to \mathcal{G}_p)$ by universal property of $\mathrm{Ker}(\mathcal{F}_p \to \mathcal{G}_p)$. This is because we can compute the composition

$$(\mathrm{Ker}(\mathcal{F} \to \mathcal{G}))_p \to \mathcal{F}_p \to \mathcal{G}_p = 0.$$

For $s_p \in (\mathrm{Ker}(\mathcal{F} \to \mathcal{G}))_p$, we can diagram chasing it...

Conversely, ... a diagram chasing argument.

2.135.1 Solution

For details see Autumn HW2 2023 Problem 3. Or this online solution.

2.136 2.7.A.

I couldn't give a complete example ...

2.136.1 Solution

A post on inverse image presheaf.

Let $X = \{p, q\}, Y = \{w\}$ with discrete topolgoy. Let $\mathscr{G}$ be a constant sheaf on Y with value group A which contains at least two elements. Compute inverse image presheaf.

Proof. For any $U \subset X$, we have

$$\pi_{\text{pre}}^{-1}\mathscr{F}(U) = \mathrm{Colim}_{V \supset \pi(U) = \{w\}}\, \mathscr{G}(V) = \mathrm{Colim}_{V = \{w\}}\, \mathscr{G}(V) = A.$$

Hence it's the inverse image pre-sheaf $\pi_{\text{pre}}^{-1}\mathscr{F}$ is a constant pre-sheaf. It's not a sheaf for exactly the same reason constant pre-sheaf isn't a sheaf for failure of glubility. □

?

2.137 2.7.D.

By definition... See [8] Definition on Page 65 of Section 2 of Chapter II.

2.138 2.7.G.

See Section 2.42.

2.139 3.1.A.

?

2.140 3.2.A.

2.140.1 Example 5

A complete description of Spec($\mathbb{R}[x]$) is given by a post HERE.

To find all prime ideals in P.I.D. $\mathbb{R}[x]$ is generated by one single element, i.e. a polynomial $f \in \mathbb{R}[x]$. While $\mathbb{R}[x]$ is U.F.D., a prime element is equivalent to an irreducible element. Therefore we need to find all irreducible polynomials in $\mathbb{R}[x]$. Notice that field extension

$$[\mathbb{C} : \mathbb{R}] = 2 \ \Rightarrow \ \deg f \leq 2$$

for any irreducible polynomial f.

In case of $\deg f = 2$, it's precisely the case where we require f to be an irreducible quadratic.

See a post HERE.

2.141 3.2.B.

For an irreducible polynomial $x^2 + ax + b \in \mathbb{R}[x]$, it cannot admit a real root. Therefore in algebraic closure of R, i.e. in $\mathbb{C}$, we can find two roots α_1, α_2 of the polynomial.

2.142 3.2.C.

?

2.143 3.2.D.

I didn't figure out the Euclid's proof for this.

For Euclid's proof, see a post HERE.

Basic ideal is consider

$$f = \prod_{i \in I} f_i + 1$$

where $\langle f_i \rangle = \mathfrak{p}_i$ for a *finite* index set I. Here $\mathfrak{p}_i \triangleleft k[x]$ is a prime ideal. Each f_i must have degree $\deg f_i \geq 1$ because otherwise $\mathfrak{p}_i$ will contain a unit. While in integral domain $k[x]$, we know degree f is again large or equal to 1, hence not a unit. Clearly it's nonzero. As a non-zero non-unit element in U.F.D. $k[x]$, it could be written as a product of irreducible elements, say g_1. It is will also generate a prime ideal, which means

$$\langle g_1 \rangle = \mathfrak{p}_i = \langle f_i \rangle \ \Rightarrow \ g_1 = u f_i$$

for some $i \in I$ and unit u. Then $f \equiv 1 \pmod{g_i}$, which means g_i will divide unit 1, contradiction.

See a post HERE, HERE.

2.144 3.2.L. Exercise

Proof. Localisation commute with quotient, therefore we plan to prove the isomophism by constructing a ring homomorphism from $\psi : \mathbb{C}[x, y]_x \to \mathbb{C}[x]_x$ and compute the kernel. The map ψ is defined as

$$\begin{aligned} \psi : \mathbb{C}[x, y]_x &\to \mathbb{C}[x]_x \\ f(x, y)/x^i &\mapsto f(x, 0)/x^i \end{aligned}$$

where $f(x, y) \in \mathbb{C}[x, y]$ and i is some integer.

Clearly we see $(xy)_x \subset \operatorname{Ker}\psi$. Conversely, let's pick an element $f(x, y)/x^i$ such that $f(x, 0)/x^i = 0 \in \mathbb{C}[x]_x$, this implies there exists $j \in \mathbb{N}$ such that

$$x^j f(x, 0) = 0 \in \mathbb{C}[x] \ \Rightarrow \ f(x, 0) = 0 \in \mathbb{C}[x].$$

stopped,...

The conclusion follows immediately if we realise

$$(y)_x = (xy)_x \subset \mathbb{C}[x, y]_x.$$

One way to interpret this is as follow: both $(x)_x$ and $(xy)_x$ are image of localisation map of a principal ideas $(x), (xy) \lhd \mathbb{C}[x, y]$. And two principal ideals are the same if they differ by a unit, say x. Or we can perform a double inclusion argument. $\square$

2.144.1 Hint

See a post HERE.

Some details are not clear, so see the following.

Proof. Two facts to recall is that we have two exact sequences

$$0 \longrightarrow (xy)_x \subset \mathbb{C}[x, y]_x \longrightarrow \mathbb{C}[x, y]_x \longrightarrow (\mathbb{C}[x, y]/(xy))_x \longrightarrow 0$$

$$0 \longrightarrow (y) \subset \mathbb{C}[x, y] \longrightarrow \mathbb{C}[x, y] \longrightarrow \mathbb{C}[x] \longrightarrow 0$$

Here the second rows will induce, by exactness of localisation

$$0 \longrightarrow (y)_x = (xy)_x \subset \mathbb{C}[x, y]_x \longrightarrow \mathbb{C}[x, y]_x \longrightarrow \mathbb{C}[x]_x \longrightarrow 0$$

And this concludes the desired isomorphism. One crutial observation I missed was $(xy)_x = (y)_x$.

□

2.145 3.2.T.

I guess it's
$$f(x+\epsilon) = f(x) + \epsilon f'(x) \in k[x,\epsilon]/(\epsilon^2)$$

?

2.146 3.5.E.

