Contents

1 Notes

1.1 Resources

HERE is a post on A Term of Commutative Algebra by Altman.
HERE is a LaTex version of Hideyuki Matsumura's Commutative Algebra
Digital Version of Atiyah's Book HERE

Part I

Introduction to Commutative Algebra Atiyah

2 Chapter 1

2.1 Ex 1.

See MAT's Page 1.
For a nilpotent element x and a unit a, we can prove that $a+x$ is again a unit.
Say $x^n = 0$ for some $n \in \mathbb{Z}$, and set

$$y = -a^{-1}x.$$

Notice that

$$(1-y)(1+y+\cdots+y^{n-1}) = 1-y^n = 1-(-a^{-1}x)^n = 1,$$

which suggests that $1-y$ is a unit. So we have $a+x = a(1-y)$ is product of unit, thus it's a unit as expected.

2.2 Ex 2.

2.2.1 (i)

$\Leftarrow$: standard application of **Ex.1**
$\Rightarrow$: We follow the hint and suppose $b_0 + b_1x + \cdots + b_mx^m$ is the inverse of f. We prove by induction on r that $a_n^{r+1}b_{m-r} = 0$:
for $r = 0$, note a_nb_m is the coefficient for x^{n+m}, which must be 0;
assume the validity for $r-1$. HERE is the way to deal with the coefficients.
Consider the coefficients of x^{n+m-1}, which must be $a_{n-1}b_m + a_nb_{m-1} = 0$. If we multiply both sides with a_n will have

$$a_{n-1}a_nb_m + a_n^2b_{m-1} = a_n^2b_{m-1} = 0$$

since $a_nb_m = 0$. In general, we can continue by the similar fashion, namely look at the coefficients of x^{n+m-i} for some natural number i:

$$\sum_{\alpha+\beta=n+m-i} a_\alpha b_\beta.$$

Multiply a^i on both sides will make all terms except the last one be 0, which forces the last term $a_n^{i+1}b_{m-i} = 0$ to be zero. In particular, let $i = m$ we have $a_n^{i+1}b_0 = 0$. Recall that b_0 is a unit, so we have $a_n^{m+1} = 0$ as desired. So $-a_nx^n$ is a nilpotent and $f + (-a_nx^n) = a_0 + \cdots + a_{n-1}x^{n-1}$ is a unit by Ex 1. So we can apply the argument again, implies that a_{n-1} is nilpotent. Until we only have the constant term, namely $a_0 = a_0 + a_1x - a_1x$ must be a unit.

2.2.2 (ii)

$\Rightarrow$: Suppose f is nilpotent, thus we have

$$\begin{aligned}0 &= f^m \\ &= (a_0 + a_1x + \cdots + a_nx^n)^m \\ &= a_0^m + (ma_0^{m-1}a_1)x + \cdots + a_1^mx^m + \cdots + (a_nx^n)^m,\end{aligned}$$

which forces, in particular, a_0 to be nilpotent. Then we know $f - a_0 = a_1 x + a_2 x^2 + \cdots + a_n x^n$ is nilpotent since all nilpotent elements form an ideal. We define it as $f_1 := f - a_0$. It has a corresponding $m_1 \in \mathbb{N}$ such that

$$0 = f_1^{m_1} = (a_1^{m_1}) x^{m_1} + \cdots$$

The coefficient for the term x^{m_1} must be zero, which proves a_1 is nilpotent. And for all finitely many coefficient we can inductive do this and conclude they're all nilpotent as desired.

$\Leftarrow$: To prove f is nilpotent, we only have to find a "large" enough number suffices to turn f into 0. Since all coefficients are nilpotent, we could find $d_0, d_1, ..., d_n \in \mathbb{Z}$ such that

$$a_i^{d_i} = 0$$

for $0 \leq i \leq n$. Denote $\sum_{j=0}^{n} d_i = D$. We could verify that

$$\begin{aligned} f^D &= (a_0 + a_1 x + \cdots + a_n x^n)^D \\ &= [a_0^D + \cdots + (a_n x^n)]^D \\ &= C_0 + C_1 x + \cdots + C_{Dn} x^{Dn} \end{aligned}$$

where $C_i = \sum \prod_{i \in I_n} a_i$ for some finite index set $I_n \subset \{0, 1, 2, ..., n\}$. The coefficient C_i is sum of some coefficient that collected various powers of x, namely terms of x^j where $j \in I_i$. Each such power will contain various a_i, but according to Pigeon Principle we've lifted them to power D, at least one of them will become 0 and then this term is zero. All such term will be zero, and their sum, C_i, will be zero. So we have $f^D = 0$ as desired.

2.2.3 (iii)

$\Leftarrow$: this is precisely the definition.

