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# 1 Notes

## 1.1 Resources

HERE is a post on A Term of Commutative Algebra by Altman.

HERE is a LaTeX version of Hideyuki Matsumura's Commutative Algebra  
Digital Version of Atiyah's Book HERE

See Old Lecture Homepages of ETH HERE. There are some courses on  
commutative algebra with solutions for homeworks.

See notes on commutative algebra by Prof. Dr. Andreas Gathmann HERE.

An informal post on errata of Atiyah HERE.

UCB Math 250B by Arthur HERE.

Website for commutative algebra community HERE.

See a very comprehensive notes by Pete Clark HERE.

See a post for Errata HERE.

Math 221 2019 HERE; HERE; 2007 HERE.

## Part I

# Introduction to Commutative Algebra by Atiyah MacDonal

## 2 Ex 1.

See MAT's Page 1.

For a nilpotent element  $x$  and a unit  $a$ , we can prove that  $a + x$  is again a unit.

Say  $x^n = 0$  for some  $n \in \mathbb{Z}$ , and set

$$y = -a^{-1}x.$$

Notice that

$$(1 - y)(1 + y + \cdots + y^{n-1}) = 1 - y^n = 1 - (-a^{-1}x)^n = 1,$$

which suggests that  $1 - y$  is a unit. So we have  $a + x = a(1 - y)$  is product of unit, thus it's a unit as expected.

## 3 Ex 2.

### 3.1 (i)

$\Leftarrow$ : standard application of **Ex.1**

$\Rightarrow$ : We follow the hint and suppose  $b_0 + b_1x + \cdots + b_mx^m$  is the inverse of  $f$ . We prove by induction on  $r$  that  $a_n^{r+1}b_{m-r} = 0$ :

for  $r = 0$ , note  $a_nb_m$  is the coefficient for  $x^{n+m}$ , which must be 0;

assume the validity for  $r - 1$ . HERE is the way to deal with the coefficients.

Consider the coefficients of  $x^{n+m-1}$ , which must be  $a_{n-1}b_m + a_nb_{m-1} = 0$ .

If we multiply both sides with  $a_n$  will have

$$a_{n-1}a_nb_m + a_n^2b_{m-1} = a_n^2b_{m-1} = 0$$

since  $a_nb_m = 0$ . In general, we can continue by the similar fashion, namely look at the coefficients of  $x^{n+m-i}$  for some natural number  $i$ :

$$\sum_{\alpha+\beta=n+m-i} a_\alpha b_\beta.$$

Multiply  $a^i$  on both sides will make all terms except the last one be 0, which forces the last term  $a_n^{i+1}b_{m-i} = 0$  to be zero. In particular, let  $i = m$  we have  $a_n^{i+1}b_0 = 0$ . Recall that  $b_0$  is a unit, so we have  $a_n^{m+1} = 0$  as desired. So  $-a_nx^n$  is a nilpotent and  $f + (-a_nx^n) = a_0 + \cdots + a_{n-1}x^{n-1}$  is a unit by Ex 1. So we can apply the argument again, implies that  $a_{n-1}$  is nilpotent. Until we only have the constant term, namely  $a_0 = a_0 + a_1x - a_1x$  must be a unit.

### 3.2 (ii)

$\Rightarrow$ : Suppose  $f$  is nilpotent, thus we have

$$\begin{aligned} 0 &= f^m \\ &= (a_0 + a_1x + \cdots + a_nx^n)^m \\ &= a_0^m + (ma_0^{m-1}a_1)x + \cdots + a_1^m x^m + \cdots + (a_nx^n)^m, \end{aligned}$$

which forces, in particular,  $a_0$  to be nilpotent. Then we know  $f - a_0 = a_1x + a_2x^2 + \cdots + a_nx^n$  is nilpotent since all nilpotent elements form an ideal. We define it as  $f_1 := f - a_0$ . It has a corresponding  $m_1 \in \mathbb{N}$  such that

$$0 = f_1^{m_1} = (a_1^m)x^{m_1} + \cdots$$

The coefficient for the term  $x^{m_1}$  must be zero, which proves  $a_1$  is nilpotent. And for all finitely many coefficient we can inductive do this and conclude they're all nilpotent as desired.

$\Leftarrow$ : To prove  $f$  is nilpotent, we only have to find a "large" enough number suffices to turn  $f$  into 0. Since all coefficients are nilpotent, we could find  $d_0, d_1, \dots, d_n \in \mathbb{Z}$  such that

$$a_i^{d_i} = 0$$

for  $0 \leq i \leq n$ . Denote  $\sum_{j=0}^n d_j = D$ . We could verify that

$$\begin{aligned} f^D &= (a_0 + a_1x + \cdots + a_nx^n)^D \\ &= [a_0^D + \cdots + (a_nx^n)^D] \\ &= C_0 + C_1x + \cdots + C_{Dn}x^{Dn} \end{aligned}$$

where  $C_i = \sum \prod_{i \in I_n} a_i$  for some finite index set  $I_n \subset \{0, 1, 2, \dots, n\}$ . The coefficient  $C_i$  is sum of some coefficient that collected various powers of  $x$ , namely terms of  $x^j$  where  $j \in I_i$ . Each such power will contain various  $a_i$ , but according to Pigeon Principle we've lifted them to power  $D$ , at least one of them will become 0 and then this term is zero. All such term will be zero, and their sum,  $C_i$ , will be zero. So we have  $f^D = 0$  as desired.

### 3.3 (iii)

$\Leftarrow$ : this is precisely the definition.

$\Rightarrow$ : We follow the hint. Suppose  $f$  is a zero-divisor, then there exists a nonzero element  $g$  in the polynomial ring such that  $fg = 0$ . We can further suppose  $g$  is the smallest degree polynomial satisfy the condition we mentioned.

$$0 = fg = (a_0 + a_1x + \cdots + a_nx^n)(b_0 + b_1x + \cdots + b_mx^m).$$

In particular we have  $a_nb_m = 0$ . Notice that we'll have

$$(a_ng)(f) = 0.$$

But polynomial  $a_n g$  is of degree less than  $g$  since  $a_n b_m = 0$ , this contradicts least degree property of  $g$ , which implies that  $a_n g = 0$ . Let's examine the equality above again:

$$\begin{aligned} 0 = fg &= (a_0 + a_1x + \cdots + a_{n-1}x^{n-1})(b_0 + b_1x + \cdots + b_mx^m) \\ &\quad + (a_nx^n)(b_0 + b_1x + \cdots + b_mx^m) \\ &= (a_0 + a_1x + \cdots + a_{n-1}x^{n-1})(b_0 + b_1x + \cdots + b_mx^m). \end{aligned}$$

This time we look at coefficient for term  $x^{n+m-1}$ , which is  $a_{n-1}b_m$  precisely. Since the product is 0, which forces in particular  $a_{n-1}b_m = 0$ . Then we can use polynomial  $a_{n-1}g$ , which is of degree less than  $g$ , but will annihilate  $f$ . We can continue in this fashion and conclude that all  $a_i g = 0$  for all  $i \in \{1, 2, \dots, n\}$ . While  $g$  is non-zero so we can find some  $b_i \neq 0$  such that

$$a_i b_j = 0$$

as expected.

### 3.4 (iv)

Solutions from StackExchange.

*For one direction, refer to Gauss's Lemma.*

In fact, we have many versions of Gauss's Lemma.

## 4 Ex 3.

Induction on  $r$ , which holds for  $r = 1$ . Suppose all results hold for  $A[x_1, \dots, x_{r-1}]$ .

$f$  is a unit in  $A[x_1, \dots, x_{r-1}][x_r] \Leftrightarrow a_0$  is a unit in  $A[x_1, \dots, x_{r-1}]$  and  $a_1, \dots, a_n \in A[x_1, \dots, x_{r-1}]$  are nilpotent.

One thing to notice is that here the coefficients are polynomial with indeterminants, so we can use the induction assumption to finish (since polynomial is of finite degree and we only have to apply finitely many times of results).

For example,  $a_0$  is a polynomial and a unit in  $A[x_1, \dots, x_{r-1}]$ , thus we could distract  $b_0 \in A[x_1, \dots, x_{r-2}]$  is a unit and finitely many polynomial  $b_1, \dots, b_r$  that are nilpotent. After applying (i) for finitely many times, we could find a "pure" coefficient in  $A$  that's a unit, along with many polynomial that're nilpotent. Since their coefficients are all nilpotent, we could repeat (ii) for finitely many times and argue that all coefficients other than the first one in  $A$  are nilpotent.

So the generalised version of theorem states that the constant term is a unit in  $A$ , and other coefficients are nilpotent.

## 5 Ex 4.

Maximal ideal is prime ideal, thus Jacobson Radical  $\supset$  Nilradical.  
So the non-trivial direction of proof starts by picking up an element  $f$  from Jacobson Radical  $\mathfrak{R}$ . By Prop 1.9 we know that  $1 - fy$  is a unit for any  $y$ .  
Now we let  $y = x$ , this gives us

$$1 - fx = 1 - a_0x - a_1x^2 - \dots - a_nx^{n+1}$$

is a unit. Use (ii) of Ex. 2, we know that all  $a_0, a_1, \dots, a_n$  are nilpotent then  $f$  is nilpotent by (iii) of Ex 2.

## 6 Ex 5.

### 6.1 Formal Power Series

Here are some remarks for Formal Power Series: see a post [HERE](#), and [HERE](#).

In the second post: Ring of formal power series over a principal ideal domain is a unique factorisation domain. A quick solution is to use Kaplansky's Criterion to determine formal power series  $R[[X]]$  is UFD. See a paper [HERE](#).

### 6.2 (i)

$\Leftarrow$ : ✓

$\Rightarrow$ : Assume  $g := b_0 + b_1x + b_2x^2 + \dots \in A[[x]]$  to be the inverse for  $f$  in the formal power series. We only need to look at the lowest power of their product

$$1 = fg = (b_0 + b_1x + \dots)(a_0 + a_1x + \dots) = b_0a_0 + (b_0a_1 + b_1a_0)x + \dots$$

The fact that this equality holds implies that we must have  $b_0a_0 = 1$ , which gives us the desired result.

### 6.3 (ii)

Suppose we have  $f^n = 0$  for some integer  $n > 0$ . If we look at the constant coefficient, we'll have  $a_0^n = 0$ , so  $a_0$  nilpotent. Then inductively prove  $a_i$  is nilpotent for all  $i \in \mathbb{N}$  by following Ex 2. (ii).

Converse is ... ???

### 6.4 (iii)

$\Leftrightarrow$ : Suppose  $f$  belongs to the Jacobson radical, so according to Prop 1.9 this is equivalently to say

$$1 - fy$$

is a unit in  $A[[x]]$  for all  $y \in A[[x]]$ . By part (ii) we know that this is the same as  $1 - a_0 b_0$  is a unit in  $A$  for any  $b_0 \in A$  where  $b_0$  is the constant coefficient for  $y \in A[[x]]$ . Recall part (ii) again we know that this is equivalently to say  $a_0$  belongs to the Jacobson Radical of  $A$ .

HERE is a good counterexample explaining the reason of why we defined product of ideal with "finiteness".

## 6.5 (iv)

??? Here I assume the underlying ring homomorphism is  $f : A \rightarrow A[[x]]$  which sends a ring element  $a \mapsto a$  in the formal power series.

**Fact:** preimage (under ring homomorphism) of a prime ideal is again a prime ideal. The proof is basically unwrap the definition.

**Fact:** how about maximal ideal? This is generally incorrect. Consider a ring homomorphism of inclusion

$$f : \mathbb{Z}[x] \rightarrow \mathbb{Q}[x] \text{ defined by } q \mapsto q.$$

We have a maximal ideal  $\langle x \rangle$  since  $\mathbb{Q}[x]/\langle x \rangle \cong \mathbb{Q}$  is a field. But the preimage, or contraction of  $\langle x \rangle$  is  $\langle x \rangle \subset \mathbb{Z}[x]$ , which isn't maximal. Another great example is HERE page 6.

??? DON'T KNOW HOW TO DO...

For a maximal ideal  $\mathfrak{m} \subset A[[x]]$ , its contraction under the inclusion map is

$$f^{-1}(\mathfrak{m}) = \mathfrak{m} \cap A.$$

According to the definition of the maximal ideal,  $\mathfrak{m}$  is composed of non-units in  $A[[x]]$ . Suppose we have

$$\mathfrak{m} \subsetneq I \subset A[[x]]$$

for some ideal  $I \subset A[[x]]$ . Then  $I = A[[x]]$  and in particular  $I$  contains a unit in  $A[[x]]$ . Apply contraction to them we have

$$f^{-1}(\mathfrak{m}) \subsetneq f^{-1}(I) \subset A.$$

According to part (i), the ideal  $I$  contains unit  $g \in A[[x]]$ , then  $f^{-1}(I)$  contains the constant coefficient  $g_0 \in A$ , which is a unit. This implies that  $f^{-1}(I) = A$ , therefore  $f^{-1}(\mathfrak{m})$  is a maximal ideal.

## 6.6 (v)

For a given prime ideal  $\mathfrak{p} \subset A$ , note that

$$(\mathfrak{p} + xA[[x]])^c = \mathfrak{p}$$

under the inclusion map. It suffices to prove that  $\mathfrak{p} + xA[[x]]$  is a prime ideal in  $A[[x]]$ . Firstly of all it's an ideal since it's clearly a subgroup and absorb multiplication from elements of  $A[[x]]$ . Suppose we have  $f, g \in A[[x]]$  where  $f = f_0 + f_1x + \dots$  and  $g = g_0 + g_1x + \dots$  such that

$$fg \in \mathfrak{p} + xA[[x]].$$

In particular we have  $f_0g_0 \in \mathfrak{p}$ , and then  $f_0, g_0 \in \mathfrak{p}$  since it's a prime ideal in  $A$ . This actually implies that

$$f = f_0 + f_1x + \dots = f_0 + x(f_1 + f_2x + \dots) \in \mathfrak{p} + xA[[x]]$$

and similarly for  $g$ . So we've proved that both  $f, g$  belongs to  $\mathfrak{p} + xA[[x]]$ , and this confirms that it's a prime ideal.

## 7 Ex 6.

HERE is one solution. HERE is another solution.

Naturally we have nilradical is a subset of Jacobson radical.

The non-trivial direction is to prove every maximal ideal is contained in the nilradical. Suppose for the sake of contradiction that there exists a maximal ideal  $I$  that's not contained in the nilradical, then by assumption we can find a non-zero idempotent. By Prop 1.9 we know that in particular we have for  $y = 1$ ,

$$1 - xy = 1 - x$$

is a unit. But then we have  $(1 - x)x = x - x^2 = 0$ , which implies that  $1 - x$  is a zero-divisor. This is a contradiction since a unit cannot be a zero-divisor.

## 8 Ex 7.

Fix a prime ideal  $I$ . We know  $A/I$  is an integral domain. To prove  $I$  is a maximal ideal, we only need to check  $A/I$  is a field. More precisely, we need to check every nonzero element in  $A/I$  has an inverse.

For a nonzero element  $a \in A$ , there exists  $n \geq 2$  such that  $a^n = a$ .

Note that we have

$$([a]^{n-1} - 1)[a] = 0 \Rightarrow [a]^{n-1} = 1$$

where  $[a]$  is the natural projection from  $a$  onto integral domain  $A/I$ . Notice that  $[a]^{n-2}[a] = 1$ , which proves that any non-zero element is a unit as desired. This implies that  $A/I$  is a field, hence every prime ideal is a maximal ideal.

## 9 Ex 8.

### 9.0.1

Nilradical is the minimal element with respect to inclusion. ??✖ This element has to retain to be prime.

**Fact:** Intersection of prime ideal isn't necessarily prime. For example, in  $\mathbb{Z}$ , we have two prime ideals  $(2), (3)$ . Their intersection is precisely  $(6)$ , which isn't prime.

**Fact:** How about maximal ideal?

- 1) As long as we have two maximal ideal with intersection as either one's proper subset, then their intersection mustn't be maximal;
- 2) since  $\mathbb{Z}$  is P.I.D, non-zero prime ideal is maximal ideal. We can take the same example as before.

### 9.1

Although we know that intersection of prime ideals are not necessarily prime ideal, here we only need to consider a special case, which is for applying Zorn's Lemma. We give the order of all ideals  $I_1 \leq I_2$  if and only if  $I_2 \subset I_1$ . This special case is to prove that for all prime ideals that are in one *chain*, their intersection is again a prime ideal. Refers [HERE](#).

??? But for this post, I don't see the reason why it requires existence of prime ideals. Zorn's Lemma only needs "for every chain it has a upper bound..."

Namely if we have a chain of prime ideals  $\{I_i\}$  where  $i \in A$  for some index set  $A$  such that

$$\dots \subset I_i \subset \dots$$

We need to prove that  $\bigcap_{i \in A} I_i$  is a prime ideal. We use contrapositive argument here. Suppose we have  $a, b \notin \bigcap_{i \in A} I_i$ , then we can find  $i_1, i_2 \in A$  such that

$$a \notin I_{i_1} \quad \text{and} \quad b \notin I_{i_2}.$$

Without loss of generality, we can assume that  $I_{i_1} \subset I_{i_2}$ . So we have  $a, b \notin I_{i_1}$ , while it's a prime ideal, then we have

$$ab \notin I_{i_1} \Rightarrow ab \notin \bigcap_{i \in A} I_i.$$

This proves that  $\bigcap_{i \in A} I_i$  is a prime ideal as expected.

### 9.2

See [1] 3.17; minimal prime over an ideal...



## 10 Ex 9.

$\Leftarrow$ : Suppose we have some prime ideals  $\{P_i\}_{i \in A}$  for some index set  $A$ , such that

$$\mathfrak{a} = \bigcap_{i \in A} P_i.$$

In order to prove that  $\text{rad}(\mathfrak{a}) = \mathfrak{a}$ , it suffices to prove  $\text{rad}(\mathfrak{a}) \subset \mathfrak{a}$ . For any  $x \in \text{rad}(\mathfrak{a})$ , this implies that  $x^n \in \mathfrak{a}$  for some integer  $n > 0$ . For any  $i \in A$ , we have

$$x^n \in P_i.$$

While this is a prime ideal, either  $x^{n-1}$  or  $x$  belongs to  $P_i$ . We can continue in this fashion and conclude that  $x$  belongs to  $P_i$ . While the index  $i$  is arbitrary, we know that

$$x \in \bigcap_{i \in A} P_i = \mathfrak{a}$$

as desired.

$\Rightarrow$ : DON'T KNOW HOW TO DO...

!!! This direction is trivial if we recall Prop 1.14. Since radical is intersection of prime ideals which containing  $\mathfrak{a}$ , the intersection of this set of prime ideals will be  $\mathfrak{a}$  given that it's radical.

HERE I present another solution in case you forget Prop 1.14.

Since  $\mathfrak{a} \neq (1)$ , then we can find a maximal ideal  $P$  that containing  $\mathfrak{a}$ . This ideal  $P$  is in particular prime, and we define

$$B := \{Q \subset R \mid Q \text{ is a prime ideal} \mid Q \supset \mathfrak{a}\}$$

as all ideals that containing  $\mathfrak{a}$ . It's non-empty since we must have  $P \in B$ .

Now we use non-trivial direction of the inclusion. For any  $x \in \text{rad}(\mathfrak{a})$ , namely we can find an integer  $n$  such that  $x^n \in \mathfrak{a}$ , it belongs to  $\mathfrak{a}$ . So we have  $x^n \in P$ , which is prime, then we have  $x \in P$  by a similar argument as above. This implies that

$$\mathfrak{a} = \text{rad}(\mathfrak{a}) \subset \bigcap_{i \in B} Q_i.$$

It suffices to prove the other direction, in which we intend to use a contrapositive argument. Suppose we have  $r \notin \text{rad}(\mathfrak{a})$ , this implies that we have

$$r^k \notin \mathfrak{a}$$

for any  $k \in \mathbb{N}$ . So  $S := \{1, r, r^2, \dots\}$  is a multiplicative set (this is defined HERE page 5). According to Prop 2.2 Page 5, we know that  $R \setminus S$  contains a prime ideal that containing  $\mathfrak{a}$ . This implies that we have  $x \notin \bigcap_{i \in B} Q_i$  as desired.