Notice that

$$\begin{aligned} D(f) \subset D(g) \;&\Leftrightarrow\; V(\langle f\rangle) \supset V(\langle g\rangle) \\ \Leftrightarrow\; f \in \sqrt{\langle f\rangle} \subset \sqrt{\langle g\rangle} \;&\Rightarrow\; f^n \in \langle g\rangle \;\text{ for some }\; n \geq 1. \end{aligned}$$

In fact, if we have $f^n \in \langle g\rangle$ for some integer n, then

$$f \in \sqrt{\langle g\rangle} \;\Rightarrow\; \langle f\rangle \subset \sqrt{\langle g\rangle} \;\Rightarrow\; \sqrt{\langle f\rangle} \subset \sqrt{\langle g\rangle}.$$

Hence the equivalence of the first two statements are checked.

Now if we assume $f^n \in \langle g\rangle$ for some integer n, then $cg = f^n$ for some $c \in A$. Hence
$$\frac{g}{1} \cdot \frac{c}{f^n} = 1,$$
which means g is an invertible element of A_f. Conversely, g being an invertible element in A_f means there exists a/f^m such that

$$g/1 \cdot a/f^m = ga/f^m = 1/1 \in A_f \;\Rightarrow\; f^l(ga - f^m) = 0 \in A$$

for some integers l, m. Hence take $n := l + m$ leads to $f^l f^m = f^l ga \in \langle g\rangle$.

2.146.1 References

Checked at Lemma 01HR of [16].

2.147 3.6.6.

This is related to the counterexample Section 2.160.

The inclusion map is given by universal property of coproduct: for a fixed $i \in I$, we have a continuous projection $\pi_i : \prod_{j\in\mathbb{N}} A_i \to A_i$ which induces $\operatorname{Spec} A_i \to \operatorname{Spec} \prod_{j\in\mathbb{N}} A_i$. More explicitly, this map on topological spaces is defined by taking pre-image of a prime ideal. Two distinct prime ideals $\mathfrak{p}, \mathfrak{q} \lhd A_i$ will give rise to different pre-image of prime ideals. Hence this map is injective.

?

And this induces, by universal property of coproduct i.e. disjoint union of two topological spaces, a map

$$\amalg_{j\in\mathbb{N}} \operatorname{Spec} A_i \to \operatorname{Spec} \prod_{j\in\mathbb{N}} A_i.$$

This map is injective for the inclusion from $\operatorname{Spec} A_i \to \amalg_{j\in\mathbb{N}} A_j$ is injective, the commutative diagram proves the injectivity.

$$\begin{tikzcd} \operatorname{Spec} A_j \arrow[r, hook] \arrow[dr, hook] & \amalg_{j\in\mathbb{N}} \operatorname{Spec} A_j \arrow[d, hook] \\ & \operatorname{Spec} \prod_{j\in\mathbb{N}} A_j \end{tikzcd}$$

2.148 4.1.1. Definition: Structure Sheaf on Base

As stated, we define

$$\begin{aligned} \mathscr{O}_{\operatorname{Spec} A}(D(f)) &:= S_{D(f)}^{-1} A \\ &:= \{g \in A \mid V(g) \subset V(f)\}^{-1} A. \end{aligned}$$

This is well-define, i.e. independent of the choice f because when we have $D(f') = D(f)$, the localisation will be the same.

Also, we claim $S_{D(f)}$ is indeed a multiplicatively closed subset of A:

- $1 \in S_{D(f)}$ for $V(1) = \emptyset \subset V(f)$ for any choice of $f \in A$.
- Assume $g_1, g_2 \in S$, then

$$D(f) \subset D(g_1 g_2) = D(g_1) \cap D(g_2) \Rightarrow V(g_1 g_2) \subset V(f).$$

2.149 Definition of Restriction Map

Here are some remarks for the restriction map. Notice that we have

$$D(f') \subset D(f) \Rightarrow S_{D(f)}^{-1} \subset S_{D(f')}.$$

?

Hence the restriction is just inclusion map.

2.149.1 References

The map is given explicitly on Page 4 on the BOOK.

2.150 4.1.A.

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ A_f \arrow[rr, dashed, bend left=10, "\exists!"] & & S_{D(f)}^{-1}A = \mathscr{O}_{\operatorname{Spec} A}(D(f)) \arrow[ll, dashed, bend left=10, "\exists!"] \end{tikzcd}$$

Here α, β denotes the localisation map. For β, we have $f \in S_{D(f)}$ for $V(f) \subset V(f)$. Since it's a multiplicatively closed set, then

$$\beta(\{1, f, f^2, ...\}) \subset (S_{D(f)}^{-1}A)^\times$$

will become units, therefore by universal property of A_f there exists a unique map from A_f to $S_{D(f)}^{-1}A$ making the diagram commute.

Conversely, by Exercise 3.5.E. Section 2.146, while $V(g) \subset V(f) \Rightarrow D(f) \subset D(g)$, then g is invertible in A_f. Hence

$$\alpha(S_{D(f)}) = \{\alpha(g) \ | \ g \in S_{D(f)}\} \subset (A_f)^\times$$

and by universal property we can factor uniquely through A_f to $\mathscr{O}_{\operatorname{Spec} A}(D(f))$. Their composition must be identity hence we're done.

2.150.1 References

See [7] (2.10) on Page 59 for details.

2.151 4.1.B.

See next section Section 2.152

2.152 4.1.C.

See KEY LEMMA 1.13. on Page 5 of a book HERE.

Previously we proved that for arbitrary i, j

$$g_j^M \cdot b_i' = g_i^M \cdot b_j'. \star$$

In this lemma, the final equality is given by the following: let $f^k = \sum_{i \in I} a_i' g_i^M$ for some index set I and denote $b = \sum_{i \in I} a_j' b_j'$. For a fixed $i \in I$ with $j \in J$

arbitrary, we have

$$\begin{aligned} g_i^M b &= g_i^M \left(\sum_{j\in I} a'_j b'_j \right) \\ &= \left(\sum_{j\in I} a'_j g_i^M b'_j \right) \\ &= \left(\sum_{j\in I} a'_j g_j^M b'_i \right) \quad \text{by applying } \star \\ &= \left(\sum_{j\in I} a'_j g_j^M \right) b'_i = f^k b'_i. \end{aligned}$$

2.153 4.1.D.

2.153.1 Hint

See Def 01HT of [16], in which a lemma 00CR build connection between localisation and colimit. For this lemma, see also "A Term"...

For relations between colimit and localisation, see this post.

2.154 4.1.E.

See Section 2.153

2.155 4.1.F.

(a) See Atiyah [2] Chapter 3 Local Property.

(b) By 2.4.A. and Section 2.154, we have sheaf $\widetilde{M}$. According to Lemma 01HV (2), we have an injection

$$\widetilde{M}(\mathcal{O}_{\operatorname{Spec} A}) = M \to \prod_{[\mathfrak{p}]\in \operatorname{Spec} A} \widetilde{M}_{[\mathfrak{p}]} = \prod_{\mathfrak{p}\in \operatorname{Spec} A} M_{\mathfrak{p}}.$$

2.156 4.3.A.

See Hartshorne [8]?

2.157 4.3.B.

See Hartshorne[8] Exercise in Section 2.2.

2.158 4.3.C.

2.159 4.3.D.

By definition of a scheme, we know the union of all affine open subschemes will cover the scheme.

Given two affine open $(X_1, \mathcal{O}_{X_1}), (X_2, ...)$. For any point $p \in X_1 \cap X_2$, we know there exists an open neighborhood U such that $p \in U \subset X_1 \cap X_2$ given both X_1, X_2 are open. While we know distinguish sets form a base, there exists some

$$p \in D(f) \subset U \subset X_1 \cap X_2.$$

And by 2.157 or 2.57 we know $D(f)$ is an affine open as expected.

Therefore we've checked that affine opens form a base for Zariski topology. ?

2.160 4.3.E.

The topology on $\amalg X_i$ is box topology or product topology? I guess product topology... See a post regrading box topology and product topology.

2.160.1 (a)

Assume I is a finite index set. For each $i \in I$, we have an affine scheme $(X_i, \mathcal{O}_{X_i})$ where $X_i = \operatorname{Spec} A_i$ for some ring A_i. According to Exercise 3.6.A, we denote $A = \prod_{i \in I} A_i$. While we proved in [8] Chap II Exercise 1.9. 2.41. that finite number of direct sum of sheaves is again a sheaf. We claim that

$$(\amalg_{i \in I} X_i, \oplus_{i \in I} \mathcal{O}_{X_i}) \simeq (\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec} A}).$$

Exercise 3.6.A. checked two topological spaces are homeomorphic via $\pi : \amalg_{i \in I} X_i \to \operatorname{Spec} A$. It remains to check isomorphism of sheaves

$$\mathcal{O}_{\operatorname{Spec} A} \to \pi_*(\oplus_{i \in I} \mathcal{O}_{X_i}).$$

The map on sheaves is given by free if we assume Chap II Proposition 2.3. of [8] and its counterpart statement (which is slightly stronger) in here 2.156. Basic idea is that we have homeomorphism of two topological spaces, then apply the theorem, which will give us a unique morphism of sheaves. Do this conversely...

2.160.2 (b)

Suppose index set I is infinite. Assume for the sake of contradiction that $\amalg_{i \in I} X_i \simeq \operatorname{Spec} A$ for some ring A. Quasi-compact is preserved by homeomorphism, the $\operatorname{Spec} A$ is quasi-compact, but I claim the left hand side is not quasi-compact. ?

See 3.6.6.

2.160.3 References

See post regarding some concrete counterexamples.

Verification for part (a), post.

For non-compactness of (b). See a solution HERE by Pink. See post, then we know the space in (b) is not quasi-compact because we can choose a covering in which each open subset $U_i = A \times \cdot \times B_i \times \cdot A$ where $B_i \subsetneq A_i$. It cannot reduce to a finite sub-cover.

See Tag 00ED, Lemma 01I5 for more rigorous approach to (a).

2.161 4.3.F.

2.161.1 (a)

2.162 4.4.A.

Suppose we're given schemes $(X_i, \mathcal{O}_{X_i})$ indexed by $i \in I$.

Then we define

$$X = \amalg_{i \in I} X_i / \sim_{ij}$$

where $\sim_{ij}$ is generated by all f_{ij}. The structure sheaf is defined by glueing sheaves.

2.162.1 References

From a special case of S-scheme, [13] Lemma 3.33 of Chapter 2 on Page 49.

From Section 01JA of [16] on Glueing Schemes.

2.162.2 Cocycle Condition

Given Cocycle Condition, we claim that both "inverse" and "identity" requirements are redundent.

- For "identity": let all index to be i
- For "inverse": See the diagram (2) of Tag 01JA, we simply let $k = i$, which will give us $\varphi_{ji} \circ \varphi_{ij} = \mathrm{id}_{U_{ij}}$. Swap i, j will give us another equality, hence we must have $\varphi_{ij} = \varphi j i^{-1}$.

For confirmation, see [7] Definition 3.9. on Page 71.

2.163 5.1.C.

The hint in the prompt was refering to Exercise 2.13. Section 2.66 Chapter 2 of Hartshorne [8].

2.164 5.1.D.

Remarks: $\Rightarrow$ is easy because affine opens form a basis, while the converse direction is non-trivial. The point is that we only know a specific choice of affine opens.

Fact: we can build a larger quasi-compact topological spaces by union finitely many quasi-compact topological spaces. See this POST.

2.165 5.1.F.

$\Rightarrow$: Let U, V be two affine opens in scheme X. We know U, V are quasi-compact, then their intersection $U \cap V$ is quasi-compact. While affine opens form a basis for topology of X, by quasi-compactness of $U \cap V$ we can cover it by finitely many affine opens.

$\Leftarrow$: Let A, B be two quasi-compact open subsets of X. By quasi-compactness we can decompose them as

$$A = \bigcup_{i=1}^{n} A_i, \ B = \bigcup_{j=1}^{m} B_j.$$

where all A_i, B_j are affine opens. Notice that

$$A \cap B = \bigcup_{1 \le i \le n, \ 1 \le j \le m} (A_i \cap B_j)$$

is a finite union of $A_i \cap B_j$, by assumption it's a finite union of affine opens. Apply Section 2.164 we conclude $A \cap B$ is quasi-compact, which proves X is quasi-separated.

2.166 5.1.G.

Zariski topology admits distinguished open as a basis. For solution see this POST.

2.167 5.1.H.

$\Rightarrow$: If we take Section 2.165 as an equivalent definition of quasi-separated, then we can take the open affine cover as the original affine opens.

$\Leftarrow$: ???

2.167.1 Hint

See POST discussing this exercise.

2.168 5.1.K.

For details, see Tag 01KH.