## 11 Ex 10.

ii)  $\Rightarrow$  iii): We prove that  $\mathfrak{R}$  is a maximal ideal. This is because if we have

$$\mathfrak{R} \subsetneq I \subset A$$

for some ideal  $I$ , then this implies that we have  $i \in I \setminus \mathfrak{R}$ . While the nilradical composed of all elements that're nilpotent, then we know that  $i$  must be a unit. This implies that  $I = A$ , which confirms that  $\mathfrak{R}$  is prime.

iii)  $\Rightarrow$  i): Since it's a field, nilradical  $\mathfrak{R}$  is a maximal ideal and is prime. While by definition

$$\mathfrak{R} = \bigcap_{i \in J} P_i$$

for all prime ideals  $P_i \subset A$  and for some index set  $J$ . By Prop 1.11 (ii), we know that  $\mathfrak{R} = P_{i_0}$  for some  $i_0 \in J$ . This implies that all prime ideals in  $J$  is the same, otherwise the equality doesn't hold. So this confirms that  $A$  has exactly one prime ideal.

i)  $\Rightarrow$  ii): We have a ring  $A$  that only has one prime ideal. While each maximal ideal is prime, then at most we have one maximal ideal. Suppose  $A = 0$ , the case is trivial. For  $A \neq 0$ , Theorem 1.3 let us conclude that we have at least one maximal ideal. So in this case we have exactly one maximal ideal, which equals to  $\mathfrak{R}$ . Each element of  $A$  is either a unit or a non-unit. The first case is done, then we consider the case for a non-unit, which is contained in a maximal ideal, namely  $\mathfrak{R}$ . And it must be nilpotent since it's also in nilradical. So we know that every element of  $A$  is either a unit or nilpotent.

### 11.1

See a post [HERE](#).

## 12 Ex 11.

i) Consider this, for any  $x \in A$ ,

$$x + 1 = (x + 1)^2 = x^2 + 2x + 1 = x + 2x + 1 = 3x + 1 \Rightarrow 2x = 0.$$

ii) According to 8 and  $n = 2 > 1$ , we can conclude that every prime ideal is maximal.

According to 8, we know that any  $x + \mathfrak{p}$  for some non-zero element  $x \in A$  is the multiplicative identity since

$$(x + \mathfrak{p})(x + \mathfrak{p}) = x^2 + \mathfrak{p} = x + \mathfrak{p} \Rightarrow x + \mathfrak{p} = 1 + \mathfrak{p}.$$

Note that we cannot have  $x^n + \mathfrak{p}$  for some  $n \geq 2$  since we're in a Boolean ring. And each  $x + \mathfrak{p}$  is equal to  $1 + \mathfrak{p}$ . Together with  $0 + \mathfrak{p}$ , we have exactly two

elements in the field  $A/\mathfrak{p}$ .

iii) We consider the ideal  $\langle x, y \rangle \subset A$  generated by two distinct elements  $x, y \in A$ . It suffices to prove that this ideal is principle.

DON'T KNOW HOW TO... HERE

The right candidate for principle ideal isn't  $xy$ , but  $x + y + xy$ .

We need to prove that  $\langle x, y \rangle = \langle x + y + xy \rangle$ . Clearly we have  $\langle x, y \rangle \supset \langle x + y + xy \rangle$ . Conversely, we have

$$x(x + y + xy) = x + xy + xy = x + 2(xy) = x,$$

$$y(x + y + xy) = xy + y + xy = 2(xy) + y = y.$$

And this completes the other direction of inclusion.

## 13 Ex 12.

Also see Spring 2023 506 HW 4 Problem 3.

Since it's a local ring, we have exactly one maximal ideal  $\mathfrak{m}$ . Suppose we have a non-zero idempotent element  $x \in A$ , then we pass it to the quotient

$$(x + \mathfrak{m})(x + \mathfrak{m}) = x^2 + \mathfrak{m} = x + \mathfrak{m} \Rightarrow x + \mathfrak{m} = 1 + \mathfrak{m}$$

given that  $A/\mathfrak{m}$  is a field. We know that  $1 - x \in \mathfrak{m}$ . Also we know that  $x \in \mathfrak{m}$  since it's a non-unit given that  $x(x - 1) = 0$  (it's a zero-divisor). But this gives us

$$1 = x - (x - 1) \in \mathfrak{m},$$

which contradicts the fact that  $\mathfrak{m}$  is a maximal ideal.

Solution HERE and HERE.

## 14 Ex 13.

Firstly we prove that the ideal  $\mathfrak{a}$  couldn't be the whole ring. Suppose on the contrary that  $\mathfrak{a} = (1)$ , then we can find a  $g \in A := K[x_f]$  such that

$$g \prod_{i \in J} f_i = 1$$

for some index set  $J$ . This implies that  $f$  is a unit in polynomial ring  $A$ . By definition of irreducible polynomial, we know that  $f$  couldn't be a constant since  $K$  is a field. According to 3, coefficients other than the constant must be nilpotent,

this contradicts the fact that  $f$  is a monic polynomial. So we must have  $\mathfrak{a} \neq (1)$ .

By Corollary 1.4, there exists a maximal ideal  $\mathfrak{m}$  that contains  $\mathfrak{a}$ . We construct a field  $K_1 := A/\mathfrak{m}$ , which contains a root for any  $f \in \Sigma$  since it's defined as 0 in this field. Repeat this process, and since each polynomial has finite degree, union of infinitely many  $K_i$  will contain all of the roots.

The rest is precisely what the problem suggested.

## 15 Ex 14.

Proof is basically Proposition 2.2 of HERE.

We can assume  $\Sigma$  is non-empty since otherwise the proposition is trivially correct. (Do we have to write this?)

We wish to apply Zorn's Lemma. Let  $i \in J$  be an index set, we consider a chain of ideals

$$\cdots \subset L_i \subset \cdots$$

where  $L_i \in \Sigma$ . We can define their union as  $L = \cup_{i \in J} L_i$ . It's a group under addition: for any  $x, y \in L$ , we can find  $x \in L_{i_1}$  and  $y \in L_{i_2}$ . We can assume  $L_{i_1} \subset L_{i_2}$ , then  $x - y \in L_{i_2} \subset L$  since  $L_{i_2}$  is an ideal. It absorb elements from the ring, fix an element  $a \in A$ ,

$$ax \in L_{i_1} \subset L$$

for arbitrary  $x \in L_{i_1}$ . So we know that in this chain of inclusion, their union is indeed an ideal. Clearly in every chain, we can form an ideal like this, and it will serve as the upper bound of the chain. Then we can apply Zorn's Lemma and conclude that there exists a maximal element in  $\Sigma$ .

✖ For any maximal element  $\mathfrak{p} \in \Sigma$ , we hope to prove it's a prime ideal. Suppose we have  $ab \in \mathfrak{p}$ , if  $a \notin \mathfrak{p}$ , then we have a proper inclusion of ideals

$$\mathfrak{p} \subsetneq \mathfrak{p} + \langle a \rangle.$$

While  $\mathfrak{p}$  is assumed to be a maximal element, then  $\mathfrak{p} + \langle a \rangle \notin \Sigma$ . This implies that we will have an element  $p + ax \in \mathfrak{p} + \langle a \rangle$  for some  $p \in \mathfrak{p}$  and  $x \in A$  such that it's not a zero-divisor. ✖

?How to prove it's prime... HERE, contrapositive is easy ✓

Now we try to prove the maximal element  $\mathfrak{p}$  with respect to inclusion is a prime ideal by resorting a contrapositive argument. Suppose we have  $a \notin \mathfrak{p}$  and  $b \notin \mathfrak{p}$ , then we can form two proper inclusion as

$$\mathfrak{p} \subsetneq \mathfrak{p} + \langle a \rangle, \quad \mathfrak{p} \subsetneq \mathfrak{p} + \langle b \rangle.$$

Since we already have  $\mathfrak{p} \in \Sigma$  as a maximal element, then both ideals  $\mathfrak{p} + \langle a \rangle$  and  $\mathfrak{p} + \langle b \rangle$  doesn't belong to  $\Sigma$ . This implies that we can find non-zero-divisors in both ideals. Then we consider their product

$$(\mathfrak{p} + \langle a \rangle)(\mathfrak{p} + \langle b \rangle) \subset \mathfrak{p} + \langle ab \rangle,$$

which will contain at least one non-zero-divisor. So we have  $\mathfrak{p} \subsetneq \mathfrak{p} + \langle ab \rangle$  is a proper inclusion, this implies that  $xy \notin \mathfrak{p}$  as desired. So we've verified that  $\mathfrak{p}$  is a prime ideal.

On page 8 of the book, we know that zero-divisors are union of annihilators

$$D = \bigcup_{x \neq 0} \text{Ann}(x).$$

So zero-divisors are union of ideals (annihilator is a special case for quotient ideal). Each ideal  $\text{Ann}(x) \in \Sigma$  since it's composed of all zero-divisors. By the previous part we've build the existence of maximal element with respect to inclusion, so we can cover all zero-divisors  $D$  with those maximal elements  $\mathfrak{p}$  that are also prime ideals. So we've proved that the set of zero-divisors in  $A$  is a union of prime ideals.

## 16 Ex 15.

### 16.1 (i)

We denote the ideal as  $\mathfrak{a} = \langle E \rangle \subset A$ . Clearly we have  $V(E) \supset V(\mathfrak{a})$  since for any prime ideal that contains  $\mathfrak{a}$  must contain  $E$  in particular. Conversely, give an arbitrary prime ideal  $\mathfrak{p} \in V(E)$ , this means  $\mathfrak{p} \supset E$  by definition. We interpret the ideal generated by  $E$  as intersection of all ideals that containing  $E$ . Hence we know that

$$\mathfrak{p} \supset \mathfrak{a}.$$

This implies that ideal  $\mathfrak{p}$  is a prime ideal that containing  $\mathfrak{a}$  so  $\mathfrak{p} \in V(\mathfrak{a})$ . Therefore we've build the first equality.

For the second equality, again we note that  $V(\mathfrak{a}) \supset V(\text{rad}(\mathfrak{a}))$ . For any given prime ideal  $\mathfrak{q} \supset \mathfrak{a}$  that contains  $\mathfrak{a}$ , while  $\text{rad}(\mathfrak{a})$  is the intersection of the prime ideals which contain  $\mathfrak{a}$  by Proposition 1.14, so we have  $\mathfrak{q} \supset \text{rad}(\mathfrak{a})$  and  $\mathfrak{q} \in V(\text{rad}(\mathfrak{a}))$ . In summary we have

$$V(E) = V(\mathfrak{a}) = V(\text{rad}(\mathfrak{a})).$$

### 16.2 (ii)

For  $\{0\}$ , any prime ideals of  $A$  would contain it so  $V(0) = X$ . Recall in the textbook we defined prime ideal  $\neq \langle 1 \rangle$ . So we cannot consider the  $A$  as a prime ideal of  $A$  itself. Hence we have  $V(1) = \emptyset$ .

### 16.3 (iii)

This is basically unwrap the definition. For a prime ideal  $\mathfrak{p}$  that contains  $\bigcup_{i \in I} E_i$ , it must contain each  $E_i$ . This implies that  $\mathfrak{p} \in V(E_i)$  for each index  $i \in I$ , therefore it belongs to  $\bigcap_{i \in I} V(E_i)$ . Conversely, pick any prime ideal  $\mathfrak{p}$  such that lives in every  $V(E_i)$  where  $i \in I$ . This just means  $\mathfrak{p}$  contains every  $E_i$  where  $i \in I$ , hence we have  $\mathfrak{p} \in V(\bigcup_{i \in I} E_i)$ . This completes another direction of inclusion and proved the equality.

### 16.4 (iv)

Now we try to prove the first equality. Clearly we have  $V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{a}\mathfrak{b})$  since  $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$ . Conversely, for any prime ideal  $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$ , it contains product of ideal  $\mathfrak{a}\mathfrak{b}$ . Since  $\mathfrak{p}$  is prime, by Lemma 2.1 on page 4 of HERE, without loss of generality, we have  $\mathfrak{p} \supset \mathfrak{a}$ . This implies that  $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$ .

Now we finish the second equality. Clearly we have  $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a} \subset \mathfrak{p}$ , this implies that  $V(\mathfrak{a}\mathfrak{b}) \supset V(\mathfrak{a}) \cup V(\mathfrak{b})$ . Now consider any prime ideal  $\mathfrak{p} \supset \mathfrak{a}\mathfrak{b}$ , by the same approach above, we can assume that this prime ideal contains  $\mathfrak{a}$ . Hence we have  $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$ . In summary we have

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}).$$

See [1] Exercise 2.23 on Page 15.



## 17 Ex 16.

### 17.1

For  $\text{Spec}(\mathbb{Z})$ , it is composed of prime ideals  $0$  and  $\langle p \rangle$  for all prime number  $p$ .

See HERE. See a post HERE.



### 17.2

For  $\text{Spec}(\mathbb{R})$ . Since  $\mathbb{R}$  is a field, then it's just one point.

See a post HERE.



### 17.3

For  $\text{Spec}(\mathbb{C}[x])$ . Since  $\mathbb{C}$  is algebraically closed, all prime ideals are of the form  $ax + b$  for some  $a, b \in \mathbb{C}$ .

See a post HERE. See a post HERE for general approach for spectrum of a polynomial ring over a field.

## 17.4

For  $\text{Spec}(\mathbb{R}[x])$ .

According to Prop 12 on Page 286 of [3], which states that in *U.F.D.* a nonzero element is prime iff it's irreducible. Hence we need to find all irreducible in  $\mathbb{R}[x]$ .

All irreducibles in  $\mathbb{R}[x]$  are degree 1 polynomials, with quadratic that has no real root. This is because any odd degree real polynomial will necessarily have a real root by intermediate value theory; and any even degree larger than 2 will contradict the fact that  $\mathbb{C}$  is algebraically closed and the degree of extension  $[\mathbb{C} : \mathbb{R}] = 2$ . See [HERE](#) and [HERE](#) for a comprehensive explanation.

See a post [HERE](#). See a related post [HERE](#).



## 17.5

For  $\text{Spec}(\mathbb{Z}[x])$ , ???

I had no idea...

See a post [HERE](#), [HERE](#), and [HERE](#).

Verified on Page 28, Example 1.8 from [5].



## 17.6

For irreducibility of  $\mathbb{R}[x, y]$ , see a post [HERE](#).

See [HERE](#) for description for  $\text{Spec}(\mathbb{R}[x, y])$ .

## 18 Ex 17.

Here we use  $D(f) := X_f$ . By definition we know Zariski topology is defined by the closed subsets. While  $D(\bullet)$  is the complement of closed set, they satisfy the basis for open sets.

### 18.1 (i)

We can compute

$$\begin{aligned} D(f) \cap D(g) &= V(f)^c \cap V(g)^c \\ &= [V(f) \cup V(g)]^c \\ &= [V(\langle f \rangle) \cup V(\langle g \rangle)]^c \\ &= [V(\langle fg \rangle)]^c \\ &= [V(fg)]^c \\ &= D(fg). \end{aligned}$$

## 18.2 (ii)

$\Rightarrow$ : Assume  $D(f) = \emptyset$ , then  $V(f) = \text{Spec } A$ . In particular, we know  $\langle f \rangle \subset \mathfrak{p}$  for any  $\mathfrak{p} \in \text{Spec } A$ , which implies

$$\langle f \rangle \subset \bigcap \mathfrak{p} = \text{Nil}(A).$$

And this implies  $f$  is nilpotent.

$\Leftarrow$ : Assume  $f$  is nilpotent, then  $f^r = 0$  for some integer  $r > 0$ . Note that

$$f^r = 0 \in \mathfrak{p} \Rightarrow f \in \mathfrak{p} \Rightarrow \langle f \rangle \subset \mathfrak{p}$$

for arbitrary prime ideal  $\mathfrak{p}$ . While we always have  $V(\langle f \rangle) \subset \text{Spec } A$ , hence we can conclude

$$V(\langle f \rangle) \subset \text{Spec } A.$$

### 18.2.1 (iii)

$\Rightarrow$ : Note  $D(f) = X$  implies  $V(f) = V(\langle f \rangle) = \emptyset$ . If  $\langle f \rangle \subsetneq A$ , then it must lie inside some maximal ideal  $\mathfrak{m} \in V(\langle f \rangle)$ , which contradicts the assumption. Hence we necessarily have  $\langle f \rangle = A = \langle 1 \rangle$ , which implies  $f$  is a unit.

$\Leftarrow$ : Say  $f$  is a unit, then  $V(f) = V(\langle f \rangle) = V(A) = \emptyset$ , which forces  $D(f) = X$  as expected.

## 18.3 (iv)

$\Rightarrow$ : Assume  $D(f) = D(g)$ , i.e.  $V(\langle f \rangle) = V(\langle g \rangle)$ . In particular, we try to interpret  $V(\langle f \rangle) \subset V(\langle g \rangle)$ . This means for any prime ideal  $\mathfrak{p} \supset \langle f \rangle$ , we have  $\mathfrak{p} \supset \langle g \rangle$ . Then by Characterisation of Radical, it follows that

$$\langle g \rangle \subset \bigcap_{\mathfrak{p} \supset \langle f \rangle, \mathfrak{p} \in \text{Spec } A} \mathfrak{p} = \sqrt{\langle f \rangle}.$$

Similarly, we can get inclusion conversely and

$$\sqrt{\langle g \rangle} \subset \sqrt{\sqrt{\langle f \rangle}} = \sqrt{\langle f \rangle} \subset \sqrt{\sqrt{\langle g \rangle}} = \sqrt{\langle g \rangle} \Rightarrow \sqrt{\langle f \rangle} = \sqrt{\langle g \rangle}.$$

$\Leftarrow$ : Now we assume  $\sqrt{\langle f \rangle} = \sqrt{\langle g \rangle}$ , which implies

$$\bigcap_{\mathfrak{p} \in \text{Spec } A, \mathfrak{p} \supset \langle f \rangle} \mathfrak{p} = \bigcap_{\mathfrak{q} \in \text{Spec } A, \mathfrak{q} \supset \langle g \rangle} \mathfrak{q}.$$

For arbitrary prime ideal  $\mathfrak{p} \supset \langle f \rangle$ , we have

$$\mathfrak{p} \supset \bigcap_{\mathfrak{p} \in \text{Spec } A, \mathfrak{p} \supset \langle f \rangle} \mathfrak{p} = \sqrt{\langle g \rangle} \supset \langle g \rangle.$$

Therefore we have  $V(\langle f \rangle) \subset V(\langle g \rangle)$ ; similarly we get the inclusion conversely. Then it follows that

$$D(f) = V(\langle f \rangle)^c = V(\langle g \rangle)^c = D(g).$$



### 18.3.1 (v)

We proceed according to the hint. Given an arbitrary open covering for space  $X = \bigcup_{i \in I} D(f_i)$  for some index set  $I$ . We can further express as

$$X = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} V(f_i)^c = \left[ \bigcap_{i \in I} V(f_i) \right]^c = \left[ V \left( \bigcup_{i \in I} \langle f_i \rangle \right) \right]^c$$

$$\Rightarrow V \left( \bigcup_{i \in I} \langle f_i \rangle \right) = \emptyset \Rightarrow \bigcup_{i \in I} \langle f_i \rangle = \langle 1 \rangle$$

See a post [HERE](#).

The correct argument is to derive the following equality

$$\emptyset = \bigcap_{i \in I} V(f_i).$$

We claim that  $\langle \sum_{i \in I} f_i \rangle = \langle 1 \rangle$ . Suppose it's not the case, then this implies there's maximal ideal  $\mathfrak{m}_0$  will contain  $\langle \sum_{i \in I} f_i \rangle$ . And this implies  $\mathfrak{m}_0 \in V(f_i)$  for each  $V(f_i)$ , which contradicts the fact that  $\emptyset = \bigcap_{i \in I} V(f_i)$ . Hence the claim is proved.

The claim told us

$$1 \in \langle \sum_{i \in I} f_i \rangle,$$

which implies we can express 1 as a *finite* linear combination of  $\{f_i\}_{i \in J}$  where  $J$  is a *finite* index set. Notice that we must have

$$\emptyset = \bigcap_{i \in J} V(\langle f_i \rangle),$$

which is equivalent to say  $\{D(f_i)\}_{i \in J}$  is a finite subcover for  $X$ .

See a post [HERE](#). Proposition 13.2 on Page 95 [1].

Last implication requires Hilbert's Nullstellensatz?



### 18.4 (vi)

One approach is to generalise the proof in previous part (v). But I encountered a problem...

See a post [HERE](#). It used the property proved in (i) to simplify the open cover, which is why my approach didn't work... Given any *basic open set*  $D_f$  with an arbitrary open cover

$$D(f) = \bigcup_{i \in I} D(f_i)$$

for some index set  $I$ . According to (i) we know  $D(f) \cap D(f_i) = D(ff_i)$ . Hence we can re-write the open covering as

$$D(f) = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} D(f_i) \cap D(f) = \bigcup_{i \in I} D(ff_i).$$



This implies, without loss of generality, we can assume the open covering  $\{D(f_i)\}$  has the property that  $f_i \in (f)$  (otherwise, take  $ff_i$  instead). With such assumption, we take the complement of the above equality gives us

$$V(f) = \bigcap_{i \in I} V(f_i) = V\left(\bigcup_{i \in I} \langle f_i \rangle\right) \Rightarrow \sqrt{\langle f \rangle} = \sqrt{\bigcup_{i \in I} \langle f_i \rangle}.$$

In particular, we have

$$f \in \sqrt{\bigcup_{i \in I} \langle f_i \rangle} \Rightarrow f^n = \sum_{i \in J}^{\infty} a_i f_i = \sum_{i \in J} a_i f_i$$

for some  $a_i \in A$  and integer  $n > 0$ . Here it's important that the index set  $J$  is *finite* since an element in sum of those ideals are finite sum of such form.

By definition of radical, it follows that

$$f \in \sqrt{\bigcup_{i \in J} \langle f_i \rangle} \Rightarrow \langle f \rangle \subset \sqrt{\bigcup_{i \in J} \langle f_i \rangle}.$$

While  $\bigcup_{i \in J} \langle f_i \rangle \subset \langle f \rangle$  by our further assumption, we have the converse inclusion of the radicals. Hence we have

$$\sqrt{\langle f \rangle} = \sqrt{\bigcup_{i \in J} \langle f_i \rangle} \Rightarrow Z(\langle f \rangle) = Z\left(\bigcup_{i \in J} \langle f_i \rangle\right) \Rightarrow D(f) = \bigcup_{i \in J} D(f_i),$$

which gives a finite subcover of  $D(f)$  as expected.

According to the post above, HERE. We can also use a homeomorphism  $\text{Spec}(A_f) \simeq X_f$  to finish the proposition.

How to prove this homeo?

## 18.5 (vii)

$\Leftarrow$ : This direction is clear by (vi).

$\Rightarrow$ : Since all basic open sets form a basis for Zariski topology on  $A$ , then for any open quasi-compact set  $U$  we can express it as a union of basic open sets

$$U = \bigcup_{i \in I} D(f_i),$$

and it suffices to prove it has a finite subcover...

no idea...

See the prompt of Exercise 13.41, 94, on Page 101 of [1].

Hence it suffices to prove the equivalence of the following statements:

*An open set  $U$  is a union of finitely many basic open sets if and only if  $X - U = V(\mathfrak{a})$  where  $\mathfrak{a}$  is finitely generated.*

✓

*Proof.* Assume  $U = \bigcup_{i=1}^m D(f_i)$ , then define  $\mathfrak{a} = \langle f_1, \dots, f_m \rangle$  should suffice to finish the claim. Because we have

$$X - U = \bigcup_{i=1}^m \mathbf{D}(f_i) = \bigcap_{i=1}^m \mathbf{V}(f_i) = \mathbf{V}(\langle f_1, \dots, f_m \rangle) = \mathbf{V}(\mathfrak{a}).$$

Conversely, say  $\mathfrak{a} = \langle f_1, \dots, f_m \rangle$ . Then according the above equality we know

$$\{\mathbf{D}(f_i)\}_{1 \leq i \leq m}$$

will cover  $U$ . □

See a lecture note [HERE](#).

## 19 Ex 18.

### 19.1 (i)

Follows immediately according to (ii).

### 19.2 (ii)

By definition we have

$$\overline{\{x\}} = \bigcap_{\mathfrak{p} \subset \mathfrak{p}_x} \mathbf{V}(\mathfrak{p}) \supset \mathbf{V}(\mathfrak{p}_x).$$

Conversely, for any prime ideal  $\mathfrak{q} \in V(\mathfrak{p})$  where  $\mathfrak{p} \subset \mathfrak{p}_x$  is arbitrary. When  $\mathfrak{p} = \mathfrak{p}_x$ , we have  $\mathfrak{p} \subset \mathfrak{q}$ . And this implies  $\mathfrak{q} \in \mathbf{V}(\mathfrak{p}_x)$ , then

$$\bigcap_{\mathfrak{p} \subset \mathfrak{p}_x} \mathbf{V}(\mathfrak{p}) = \mathbf{V}(\mathfrak{p}_x) \Rightarrow \overline{\{x\}} = \mathbf{V}(\mathfrak{p}_x)$$

See Exercise 13.16 [1] on Page 98.

???

### 19.3 (iii)

We use contrapositive argument. We wish to prove that if  $y$  lies in any neighborhood of  $x$  and vice versa, then  $x = y$ .

*A potentially wrong approach...* Negation of the later part gives us for any open set  $O$ , we have either  $a, b \in O$  or  $a, b \notin O$ . Take  $O := \mathbf{D}(a)$  in the second case will forces  $b \in \mathbf{V}(a)$ . Similarly we know the other inclusion, which gives us  $a = b$ .

See [HERE](#) for a solution, which used a middle claim... See Kolmogorov space from Wiki.

Wrong contrapositive ???

Is this correct?

✓

## 20 Ex 19.

### 20.1 Equivalent Definition for Irreducible

Take  $X$  a topological space. T.F.A.E.

- (a) Space  $X$  cannot be expressed as a union of two closed proper subset.
- (b) Every pair of non-empty open sets in  $X$  intersect.
- (c) Every non-empty open set is dense in  $X$ .

(a)  $\Leftrightarrow$  (b): Contrapositive. Suppose there exist  $U_1, U_2$  that do not intersect in  $X$ , then  $X$  is reducible given

$$X = (X \setminus U_1) \cup (X \setminus U_2)$$

where both  $X \setminus U_1, X \setminus U_2$  are non-empty and closed. Conversely, suppose  $X$  is reducible. Then  $X = V_1 \cup V_2$  where  $V_1, V_2$  are non-empty closed subsets of  $X$ . Then both  $X \setminus V_1, X \setminus V_2$  will be non-empty proper open subsets of  $X$ . But they won't have intersection given

$$(X \setminus V_1) \cap (X \setminus V_2) = X \setminus [V_1 \cup V_2] = \emptyset.$$

(a)  $\Leftrightarrow$  (c): Contrapositive. Assume a non-empty open set  $U$  isn't dense in  $X$ , i.e.  $\overline{U} \subsetneq X$ . Therefore we have

$$X = \overline{U} \cup (X \setminus U),$$

which contradicts the assumption that  $X$  is irreducible. Conversely, suppose  $X$  is reducible. Then  $X = V_1 \cup V_2$  where  $V_1, V_2$  are non-empty proper closed subsets of  $X$ . But immediately we have  $(X \setminus V_1)$  is a non-empty open set in  $X$ , and it's not dense in  $X$  since

$$\overline{X \setminus V_1} \subset V_2 \subsetneq X.$$

### 20.2

*Proof.* Assume  $\mathfrak{q} := \text{Nil}(A)$  is a prime ideal, we wish to show that  $\text{Spec}(A)$  is irreducible. By one characterisation of nilradical, we have

$$\mathfrak{q} = \bigcap_{\mathfrak{p} \in \text{Spec } A} \mathfrak{p} \Rightarrow \text{Spec } A \subset \mathbf{V}(\mathfrak{q}) \Rightarrow \text{Spec } A = \mathbf{V}(\mathfrak{q}).$$

Furthermore, recall in previous exercise 19 we can express the variety as a closure of a point.

$$\text{Spec } A = \mathbf{V}(\mathfrak{q}) = \overline{\{\mathfrak{q}\}}.$$

While a single point  $\mathfrak{q}$  is irreducible, then we can apply a result from 21 to conclude that  $\text{Spec } A$  is irreducible in itself, hence irreducible.

See HERE  
in case  
you're un-  
comfort-  
able about  
the previous  
claim...

Now we have the deal with the other direction. We proceed by contrapositive. Assume  $\mathfrak{b} := \text{Nil}(A)$  is not a prime ideal, though it's an ideal anyway. Hence there exist some  $xy \in \mathfrak{b}$  but  $x \notin \mathfrak{b}$  and  $y \notin \mathfrak{b}$ . Now we can use results from previous exercise 17 18. While  $xy$  is nilpotent, then  $\mathbf{V}(xy) = \text{Spec } A$ ; while  $x, y$  aren't nilpotent, then  $\mathbf{V}(x), \mathbf{V}(y) \subsetneq \text{Spec } A$ . Notice we have two proper closed subsets

$$\mathbf{V}(x) \cup \mathbf{V}(y) = \mathbf{V}(xy) = \text{Spec } A,$$

which proves  $\text{Spec } A$  is reducible as expected. □

See a post [HERE](#). See a blog [HERE](#).

See a Lemma on Page 2 of a note [HERE](#).



## 21 Ex 20.

### 21.1 (i)

Both  $Y$  and  $\bar{Y}$  are topological space with subspace topology given by  $X$ . For any pair of non-empty open subset  $U_1, U_2 \in \bar{Y}$ , there exist  $W_1, W_2$  open in  $X$  such that

$$U_1 = W_1 \cap \bar{Y}, U_2 = W_2 \cap \bar{Y}.$$

While  $Y$  is irreducible, then  $(W_1 \cap Y) \cap (W_2 \cap Y)$  will intersect in  $Y$ . By definition of subspace topology in  $Y$ , we know both  $(W_1 \cap Y), (W_2 \cap Y)$  are open. Furthermore, we note

$$\emptyset \subsetneq [(W_1 \cap Y) \cap (W_2 \cap Y)] \cap Y \subset (U_1 \cap U_2) \cap \bar{Y}.$$

Thus it follows that any pair of two open sets in  $\bar{Y}$  will intersection in  $\bar{Y}$ , which means  $\bar{Y}$  is irreducible.

See Stacks Project [HERE](#).



### 21.2 (ii)

I have no idea what is needed to prove? Isn't this equivalent to some sort of Axiom of Choice or Zorn?



See [1] Lemma 16.50 on Page 125, "*irreducible component*". Also see a post [HERE](#).



Use Zorn's Lemma... Order all irreducible spaces by inclusion. In any *chain* of irreducible spaces, we claim the union of irreducible spaces are again irreducible. Then any chain is bounded above and apply Zorn's Lemma gives the existence of a maximal irreducible space. Therefore any irreducible space is contained in a maximal one, for which we call *irreducible component* in (iii) later on.

### 21.3 (iii)

Say  $A \subset X$  is an irreducible component, which means maximal with respect to inclusion. It follows immediately  $A$  is closed given

$$A \subset \overline{A} \Rightarrow A = \overline{A}.$$

For any point  $x \in X$ , note that  $\{x\}$  is irreducible. Hence it lies in some irreducible component. This means all irreducible components will cover  $X$ .

For a Hausdorff space  $X$ , irreducible component is just a singleton  $\{x\}$  for any  $x \in X$ .

See [1] Lemma 16.50 on Page 125; See an official solution [HERE](#).



### 21.4 (iv)

According to 19, we know  $\mathbf{V}(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$  is closed for being a closure of a single point by (ii). It's maximal given  $\mathfrak{p}$  is minimal.

See a post [HERE](#).



## 22 Ex 21.

### 22.1 (i)

For any  $f \in A$  we unwrap the definition as follows

$$\begin{aligned} \phi^{*-1}(\mathbf{D}(f)) &= \{\alpha \in Y \mid \phi^*(\alpha) \in \mathbf{D}(f)\} \\ &= \{\alpha \in Y \mid \phi^{-1}(\alpha) \notin \mathbf{V}(f)\} \\ &= Y - \{\alpha \in Y \mid \phi^{-1}(\alpha) \in \mathbf{V}(f)\} \\ &= Y - \{\alpha \in Y \mid f \subseteq \phi^{-1}(\alpha)\} \\ &= Y - \{\alpha \in Y \mid \phi(f) \subseteq \alpha\} \\ &= Y - \{\alpha \in Y \mid \alpha \in \mathbf{V}(\phi(f))\} \\ &= \{\alpha \in Y \mid \alpha \notin \mathbf{V}(\phi(f))\} \\ &= Y - \mathbf{V}(\phi(f)) \\ &= \mathbf{D}(\phi(f)). \end{aligned}$$

Hence the induced map  $\phi^*$  is continuous.

### 22.2 (ii)

Given  $\mathfrak{a} \subset A$  is an ideal. We can compute the pre-image as

?

### 22.3

See a solution [HERE](#).

## 23 Finitely Generated Modules

### 23.1 Pathological Examples

A finitely generated module (we'll denote as f.g. module) might contain a submodule that's not f.g. See a post [HERE](#), [HERE](#). See Keith's notes on Modules [HERE](#).

! But things are different when we think about finite ring maps. See definition defined on Atiyah Page 30. See Stacks Project's description on finite ring maps [HERE](#).

## 24 Ex 2.2 Page 20

i) For any element  $a \in \text{Ann}(M + N)$ , namely for any element such that

$$a(M + N) = 0.$$

This implies that in particular,  $aM = 0$  and  $aN = 0$  given  $M, N \subset M + N$  by definition of sum of modules. Hence we have  $a \in \text{Ann}(M) \cap \text{Ann}(N)$ .

Conversely, if we have  $a \in \text{Ann}(M)$  and  $a \in \text{Ann}(N)$ . This implies that for any finite sum

$$a(m_1 + \cdots + m_i + n_1 + \cdots + n_j) = 0,$$

so we have  $a(M + N) = 0$  for any  $a \in A$ . This completes the other direction and therefore we have the equality

$$\text{Ann}(M + N) = \text{Ann}(M) \cap \text{Ann}(N).$$

ii) For any element  $x \in (N : P)$ , by definition  $xP \subset N$ . Now we try to interpret annihilator of  $(N + P)/N$ , they're all ring elements  $x \in A$  such that

$$x(N + P)/N = 0 \Rightarrow x(N + P) \subset N.$$


If we have  $xP \subset N$ , then naturally we have  $x(N + P) \subset N$ , which completes one direction of inclusion. Conversely, if we're given  $x \in \text{Ann}((N + P)/N)$ , equivalently we know that  $x(N + P) \subset N$ , while  $xN \subset N$  given that  $N$  is module. We must have  $xP \subset N$ , hence  $x \in (N : P)$ . So in summary we have

$$(N : P) = \text{Ann}((N + P)/N).$$

## 25 Prop 2.3 Remark

It works well without assumption of finitely generated? See this post discuss why every module is a quotient of free module [HERE](#).

## 26 Remark ii) Page 25

 This is an exercise 2 in the section of tensor product on Dummit and Foote Page 375. It's meant to illustrate the fact that tensor product notation  $x \otimes y$  is ambiguous unless we specify which tensor product it lives in.

On the book we've already seen that  $2 \otimes 1 = 0$  in  $\mathbb{Z}$ -module  $\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$ .  
Now we wish to prove it's non-zero in  $2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$ .

## 27 Proposition 2.4

*Generalized Version of the Cayley–Hamilton Theorem*

HERE is a useful paraphrase and some errata.

See an excellent explanation HERE.

➔ A counterexample of Nakayama's Lemma HERE. Also see HERE, explaining why  $(\mathbb{Q}, +)$  is not finitely generated; and HERE, explaining  $\mathbb{Q}$  is not a finitely generated  $\mathbb{Z}$ -module.

*Proposition 2.4 Proof*

Let  $x_1, \dots, x_n$  be a set of generators of  $M$ . For each element in the ideal  $a \in \mathfrak{a}$ , we have to consider it as an endomorphism, i.e., we consider  $a := \psi(a) \in \text{Hom}_A(M, M)$ . Then for each  $x_i$  where  $1 \leq i \leq n$ , we have (here all  $a_{ij} := \psi(a_{ij})$ )

$$\begin{aligned}\phi(x_i) &= a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = \sum_{j=1}^n a_{ij}x_j \\ \Rightarrow (-a_{i1})x_1 + \dots + (\phi - a_{ii})x_i + \dots + (-a_{in})x_n &= 0 \\ \Rightarrow \sum_{j=1}^n (\delta_{ij}\phi - a_{ij})x_j &= 0.\end{aligned}$$

for all  $a_{ij} \in \mathfrak{a}$ . And we can write them in the form of a matrix as

$$\begin{pmatrix} \phi - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \phi - a_{22} & \dots & -a_{2n} \\ \dots & & & \\ -a_{n1} & -a_{n2} & \dots & \phi - a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = 0.$$

Moreover, we denote the first matrix as  $A$  and the later column vector as  $v$ . Now we can apply adjugate matrix on that and have

$$\begin{aligned}\text{adj}(A)A &= \det A I = 0 \\ \Rightarrow \text{adj } Av &= \det A Iv = 0.\end{aligned}$$



Note that this implies the matrix  $\det AI$  annihilates all generators, then it must be zero endomorphism. So we have  $\det AI = 0 \in \text{Hom}_A(M, M)$ . This implies  $\det A = 0$  as a scalar, so we can expand the determinant to get the desired result

$$\phi^n + \psi(b_1)\phi^{n-1} + \cdots + \psi(b_n) = 0$$

where  $b_i \in \mathfrak{a}$  since it's the product of some  $a_i$ .

## 28 Corollary 2.13

HERE is a post discussing this.

## 29 Corollary 2.7

Recall that element in product of an ideal and the module is finite sum of ...  
See the post HERE.

## 30 Proposition 2.8

See this post HERE and HERE.

### 30.1 Exercise 2.20

Fix an exact sequence of  $B$ -module

$$0 \longrightarrow N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \longrightarrow 0.$$

Let's consider this after tensor with  $M_B$ . Note that we have

$$0 \longrightarrow N_1 \otimes M_B \xrightarrow{g_1 \otimes 1} N_2 \otimes M_B \xrightarrow{g_2 \otimes 1} N_3 \otimes M_B \longrightarrow 0$$

$$\Rightarrow 0 \rightarrow N_1 \otimes_B B \otimes_A M \xrightarrow{g_1 \otimes 1} N_2 \otimes_B B \otimes_A M \xrightarrow{g_2 \otimes 1} N_3 \otimes_B B \otimes_A M \rightarrow 0$$

$$\Rightarrow 0 \longrightarrow N_1 \otimes_A M \xrightarrow{g_1 \otimes 1} N_2 \otimes_A M \xrightarrow{g_2 \otimes 1} N_3 \otimes_A M \longrightarrow 0,$$

which is exact since  $N_1, N_2, N_3$  inherits  $A$ -module structure and  $M$  is flat as an  $A$ -module. And this proves that  $M_B$  is flat as  $A$ -module.