2.169 5.2.A.

Proof. See 3.2.1. When we say two functions f, g agree at all points, this means

$$f \equiv g \pmod{p}$$

for all $p \in \Gamma(X, \mathcal{O}_X)$.

To prove $f = g$, we wish to use the Identity Axiom of Sheaf. Fix an open covering $\{U_i\}_{i\in I}$ of X where I an index set. It suffices to check $f\mid_{U_i} = g\mid_{U_i}$.

We can identify them by injective map

$$\mathcal{O}_X(U_i) \longhookrightarrow \prod_{p\in U_i} \mathcal{O}_{X,p}$$

This forces $f = g$ when they have same ???

□

2.169.1 References

See this POST: but is the definition of value of function correct?

See another POST.

2.170 5.2.C.

Localisation of reduced ring is again reduced for reduce is a local property.

For $\mathbb{A}^n_k$: polynomial ring $k[x_1, ..., x_n]$ is reduced.

For $\mathbb{P}^n_k$: it's similar, since taking subring will preserve reducedness.

2.171 5.3.1. Proposition

2.172

We'll try to interpret the equality in the proof by applying Hartshorne [8] Chap II Exercise 2.16.

$$\begin{aligned}
D_B(g) &= \operatorname{Spec} B_g \\
&= \operatorname{Spec} B_g \cap \operatorname{Spec} A_f \\
&= (\operatorname{Spec} B)_g \cap \operatorname{Spec} A_f \quad \text{by Hart Ex 2.16} \\
&= D_{A_f}(g') \quad \text{by Hart Ex 2.16} \\
&= (\operatorname{Spec} A_f)_{g'} \\
&= \operatorname{Spec} A_{fg''}.
\end{aligned}$$

Here g'' is specified as in the textbook [19], and g' is defined as

$$g \in B_g \to g' \in \Gamma(\operatorname{Spec} A_f, \mathcal{O}_X|_{\operatorname{Spec} A_f}) = A_f.$$

For $\operatorname{Spec} B_g = \operatorname{Spec}(A_f)_{g'}$, we need [8] Hartshorne Chap II Exercise 2.16.

2.172.1

Or we could do this...

Consider the following commutative diagram with corresponding ideals and elementes as labeled.

$$\begin{tikzcd}
& g \in \mathfrak{p}' \triangleleft B \arrow[dl] \arrow[dr] & \\
g' \in \mathfrak{q} \triangleleft A_f \arrow[rr] & & g/1 \in \mathfrak{p} \triangleleft B_g
\end{tikzcd}$$

By commutativity of the diagram, given a prime ideal $\mathfrak{p}$ then it corresponding uniquely to $\mathfrak{p}'$ and $\mathfrak{q}$; given $g/1$, then there exist g and g'.

$$\begin{aligned}
\operatorname{Spec} B_g &\cong \{\mathfrak{p}' \triangleleft B \mid g \notin \mathfrak{p}'\} \\
&\cong \{\mathfrak{q} \triangleleft A_f \mid g' \notin \mathfrak{q}\} \\
&\cong \operatorname{Spec} A_f \setminus \{[\mathfrak{p}] \ : \ g' \in \mathfrak{p}\} = \operatorname{Spec}(A_f)_{g'}.
\end{aligned}$$

2.173

*Let X be a scheme with **affine** open subsets $V \subset U \subset X$. Given $f \in \Gamma(U, \mathcal{O}_X)$ such that $\mathbf{D}_U(f) \subset V \subset U \subset X$. Then $\mathbf{D}_U(f) = \mathbf{D}_V(f|_V)$.*

?

2.174 5.3.2. Affine Communication Lemma.

See this POST.

Question: why stalk-local implies affine-local?

2.175 5.3.A.

Show that locally Noetherian schemes are quasi-separated.

Proof. According to Section 2.167, it suffices to show the existence of a cover of X by affine open satisfies certain property. ☐

2.176 6.1.2. Theorem.

For the claim $\widetilde{M}|_{\mathrm{Spec}\,A_f} = \widetilde{M_f}$ See Theorem 1.7. Chap 5 on Page 160 [13].

The isomorphism of two sheaves of modules

$$\widetilde{M_i}|_{\mathbf{D}(f_i)\cap\mathbf{D}(f_j)} \simeq \widetilde{M_j}|_{\mathbf{D}(f_i)\cap\mathbf{D}(f_j)}$$

yield isomorphisms $\phi_{ij} : (M_i)_{f_j} \simeq (M_j)_{f_i}$ of $A_{f_if_j}$-modules. Notice that M_i admits A_{f_i}-module structure as assumed. Therefore,

$$\widetilde{M_i}|_{\mathbf{D}(f_i)\cap\mathbf{D}(f_j)\subset\mathrm{Spec}\,A} = \widetilde{M_i}(\mathbf{D}_{\mathrm{Spec}\,A_{f_i}}(\overline{f_j})) = (M_j)_{f_j}$$

where $\overline{f_i}$ denotes the image of f_i under restriction from $\Gamma(\mathrm{Spec}\,A_{f_i}, \mathcal{O}_{\dots})$ to $\Gamma(\mathrm{Spec}\,A_{f_i}|_{\mathrm{Spec}\,A_{f_j}}, \mathcal{O}_{\dots})$. Here the second equality applied result of Ex 2.16. Chap 2 [8].

Todo List

1. Define β
2. Check injective, and $\mathrm{Im}\,\beta = \mathrm{Ker}\,\gamma_{f_1}$.
3. Apply five lemma to conclude the isomorphism between $M_{f_i} \simeq M_1$.

Then the induced map in the category of quasi-coherent sheaves will be an isomorphism as expected.

To define β, it suffices to define $M_1 \to (M_i)_{f_1}$.

$$\widetilde{M_1} = \mathcal{F}|_{\mathbf{D}(f_i)} = \mathcal{F}|_{D(f_i)\cap D(f_1)} = \dots = \widetilde{M_i}|_{\mathrm{Spec}\,A_{f_1}}.$$

???

2.176.1 Remarks

The part for verifying the first condition of Affine Communication Lemma is clear. We'll focus on checking the second condition. Here we're given information for $\mathrm{Spec}\,A_{f_i}$ for $1 \le i \le n$. By definition of property P, we have $\mathscr{F}$ is $\mathscr{O}_{\mathrm{Spec}\,A}$-module and $\mathscr{F}|_{D_{\mathrm{Spec}\,A}(f_i)} = \widetilde{M_i}$. We wish to prove $\mathrm{Spec}\,A$ has the property P.