## 31 Ex 1.

According to Qing Liu's book Remark 1.3, that the tensor product of modules is generated by all elements of the form  $a \otimes b$  and any element in the tensor product can be written as  $\sum_{\text{finite}} a_i \otimes b_j$  a finite sum of tensor products. It suffices to prove for any  $a \in \mathbb{Z}/m\mathbb{Z}$  and  $b \in \mathbb{Z}/n\mathbb{Z}$ , their tensor product is zero.

Since  $m, n$  are coprime, then we have  $mx + ny = 1$  for some integers  $x, y \in \mathbb{Z}$ . In  $\mathbb{Z}$  we have 1, then this gives us

$$\begin{aligned} a \otimes b &= (mx + ny)a \otimes b \\ &= mxa \otimes b + (ny)a \otimes b \\ &= mxa \otimes b + a \otimes nyb \\ &= 0 \otimes b + a \otimes 0 \\ &= 0. \end{aligned}$$

Since for every element in the tensor product we have proved it's zero, then the module is zero.

## 32 Ex 2.

For the exact sequence we can tensor it with  $M$  as

$$\begin{aligned} 0 \rightarrow \mathfrak{a} \rightarrow A \rightarrow A/\mathfrak{a} \rightarrow 0, & \quad (1) \\ \Rightarrow \mathfrak{a} \otimes M \rightarrow A \otimes M \rightarrow (A/\mathfrak{a}) \otimes M \rightarrow 0, & \quad (2) \end{aligned}$$

which is exact by Proposition 2.18. The first map in (2) is  $\iota \otimes 1$  where  $\iota : \mathfrak{a} \rightarrow A$  is inclusion. We have to figure out the image of the first map in (2),

$$\iota \otimes 1(\mathfrak{a} \otimes M) = \mathfrak{a} \otimes M.$$

Now we claim that  $\mathfrak{a} \otimes_A M \cong \mathfrak{a}M$ . We'll prove this by using universal property. ???

We wish to prove this by universal property. Let  $N$  be any  $A$ -module with an  $A$ -module homomorphism  $f : \mathfrak{a} \times M \rightarrow N$ .

$$\begin{array}{ccc} \mathfrak{a} \times M & & \\ T \downarrow & \searrow f & \\ \mathfrak{a}M & \xrightarrow{\bar{f}} & N \end{array}$$

We need to verify that for any  $f$ , there exist  $A$ -linear map  $\bar{f}$  that factors through  $\mathfrak{a}M$ . And by universal property of tensor product we'll know that

$$\mathfrak{a}M \cong \mathfrak{a} \otimes_A M.$$

And this gives us

$$A \otimes_A M / \mathfrak{a} \otimes_A M \cong M / \mathfrak{a}M \cong (A/\mathfrak{a}) \otimes_A M.$$

HERE is a post explaining an isomorphism.

### 33 Ex 3.

HERE is one solution.

HERE is a post, discussing an important isomorphism.

We follow the hint. Let  $\mathfrak{m} \subset A$  be the maximal ideal of local ring  $A$ , then we can define  $k := A/\mathfrak{m}$  as its residue field. According to 32, we can define a module as

$$M_k := k \otimes_A M \cong M/\mathfrak{m}M.$$

Notice that  $M$  is finitely generated and  $\mathfrak{m}$  lies inside Jacobson Radical  $\mathfrak{R}$ , so assumption of Nakayama's Lemma are verified. If we have  $M/\mathfrak{m}M = 0$ , then we can conclude that

$$M = 0.$$

Now we back to the problem itself, observe that

$$\begin{aligned} M \otimes_A N &= 0 \\ \Rightarrow (k \otimes_k k) \otimes_A M \otimes_A N &= 0 \\ \Rightarrow [k \otimes_k (k \otimes_A M)] \otimes_A N &= 0 \text{ by Qing Liu Prop 1.10 on Page 3} \\ \Rightarrow (k \otimes_A M) \otimes_k k \otimes_A N &= 0 \\ \Rightarrow \Rightarrow (k \otimes_A M) \otimes_k (k \otimes_A N) &= 0 \\ \Rightarrow M_k \otimes_k N_k &= 0. \end{aligned}$$

Notice that since  $M_k$  is annihilated by  $\mathfrak{m}$ , then it's inherits  $A/\mathfrak{m}$ -module structure, which implies it's a vector field. While  $M, N$  finitely generated, so we have  $M_k = M/\mathfrak{m}M$  is a finite dimensional vector space. Since all vector spaces are free, so we can write them as

$$M_k \otimes_k N_k = (A)^{\text{rank } M \cdot \text{rank } N}.$$

The fact that this free module equals to 0 will force each component to be 0, i.e. either  $M_k = 0$  or  $N_k = 0$ , then by the previous argument we know we must have either  $M = 0$  or  $N = 0$  as expected.

### 34 Ex 4.

Fix an injective  $A$ -module  $f : N_1 \rightarrow N_2$ . Since  $M = \oplus_{i \in I} M_i$  is flat, so we have

$$\begin{aligned} f \otimes 1 : N_1 \otimes M &\rightarrow N_2 \otimes M \\ \Rightarrow f \otimes 1 : \oplus_{i \in I} (N_1 \otimes M_i) &\rightarrow \oplus_{i \in I} (N_2 \otimes M_i) \end{aligned}$$

is injective. We define its component function as

$$(f \otimes 1)_i := N_1 \otimes M_i \rightarrow N_2 \otimes M_i$$

for arbitrary  $i \in I$ . Equivalently we can express the function as

$$f \otimes 1 = \oplus_{i \in I} (f \otimes 1)_i.$$

It suffices to prove the following claim: we claim that  $f \otimes 1$  is injective if and only if every  $(f \otimes 1)_i$  is injective for all  $i \in I$ .

$\Leftarrow$ : For distinct  $\mathbf{x}, \mathbf{y} \in \oplus_{i \in I} (N_1 \otimes M_i)$ , then we can find at least one  $i_0 \in I$  such that  $\mathbf{x}_{i_0} \neq \mathbf{y}_{i_0}$ , then we must have  $(f \otimes 1)_{i_0}(\mathbf{x}_{i_0}) \neq (f \otimes 1)_{i_0}(\mathbf{y}_{i_0})$  give each component function is injective. And this will force  $(f \otimes 1)(\mathbf{x}) \neq (f \otimes 1)(\mathbf{y})$  as expected.

$\Rightarrow$ : Given an index  $j \in I$ . For distinct  $\mathbf{x}_j, \mathbf{y}_j$ , we can embed it into  $\mathbf{x} = \mathbb{1}_{\{j\}} \mathbf{x}_j$  (we adopt the notation of indicator function HERE, this just means an element of  $N_1 \otimes M$  that are nonzero only in  $j$ -th coordinate) and similarly for  $\mathbf{y}_j$ . Given that  $f \otimes 1$  is injective, so we have

$$\begin{aligned} (f \otimes 1)(\mathbf{x}) &\neq (f \otimes 1)(\mathbf{y}) \\ \Rightarrow (f \otimes 1)_j(\mathbf{x}_j) &\neq (f \otimes 1)_j(\mathbf{y}_j), \end{aligned}$$

which proves that  $(f \otimes 1)_j$  is injective.

## 35 Ex 5.

HERE is a post discussing the structure of polynomial ring, and write it into direct sum of modules.

For a polynomial ring, we can write it as direct sum of distinct powers of indeterminate

$$A[x] \cong A \oplus A\{x\} \oplus A\{x^2\} \oplus \cdots \cong \oplus_{i \in \mathbb{N}} A\{x^i\} \cong \oplus_{i \in \mathbb{N}} A.$$

According to 34, it remains to check each module  $A$  is flat for  $i \in \mathbb{N}$ . Give an injective  $A$ -module homomorphism  $f : N_1 \rightarrow N_2$ , we have

$$f : N_1 \otimes_A A \rightarrow N_2 \otimes_A A \Rightarrow f : N_1 \rightarrow N_2$$

is injective, which proves that  $A[x]$  is flat as desired.

## 36 Ex 6.

$\Leftarrow$  Potential typo: here I assume  $M$  is an  $A$ -module.

Notice that polynomial ring  $A[x] \cong \oplus_{i \in \mathbb{N}} A_i$  where  $A_i \cong A$ . So we have

$$(\oplus_{i \in \mathbb{N}} A_i) \otimes_A M \cong \oplus_{i \in \mathbb{N}} M \cong M[x].$$

### 37 Ex 7.

Let  $\mathfrak{p}$  be a prime ideal in  $A$ . Show that  $\mathfrak{p}[x]$  is a prime ideal in  $A[x]$ . If  $\mathfrak{m}$  is a maximal ideal in  $A$ , is  $\mathfrak{m}[x]$  a maximal ideal in  $A[x]$ ?

Refers to Dummit and Foote, Section 9.1 of Polynomial Ring, on Page 296. "However, the ideal generated by  $\mathfrak{m}$  and  $x$  is maximal..."

► Proposition 4 of Chapter 7.2 on Page 235 of Dummit and Foote;

► Proposition 2 of Chapter 9.1 on Page 296 of Dummit and Foote.

Since we have, by Prop 2 above

$$A[x]/\mathfrak{p}[x] \cong A/\mathfrak{p}[x].$$

Note that  $A/\mathfrak{p}$  is an integral domain given  $\mathfrak{p}$  is a prime ideal in  $A$ . According to Prop 4 above, we know that  $A/\mathfrak{p}[x]$  is an integral domain. This proves that  $\mathfrak{p}[x]$  is a prime ideal in  $A[x]$ .

When  $I \subset A$  is a maximal ideal. See contents after Prop 2 above.

$$\mathbb{Z}[x]/\langle 2\mathbb{Z}[x] \rangle \cong \mathbb{Z}/2\mathbb{Z}[x].$$

It's not a field, so it's never a maximal ideal.

► See HERE, HERE, and HERE.

Note that  $\langle 2, x \rangle$  is maximal ideal in  $\mathbb{Z}[x]$  since we have

$$\mathbb{Z}[x]/\langle 2, x \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

We can define a map explicitly as

$$\begin{aligned} \varphi : \mathbb{Z}[x] &\rightarrow \mathbb{Z}/2\mathbb{Z} \\ a_0 + a_1x + \cdots + a_nx^n &\mapsto [a_0] + [a_1] \cdot 0 + \cdots + [a_n] \cdot 0^n. \end{aligned}$$

And we can verify that the kernel is exactly  $\langle 2, x \rangle$ .

Or we can interpret this isomorphism by applying Third Isomorphism. See HERE, and HERE.

### 38 Ex 8.

i) Use Prop 2.19 characterisation (iii) twice, together with the associativity of tensor product.

We start with an injective  $A$ -module homomorphism  $f : L' \rightarrow L$ . Since  $M$  is flat as  $A$ -module, so we have

$$f \otimes_A 1_M : L' \otimes_A M \rightarrow L \otimes_A M$$

is injective. Again, we can apply the assumption that  $N$  is flat as an  $A$ -module, which gives us

$$\begin{aligned} f \otimes_A 1_M \otimes_A 1_N : L' \otimes_A M \otimes_A N &\rightarrow L \otimes_A M \otimes_A N \\ \Rightarrow f \otimes_A (1_{M \otimes_A N}) : L' \otimes_A (M \otimes_A N) &\rightarrow L \otimes_A (M \otimes_A N), \end{aligned}$$

which confirms that  $M \otimes_A N$  is flat.

ii) Needs change the order when we applying flatness.

Let's start with an injective  $B$ -module homomorphism  $g : L' \rightarrow L$ . Since  $N$  is flat  $B$ -module, then we know that

$$g \otimes_B 1_N : L' \otimes_B N \rightarrow L \otimes_B N$$

is injective. And since  $B$  is  $A$ -algebra, then we can regard this homomorphism as  $A$ -module homomorphism, together with the fact that  $B$  is flat  $A$ -algebra, we have injective maps

$$\begin{aligned} F := g \otimes_B 1_N \otimes_A 1_B : L' \otimes_B N \otimes_A B &\rightarrow L \otimes_B N \otimes_A B \quad \text{this induces injective map} \\ \Rightarrow F' : L' \otimes_B B \otimes_A N &\rightarrow L \otimes_B B \otimes_A N \quad \text{this induces injective map} \\ \Rightarrow F'' : L' \otimes_A N &\rightarrow L \otimes_A N, \end{aligned}$$

which gives us  $N$  is a flat  $A$ -module. Notice that the reason we have apply associativity of Exercise 2.15 from book is because both  $N, B$  are  $(A, B)$ -bimodule.

### 39 Ex 9.

Notice that we have  $M/M' \cong M''$ . And we have a fact from Dummit and Foote, Chapter 10.3 Exercise 7, Page 356.

### 40 Ex 10.

♦ *Surjectivity could be interpreted as cokernel is trivial.*

↪ *I had some misunderstanding about the proposition initially. Need counterexample for help.*

♣ *Elegant solution, just see the post [HERE](#), which relies on the fact that tensoring preserve cokernel by right exactness, a more detailed version is [HERE](#); also see the post [HERE](#) for the solution.*

*Solution 1* According to Corollary 2.7, since  $N$  is finitely generated and  $\mathfrak{a} \subset \mathfrak{R}$  in Jacobson radical, we have

$$u(M) + \mathfrak{a}N = N \Rightarrow u(M) = N.$$

Clearly we have  $u(M) + \mathfrak{a}N \subset N$ , it suffices to check the other direction of inclusion. For arbitrary  $x \in N$ , we can find  $n_0 \in N$  such that

$$x \in n_0 + \mathfrak{a}N.$$

Notice that since  $M \rightarrow M/\mathfrak{a}M \rightarrow N/\mathfrak{a}N$  is surjective and we denote the composition map as  $h$ , then we know that there exists  $m_0$  such that  $h(m_0) = n_0 + \mathfrak{a}N$ . This gives us

$$x \in n_0 + \mathfrak{a}N = u(m_0) + \mathfrak{a}N \subset u(M) + \mathfrak{a}N.$$

*I couldn't proceed with this proof, because I don't know if the diagram is commute or not???*

*Solution 2* In general for cokernel, we have an exact sequence as

$$M \rightarrow N \rightarrow \text{coker}(u) := N/u(M) \rightarrow 0$$

According to Proposition 2.18, we know that we can tensor this sequence with  $(A/\mathfrak{a})$  as

$$\begin{aligned} M \otimes_A (A/\mathfrak{a}) &\rightarrow N \otimes_A (A/\mathfrak{a}) \rightarrow \text{coker}(u) \otimes_A (A/\mathfrak{a}) \rightarrow 0 \\ \Rightarrow M \otimes_A (A/\mathfrak{a}) &\rightarrow N \otimes_A (A/\mathfrak{a}) \rightarrow \text{coker}(u) \otimes_A (A/\mathfrak{a}) \rightarrow 0 \\ \Rightarrow M/\mathfrak{a}M &\rightarrow N/\mathfrak{a}N \rightarrow \text{coker}(u)/\mathfrak{a} \text{coker}(u) \rightarrow 0. \end{aligned}$$

Notice that since the induced map  $u' : M/\mathfrak{a}M \rightarrow N/\mathfrak{a}N$  is surjective, so  $\text{coker}(u) = \mathfrak{a} \text{coker}(u)$ , it's finitely generated since it's a quotient of finitely generated module  $N$ . While  $\mathfrak{a}$  lives inside Jacobson radical, by Nakayama's Lemma, we can conclude  $\text{coker}(u) = 0$ , which is equivalently to say that  $u : M \rightarrow N$  is surjective.

See a solution [HERE](#), and [HERE](#).

**Counterexample** Converse is true in general. Why does the ideal  $\mathfrak{a}$  have to lie in Jacobson radical?

Take a non-local ring  $\mathbb{Z}$  and it acts on itself. Consider this diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z}/3\mathbb{Z} & \xrightarrow[\cong]{-----} & \mathbb{Z}/3\mathbb{Z} \end{array}$$

There's another problem of similar taste.

*Let  $I$  be a nilpotent ideal in a commutative ring  $R$ . Let  $M, N$  be  $R$ -modules with  $\varphi : M \rightarrow N$  an  $R$ -module homomorphism. Show that if the induced map  $\bar{\varphi} : M/IM \rightarrow N/IN$  is surjective, then  $\varphi$  is surjective.*

See this post [HERE](#).

But here we don't have the fin.gen. assumption, which forbids us to use Nakayama.

## 41 Ex 11.

### 41.0.1 i)

According to the hint, we have induced isomorphisms as

$$\begin{aligned} 1 \otimes_A : (A/\mathfrak{m}) \otimes_A A^m &\rightarrow (A/\mathfrak{m}) \otimes_A A^n \\ \Rightarrow (1 \otimes_A)' : (A/\mathfrak{m} \otimes_A A)^m &\rightarrow (A/\mathfrak{m} \otimes_A A)^n \\ \Rightarrow (1 \otimes_A)'' : (A/\mathfrak{m})^m &\rightarrow (A/\mathfrak{m})^n. \end{aligned}$$

Notice that we have the last isomorphism between vector space, while in vector space of finite dimension, they're isomorphic if and only if dimension is the same. And this case we can deduce that  $m = n$  as expected.

### 41.0.2 ii)

## 42 Ex 12.

We follow the hint, wishing to prove that  $\text{Ker}(\phi)$  is a direct summand of  $M$ . While  $M$  is finitely generated, then so will be its direct summand.

Let  $e_1, \dots, e_n$  be basis of  $A^n$ , while  $\phi$  is surjective, we define

$$u_i := \phi^{-1}(e_i) \in M$$

for all integers  $1 \leq i \leq n$ . And we define the module all such elements generated inside  $M$  as

$$U = \langle u_1, \dots, u_n \rangle.$$

Clearly we have  $U + \text{Ker}(\phi) \subset M$ .

And we must have  $U \cap \text{Ker}(\phi) = 0$  since otherwise we'll have a nontrivial element  $m = m_1 u_1 + \dots + m_n u_n \in M$ . Here nontriviality of  $m$  enforces that  $m_1, \dots, m_n$  are not all 0. Hence we know

$$0 = \phi(m) = \phi(m_1 u_1 + \dots + m_n u_n) = m_1 e_1 + \dots + m_n e_n.$$

But  $m_1 e_1 + \dots + m_n e_n \neq 0$  since  $m_1, \dots, m_n$  are not all trivial and  $\{e_1, \dots, e_n\}$  is a basis for  $A^n$ .

To prove that  $M = U \oplus \text{Ker}(\phi)$ , it suffices to check  $M \subset U + \text{Ker}(\phi)$ . For any element  $m \in M$ , its image is  $a := (a_1, \dots, a_n) := \phi(m) \in A^n$ . If we have  $\phi(m) = 0$ , then we can write it as  $m = 0 + m \in U + \text{Ker}(\phi)$  since  $m \in \text{Ker}(\phi)$ .

If it's not 0, i.e. we know  $\phi(m) = a_1 e_1 + \dots + a_n e_n \neq 0$  for some nontrivial coefficients  $a_1, \dots, a_n \in A$ . Furthermore, we'll get  $m \in U$  by definition of  $U$ , which means we can write  $m = m + 0 \in U + \text{Ker}(\phi)$ .

In summary, we've proved that  $M \subset U + \text{Ker}(\phi)$  as desired.

► See a post [HERE](#) and [HERER](#), which elegantly uses projectiveness of free module  $A^n$  in exact sequence.

See a proof [HERE](#).

"Homomorphism preserve finitely generated", see this post [HERE](#).



### 43 Ex 13.

Clearly we can verify that

$$p \circ g = \text{id}_N,$$

which implies that  $g$  is injective. So we can consider the exact sequence

$$0 \longrightarrow N \xrightarrow[g]{p} N_B \longrightarrow g(N) \longrightarrow 0$$

We know that this sequence splits, so we know  $N_B = N \oplus g(N) \cong$ .

See this solution [HERE](#).

See the post [HERE](#), and [HERE](#).

### 44 Ex 14.

*Direct limits*

### 45 Ex 15.

Since  $C$  is direct sum of all modules and  $\mu_i$  is the restriction of surjective map  $\mu : C \rightarrow C/D$ . For any element  $x \in M := C/D$ , we can express its preimage

$$\mu^{-1}(x) = \bigoplus_{I_0} x_i \in C = \bigoplus_{i \in I} M_i$$

where  $x_i \in M_i$  and  $I_0$  is a *finite* index set.

For the first two distinct components  $x_1, x_2$ , by definition of the direct system we can find  $k \in I$  such that  $1 \leq k$  and  $2 \leq k$ . Furthermore, we have

$$x_1 - \mu_{1k}(x_1) \in D, \quad x_2 - \mu_{2k}(x_2) \in D.$$

But this implies we can replace  $x_1, x_2$  in  $\mu^{-1}(x)$  by 0 and write  $\mu_{1k}(x_1) + \mu_{2k}(x_2)$  at  $k$ -th coordinate, i.e. we define an element

$$y := 0 \oplus 0 \oplus (\oplus_{I_0 \setminus \{1,2\}} x_i) + \underbrace{(0, \dots, 0, \mu_{1k}(x_1) + \mu_{2k}(x_2), 0, \dots, 0)}_{k\text{-th coordinate}}.$$

See [HERE](#) for the horizontal curly braces.

We claim that  $\mu(y) = \mu(x)$  by construction.

The difference is that  $y$  is direct sum whose nonzero components are fewer than  $|I_0|$ . And we can inductively do this process, given  $I_0$  is finite, and reach to one point that there's only one nonzero component as expected.

See [HERE](#) for a discussion post on MathStackExchange.

If we have  $\mu_i(x_i) = 0$ , more precisely we have

$$\mu_i(x_i) \in D = \langle x_i - \mu_{ij}(x_i) \rangle_{i,j \in I}.$$

Also see "A Term..." Corollary 7.5 on Page 53.

## 46 Ex 16.

Given the construction and verify the universal property. See Theorem 7.4 of A Term of Commutative Algebra...

## 47 Ex 17.

In "A Term of Commutative Algebra", Ex 7.2 on Page 52 proved that

$$\varinjlim M_i = \bigcup M_i.$$

The third equality holds, since the inclusion...

See a post [HERE](#).

## 48 Ex 18.

$$\begin{array}{ccccc} M_i & \xrightarrow{\mu_{ij}} & M_j & \xrightarrow{\mu_j} & M \\ \phi_i \downarrow & & \phi_j \downarrow & & \downarrow \exists! \phi \\ N_i & \xrightarrow{v_{ij}} & N_j & \xrightarrow{v_j} & N \end{array}$$

Notice that we're given another cone under  $\{M_i\}$  with nadir  $N$  by composing  $v_i \circ \phi_i : M_i \rightarrow N$ . By universal property of  $M$  there exists a unique map  $\phi : M \rightarrow N$  as expected.

## 49 Ex 19.

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_i & \xrightarrow{\phi_i} & N_i & \xrightarrow{\psi_i} & P_i \longrightarrow 0 \\ & & \alpha_{ij} \downarrow & & \beta_{ij} \downarrow & & \gamma_{ij} \downarrow \\ 0 & \longrightarrow & M_j & \xrightarrow{\phi_j} & N_j & \xrightarrow{\psi_j} & P_j \longrightarrow 0 \\ & & \alpha_j \downarrow & & \beta_j \downarrow & & \gamma_j \downarrow \\ 0 & \longrightarrow & \varinjlim M & \xrightarrow{\phi} & \varinjlim N & \xrightarrow{\psi} & \varinjlim P \longrightarrow 0 \end{array}$$

Note that we'll decide the index  $i, j$  along the proof.

According to the assumption, the first two rows are exact, we wish to show the last row is exact. Here  $\phi, \psi$  are given by applying 48. Furthermore, 48 tells us the diagram is commute.

We claim that  $\text{Im } \phi \subset \text{Ker } \psi$ . Pick an element  $x \in \varinjlim M$ , by 45 there exists an index  $i$  such that  $\alpha(x_i) = x$  where  $x_i \in M_i$ . While the diagram commute, we can chase the  $x_i$  under the map in either ways

$$\begin{aligned}\psi \circ \phi \circ \alpha_i(x_i) &= \gamma_i \circ \psi_i \circ \phi_i(x_i) = \gamma_i(0) = 0 \\ \Rightarrow \psi \circ \phi(x) &= 0,\end{aligned}$$

which proves the claim.

Conversely, we need to show  $\text{Im } \phi \supset \text{Ker } \psi$ . Start by picking an element  $y \in \text{Ker } \psi$ , then apply 45. Hence there exists  $j$  such that  $\beta_j(y_j) = y$  for some  $y_j \in N_j$ . Given the diagram is commute,  $y_i \in \text{Ker } \psi_j = \text{Im } \phi_j$ . This implies there exists some  $y' \in M_j$  such that  $\phi_j(y') = y_j$ . While the diagram involving  $M_j, N_j, \varinjlim M, \varinjlim N$  commute, we know  $\phi(\alpha_j(y')) = y$ , which proves the claim.

## 50 Ex 20.

Follow the hint, using universal property...

## 51 Ex 21.

The colimit  $\varinjlim A$  is a  $\mathbb{Z}$ -module. To be defined as a ring, we need to define multiplication, unit.

Firstly, we define the multiplication of  $a, b \in \varinjlim A$ . Pick two index such that  $\alpha_i(a_i) = a$ ,  $\alpha_j(b_j) = b$  by invoking 45. According to the definition of filtered set, we can pick another index  $k$  such that  $i, j \leq k$ . While the diagram is commute, we also have  $\alpha_k \circ \alpha_{ik}(a_i) = a$ ,  $\alpha_k \circ \alpha_{jk}(b_j) = b$ . And we define

$$a \cdot b := \alpha_{ik}(a_i) \cdot_{A_k} \alpha_{jk}(b_j)$$

where the multiplication is taken from  $A_k$ . It is well-defined because the diagram is commute, the value is independent of choice of index. Also we can define  $1 := \alpha_k(1_k)$ , which is also well-defined for the diagram is commute. See a post [HERE](#).

## 52 Ex 27. Absolutely Flatness

### 52.0.1

i)  $\Rightarrow$  ii): I have no idea what the hint was...  
See a post [HERE](#), and a solution [HERE](#).

Let's start with the exact sequence

$$0 \longrightarrow \langle x \rangle \longrightarrow A \longrightarrow A/\langle x \rangle \longrightarrow 0$$

Since  $A$  is absolutely flat, we know the functor  $- \otimes_A A/\langle x \rangle$  is an exact functor. Apply this exact functor gives us

$$0 \longrightarrow \langle x \rangle \otimes_A A/\langle x \rangle \longrightarrow A \otimes_A A/\langle x \rangle \longrightarrow A/\langle x \rangle \otimes_A A/\langle x \rangle \longrightarrow 0$$

Now we can apply quotient isomorphism we proved in 32, which gives us

$$\begin{aligned} \langle x \rangle \otimes_A A/\langle x \rangle &\simeq \langle x \rangle / \langle x \rangle^2 = \langle x \rangle / \langle x^2 \rangle. \\ A/\langle x \rangle \otimes_A A/\langle x \rangle &\simeq A/\langle x \rangle / \langle x \rangle A/\langle x \rangle \simeq A/\langle x \rangle. \end{aligned}$$

More precisely, we get a short exact sequence as follows

$$0 \longrightarrow \langle x \rangle / \langle x \rangle^2 \longrightarrow A/\langle x \rangle \longrightarrow A/\langle x \rangle \longrightarrow 0$$

Then we can apply first isomorphism theorem to enforce  $\langle x \rangle / \langle x \rangle^2 = 0$ .

??? What are the map induced from  $A/\langle x \rangle \rightarrow A/\langle x \rangle$ , it should be identity but I don't know...

### 52.0.2

ii)  $\Rightarrow$  iii):

### 52.0.3

iii)  $\Rightarrow$  i): See a post [HERE](#) without using Tor functor...

## 53 Ex 28.

### 53.0.1 Boolean Ring

Clearly a Boolean ring is *absolutely flat* by applying characterisation of absolutely flatness in 52.

### 53.0.2 Ring from Chapter 1 Exercise 7

The requirement of Exercise 7 in Chapter 1 8 is  $x = x^n$  for any  $x \in A$  with an integer  $n \geq 2$ .

??? Didn't complete the proof. See [HERE](#), and [HERE](#).

Note that in general we have  $\langle x \rangle \supset \langle x^2 \rangle$ . Conversely, we observe  $ax = ax^n = ax^{n-2}x^2 \in \langle x^2 \rangle$ , which proves  $\langle x \rangle = \langle x^2 \rangle$ .

### 53.0.3 Homomorphic image

Let  $f : A \rightarrow B$  be a ring homomorphism where  $A$  is *absolutely flat*, then  $f(A)$  is *absolutely flat*.

We use characterisation from 52. For any principal ideal  $\mathfrak{J} \subset f(A)$ , we can express it as  $\mathfrak{J} = \langle f(a) \rangle \subset f(A)$  where  $a \in A$ . Note that the surjective map  $A \rightarrow f(A)$  will map an ideal in  $A$  to an ideal in  $f(A)$ , therefore we have

$$\langle f(a) \rangle = f(\langle a \rangle) = f(\langle a^2 \rangle) = \langle f(a)f(a) \rangle,$$

which proves that  $f(A)$  is idempotent.

One crucial step is the first and the last equality (without surjectivity assumption we'll only get  $\langle f(a) \rangle \supset f(\langle a \rangle)$  and similarly for the other one). See a post [HERE](#), be careful see [HERE](#)!

### 53.0.4 Absolutely flat local ring, non-unit...

Assume local ring  $(A, \mathfrak{m}, k)$  is absolutely flat. Didn't work out... See "A Term of Commutative Algebra" Ex 10.26. Take a non-unit  $x \in A$ . Since  $\langle x \rangle = \langle x^2 \rangle$  we have  $x = ax^2$  for some  $a \in A$ . This implies  $x(ax - 1) = 0$ , while  $x$  is a non-unit so  $ax - 1 \neq 0$ . So we know a non-unit must be a zero-divisor.

Now let's consider the ideal  $\langle x \rangle$ , it's not the whole ring given  $x$  is assumed to be a non-unit. So it must lie inside the maximal ideal of the local ring, i.e.

$$\langle x \rangle \subset \mathfrak{m}$$

Recall the characterisation of Jacobson radical, we know that  $ax - 1$  is a unit in  $A$ . Therefore we have

$$x = (ax - 1)^{-1}(ax - 1)x = 0 \Rightarrow \mathfrak{m} = 0.$$

And this proves that  $A$  is a field.

??? In fact, we can say more about the converse. Any field is a local ring and absolutely flat.

### 53.0.5

Apart from this problem, we found a post [HERE](#), which is about "nilradical of an absolutely flat ring is trivial".

## 54 Example

Warning: localisation might not produce a local ring, for example see [HERE](#).  
If we localise at a prime ideal  $\mathfrak{p}$  then it's local. See [HERE](#).

## 55 Proposition 3.7

Localisation commutes with tensor product. See [HERE](#) for a pure tensor manipulation. See [HERE](#) for a post, using universal property of tensor product.

## 56 Proposition 3.11

For i), I have some doubts in the last step of [HERE](#).

For iv), in the proof we have " $S^{-1}A/S^{-1}\mathfrak{p} \simeq \overline{S}^{-1}(A/\mathfrak{p})$ ". In fact we if we apply exact functor  $S^{-1}(-)$  to first isomorphism exact sequence we'll get it, with  $S^{-1}$  but not  $\overline{S}^{-1}$ ???

See a post discussing this [HERE](#). In the second answer, it used Universal Property of Localisation to prove the isomorphism...

For v), about extension of ideal, which really should be discussed in Chap 1. [HERE](#) is a post, discussion the definition.

For a more detailed treatment, see [The Rising Sea 1.3.F. EXERCISE](#).

## 57 Proposition 3.14

One might wonder if the condition of finitely-generated is omitted what will happen?

Let  $A = k[x]$  where  $k$  is a field, and define

$$M = \bigoplus_{i=2}^{\infty} \frac{k[x]}{x^i} = \frac{k[x]}{x^2} \oplus \frac{k[x]}{x^3} \oplus \dots$$

It's not finitely-generated as  $k[x]$ -module.

Let  $S = \{1, x, x^2, \dots\}$ .

Notice that by definition  $S^{-1}M = 0$  given that for any  $m \in M$  we can always choose a large enough  $s \in S$  such that  $sm = 0$ .

On the other hand we have  $\text{Ann}(M) = 0$ , which gives us

$$0 = S^{-1}(\text{Ann}(M)) \neq \text{Ann}(S^{-1}M) = \text{Ann}(0) = k[x]$$

## 58 Proposition 3.16

Question on the last equality. Purely based on the contents we knew

$$\mathfrak{q}^c \supset \mathfrak{p}^{ec} = \mathfrak{p}.$$

Denote the ring homomorphism from  $A \rightarrow B$  as  $f$ . Conversely, we proceed by a contrapositive argument. Suppose we have  $x \in \mathfrak{q}^c \setminus \mathfrak{p}$ , then  $f(x) \in \mathfrak{q}$  and  $f(x) \in f(A - \mathfrak{p}) = S$  by definition. But it was ensured that  $\mathfrak{q} \cap S = \emptyset$ , contradiction.

### 58.1 Side Notes

There are some examples from Reid's Undergraduate C.A, see P41.

### 59 Ex 1.

$\Leftarrow$ : Assume the existence of an element  $s \in S$  such that  $sM = 0$ . For any element  $m_0/s_0 \in S^{-1}M$ , we claim

$$m_0/s_0 = 0/1 \text{ given } s(m_0 \cdot 1 - s_0 \cdot 0) = sm_0 = 0.$$

Since every element is zero, hence  $S^{-1}M = 0$ .

$\Rightarrow$ : Since  $M$  is finitely-generated, let  $m_1, \dots, m_n$  be its generators for some  $n \in \mathbb{Z}$ . For each  $m_i$ , there's some  $s_i \in S$  such that  $s_i m_i = 0$  by assumption. Now we define

$$s = \prod_{i=1}^n s_i$$

and claim that  $sM = 0$ . This because for any  $m \in M$ , we can express it as  $a_1 m_1 + \dots + a_n m_n = m \in M$  for some coefficients  $a_i \in A$ . Furthermore, we notice

$$sm = s(a_1 m_1 + \dots + a_n m_n) = a_1 sm_1 + \dots + a_n sm_n = 0,$$

which proves  $sM = 0$ .

$\nRightarrow$  "finitely-generated  $A$ -module  $M$ " is important? Any counterexample???

### 60 Ex 2.

We recall the characterisation of Jacobson radical in a ring. For any element  $x \in \text{Jac}()$

### 61 Ex 3.

$$\begin{array}{ccccc} A & \xrightarrow{f_1} & S^{-1}A & \xrightarrow{f_2} & U^{-1}(S^{-1}A) \\ & \searrow g_1 & \downarrow f_3 & \swarrow \exists! \phi & \nearrow \exists! \psi \\ & & (ST)^{-1}A & & \end{array}$$

Firstly, we claim that  $f_2 \circ f_1$  will send every element of  $ST$  to a unit in  $U^{-1}(S^{-1}A)$ . Hence by universal property there's a unique map

$$\psi : (ST)^{-1}A \rightarrow U^{-1}(S^{-1}A).$$

Proof for the first claim. Element of  $ST$  is like  $st$  for some  $s \in S$  and  $t \in T$ . Under  $f_1$ ,  $s/1$  is a unit in  $S^{-1}A$  and  $t/1 \in U$ . Furthermore, under  $f_2$ , element  $f_2(t/1)$  will be a unit in  $U^{-1}(S^{-1}A)$ . And we have  $f_2 \circ f_1(st) = \frac{st/1}{1}$  is a unit for there exists

$$\frac{1/s}{t/1} \in U^{-1}(S^{-1}A) \text{ such that } \frac{st/1}{1} \cdot \frac{1/s}{t/1} = 1.$$

Secondly, we claim that  $f_3$  will send every element of  $U$  to unit in  $(ST)^{-1}A$ , which give rise to the unique existence of

$$\phi : U^{-1}(S^{-1}A) \rightarrow (ST)^{-1}A.$$

Proof for the second claim. Note that the map

$$f_3 : S^{-1}A \rightarrow (ST)^{-1}A$$

is a ring inclusion. Clearly  $t/1 \in U$  will be a unit under  $f_3$  in  $(ST)^{-1}A$  given  $1/t \in (ST)^{-1}A$  and  $t/1 \cdot 1/t = 1$ .

Both maps are unique and making the diagram commute, hence they're isomorphisms.

See Stack Project Prop 10.9.10 [HERE](#). See this page, for a discussion between isomorphisms of localisations as either ring or module...

## 62 Ex 4.

My approach was to produce two unique morphisms by universal property of  $S^{-1}B$  and  $T^{-1}B$ , respectively.

$$\begin{array}{ccc} & & S^{-1}B \\ b \mapsto b/1 \nearrow & & \uparrow \\ B & & \exists! \\ b \mapsto b/f(1) \searrow & & \downarrow \\ & & T^{-1}B \end{array}$$

But it seems to be difficult to write in valid proof....

See a post discuss a more straight-forward approach [HERE](#), and [HERE](#). In the first post, there's an obvious map to define and module homomorphism together with surjectivity easy to check. For the injectivity, we can check...



## 63 Ex 5.

I encountered a problem, wishing to prove  $x/1 \neq 0/1$  in the localised ring.

### 63.1 A potentially incorrect solution

We wish to prove that

$$\mathrm{Nil}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Spec} A} \mathfrak{p} = 0.$$

For the localised ring  $A_{\mathfrak{p}}$ , by Prop 3.11 (i) and Coro 3.13, we know every ideals are extended ideals and

$$0 = \mathrm{Nil}(A_{\mathfrak{p}}) = \bigcap_{\mathfrak{q} \in \mathrm{Spec} A, \mathfrak{q} \subset \mathfrak{p}} \mathfrak{q}^e \supset \left( \bigcap_{\mathfrak{q} \in \mathrm{Spec} A, \mathfrak{q} \subset \mathfrak{p}} \mathfrak{q} \right)^e$$

where the last inclusion is given by Exercise 1.18 on Page 10 of Atiyah's textbook. But the only way that an extended ideal to be zero is the ideal

$$\bigcap_{\mathfrak{q} \in \mathrm{Spec} A, \mathfrak{q} \subset \mathfrak{p}} \mathfrak{q} = 0.$$

Therefore we have

$$\mathrm{Nil}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Spec} A} \mathfrak{p} \subset \bigcap_{\mathfrak{q} \in \mathrm{Spec} A, \mathfrak{q} \subset \mathfrak{p}} \mathfrak{q} = 0.$$

See a post [HERE](#). For examples...

See a post [HERE](#). It used Corollary 3.12.

### 63.2 Counterexample

*"Being an integral domain is not a local property."* See a post [HERE](#), and [HERE](#).

Possible counterexamples including  $\mathbb{Z}/6\mathbb{Z}$ ,  $\mathbb{C}[x, y]/\langle xy \rangle$ , etc. But I'm not sure about the what will be the localisation...

One very simple way to construct a counterexample to take product of two integral domains. Say  $R = \mathbb{Q} \times \mathbb{Q}$ . According to Milne's Notes on Algebraic Number Theory Page 15, "Ideals in product of rings", we know that prime ideals in  $R$  is of the form  $\mathfrak{q} = \mathfrak{p} \times \mathbb{Q}$  or  $\mathbb{Q} \times \mathfrak{p}$  where  $\mathfrak{p} \subset \mathbb{Q}$  is a prime ideal. We can see that  $R_{\mathfrak{q}}$  is always an integral domain. However,  $R$  is not an integral domain for  $(1, 0), (0, 1)$  are nontrivial zero-divisors.