?? but as the statement of affine communication lemma, the information we get is $\mathrm{Spec}\,A_{f_i} \to X$, but even in that case, we just restrict and regard $\mathscr{F}$ as a sheaf on $\mathrm{Spec}\,A$

The text indicated that

$$\mathscr{F}|_{\mathbf{D}(f_i)\cap\mathbf{D}(f_j)} \simeq \widetilde{M_i}|_{\mathbf{D}(f_i)\cap\mathbf{D}(f_j)}.$$

More precisely, this could be written as

$$\begin{aligned} &\phi_i : \mathscr{F}|_{\mathbf{D}_{\mathrm{Spec}\,A}(f_i)} \simeq \widetilde{M_i} \\ \Rightarrow\ &\mathscr{F}|_{\mathbf{D}_{\mathrm{Spec}\,A}(f_i)\cap\mathbf{D}_{\mathrm{Spec}\,A}(f_j)} \simeq \widetilde{M_i}|_{\mathbf{D}_{\mathrm{Spec}\,A}(f_i)\cap\mathbf{D}_{\mathrm{Spec}\,A}(f_j)} \\ \Rightarrow\ &(F|_{\mathbf{D}_{\mathrm{Spec}\,A}f_i})|_{\mathbf{D}_{\mathrm{Spec}\,A_{f_i}}(\overline{f_j})} \simeq \widetilde{M_i}|_{\mathbf{D}_{\mathrm{Spec}\,A_{f_i}}(\overline{f_j})}. \end{aligned}$$

Here we denote $\overline{f_j} = (\mathrm{Res}_{..} : \mathrm{Spec}\, A \to \mathrm{Spec}\, A_{f_i})(f_j)$, i.e. we denote $\overline{f_j}$ as the image of f_j under restrction to $\mathrm{Spec}\, A_{f_i}$. Notice that $\widetilde{M_i}$ is a sheaf on $\mathrm{Spec}\, A_{f_i}$, which gives us

$$\widetilde{M_i}|_{\mathbf{D}_{\mathrm{Spec}\, A_{f_i}}(\overline{f_j})} = (M_i)_{\overline{f_j}}.$$

So a more precise way to describe ϕ_{ij} is by indicating that the function f_j is the image $\overline{f_j}$, or ?

$$\phi_{ij} : (M_i)_{\overline{f_j}} \to (M_j)_{\overline{f_i}}.$$

In textbook it was regarded as $A_{f_i f_j}$-module. More precisely, it localised twice.

The assumption of condition 2 give us existence of $\mathscr{F}$ (on $\mathrm{Spec}\, A$, if not then restrict), we need to provide the existence of A-module M such that $\widetilde{M} \simeq \mathscr{F}$. We define it as the kernel of the exact sequence as indicated in the textbook.

What we need to do is to glue sheaves from each M_i to M. In other words, we're trying to check isomorphism of sheaves on distinguished opens.

$$\forall\, \mathrm{Spec}\, A_{f_i}, \widetilde{M}|_{\mathrm{Spec}\, A_{f_i}} = \mathscr{F}|_{\mathrm{Spec}\, A_{f_i}} = \widetilde{M_i} \;\Leftrightarrow\; \widetilde{M} = \mathscr{F}.$$

To define β, we recall that we've defined ϕ_{ij}, then we can use them to define

$$\begin{aligned} \beta : M_1 &\to (M_1)_{f_1} \times \cdots \times (M_n)_{f_1} \\ m &\mapsto ((m/1), \phi_{12}((M_1 \to (M_1)_{f_2})), ..., \phi_{1n}((M_1 \to (M_1)_{f_n})) \end{aligned}$$

$$\begin{tikzcd} & M_1 \arrow[dl] \arrow[dr, dashed] & \\ (M_1)_{f_2} \arrow[rr, "\phi_{12}"] & & (M_2)_{f_1} \end{tikzcd}$$

Clearly this map is injective. Remains to check $\mathrm{Im}\, \beta = \mathrm{Ker}\, \gamma_{f_1}$. How?

Or another approach is to identify (6.1.2.3) and (6.1.2.4) term by term. The later (6.1.2.4) is exact and β will be obvious to define.

2.176.2 References

See NOTES. Or Proposition-Definition 5.1. of Algebraic Geometry II on Page 20 [**?**].

2.177 6.1.B.

Here is a POST discussing a non-example of quasi-coherent sheaf. And HERE.

2.178 6.2.1. Definition: Distinguished Affine Base

See this POST.

2.179 6.2.D.

$\Rightarrow$: Suppose $\mathscr{O}_X$ module $\mathscr{F}$ is quasi-coherent, then the above diagram gives us $\Gamma(\operatorname{Spec} A, \mathscr{F}) = M$ for some A-module M.

$\Leftarrow$:

2.180 6.2.G.

2.181 6.4.A.

2.181.1 (i)

Localisation (with respect to f) is exact. Hence the following is exact

$$(R^{\oplus n})_f = (R_f)^{\oplus n} \to M_f \to 0.$$

Another way to interpret this is by noticing that the generator $m_1, ...m_n \in M$ will be send to generators of M_f as A_f-module.

2.181.2 (ii)

We can define $\phi : M \to \prod_{1\leq i\leq n} M_{f_i}$ by $m \mapsto (m/1, m/1, ..., m/1)$, i.e. use universal property of product with natural map of localisation.

Now we have M_{f_i} is f.g. as A_{f_i}-module hence we can assume there's a surjective for each i

$$(A_{f_i})^{\oplus l_i} \to M_{f_i} \to 0$$

and we denote $L = \sum_{1\leq i\leq n} l_i$.

$$\begin{tikzcd}
A^{\oplus L} \arrow[r, dashed] \arrow[d, hook, "\psi"'] & M \arrow[d, hook, "\phi"] \\
\prod_{1\leq i\leq n} A_{f_i}^{\oplus l_i} \arrow[r, two heads, "\prod \gamma_i"'] & \prod_{1\leq i\leq n} M_{f_i}
\end{tikzcd}$$