## 64 Ex 6.

Apply Zorn's Lemma. Since  $A \neq 0$ , hence it has identity. Then  $\{1\}$  would be a multiplicatively closed subset. Hence  $\Sigma$  is nonempty. Each *chain* is bounded above by  $A \setminus \{\text{zerodivisors}\}$ . ❌!!! This is a wrong way to apply Zorn. Because I was confused how to find the correct upper bound?...

The set  $\Sigma$  is nonempty and ordered by inclusion as illustrated above. Fix a chain, the upper bound is the union of all multiplicatively closed subsets

$$S = \bigcup S_i$$

where  $S_i$  are all multiplicatively closed subsets in the chain. Clearly, union of multiplicatively closed subsets  $S$  is again multiplicatively closed and crucially we know  $0 \notin S$ . Therefore each chain is bounded above, then we can apply Zorn's Lemma to conclude the existence of a maximal element  $S \in \Sigma$ . *This proof is exactly the same as proper ideal lies in some maximal ideal...*

$\Rightarrow$ : Use contrapositive for "minimal" (Is this legal??? In fact we can contradict either three properties including minimal, prime, or ideal...). Suppose on the contrary that we have a prime ideal  $\mathfrak{p} \subsetneq A - S$ , then  $A - \mathfrak{p} \in \Sigma$  is a multiplicatively closed subset such that  $0 \notin A - \mathfrak{p}$ . Clearly we have  $A - \mathfrak{p} \supsetneq S$ , and this contradicts the assumption that  $S$  is the maximal element in  $\Sigma$ .

$\Leftarrow$ : Assume  $A - S$  is a minimal prime ideal. Hence  $S$  is a multiplicatively closed set such that  $0 \notin S$ . Furthermore, multiplicatively closed set  $S$  is maximal given

### 64.1

One concern was about contrapositive, is that legal? Another problem I encountered was taking complement of multiplicatively closed subset. It will not necessarily give us an ideal. For example, consider  $\mathbb{Q}[x]$  with a multiplicatively closed subset  $S_0 = \{1, x, x^2, \dots\}$ , the complement isn't even closed under addition for we have  $x + 1/2, x - 1/2$ . Or another even simpler example, take  $S_1 = \{1\}$  in  $\mathbb{Z}$ .

However, "complement of maximal multiplicatively closed subset *is* a prime ideal." See this post [HERE](#).



### 64.2

A post [HERE](#). A post [HERE](#), see this post "conversely".

### 64.3

Note that on "A Term..." there's another approach, requiring saturation...

## 65 Ex 7.

### 65.1 (i)

$\Leftarrow$ : Assume  $A - S$  is a union of prime ideals. Then  $S$  is a intersection of multiplicatively closed subset, which is again a multiplicatively closed subset. We only need to prove that

$$xy \in S \Rightarrow x \in S \text{ and } y \in S.$$

This is true since  $xy \notin \mathfrak{p}_i$  for each prime ideals implies  $x \notin \mathfrak{p}_i$  and  $y \notin \mathfrak{p}_i$ . While  $\mathfrak{p}_i$  is arbitrary, we can conclude that  $x \in S$  and  $y \in S$ .

Verified at [1] Exercise 3.24.

$\Rightarrow$ : Assume  $S$  is saturated.

We note that  $A - S$  contains no unit, so we can cover it by  $\{\mathfrak{p}_i\}_{i \in I}$  for some index  $I$  in which every  $\mathfrak{p}_i$  is a maximal ideal. Now we need to prove  $\mathfrak{p}_i \cap (A - S)$  is an ideal and prime.

The above approach is potentially wrong... I couldn't work out. Because I couldn't prove  $A$  is an additive Abelian group. See a post [HERE](#).

### 65.2

*Potentially wrong...* It turned out I choose the wrong covering. See a post [HERE](#). It has some slight problems...

Start with fact that any  $x \in A - S$  is a non-unit, under the ring homomorphism  $f : A \rightarrow S^{-1}A$  the element  $f(x)$  will still be a non-unit (???). Hence we can find a maximal ideal  $\mathfrak{q} \subset S^{-1}A$  contains  $f(x)$ . In particular, it's prime. By correspondence theorem we know its preimage  $f^{-1}(\mathfrak{q})$  will have trivial intersection with  $S$ . Hence

$$x \in f^{-1}(\mathfrak{q}) \subset A - S$$

and this proves that  $A - S$  is a union of prime ideals.

I have some doubts towards the above argument. Ring homo will map unit to unit. But ring homo won't necessarily map non-unit to non-unit, for example take inclusion map  $\mathbb{Z} \rightarrow \mathbb{Q}$ . But is the argument right in this specific case?

### 65.3 a verified approach

According to a post [HERE](#). One important claim is that we can find a maximal ideal containing  $\langle x \rangle$  and disjoint from  $S$ . This is possible, see Eisenbud's "Commutative Algebra...", Page 70, construction of a prime ideal.

Or see "A Term..." Proposition 3.9, Exercise 3.24. See [HERE](#).

### 65.4 Examples

Assumption of  $S$  being saturated is necessary. Note that  $\{1\} \subset \mathbb{Z}$  is a multiplicatively closed subset that's not *saturated*, whereas  $\{-1, 1\}$  is saturated.

And clearly  $\mathbb{Z} \setminus \{1\}$  isn't a union of prime ideals for we have  $1, -1$  inside that could never be covered by prime ideals. Indeed, we note  $\mathbb{Z} \setminus \{-1, 1\}$  is a union of primes. (?)

## 65.5 (ii)

Let  $\sum$  denotes the set of all *saturated* multiplicatively closed subset in  $A$  that containing  $S$ . Note that in each *chain*  $\{S_i\}_{i \in I}$  where  $I$  is some index set, the intersection

$$A := \bigcap_{i \in I} S_i$$

is again saturated multiplicatively closed subset: clearly it contains 1; multiplicatively closed for each  $S_i$  is so; and saturated for the similar reason. Also it must contain  $S$ , so  $A \in \sum$ .

Hence each chain in  $\sum$  is bounded below, Zorn's Lemma gives us the existence of minimal element. Suppose we have two distinct minimal element  $A_1, A_2$ , i.e. we have  $A_1 - A_2 \neq \emptyset, A_2 - A_1 \neq \emptyset$ . Pick  $x \in A_1 - A_2$  and  $y \in A_2 - A_1$ . Then  $xy \in A_1 \cap A_2$ , while  $A_1$  is saturated then  $x \in A_2$ , contradiction! Therefore there's only one minimal element, i.e. the smallest one, in  $\sum$  and we can denote it as  $\overline{S}$ .

Let the set  $\{\mathfrak{Q}_i\}_{i \in J}$  denotes the set of all prime ideals in  $A$  that do not meet  $S$  for some index set  $J$ . According to (i),

$$A - \bigcup_{i \in J} \mathfrak{Q}_i$$

is a saturated multiplicatively closed subset. Clearly we have

$$S \subset A - \bigcup_{i \in J} \mathfrak{Q}_i,$$

while  $\overline{S}$  is smallest saturated one we know

$$\overline{S} \subset A - \bigcup_{i \in J} \mathfrak{Q}_i.$$

Also we must have the inclusion in converse direction. Otherwise, say  $A - \bigcup_{i \in J} \mathfrak{Q}_i \subsetneq \overline{S}$ , we'll get a strictly larger set of prime ideals that don't meet  $S$  indexed by  $J_0$ . This is because  $\overline{S}$  is saturated we can apply (i) above. And this is a contradiction for definition of  $J$ . Hence we can conclude

$$\overline{S} = A - \bigcup_{i \in J} \mathfrak{Q}_i.$$

See a verified post [HERE](#).



## 65.6 Compute a specific example, wrong

For the case where  $\mathfrak{a} = A$ , we have  $1 + \mathfrak{a} = A$  and  $\overline{S} = A$ .

Now we can assume  $1 \notin \mathfrak{a}$ . Note that  $\langle 1 + \mathfrak{a}, \mathfrak{a} \rangle = \langle 1 \rangle = A$ . So the set of all prime ideals in  $A$  that do not meet  $S$  lies strictly inside  $\mathfrak{a}$ .

For any prime ideal  $\mathfrak{p} \subsetneq \mathfrak{a}$ , we claim that  $\mathfrak{p} \cap (1 + \mathfrak{a}) = \emptyset$ . Suppose we have

$$1 + a = p$$

for  $a \in \mathfrak{a}$  and  $p \in \mathfrak{p} \subset \mathfrak{a}$ , then  $1 \in \mathfrak{a}$ , contradiction. Therefore we know that any prime ideal lies strictly inside  $\mathfrak{a}$  are exactly the set of prime ideals that do not meet  $S$ . Hence we can conclude

$$\overline{S} = \bigcup_{\mathfrak{p} \in \text{Spec } A, \mathfrak{p} \subsetneq \mathfrak{a}} \mathfrak{p}.$$

Above argument is wrong! I made a mistake in the original proof ✗  
 ??? Recall the localisation, we have

$$\overline{S} = A - \left( \bigcup_{\mathfrak{q} \in \text{Spec}(S^{-1}A)} f^{-1}(\mathfrak{q}) \right)$$

where  $f : A \rightarrow S^{-1}A$ .

## 65.7 Compute a specific example, verified

See HERE. And [1] Exercise 3.25.

The correct approach is to consider all primes that containing  $\mathfrak{a}$ .

$$\overline{S} = A - \left( \bigcup_{\mathfrak{p} \in \text{Spec}(A), \mathfrak{a} \subset \mathfrak{p}} \mathfrak{p} \right).$$

## 66 Ex 8.

### 66.1

For (i)  $\Rightarrow$  (ii). Fix  $t \in T$ . Then  $t/1$  is a unit in  $S^{-1}A$  with the inverse being  $\phi^{-1}(1/t) \in S^{-1}A$  given  $\phi$  is bijective. Notice that under ring homomorphism

$$\phi(t/1 \cdot \phi^{-1}(1/t)) = \phi(t/1) \cdot 1/t = t/1 \cdot 1/t = 1 \Rightarrow t/1 \cdot \phi^{-1}(1/t) = 1 \in S^{-1}A.$$

### 66.2

For (ii)  $\Rightarrow$  (iii). Fix an element  $t \in T$ . By (ii) we know there's an element  $a/s \in S^{-1}A$  for some  $a \in A$  and  $s \in S$  such that  $1 = t/1 \cdot a/s = ta/s$ . By definition there's some  $s' \in S$  such that

$$s'(ta - s) = 0 \in A \Rightarrow (s'a)t = s's \in S.$$

Notice that  $s'a \in A$ , so we can define  $x = s'a$  hence  $xa \in S$  as expected.

### 66.3

For (iii)  $\Rightarrow$  (iv). For any  $t \in T$ , there exists some  $x \in A$  such that

$$xt \in S \subset \overline{S} \Leftrightarrow x, t \in \overline{S}.$$

### 66.4

For (iv)  $\Rightarrow$  (v). Take contrapositive. For any prime ideal  $\mathfrak{p}$  that doesn't meet  $S$ , then

$$\mathfrak{p} \subset A \setminus \overline{S} \subset A \setminus T.$$

And this is saying that  $\mathfrak{p}$  won't meet  $T$ .

### 66.5

For (v)  $\Rightarrow$  (i). ?? No idea. I tried two approaches. One is use local property, which leads to localise twice. Another is to verify bijectivity element-wise.

### 66.6

For (v)  $\Rightarrow$  (iv). Okay. Use contrapositive and characterisation of saturation in Section 65. Say  $T$  isn't completely contained in  $\overline{S}$ , this means it meet some prime ideal that doesn't meet  $S$ . And this contradicts the (iv) as expected.

### 66.7

For (iii)  $\Rightarrow$  (ii). For any  $t \in T$ , we claim  $t/1$  is a unit in  $S^{-1}A$ . Since we can find  $x \in A$  such that  $xt \in S$ , and for  $x/xt \in S^{-1}A$ ,

$$t/1 \cdot x/xt = 1.$$

### 66.8

For (iii)  $\Rightarrow$  (i). Since we've checked equivalence between (ii) and (iii), we will use (ii) as an assumption.

Firstly we'll try to prove (iii) implies injectivity of  $\phi$ . For  $a_1/s_1, a_2/s_2 \in S^{-1}A$  such that

$$a_1/s_1 = \phi(a_1/s_1) = \phi(a_2/s_2) = a_2/s_2 \Rightarrow \exists t_1 \in T, t_1(a_1s_2 - s_1a_2) = 0 \in A.$$

By (iii) we know there exists some  $x \in A$  such that  $xt_1 \in S$ , so  $xt_1(a_1s_2 - s_1a_2) = 0$  will imply

$$a_1/s_1 = a_2/s_2 \in S^{-1}A,$$

which proves injectivity.

Secondly we'll prove (ii) implies surjectivity of  $\phi$ . Indeed, by (ii) we know  $(t/1)^{-1} = a_0/s_0 \in S^{-1}A$  for some  $a_0 \in A$  and  $s_0 \in S$ . Let's start by picking an element  $a/t \in T^{-1}A$ . And we claim that

$$\frac{aa_0}{s_0} \xrightarrow{\phi} \frac{a}{t}.$$

Therefore we have to check  $\phi(aa_0/s_0) = aa_0/s_0 = a/t$ . Recall that by definition we know

$$a_0/s_0 \cdot t/1 = 1 \Rightarrow \exists s' \in S, s'(a_0t - s_0) = 0.$$

Then we have

$$s'(aa_0t - s_0a) = 0 \Rightarrow \frac{aa_0}{s_0} = \frac{a}{t}.$$

Hence we've proved that  $\phi$  is bijective.

## 66.9 Comments

In summary, we've checked (v) and (iv) are equivalent; (i), (ii), and (iii) are equivalent. With an extra implication Section 66.3.

I couldn't finish the direction of (v) implies (i).

But there's certainly a valid way to finish this. In Matsumura [6] or [1], there's another equivalent definition of *saturation*, which is explicit in terms of set-builders. In that way, we can see (iv) implies (ii) easily.

Now we only need to verify two definitions are equivalent. See [1] Exercise 3.25.

Is there a direct proof?

## 66.10

See a post [HERE](#) and [HERE](#).

In fact, according to Martin's hint, we should try to prove equivalence between (ii) and (v). In previous parts we've managed to verify (ii)  $\Rightarrow$  (v).

Now we try to verify (v)  $\Rightarrow$  (ii). Suppose on the contrary that there exists some  $t_0 \in T$  such that  $t_0/1$  isn't a unit in  $S^{-1}A$ . Then there exists a maximal ideal such that

$$\langle t_0/1 \rangle \subset \mathfrak{M} \triangleleft_{\max} S^{-1}A.$$

While we know every ideal in  $S^{-1}A$  is *extended* ideal, by characterisation in Chapter 1 Prop 1.17., we know

$$\mathfrak{M}^{ce} = \mathfrak{M}.$$

Denote  $\mathfrak{m} = \mathfrak{M}^c$ , and we have  $S^{-1}\mathfrak{m} = \mathfrak{M}$ . In particular, we have  $t/1 \in S^{-1}\mathfrak{m}$ , in which  $\mathfrak{m}$  is a prime ideal in  $A$ . Notice that we have

$$t = f^{-1}(t/1) = (t/1)^c \in \mathfrak{M}^c = \mathfrak{m}^{ec} = \mathfrak{m}$$

where  $f : A \rightarrow S^{-1}A$ . Then assumption (v) will force  $\mathfrak{m}$  meet  $s$ . Apply Prop 3.11 (ii) we know

$$\mathfrak{M} = \mathfrak{m}^e = (1),$$

which is a contradiction.

is the first equality right?

## 66.11

Some concerns are here. Mainly for the equality mentioned above. We know  $t/1 \in \mathfrak{M} = \mathfrak{m}^e$  is element generated by some finite linear (with coefficients in  $S^{-1}A$ ) combination of elements from  $\mathfrak{m}$ . One thing worth noting is that  $\mathfrak{m}^e = S^{-1}\mathfrak{m}$ , so  $t/1 \in m_0/s_0$  for some  $m_0 \in \mathfrak{m}$  and  $s_0 \in S$ . But this isn't enough to argue  $t \in \mathfrak{m}$ ???

But thanks to definition of  $(\cdot)^c$ , which is essentially inverse  $f$ ...

## 67 Ex 9.

*The set  $S_0$  of all non-zero-divisors in  $A$  is a saturated multiplicatively closed subset of  $A$ . Hence the set  $D$  of zero-divisors in  $A$  is a union of prime ideals (see Chapter 1, Exercise 14). Show that every minimal prime ideal of  $A$  is contained in  $D$ . [Use Exercise 6.]*

*The ring  $S_0^{-1}A$  is called the **total ring of fractions** of  $A$ . Prove that*

- *$S_0$  is the largest multiplicatively closed subset of  $A$  for which the homomorphism  $A \rightarrow S_0^{-1}A$  is injective.*
- *Every element in  $S_0^{-1}A$  is either a zero-divisor or a unit.*
- *Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions (i.e.,  $A \rightarrow S_0^{-1}A$  is bijective).*

### 67.1

Clearly we have  $S_0$  is saturated multiplicatively closed. Since we can easily prove

$$ab \notin S_0 \Leftrightarrow a \notin S_0 \vee b \notin S_0.$$

### 67.2

By Section 65, we know  $A - S_0 = D$  is a union of prime ideals. Compare this result with Chapter 1 Exercise 14.

Let  $\mathfrak{q}_1 \triangleleft A$  be a minimal prime ideal, then  $A - \mathfrak{q}_1$  is a multiplicatively closed subset such that  $A - \mathfrak{q}_1 \in \Sigma$  by 64. Here  $\Sigma$  denotes the same thing as in Exercise 6 64.

### 67.3

I couldn't proceed. See a post [HERE](#).

See [1], Corollary 11.4. Very elegant proof using localisation.

However, prime ideals might well contain non-zero-divisors. For example, in  $\mathbb{Z}$  we have  $(3), (5), \dots$  But 3 is clearly a non-zero-divisor.

I don't know if there's a proof that solely relies on result of Exercise 6 according to the hint.



### 67.4 (i)

First of all, we need to check that the homomorphism  $A \rightarrow S_0^{-1}A$  is indeed injective. Suppose we have two distinct element  $a, b \in A$  such that

$$a/1 = b/1 \in S_0^{-1}A \Leftrightarrow \exists s \in S_0, s(a \cdot 1 - b \cdot 1) = 0.$$

While  $a - b \neq 0$  would suggest that  $s$  is a zero-divisor, contradiction. Therefore we must have  $a/1 \neq b/1$  and the homomorphism is injective.

Suppose  $S_1 \supsetneq S_0$  is a multiplicatively closed subset such that  $A \rightarrow S_1^{-1}A$  is injective. Notice that by definition it must contain a zero-divisor  $q_1 \in S_1$ . This means there exists  $q_2 \neq 0$  such that  $q_1 q_2 = 0$ . But this implies

$$q_2/1 = 0/1 \in S_1^{-1}A.$$

Therefore  $A \rightarrow S_1^{-1}A$  isn't injective anymore.

### 67.5 (ii)

Take an element  $a_0/s_0 \in S_0^{-1}A$ . Suppose for the sake of contradiction that it's a non-zero-divisor and a non-unit.

Then we claim  $a_0 \notin S_0$ . Otherwise, we'll have  $s_0/a_0 \in S_0^{-1}A$  such that  $a_0/s_0 \cdot s_0/a_0 = 1 \in S_0^{-1}A$ .

Hence  $a_0$  must be a zero-divisor. Then there exists a non-zero element  $a_1 \in A$  such that  $a_0 a_1 = 0$ . Suppose  $a_1/1 = 0 \in S_0^{-1}A$ , then there exists  $s_2 \in S_0$  such that  $s_2 a_1 = 0$ . While  $a_1 \neq 0$ , it contradicts the fact that  $s_2$  is a non-zero-divisor. Hence we've found a nonzero element such that

$$\frac{a_0}{s_0} \cdot \frac{a_1}{1} = \frac{0}{s_0} = \frac{0}{1} = 0 \in S_0^{-1}A.$$

And this contradicts the assumption that  $a_0/s_0$  is a non-zero-divisor. Hence we can conclude that element in  $S_0^{-1}A$  is either a zero-divisors or a unit.

### 67.6 (iii)

Notice that when  $1 \neq 0$ , then every unit must be a non-zero-divisor. Therefore we have, in general,

$$\{\text{zero-divisors}\} \subset \{\text{non-units}\}.$$

See a post [HERE](#).

It suffices to check  $A \rightarrow S_0^{-1}A$  is surjective. Consider an element  $a_0/s_0 \in S_0^{-1}A$ . Notice that if  $s_0$  is a non-unit, then it's a zero-divisor which is a contradiction given  $s_0 \in S_0$ . Hence it must be a unit. Then we have  $a_0 s_0^{-1} \in A$  is a pre-image of  $a_0/s_0$ , which proves that the homomorphism  $A \rightarrow S_0^{-1}A$  is surjective.

See "A  
Term" Ex-  
ercise 11.26!

## 68 Ex 19.

### 68.1 (i)

Direct application of Proposition 3.8 on Page 40 of [2].

verified  
HERE

## 69 Ex 20.

i) Almost by definition.

ii) For every prime ideal in  $B$ , we can express it as  $\mathfrak{p}_1^e, \mathfrak{p}_2^2$  where  $\mathfrak{p}_1, \mathfrak{p}_2 \triangleleft A$  is a prime ideal. Suppose they have the same image, then

$$\mathfrak{p}_2 = \cdots = f^*(\mathfrak{p}_1^e) = \mathfrak{p}_1^{ec} = \mathfrak{p}_1$$

proves the injectivity.

Counterexample. Let  $A = \mathbb{Q}$  and  $B = \mathbb{C}$  with  $f$  being an inclusion. Then  $f^*$  is injective for  $B$  is a field. But  $\langle i \rangle$  is not an extended ideal for  $\langle i \rangle^{ce} = \langle 0 \rangle^e = 0$ .

?

## 70 Example 3) Page 51

My question was about that quotient ring...

See a post [HERE](#).

See a post discussing the quotient ring  $k[x, y, z]/\langle xy - z^2 \rangle$  [HERE](#).

## 71 Prop 4.2

For the original proof. Note that  $A/\mathfrak{a}$  is a local ring, then elements in  $A/\mathfrak{a}$  is either a unit or a nonunit. In case of a nonunit, it lies in the nilradical (intersection of all prime ideals).

Another proof is [HERE](#).

Proof is based on a result from Exercise 10 Chapter 1 of [2].

## 72 Lemma 4.3.

Change "for some  $i$ " to "for any  $i$ "

?

Notice that radical commute with finite intersection of ideals, see [HERE](#).  
How about infinite intersection of ideals?

Verified.

## 73 Lemma 4.4.

(i) Notice

$$1 \in (\mathfrak{q} : x) = \{y \in A \mid y(x) \subset \mathfrak{q}\}.$$

(ii) Clearly we have  $\mathfrak{q} \subset (\mathfrak{q} : x)$ . Conversely, for  $y \in (\mathfrak{q} : x)$ , then  $yx \in \mathfrak{p}$ . While  $x \notin \mathfrak{p}$ , it follows that  $y \in \mathfrak{q}$ .

## 74 Proposition 5.1

For iii)  $\Rightarrow$  iv), notice that  $C$  is a subring and  $1 \in C$  in particular, which helped us to prove the faithful property.

For iv)  $\Rightarrow$  i), see Gong Ting's notes [HERE](#) on Page 30, in which the last part used "determinant trick".

## 75 Corollary 5.4.

This is in fact iff.

If we know ring extensions  $A \subset B \subset C$ . Assume  $A \subset C$  is integral then  $A \subset B$  is integral for  $b \in B \subset C$ ; and  $B \subset C$  is integral for  $c \in C$  we could use exactly the same monic polynomial.

The only nontrivial direction is to prove  $A \subset B$ ,  $B \subset C$  are integral, then the tower  $A \subset C$  is integral.

For a more complete description, see Gathmann's Notes Lemma 9.6. on Page 81. For the converse direction of the statement, namely if we know  $A \subset C$  then ... It's very important  $A, B$  are both rings. See Stacks Project: finite ring map!

Checked!

## 76 Proposition 5.6.

### 76.1 (i)

Here I present a re-write of the proof.

$$A \xhookrightarrow{\iota} B \xrightarrow{\pi_2} B/\mathfrak{b}$$

Composition of the map has kernel  $\text{Ker}(\pi_2 \circ \iota) = \mathfrak{b}^c$ , which induces an injective (well-defined) map  $f : A/\mathfrak{a} \rightarrow B/\mathfrak{b}$  so that the following diagram commute

$$\begin{array}{ccc} A & \xhookrightarrow{\iota} & B \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ A/\mathfrak{a} & \xrightarrow{f} & B/\mathfrak{b} \end{array}$$

Suppose we have a monic polynomial such that for  $x \in B$ ,

$$x^n + \iota(a_1)x^{n-1} + \cdots + \iota(a_n) = 0 \in B$$

for  $a_i \in A$  where integer  $1 \leq i \leq n$ . By the commutativity of the diagram we have

$$\begin{aligned} \pi_2(0_A) &= 0 = \pi_2(x)^n + \pi_2 \circ \iota(a_1)\pi_2(x)^{n-1} + \cdots + \pi_2 \circ \iota(a_n) \\ &= \pi_2(x)^n + f \circ \pi_1(a_1)\pi_2(x)^{n-1} + \cdots + f \circ \pi_1(a_n) \end{aligned}$$

with all coefficients in  $f(A/\mathfrak{a}) \subset B$ . Notice that  $\pi_2$  is surjective, this means for any element of  $B/\mathfrak{b}$  we can find a monic polynomial in  $f(A/\mathfrak{a})[X]$  admits it as a solution. Therefore we know  $B/\mathfrak{b}$  is integral over  $A/\mathfrak{a}$ .

Also see Gathmann's Notes Lemma 9.7. for extra information regarding inheritance of integral.

## 76.2 Pathological

Question: If we know integral extension of quotient rings  $A/\mathfrak{a} \subset B/\mathfrak{b}$ , do we know  $A \subset B$  is integral? No. Consider a non-integral extension  $R \subset R[x]$ . Ideal  $\langle x \rangle$  contract as  $\langle x \rangle^c = 0$ , so

$$R = R/(I \cap R) \subset R[x]/\langle x \rangle = R$$

gives us an integral extension.

See my remarks on Gathmann's Notes Lemma 9.7.

## 76.3 (ii)

Here it's important that  $S$  is un-changed! See an example where the implication will fail if localising with respect to different multiplicatively closed subset: Gathmann Commutative Algebra Exercise 9.8. (b) on Page 82.

Question: does there exist  $S^{-1}B$  integral over  $S^{-1}A$ , but  $B$  not integral over  $A$ ?

## 77 Proposition 6.2.

According to Errata, last line of the proof should be  $M_n = N$ .

## 78 Corollary 6.4

One might wonder if  $\bigoplus_{i \in I} M_i$  is also Noetherian given the index set  $I$  is not necessarily finite with each  $M_i$  being Noetherian. Consider this example 3) on Page 76

$$k[x_1, x_2, \dots] = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \dots$$

Not sure if the above example works well as a *counterexample*???

See a post [HERE](#). Once we allow infinite index set, at least countably infinite, we'll get  $M_1 \subsetneq M_1 \oplus M_2 \subsetneq \dots$

## 79 Proposition 6.7

We cannot interpret this inclusion set-theoretically? Any element  $n_0 + N_i$  must also be a.... I cannot resolve the coset  $+N_i$  problem...

The inclusion  $N_{i-1}/N_i \subset M_{i-1}/M_i$  is given by noticing  $N_i$  is *exactly* kernel of the map (by definition the kernel is exactly  $M_i \cap N = N_i$ )

$$N_{i-1} \hookrightarrow M_{i-1} \xrightarrow{\pi} M_{i-1}/M_i$$

And this induces an injective from  $N_{i-1}/N_i \rightarrow M_{i-1}/M_i$ , for which we can interpret it as set inclusion.

## 80 Ex 1.

### 80.1 (i)

Since  $M$  is Noetherian, then it's finitely generated in particular. Then we can apply one form of Nakayama's Lemma. One approach is to use induction on number of generators of  $M$  ???

I didn't get the approach suggested by the hint HERE. One crucial observation is the ascending chain of modules

$$\text{Ker}(u) \subset \text{Ker}(u^2) \subset \text{Ker}(u^3) \subset \dots$$

### 80.2 (ii)

Couldn't understand hint from both the textbook and the post HERE.

A post HERE.

According to Artinian assumption there exists  $n \in \mathbb{Z}$  such that

$$\text{Im}(u) \supset \text{Im}(u^2) \supset \dots \supset \text{Im}(u^n) = \text{Im}(u^{n+1}) = \dots$$

And we only need to prove the claim  $M = \text{Im}(u^n) + \text{Ker}(u^n)$ ...

Another cute argument is to consider any  $m \in M$ . There exists  $m' \in M$  such that

$$u^n(m) = u^{n+1}(m').$$

By injectivity of  $u$  we must have  $u^{n-1}(m) = u^n(m')$ . Inductively, we know  $m = u(m')$ . Hence  $\text{Im}(u) = M$ .

### 80.3 Remarks

For (ii), assumption of Artinian is necessary. Since for non-Artinian  $\mathbb{Z}$ -module  $\mathbb{Z}$ , we have  $\times 2 : \mathbb{Z} \rightarrow \mathbb{Z}$  injective but not surjective.

## 81 Ex 2.

Apply argument of Prop 6.2 in Atiyah's [2]. Verified at [1] Exercise 16.32.



## 82 TOPOLOGIES AND COMPLETIONS

### 82.1 Linear Topology

See a note about  $\mathfrak{I}$ -adic topology.

### 82.2 Remarks on Page 101

The reason why  $G$  is Hausdorff. One equivalent condition for  $G$  being Hausdorff is the diagonal element in  $G \times G$  is closed. Because for point  $(x, y)$  that's off-diagonal, i.e.  $x \neq y$ , we can always find  $U \times V$  (by applying Hausdorff in  $G$ ) that's open in  $G \times G$  with trivial intersection to the diagonal. See a post [HERE](#).

## 83 Lemma 10.1

### 83.1 Details of (i)

could be found [HERE](#) and [HERE](#).

### 83.2 Details of (ii)

Details could be found at the same link [HERE](#), the first answer.

### 83.3 Details for (iii)

For each element  $g + H \in G/H$ , it's the preimage of the continuous map

$$g + H \xrightarrow{-g} H$$

While  $H$  is closed, hence  $g + H$  is closed as expected. In fact, this translation map is a homeomorphism, see Atiyah's comment before Lemma 10.1.

### 83.4 Details for (iii), without incurring (ii)

???potentially *incorrect*!

Conversely, assume  $H = 0$ . Pick distinct points  $x_1, x_2 \in G$ , then we have  $x_1 - x_2 \neq 0$  in  $G$ . While  $H = \cap_{i \in I} U_i$  where each  $U_i$  are all 0-neighborhood. Hence we can find a 0-neighborhood  $U_1$  such that

$$x_1 - x_2 \notin U \Rightarrow x_1 \notin x_2 + U_1.$$

Similarly, we can find another  $U_2$  such that

$$x_2 - x_1 \notin U \Rightarrow x_2 \notin x_1 + U_2.$$

Therefore  $x_1 + U_2 \setminus U_1$  and  $x_2 + U_1 \setminus U_2$  are two disjoint neighborhoods that contains  $x_1, x_2$  respectively, which proves  $G$  is Hausdorff.

### 83.5 Details for (iv)

$\Rightarrow$ : Assume  $G$  is Hausdorff but  $H \neq 0$ ... Pick an element  $y \in H$ ... See a post [HERE](#).

Part II

# A Course in Commutative Algebra Kemper

## 84 Exericse 8.3

See a post [HERE](#) and [HERE](#).



## Part III

# A Term of Commutative Algebra

### 85 Definition 3.11

Let  $R$  be a ring. Then the group of units  $R^\times$  is a saturated multiplicative sets. This is easy to see: if we have  $ab$  is a unit, then let  $u = (ab)^{-1}$ . So we have  $a(bu) = 1$ , which means  $bu$  is inverse of  $a$  and  $a$  is a unit. Similarly we know  $b$  is a unit. For the other direction is easy.

For the set of non-zero-divisors  $S_0$ .

$$a \in S_0 \wedge b \in S_0 \Leftrightarrow ab \in S_0.$$

Assume  $ab \in S_0$  and  $a \notin S_0$  or  $b \notin S_0$ . Without loss of generality, assume  $a \notin S_0$ . This implies there exists a non-zero element  $v \in R$  such that  $va = 0$ . However, associativity gives us a contradiction that  $ab$  is a zero-divisor for

$$v(ab) = (va)b = 0.$$

Conversely, use contrapositive. Argument follows similarly.

### 86 Definition 3.17

We can easily generalise the argument in 9 to conclude the existence of minimal prime over a given ideal.

### 87 Ex 3.16

Notice that in general intersection of two prime ideals are not necessarily prime. For example, take intersection of  $\langle 2 \rangle, \langle 3 \rangle \subset \mathbb{Z}$  will give us  $\langle 6 \rangle$ , which isn't a prime ideal. However, taking intersection in a *chain* of prime ideals will give us, again, a prime ideal.

### 88 7.1 Filtered Categories

See a great note discussing differences colimit and filtered colimit.

### 89 Exercise 11.5

$$\begin{array}{ccc} R' \times R'' & \longrightarrow & R' \times R'' / R'' \\ & \searrow \psi & \downarrow \\ & & B \end{array}$$

Notice that we have  $R'' \subset \text{Ker } \psi$ , then  $\psi$  descends to a unique map on the quotient from  $R' \times R''/R'' \rightarrow B$ . And this map serves to prove the UMP of  $R' \times R''/R'' = R'$  as  $S^{-1}R$ . Hence we have  $R' \sim S^{-1}R$ . But is this the equality set-theoretic?

## 90 Corollay (11.4).

Notice that isomorphism will preserve units.

$$\begin{array}{ccc} R & \xrightarrow{\varphi_S} & S^{-1}R \\ & \searrow \text{id}_R & \downarrow \rho \\ & & R \end{array}$$

UMP tells us there exists a unique map  $\rho : S^{-1}R \rightarrow R$  such that  $\text{id}_R = \rho\varphi_S$ . Notice that the inverse of  $\varphi_S$  will also satisfy this requirement, by the uniqueness we know it must be  $\rho$ .

## 91 11.25 Exercise

*Proof.* Here I interpret  $R_\lambda$  as for some index  $\lambda$  in certain index set, not as a localisation.

Homomorphic image of multiplicatively closed subset is again multiplicatively closed. And union of multiplicatively closed subset of a ring is again multiplicatively closed. Therefore  $S$  defined in prompt is multiplicatively closed.

Now we fix a morphism from  $R \rightarrow A$  where  $A$  is arbitrary ring. This data is equivalent to a co-cone over  $A$ . And we can build...

□

?

See solution, it's much cleaner

### 91.1 Another Remark

See nLab [HERE](#). And a post, and post. It could be easier to think about it as following. But there's a slight problem I couldn't fix: when commuting  $\lim$  and  $\otimes$  I couldn't figure out which is the correct multiplicative closed to localise.

*Proof.* We could interpret localisation as a tensor product, which is a left adjoint. Hence it must commute with taking colimit. □

## 92 Exercise 12.6

*Proof.* Firstly we need to check  $S$  is a filtered category. We check the following conditions:

- Since  $1 \in S$  then it's non-empty. Objects are elements of  $S$ , Hom set is specified as the prompt.

- For any  $s_1, s_2 \in S$ , have  $s_1 s_2 \in S$  with morphisms

$$s_1 : s_2 \rightarrow s_1 s_2, \quad s_2 : s_1 \rightarrow s_1 s_2.$$

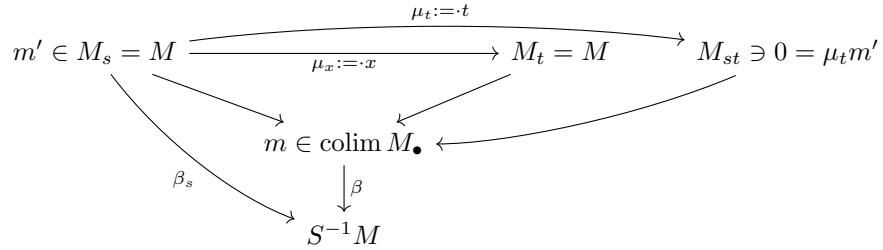
- For two parallel morphisms  $\alpha, \beta : s_1 \rightarrow s_2$  where  $\alpha, \beta \in R$  and  $s_1, s_2 \in S$ . We have a morphism  $s_1 : s_2 \rightarrow s_1 s_2$  such that the composition (given by multiplication)

$$s_1 \alpha : s_1 \rightarrow s_1 s_2 = s_1 \beta : s_1 \rightarrow s_1 s_2$$

matches by definition.

Therefore  $S$  with objects being elements of  $S$  with Hom set defined as above is a filtered category. The morphism compose by multiplication.

The following is based on solution of diagram chasing:



Correct, see solution, but I stopped for I was confused by  $M_s := M$

This proves injectivity. Surjectivity follows, see solution on Page 293.  $\square$

## 92.1 Remarks

This is closely related to Rising Sea Exercise 4.1.E., Definition 4.3.7., and Stacks Project affine scheme.

What we've done is to interpret a usual localisation as a *filtered* colimit.

?

## 92.2 References

See [HERE](#) and [HERE](#).

There is a subtle distinction prevents us to apply it directly as a proof:  $M_\bullet$  is considered as  $M$  in the exercise but as a localisation in statement of Tag 00CR of [?].

*Proof.* The pre-order specified on  $S$  makes it into a filtered category exactly by the argument above.

One thing we need to modify is to re-define  $\beta_\bullet$  as inclusion maps for any  $f \in S$

$$\beta_f : M_f \rightarrow S^{-1}M \text{ by } m/f^e \mapsto m/f^e$$

for some  $m \in M$  and integer  $e$ . Then universal property of colimit will induce a map

$$\beta : \operatorname{colim}_{f \in S} M_f \rightarrow S^{-1}M.$$

We can check  $\beta$  is surjective and injective by similar argument as in the solution of previous exercise. One important reason we can do this by diagram chasing is because we're talking about *filtered* colimit.

- Surjective. Pick an element of  $S^{-1}M$ , it's form  $m/f_0$  implies it lives in some  $M_{f_0}$ . Since the diagram is commute, we know it's projection into the colimit would be the preimage we're looking for.
- Injective. Pick  $m \in \text{colim}_{f \in S} M_f$  such that  $\beta m = 0 \in S^{-1}M$ . Firstly, by Corollary 7.5, we know there's some  $m' \in M_f$  for some  $f$ .

$$\begin{array}{ccc} M_f \ni m' & \xrightarrow{\quad} & m \in \text{colim}_{\bullet \in S} M_{\bullet} \\ & \searrow \beta_f & \downarrow \beta \\ & & \beta m = 0 \in S^{-1}M \end{array}$$

By commutativity of the above diagram we know  $m' = \beta_f(m') = \beta m = 0$  for  $\beta_f$  is just inclusion. By definition of localisation, we know there exists some  $g \in S$  such that  $gm' = 0 \in M$ . Since  $m' \in M_f$ , we consider

$$\begin{array}{ccc} m' \in M_f & \xrightarrow{\mu_g := g} & 0 = gm' \in M_{fg} \\ & \searrow & \swarrow \\ & m \in \text{colim}_{\bullet \in S} M_{\bullet} & \end{array}$$

While  $\mu_g(m') = gm' = 0 \in M_{fg}$  and the diagram is commute, we have  $m$  is a homomorphic image of 0 under a module homomorphism. Hence  $m = 0$  therefore  $\beta$  is injective.