Now we wish to check ϕ is injective. It suffices to check $\operatorname{Ker}\phi \subset 0$. For any $m \in M$ such that $\phi(m) = (m/1, ..., m/1) = 0$, we wish to show $m = 0$. Notice that $m/1 = 0 \in M_{f_i}$ implies $f_i^{k_i} m = 0 \in M$ for some integer k_i. Therefore we can pick $K = \sum k_i$ (or other finite integer large enough)

$$m = 1m = (a_1 f_1 + ... + a_n f_n)m = (a_1 f_1 + ... + a_n f_n)^K m = 0.$$

For any $m \in M$, we identify it with

$$\phi(m) = (\pi_1 \circ \phi(m), ..., \pi_n \circ \phi(m)) \in \prod_{1\leq i\leq n} M_{f_i}$$

where π_i be projection to i-th component. Then $\pi_i \circ \phi(m) \in M_{f_i}$, and under each surjection γ_i it will correspond to finitely many generators $b_{i,j}/f_i^\bullet \in M_{f_i}$ where $1 \leq j \leq l_i$. For each $1 \leq i \leq n$, we know

$$\begin{aligned} M_{f_i} &= A_{f_i} \langle b_{i,1}/f_i^\bullet, b_{i,2}/f_i^\bullet, ..., b_{i,l_i}/f_i^\bullet \rangle \\ &= A_{f_i} \langle b_{i,1}, b_{i,2}, ..., b_{i,l_i} \rangle \end{aligned}$$

Or we can say there's a surjective map from $A^{\oplus l_i} \to M_{f_i}$, and we can inductively build a surjective map from $A^{\oplus L} \to \prod_{1 \leq i \leq n} M_{f_i}$. Restrict the codomain of this map will give us a surjection that proves M is finitely generated A-module

$$A^{\oplus L} \to M.$$

2.181.3 Comments

Localisation commute with arbitrary direct sum HERE. See Tag 01OR, and Zariski-local properties of modules and algebras Tag 00EO. See this POST. For official reference, see Tag 00EN. Here the actual proof relies on Lemma 1 of Tag 00EO.

Lemma: If each $M_{f_i} = 0$, then $M = 0$.

Proof. For $\mathfrak{p} \in \operatorname{Spec} A$, we can cover it by some $\mathbf{D}(f_i)$. In particular, $f_i \notin \mathfrak{p} \triangleleft A$.

A result about further localisation Tag 02C6 will help us as

$$\begin{aligned} (M_{f_i})_\mathfrak{p} &= \overline{\{A - \mathfrak{p}\}^{-1}}(\{1, f_i, f_i^2, ...\}^{-1} M) \\ &= (\{A - \mathfrak{p}\}\{1, f_i, f_i^2, ...\})^{-1} M \\ &= (\{A - \mathfrak{p}\})^{-1} M = M_\mathfrak{p}. \end{aligned}$$

Given the assumption that $M_{f_i} = 0$, then $M_\mathfrak{p} = 0$ for arbitrary $\mathfrak{p} \in \operatorname{Spec} A$. And now we're ready to apply "being 0 is a local property" Tag 00HN to finish the proposition.

□

2.182 6.4.6. Lemma:

See this POST with an elegant solution. Note $\tilde{f}$ is defined by simply restriction. In this post, it's worth mention Tag 07JX.

See Ex. 5.18. [1]. This approach requires Schanuel's Lemma 5.17. [1].

2.183 6.4.B.

2.184 6.4.C.

Similary to Section 2.183, for references see Tag 00EN.

2.185 6.6.B.

For the counterexample, see HERE and HERE for a concrete description of computing localisation of quotient rings in a simple setting. And in general we have this example HERE.

Support of a module need not be closed in general.

Proof. According to the hint, we take $A = \mathbb{Z}$ and

$$M = \oplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}.$$

Notice that in Principal Ideal Domain $\mathbb{Z}$, we have for two primes $p, q \in \mathbb{Z}$,

$$(\mathbb{Z}/p\mathbb{Z})_{(q)} = \begin{cases} 1 \\ \mathbb{Z}/p\mathbb{Z} \end{cases}$$

Because when $p \neq q$, then... $\square$

For a more general treatment, see Example 6.25. of Gathmann's Notes in Commutative Algebra P58. And HERE.

2.186 6.6.I.

Proof. Kernel K is a nonzero submodule of M over a Noetherian ring A. Therefore it admits an associate prime $\mathfrak{p} \triangleleft A$ that is the annihilator of some element $m \in K \subset M$. Notice the injection

$$A/\mathfrak{p} \to K \to M,$$

then $\mathfrak{p} \in \operatorname{Ass} M$. Then under the natural map,

$$m \mapsto m/1 \in M_{\mathfrak{p}}$$

is nonzero. $\square$

2.187 6.6.16.

As $\mathbb{Z}$-module $\mathbb{Z}$ contains 0 as zero-divisor. Therefore

$$\operatorname{Ass}_{\mathbb{Z}} \mathbb{Z} = \{0\}.$$

On the other hand, we have

$$\operatorname{Ass}_{\mathbb{Z}} \mathbb{Z} \cup \operatorname{Ass}_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \subset \{0, (2)\}.$$

2.188 6.6.J.

See Lemma (17.16). [1]. Such a cute proof...

2.189 6.6.L.

Recall that the union of all associate primes consisits only zerodivisors. While $A = k[x, y]$ is an integral domain, any filtration must contains some nonzero primes, which cannot be associate prime.

2.190 7.2.A.

MORPHISMS OF RINGED SPACES GLUE.

Proof. For topological map f, the existence is clear (existence and the restriction will behave as desired). It remains to define the morphism of sheaves

$$f^\sharp : \mathscr{O}_Y \to \pi_*\mathscr{O}_X$$

and check that $(\pi|_{U_i})^\sharp : \mathscr{O}_Y \to (\pi|_{U_i})_*\mathscr{O}_{U_i} = \pi_i^\sharp : \mathscr{O}_Y \to \pi_{i,*}\mathscr{O}_{U_i}$.

By assumption, we have $\{\pi_i : U_i \to Y\}_{i\in\bullet}$ with $\pi_i^\sharp : \mathscr{O}_Y \to \pi_{i,*}\mathscr{O}_{U_i}$ for each $i \in \bullet$. Also recall that

$$\begin{gathered}(\pi_i|_{U_i\cap U_j})^\sharp : \mathscr{O}_Y \to (\pi_i|_{U_i\cap U_j})_*\mathscr{O}_{U_i\cap U_j} = \\ (\pi_j|_{U_i\cap U_j})^\sharp : \mathscr{O}_Y \to (\pi_j|_{U_i\cap U_j})_*\mathscr{O}_{U_i\cap U_j}.\end{gathered}$$

Therefore for any open subset $W \subset Y$, we have $(\pi_i|_{U_i\cap U_j})^\sharp(W) = (\pi_j|_{U_i\cap U_j})^\sharp(W)$. In particularly, we have

$$\begin{aligned}(\pi_i|_{U_i\cap U_j})_*\mathscr{O}_{U_i\cap U_j}(W) &= \mathscr{O}_{U_i\cap U_j}((\pi_i|_{U_i\cap U_j})^{-1}(W)) \\ &= \mathscr{O}_{U_i\cap U_j}(\pi_i^{-1}(W)\cap U_i\cap U_j) \\ &= \mathscr{O}_X(\pi_i^{-1}(W)\cap U_i\cap U_j) \\ &= \mathscr{O}_X(\pi_j^{-1}(W)\cap U_i\cap U_j) \\ &= (\pi_j|_{U_i\cap U_j})_*\mathscr{O}_{U_i\cap U_j}(W).\end{aligned}$$

Now we need to define $\pi^\sharp : \mathscr{O}_Y \to \pi_*\mathscr{O}_X$. For every open subset $W \subset Y$, we have to define a map

$$\pi^\sharp(W) : \mathscr{O}_Y(W) \to \pi_*\mathscr{O}_X(W).$$

For every $s \in \mathscr{O}_Y(W)$, we define(at this moment, we've constructed topologically the map π)

$$\begin{aligned}s_i := \pi_i^\sharp(W)(s) &= (\mathscr{O}_Y(W) \to \pi_{i,*}\mathscr{O}_{U_i}(W))(s) \\ &\in \mathscr{O}_X(\pi_i^{-1}(W)\cap U_i) = \mathscr{O}_X(\pi^{-1}(W)\cap U_i)\end{aligned}$$

We can check

$$\begin{aligned}
s_i|_{\pi^{-1}(W)\cap U_i\cap\pi^{-1}(W)\cap U_j} &= (\mathscr{O}_X(\pi^{-1}(W)\cap U_i)\to\mathscr{O}_X(\pi^{-1}(W)\cap U_i\cap U_j))(s_i)\\
&= (\mathscr{O}_X(\pi_i^{-1}(W)\cap U_i)\to\mathscr{O}_X(\pi_i^{-1}(W)\cap U_i\cap U_j))(s_i)\\
&= (\mathscr{O}_Y(W)\to\mathscr{O}_X(\pi^{-1}(W))\to\mathscr{O}_X(...))(s)\\
&= (\mathscr{O}_X(\pi_j^{-1}(W)\cap U_j)\to\mathscr{O}_X(\pi_j^{-1}(W)\cap U_i\cap U_j))(s_j)\\
&= s_j|_{\pi^{-1}(W)\cap U_i\cap\pi^{-1}(W)\cap U_j}
\end{aligned}$$

Notice that we have $\{\pi^{-1}(W)\cap U_i\}_{i\in\bullet}$ covers $\pi^{-1}(W)$ and $\mathscr{O}_X$ is a sheaf. Therefore there exist $S\in\mathscr{O}_X(\pi^{-1}(W))=\pi_*\mathscr{O}_X(W)$ such that $S|_{U_i}=s_i$. Hence for every $W\subset Y$ we've defined the map $\pi^\sharp$. And we can check that for all open subsets $W\subset U_i$,

$$(\pi|_{U_i})^\sharp(W)=\pi_i^\sharp(W)\Rightarrow(\pi|_{U_i})^\sharp=\pi_i^\sharp.$$

???

$\square$

2.190.1 Comments

See Prop 3.5. [7] on Page 70. For details, see “Glueing morphisms of sheaves together - can I just do this?” and “morphism of ringed spaces glue”.

2.191 7.2.E.

See Hartshorne [8] Page 72 for details.

2.192 7.3.A.

For a local homomorphism, we have

$$\mathfrak{m}_q\subset\mathscr{O}_{Y,q}\to\mathscr{O}_{X,p}\to\kappa(p).$$

It will induce a map $\kappa(q)\to\kappa(p)$.

2.193 7.3.B.

2.194 7.3.D.

!

2.195 7.3.G.

See [7] Proposition 3.4. Chap 3 on Page 69.
See Tag 01I1.

2.196 7.3.6.

For "fun question": is that maximalSpec?

2.197 7.3.L.

Nice hint. See Hartshorne Chap 2 Exercise 2.7. [8].

2.198 7.4.

Skipped...

2.199 7.5.

2.200 8.1.1.

2.200.1 Remarks

See a post discussing "Does "local on the target" mean the same thing as "local on the base"?" HERE.

2.201 8.1.A.

Proof. Notice that $X \times_S Y' \simeq X \times_Y Y \times_S Y'$ and $X' \times_{Y'} X \times_S Y' \simeq X' \times_{Y'} Y' \times_S X \simeq X \times_S X'$. Therefore apply (ii) and (i) we can conclude $X \times_S X' \to Y \times_S Y'$ has property P. $\square$

2.202 8.1.B.

Proof. Requirement (i) is clear.

For (ii), we have the following diagram

$$\begin{tikzcd} X \times_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "\simeq"] & Y \end{tikzcd}$$

Observe that $X \times_Y Y' \simeq Y \times_Y Y' \simeq Y'$. Hence (ii) is satisfied.

For (iii), part (a) is clear. It remains to consider part (b). And for this part we refer to [8] Exercise 2.17. (a) on Page 81 of Chapter 2. $\square$

2.203 8.2.B.

2.203.1 (a)

See this post HERE on StackExchange. To show a ring morphism is not integral, we can consider the lying over property.

2.204 8.3.B.

2.204.1 (a)

See 5.3.4. [19].

Proof. Notice that any morphism of schemes consists of a continuous map such that

$$\pi^{-1}(U) \subset X$$

is open. Then we apply Exercise 5.1.C., which shows that X is a Noetherian topological space. Apply Exercise 3.6.U. shows that any open subset, including $\pi^{-1}(U)$ is quasi-compact. And this proves that this morphism π is quasi-compact. □

?

2.204.2 (b)

2.205 8.3.D.

Clear. Because affine scheme is both quasi-compact and quasi-separated.

2.206 8.3.F.

For example (3), see Exercise 9.12. by Gathmann notes on Commutative algebra [6], Page 83.

2.207 8.3.H.

Apply 8.2.C. and use the definition "for all".

2.208 23.1.C.

Almost by definition...

2.208.1 References

See this POST, and a detailed POST.
For details, see [15] Theorem 7.2. on Page 405.

2.209 23.4.F.