□

## 93 12.17

(1) Notice that  $S^{-1} \text{Ann}(M) \subset \text{Ann}(S^{-1}M)$  is by definition.

Now we assume  $M$  is f.g. and try to prove the equality.

We can prove that  $r_1/s_1 \in \text{Ann}(M) \triangleleft R$  is

## 94 Ex 13.41

Let  $R$  be a ring,  $X := \text{Spec}(R)$ , and  $U$  an open subset. Show  $U$  is quasi-compact if and only if  $X - U = V(\mathfrak{a})$  where  $\mathfrak{a}$  is finitely generated.

One equivalent statement for the latter part is " $U$  is a finite union of basic open sets". See Atiyah's Exercise 17,18, of Chapter 1.



*Proof.* Assume  $U$  is quasi-compact. Let  $\{D(f_i)\}_{i \in I}$  be an open cover for  $U$  where  $I$  is an index set. Given  $U$  is quasi-compact, there exists a *finite* index

set  $J \subset I$  such that  $\{D(f_i)\}_{i \in J}$  is a subcover. Since  $U$  is open, then there exists a closed subset

$$V(f) = X - U$$

for some  $f \in R$  and we have

$$U = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in J} D(f_i) \Rightarrow V(f) = \bigcap_{i \in J} V(f_i) = V\left(\bigcup_{i \in J} \langle f_i \rangle\right).$$

Now we define  $\mathfrak{a} = \bigcup_{i \in J} \langle f_i \rangle$ , which is finitely-generated as desired.

Conversely, assume we have  $U = X - \mathbf{V}(\mathfrak{a})$  for a finitely-generated ideal  $\mathfrak{a} = \langle f_1, \dots, f_m \rangle \subset R$  where  $m \in \mathbb{Z}_+$ . And this implies

$$U = X - \mathbf{V}(\mathfrak{a}) = \bigcup_{1 \leq i \leq m} D(f_i).$$

While each  $D(f_i)$  is quasi-compact by (vi) of 18, we know  $U$  is quasi-compact.  $\square$

## 95 15.13. Definition: Catenary

See a post on local ring of infinite dimension [HERE](#).

## 96 Lemma 16.12

Since we have  $M_1 \subset M_2$ , the equality  $(M_1 + N)/N = (M_2 + N)/N$  implies there are some  $m_1, m_2 \in M$  and  $n_1, n_2 \in N$  such that

$$(m_2 + n_2) + N = (m_1 + n_1) + N \Rightarrow m_2 - m_1 \in N.$$

## 97 Exercise 17.4

Kernel of the quotient map  $R \rightarrow R/\mathfrak{a}$  is  $\mathfrak{a}$ , and it lives inside  $\text{Ann}(M)$  just to ensure  $M$  accepts an  $R/\mathfrak{a}$ -module structure that's defined by

$$(r + \mathfrak{a})m = rm.$$

And it's well-defined.

$$(R/\mathfrak{a})/(\mathfrak{p}/\mathfrak{a}) \simeq R/\mathfrak{p} \hookrightarrow M$$

Notice the isomorphism enables us to use the characterisation that  $M$  has a submodule isomorphic to  $R/\mathfrak{p}$  to deduce that under the quotient map

$$\kappa(\text{Ass}_R(M)) \subset \text{Ass}_{R'}(M).$$

Argument for the converse inclusion is similar.

?

## 98 Proposition 17.7

Here I present my writing. My initial guess was to define  $N$  explicitly as

$$N = \bigoplus_{\mathfrak{p} \in \Psi} R/\mathfrak{p}.$$

But does this work?

?

*Proof.* We try to eliminate some trivial cases. If  $M = 0$ , here the commutative ring  $R$  is nontrivial, so there cannot exist an injective morphism

$$R/\mathfrak{p} \rightarrow M$$

for any prime ideal  $\mathfrak{p}$ . It follows that  $\text{Ass}(M) = \emptyset$ . In this case the statement is vacuously true.

So we may assume  $M \neq 0$  for the rest of the proof.

- For  $\Psi = \emptyset$ , we can take  $N = M$ .
- For  $\Psi = \text{Ass}(M)$ , we can take  $N = 0$ .

So in the following proof we'll assume  $\Psi$  is a non-empty proper subset of  $\text{Ass}(M)$ .

We wish to define  $N$  by using Zorn's Lemma. Consider a set composed of submodules  $N_\lambda$  of  $M$  such that

$$\mathcal{S} = \{N_\lambda \subset M \mid \text{Ass}(N_\lambda) \subset \text{Ass}(M) - \Psi\}.$$

Although some formulations of Zorn's Lemma doesn't require non-empty condition, we can prove it here. We claim that  $\mathcal{S}$  is non-empty. According to the assumptions we've made, the set  $\text{Ass}(M) - \Psi$  is non-empty. Pick a prime ideal  $\mathfrak{p} \in \text{Ass}(M) - \Psi$ . Then we can define  $N_0 = \text{Im}(R/\mathfrak{p} \rightarrow M)$ , then  $N_0 \in \mathcal{S}$  because by Lemma 17.5 we know  $\text{Ass}(N_0) = \{\mathfrak{p}\}$ . Hence  $\mathcal{S}$  is non-empty.

Now we consider a chain  $\mathcal{S}_0 \subset \mathcal{S}$ . We claim that

$$N'' = \bigcup_{N_\lambda \in \mathcal{S}_0} N_\lambda \in \mathcal{S}.$$

It's clearly a submodule of  $M$ . And we know (by the first paragraph of the original proof of Prop 17.7 on Page 131 of [1])

$$\text{Ass}(N'') = \bigcup_{N_\lambda \in \mathcal{S}_0} \text{Ass}(N_\lambda) \subset \text{Ass}(M) - \Psi.$$

Hence we know every chain has an upper-bound in  $\mathcal{S}$ , and there exists a maximal element for which we denote as  $N \subset M$  such that  $\text{Ass}(N) \subset \text{Ass}(M) - \Psi$ .

According to Prop 17.6, we know

$$\Psi \subset \text{Ass}(M) \subset \text{Ass}(N) \cup \text{Ass}(M/N).$$

While we have  $\text{Ass}(N) \subset \text{Ass}(M) - \Psi$ , it follows that  $\Psi \subset \text{Ass}(M/N)$ . Moreover, union  $\text{Ass}(M/N)$  to the above inclusions we'll have

$$\Psi \subset \text{Ass}(M) \subset \text{Ass}(N) \cup \text{Ass}(M/N) \subset (\text{Ass}(M) - \Psi) \cup \text{Ass}(M/N) = \text{Ass}(M).$$

The last equality comes from the fact that  $\Psi \subset \text{Ass}(M/N)$ . Hence we know

$$\text{Ass}(M) = \text{Ass}(N) \cup \text{Ass}(M/N).$$

So it suffices to check  $\Psi \supset \text{Ass}(M/N)$ . Because in that way we'll get

$$\text{Ass}(M) = \text{Ass}(N) \cup \Psi = \text{Ass}(N) \sqcup \Psi \Rightarrow \text{Ass}(N) = \text{Ass}(M) - \Psi.$$

Then we can follow the original proof. Follow the notation starting from paragraph 3. While  $N'/N \simeq R/\mathfrak{p}$ , so we must have  $N' \supsetneq N$ . In particular, notice that by Prop 17.6 we have

$$\text{Ass}(N') \subset \text{Ass}(N) \cup \text{Ass}(N'/N) = \text{Ass}(N) \cup \{\mathfrak{p}\}.$$

Now we consider where does  $\mathfrak{p}$  belongs to. The maximality of  $N$  forces

$$\text{Ass}(N') \not\subset \text{Ass}(M) - \Psi \Rightarrow \text{Ass}(N') \cap \Psi \neq \emptyset \Rightarrow \exists \mathfrak{q} \in \text{Ass}(N') \cap \Psi.$$

for  $\text{Ass}(N')$ ,  $\Psi \subset \text{Ass}(M)$ . Recall that  $\text{Ass}(N) \subset \text{Ass}(M) - \Psi$ , then there exists a prime ideal

$$\mathfrak{q} \in \text{Ass}(N') \cap \Psi \subset (\text{Ass}(N) \cup \{\mathfrak{p}\}) \cap \Psi \subset \{\mathfrak{p}\} \cap \Psi \Rightarrow \mathfrak{q} = \mathfrak{p} \in \Psi.$$

And this proves the desired inclusion  $\text{Ass}(M/N) \subset \Psi$ .

□

## 99 Proposition 17.8

One remark on  $\mathfrak{p} \cap S = \emptyset$ . Assume on the contrary that  $S$  meet  $\mathfrak{p}$ , it follows that  $S^{-1}\mathfrak{p}$  contains 1 therefore cannot be a prime ideal.

For the quotient  $(\mathfrak{a} + \mathfrak{p})/\mathfrak{a}$ . Ideal  $\mathfrak{p}$  is assumed to be f.g. module  $\mathfrak{a} := \text{Ann } M$ . Hence  $(\mathfrak{a} + \mathfrak{p})/\mathfrak{a}$  is f.g., because for any element  $(a + p) + \mathfrak{a} = p + \mathfrak{a}$  we can take the same set of generators.

For  $x = a + \sum a_i x_i$ . Here we wish to prove  $\mathfrak{p} \subset \mathfrak{b}$ . Pick an element  $x \in \mathfrak{p}$ , note  $\mathfrak{p} \subset \mathfrak{a} + \mathfrak{p}$  where  $\mathfrak{a} := \text{Ann } M \subset \text{Ann}(sm) =: \mathfrak{b}$ . Hence we can write  $x$  in two parts as

$$x = a + \sum a_i x_i$$

where  $a \in \mathfrak{a}$  and  $x_i \in \mathfrak{b}$  by the previous construction of  $\mathfrak{b}$ . Then we know element  $x \in \mathfrak{b}$  actually.

For the third paragraph, in which we wish to prove  $\mathfrak{b} \subset \mathfrak{p}$ . Denote  $\varphi_S : R \rightarrow R_{\mathfrak{p}}$ . Notice that

$$b = \varphi_S^{-1}(b/1) \in \mathfrak{p}^S.$$

By definition of saturation there exists some  $s_0 \in S = R - \mathfrak{p}$  such that  $s_0 b \in \mathfrak{p}$ , hence  $b \in \mathfrak{p}$  for it's a prime ideal (by the virtual of being pre-image of a prime ideal).

my question was about the quotient module?

## 100 Definition (18.1).

Notice that the definition is for a module  $Q$  to be  $\mathfrak{p}$ -primary in module  $M$ .

This is very general. When we set  $Q \triangleleft M = R$  as an ideal, and if we further assume ring  $R$  is Noetherian we'll have

$$Q \text{ is } \mathfrak{p}\text{-primary} \Leftrightarrow \text{Ass}(M/Q) = \{\mathfrak{p}\}.$$

This is precisely the statement of Theorem 7.8. on Page 105 of [?].

Therefore it's indeed a generalisation of the definition of primary of modules based on the rings.

## 101 22.1 Definition

I'm uncertain about the definition of the topology. See Wiki's entry  $\mathfrak{I}$ -adic topology, and a detailed note [HERE](#).

See Exercise 7. Section 6.5. of [?] for adic topology.



## Part IV

# Undergraduate Commutative Algebra

## 102 5.1 Weak Nullstellensatz

See [2] Corollary 5.24 on Chapter 5.

## 103 7.1 Proposition

For (v), see Atiyah [2] Exercise 3 of Chapter 3 on taking localisation twice; see "A Term" [1] Proposition 11.16.

We get a ring map  $(\cdot)_P : M_Q \rightarrow (M_Q)_P = M_P$ , and if  $M_Q = 0$  then  $M_P = 0$ . Taking contrapositive is what we're looking for.

## 104 Definition 7.3

In (ii), we have an equivalent definition. Assume  $M$  contains a submodule isomorphic to  $A/\mathfrak{P}$ , i.e.

$$A/\mathfrak{P} \xrightarrow{f \simeq} N \hookrightarrow M$$

Now we take an element  $x := f(1 + \mathfrak{P}) \in N \subset M$ . Ring  $R$  acts on  $N$  as

$$r \cdot m := r f^{-1}(n) \in A/\mathfrak{P}.$$

Therefore we have

$$\forall y \in \mathfrak{P} \Leftrightarrow yx = 0 \in A/\mathfrak{P},$$

which implies  $\text{Ann}(x) = \mathfrak{P}$ .

Conversely, we use first isomorphism theorem of module as

$$\begin{array}{ccc} A & \xrightarrow{\cdot x} & \langle x \rangle \\ \downarrow & \nearrow \simeq & \\ A/\mathfrak{P} & & \end{array}$$

### 104.1 Remarks

I'm not sure for the first part...

See "A Term" [1] Proposition 17.3

verified.

## 105 Proposition 7.4

### 105.1 (a)

Element  $y \in Ax$ , which is a cyclic module, i.e.  $\langle x \rangle \subset M$ . Clearly we have  $\mathfrak{P} \subset \text{Ann}(y)$  for any thing annihilates  $x$  will also annihilate  $y$ . Conversely, we wish to show  $\text{Ann}(y) \subset \mathfrak{P}$ . For any element  $\alpha \in \text{Ann}(y) \subset A$ , we have  $\alpha y = 0$  for any nonzero  $y \in Ax$ . In proof of (a) it claim the isomorphism of rings between  $\phi : Ax \rightarrow A/\mathfrak{P}$ , which is an integral domain. Therefore we must have  $\phi(\alpha) = 0$ , which gives us  $\text{Ann}_A(\phi(y)) \subset \mathfrak{P}$ . One identity to note is

$$\text{Ann}_A(y) = \text{Ann}_A(\phi(y)) \Rightarrow \text{Ann}_A(y) = \mathfrak{P}.$$

??? Original proof is sketchy

### 105.2 (b)

Uncanny tendency of "maximal" element to be prime ideal.

### 105.3 (c)

Direct usage of (b), and apply maximal element characterisation of Noetherian ring.

Is there any counterexample when  $A$  isn't necessarily Noetherian?

### 105.4 (d)

This is where we could use definition that there exists an injective map  $A/\mathfrak{p} \rightarrow M$ .

## 106 Proposition 7.8

Here's another presentation of the original proof, with a different order.

*Proof.*  $\Rightarrow$ : Note that since  $Q$  is  $P$ -primary, ideal  $P$  is prime and therefore  $Q \subset P \subsetneq A$ . Then  $A$ -module  $A/Q \neq 0$ . While  $A$  is Noetherian, apply Proposition 7.4. (c) gives us

$$\text{Ass}(A/Q) \neq \emptyset.$$

By definition this implies there exists a non-zero element  $x \in A/Q$  such that

$$\text{Ann } x = \mathfrak{q} \in \text{Spec } A.$$

Now we can apply the original proof, which yield that  $\text{Ann } x = P$  and it's a prime.

$\Leftarrow$ : The set of minimal primes  $P' \supset \text{Ann } x$  is the same as set of minimal elements in  $\text{Supp } M$ . This is because  $A/Q$  as an  $A$ -module is f.g., so is the subring  $M \subset A/Q$ . So we can apply Proposition 7.1 (iv) on Page 96 of [?].

?

□

# Part V

## Commutative Ring Theory

### Matsumura

#### 107 Endomorphism

On Page 7, we defined product of  $\lambda, \mu$  as  $\lambda(\mu(x))$  instead of  $\lambda(x) \cdot \mu(x)$ . This is because we cannot define  $\lambda(x) \cdot \mu(x)$ , both are elements of module.

#### 108 Support of Module on Page 25

$\exists i$  such that  $\omega_i \neq 0$  in  $M_{\mathfrak{p}} \Leftrightarrow \exists i$  such that  $\text{Ann}(\omega_i) \subset \mathfrak{p}$

Notice that we have

$$\begin{aligned} \omega_i/1 &= 0/1 \in M_{\mathfrak{p}} \\ \Leftrightarrow \exists q \in A - \mathfrak{p}, q(\omega_i \cdot 1 - 1 \cdot 0) &= 0 \in M \\ \Leftrightarrow \text{Ann}(\omega_i) &\not\subset \mathfrak{p}. \end{aligned}$$

Negate both statements we'll get the equivalence as expected.

#### 109 Example 1.

See [4] Proposition 8.8 on Page 96.

## Part VI

# Online Courses

This is for the course Math 614 [HERE](#).

### 110 Spec Functor

See (7) of worksheet 2.

(b) is correct. But (c) is incorrect. For example, take two non-isomorphic fields. See a post [HERE](#). This is for Gathmann's notes [HERE](#).

### 111 Exercise 8.8.

#### 111.1 (i)

For  $\mathbb{R}[x, y]/(y - x^2)$ , we know it's  $R[x]$  hence UFD.

#### 111.2 (ii)

For  $\mathbb{R}[x, y]/(xy - 1)$ , we know

$$\mathbb{R}[x, y]/(xy - 1) \sim k[y]_y = k[y, y^{-1}].$$

*Localisation of UFD is UFD* See a post [HERE](#): Localisation preserves UFD, using kaplansky's Criterion. See Ben's notes on UFD and kaplansky's Criterion [HERE](#); Pete Clark's notes on Page 237 [HERE](#); and a post [HERE](#).

One fact to note is that being an integral domain isn't a local property.  $R$  might not be an integral domain although the localisations  $R_{\mathfrak{p}}$  are integral domains for all maximal ideals  $\mathfrak{p} \subseteq R$ .

#### 111.3 (iii)

For  $\mathbb{C}[x, y]/(x^2 + y^2 - 1)$ , we have the isomorphism according a post [HERE](#) that it's in fact a Laurent

$$\mathbb{C}[x, y]/(x^2 + y^2 - 1) \sim \mathbb{C}[e^{it}, e^{-it}]$$

See a post [HERE](#).

One thing to notice is that  $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$  is not UFD but  $\mathbb{C}[x, y]/(x^2 + y^2 - 1)$  is PID.

## 112 Lemma 9.6.

Both statement (i) and (ii) are in fact iff. Assume  $R \subset R''$  are finite, i.e.

$$R'' = R\langle s \in S \rangle$$

for a finite subset  $S \subset R''$  of generators. Every element  $x \in R' \subset R''$  could be express as a  $R$ -linear (coefficients in  $R$ ) combination with generators in  $S$ . Therefore  $R \subset R'$  is finite. For  $R' \subset R''$ , pick  $x \in R''$  with expression and consider the coefficient of  $R$  in  $R'$ . Generating set  $S$  won't change, hence  $R''$  could be regarded as a f.g.  $R'$ -module.

Argument for (ii) is similar, see Atiyah [2] Corollary 5.4.

## 113 Lemma 9.7.

### 113.1 Counterexamples: Two Finite Towers

Question on a partial "converse" of (a): If we know  $R/(I \cap R) \subset R'$  is integral, do we know  $R \subset R'$  is integral? No. See remarks on Proposition 5.6. of Atiyah's [2].

## 114 Exercise 9.8.

### 114.1 (a)

No.

Let  $\beta = 1/2\sqrt[3]{3}$ , I claim it's not integral over  $\mathbb{Z}$  for any polynomial  $f \in \mathbb{Z}[x]$  admits  $\beta$  as a root, look at the denominator will lead a contradiction.

And then apply Prop 9.5., which is "iff"!

?

#### 114.1.1 Hint

See a post [HERE](#), [HERE](#).

## 115 References

- [1] Allen Altman and Steven Kleiman. *A Term of Commutative Algebra*. 04 2021.
- [2] M.F. Atiyah and I.G. MacDonald. *Introduction To Commutative Algebra*. Addison-Wesley series in mathematics. Avalon Publishing, 1994.
- [3] David Steven Dummit and Richard M Foote. *Abstract algebra*, volume 3. Wiley Hoboken, 2004.
- [4] Gregor Kemper. A course in commutative algebra, 2009.
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- [6] Hideyuki Matsumura. *Commutative ring theory*. Number 8. Cambridge university press, 1989.