See remarks of Section 2.45.

Part III

Algebraic Geometry Qing Liu

2.210 Theorem 2.4.

Notice that

$$N_2' = N'/N_1 \subset N/N_1 = N_2.$$

2.211 Lemma 2.9.

$$\begin{tikzcd}
\mathcal{F}(V_x) \arrow[rr, "\text{Res}"] \arrow[dr] & & \mathcal{F}(U_x) \arrow[dl] \\
& s_x \in \mathcal{F}_x &
\end{tikzcd}$$

Assume $s_x = 0$, while in Category of Abelian groups the homomorphism is exactly group homomorphism. So there must exist some U_x such that $s\mid_{U_x} = 0$ as a pre-image of s_x.

2.212 Exercise 2.7.

Let $\mathcal{B}$ be a base of open subsets on a topological space X. Let $\mathcal{F}, \mathcal{G}$ be two sheaves on X. Suppose that for every $U \in \mathcal{B}$ there exists a homomorphism $\alpha(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ which is compatible with restrictions. Show that this extends in a unique way to a homomorphism of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$. Show that if $\alpha(U)$ is surjective (resp. injective) for every $U \in \mathcal{B}$, then α is surjective (resp. injective).

Proof. There are two definitions of sheaf on the whole space in terms of sheaf on base. One is to define section of an open subset $W \subset X$ (open subset W is not necessarily in base $\mathcal{B}$) by *compatible germs*, another approach is to define that by *limit*.

To construct the map, we just take the definition of *limit*. And the universal property will induce a unique map $\alpha(W)$ for arbitrary open subset $W \subset X$, which implies there exists a unique morphism $\alpha : \mathcal{F} \to \mathcal{G}$ as we expected.

Notice that for sheaves on base, the stalk at a point $x \in X$ is the same as the sheaf on whole space.

$$\operatorname*{colim}_{x \in W \subset X,\ W \text{ open}} = \mathcal{F}_x = \operatorname*{colim}_{x \in U \subset X,\ U \in \mathcal{B}}.$$

Another way to see this (compatible germs) is by 2.5.B. EXERCISE. [19]. Therefore we have the following diagram, which maps to the same stalk.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, two heads] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\mathcal{F}(U \cap U_x) \arrow[r] \arrow[d] & \mathcal{G}(U \cap U_x) \ni t|_{U \cap U_x} \arrow[d] \\
s \in \mathcal{F}(W) \arrow[r, two heads] \arrow[d] & \mathcal{G}(W) \ni t|_W \arrow[d] \\
(W, s) \in \mathcal{F}_x \arrow[r] & \mathcal{G}_x \ni t_x
\end{tikzcd}$$

For surjective. We wish to prove the induced map on stalks $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for any $x \in X$. While $\mathcal{B}$ covers the whole space, we can pick some $U \in \mathcal{B}$ such that $x \in U$. Given any $t_x \in \mathcal{G}_x$, i.e. we have $t \in \mathcal{G}(U_x)$ for some open subset $x \in U_x \subset X$. In case that $U \cap U_x$ isn't in the base, we can further restrict to $t|_W$ for some $W \in \mathcal{B}$. By commutativity of the diagram we know there's some $s \in \mathcal{F}(W)$ such that $s_x = (W, s)$ will be mapped to t_x. Hence the map induced on stalk $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\mathcal{F}(W) \arrow[r, hook] \arrow[d] & \mathcal{G}(W) \arrow[d] \\
\mathcal{F}_x \arrow[r] & \mathcal{G}_x
\end{tikzcd}$$

For injective. Now we start by picking two elements $a_x, b_x \in \mathcal{F}_x$, where $a \in \mathcal{F}(U_a)$ and $b \in \mathcal{F}(U_b)$. Then we pick an open subset $W \in \mathcal{B}$ such that $B \subset U \cap U_a \cap U_b$. The $\alpha(W)$ is injective by assumption. Then $a|_W$ and $b|_W$ will be mapped to the same element in $\mathcal{G}_x$. By commutativity this proves the injective of α_x.

□

2.213 Proposition 3.1.

2.213.1 (b)

For the proof of isomorphism, it implicitly used Corollary 7.5 of [2].

2.214 Lemma 3.35.

2.214.1 (a)

Take n to be the largest index such that $a_n \notin I^h$, and choose b_m similarly. They must exist for $a \notin I^h$ by assumption. And in general, if we have $a \notin I^h$, we can always subtract all homogeneous part in the tail of a to get a new a'. And denote $a' = a$.

Part IV

Gathmann Notes

2.215 Exercise 2.40.

See a post HERE.

Part V

Ulrich Gortz, Torsten Wedhorn

2.216 Proposition 2.27.

$$\begin{tikzcd}
s_V \in \mathcal{G}(V) \arrow[rr, "\phi_V"] \arrow[d] & & f_*\mathcal{F}(V) \arrow[d] \\
\mathcal{G}(f(U)) \arrow[r, "\phi_{f(U)}"] \arrow[d] & f_*\mathcal{F}(f(U)) \ni & \psi_U^\sharp(s) := \phi_V(s_V)|_{f(U)} \\
s \in \operatorname{colim} \mathcal{G}(?) & &
\end{tikzcd}$$

2.217 Exercise 2.11.

2.217.1 (a)

Pick $a, b \in X$ such that $u(a) = u(b)$. We wish to prove $a = b$.

By [1] Corollary 7.5. (1), the element $u(a) \in Y$ admits some $l \in I$ such that there exists $y \in Y_l$ such that

$$\beta_l(y) = u(a).$$

Suppose $l \geq i$, then we're done by diagram chase. Suppose $l < i$, by commutativity of the diagram we can map the element y under $\beta_{li} : Y_l \to Y_i$. And then apply a diagram chase argument.

2.217.2 (b)

Pick an element $y \in Y$. It corresponds to some element Y_l for some l by Corollary 7.5. (1), map it to Y_i if necessary. Then apply diagram chase because we know at least u_i is surjective.

2.218 Proposition 3.4.

2.218.1 Second Proof

Here, we define $f(x) = \mathfrak{p}$ as the kernel of the morphism

$$A \to \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x} \to \kappa(x),$$

which induces an embedding

$$\begin{tikzcd}
A \arrow[d] \arrow[dr] & \\
A/\mathfrak{p} \arrow[r, hook, dashed] & \kappa(x)
\end{tikzcd}$$

2.219 Prop 3.5.

See . See 7.2.A. [19].

Chapter 3

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