$\Rightarrow$: We follow the hint. Suppose f is a zero-divisor, then there exists a nonzero element g in the polynomial ring such that $fg = 0$. We can further suppose g is the smallest degree polynomial satisfy the condition we mentioned.

$$0 = fg = (a_0 + a_1x + \cdots + a_nx^n)(b_0 + b_1x + \cdots + b_mx^m).$$

In particular we have $a_nb_m = 0$. Notice that we'll have

$$(a_ng)(f) = 0.$$

But polynomial a_ng is of degree less than g since $a_nb_m = 0$, this contradicts least degree property of g, which implies that $a_ng = 0$. Let's examine the equality above again:

$$\begin{aligned}0 = fg &= (a_0 + a_1x + \cdots + a_{n-1}x^{n-1})(b_0 + b_1x + \cdots + b_mx^m) + (a_nx^n)(b_0 + b_1x + \cdots + b_mx^m)\\ &= (a_0 + a_1x + \cdots + a_{n-1}x^{n-1})(b_0 + b_1x + \cdots + b_mx^m).\end{aligned}$$

This time we look at coefficient for term x^{n+m-1}, which is $a_{n-1}b_m$ precisely. Since the product is 0, which forces in particular $a_{n-1}b_m = 0$. Then we can use polynomial $a_{n-1}g$, which is of degree less than g, but will annihilate f. We can continue in this fashion and conclude that all $a_ig = 0$ for all $i \in \{1, 2, ..., n\}$. While g is non-zero so we can find some $b_i \neq 0$ such that

$$a_ib_j = 0$$

as expected.

2.2.4 (iv)

Solutions from StackExchange.
For one direction, refer to Gauss's Lemma.
In fact, we have many versions of Gauss's Lemma.

2.3 Ex 3.

Induction on r, which holds for $r = 1$. Suppose all results hold for $A[x_1, ..., x_{r-1}]$.

f is a unit in $A[x_1, ..., x_{r-1}][x_r] \Leftrightarrow a_0$ is a unit in $A[x_1, ..., x_{r-1}]$ and $a_1, ..., a_n \in A[x_1, ..., x_{r-1}]$ are nilpotent.

One thing to notice is that here the coefficients are polynomial with indeterminants, so we can use the induction assumption to finish(since polynomial is of finite degree and we only have to apply finitely many times of results).

For example, a_0 is a polynomial and a unit in $A[x_1, ..., x_{r-1}]$, thus we could distract $b_0 \in A[x_1, ..., x_{r-2}]$ is a unit and finitely many polynomial $b_1, ...b_?$ that are nilpotent. After applying (i) for finitely many times, we could find a "pure" coefficient in A that's a unit, along with many polynomial that're nilpotent. Since their coefficients are all nilpotent, we could repeat (ii) for finitely many times and argue that all coefficients other than the first one in A are nilpotent.

So the generalised version of theorem states that the constant term is a unit in A, and other coefficients are nilpotent.

2.4 Ex 4.

Maximal ideal is prime ideal, thus Jacobson Radical $\supset$ Nilradical.
So the non-trivial direction of proof starts by picking up an element f from Jacobson Radical $\mathfrak{R}$. By Prop 1.9 we know that $1 - fy$ is a unit for any y.
Now we let $y = x$, this gives us

$$1 - fx = 1 - a_0x - a_1x^2 - \cdots - a_nx^{n+1}$$

is a unit. Use (ii) of Ex. 2, we know that all $a_0, a_1, ..., a_n$ are nilpotent then f is nilpotent by (iii) of Ex 2.

2.5 Ex 5.

2.5.1 (i)

$\Leftarrow$: ✔
$\Rightarrow$: Assume $g := b_0 + b_1x + b_2x^2 + \cdots \in A[[x]]$ to be the inverse for f in the formal power series. We only need to look at the lowest power of their product

$$1 = fg = (b_0 + b_1x + \cdots)(a_0 + a_1x + \cdots) = b_0a_0 + (b_0a_1 + b_1a_0)x + \cdots.$$

The fact that this equality holds implies that we must have $b_0a_0 = 1$, which gives us the desired result.

2.5.2 (ii)

Suppose we have $f^n = 0$ for some integer $n > 0$. If we look at the constant coefficient, we'll have $a_0^n = 0$, so a_0 nilpotent. Then inductively prove a_i is nilpotent for all $i \in \mathbb{N}$ by following Ex 2. (ii).

Converse is ... ???

2.5.3 (iii)

$\Leftrightarrow$: Suppose f belongs to the Jacobson radical, so according to Prop 1.9 this is equivalently to say

$$1 - fy$$

is a unit in $A[[x]]$ for all $y \in A[[x]]$. By part (ii) we know that this is the same as $1 - a_0 b_0$ is a unit in A for any $b_0 \in A$ where b_0 is the constant coefficient for $y \in A[[x]]$. Recall part (ii) again we know that this is equivalently to say a_0 belongs to the Jacobson Radical of A.

HERE is a good counterexample explaining the reason of why we defined product of ideal with "finiteness".

2.5.4 (iv)

??? Here I assume the underlying ring homomorphism is $f : A \to A[[x]]$ which sends a ring element $a \mapsto a$ in the formal power series.

Fact: preimage (under ring homomorphism) of a prime ideal is again a prime ideal. The proof is basically unwrap the definition.

Fact: how about maximal ideal? This is generally incorrect. Consider a ring homomorphism of inclusion

$$f : \mathbb{Z}[x] \to \mathbb{Q}[x] \quad \text{defined by} \quad q \mapsto q.$$

We have a maximal ideal $\langle x \rangle$ since $\mathbb{Q}[x]/\langle x \rangle \cong \mathbb{Q}$ is a field. But the preimage, or contraction of $\langle x \rangle$ is $\langle x \rangle \subset \mathbb{Z}[x]$, which isn't maximal.
Another great example is HERE page 6.

??? DON'T KNOW HOW TO DO...

For a maximal ideal $\mathfrak{m} \subset A[[x]]$, its contraction under the inclusion map is

$$f^{-1}(\mathfrak{m}) = \mathfrak{m} \cap A.$$

According to the definition of the maximal ideal, $\mathfrak{m}$ is composed of non-units in $A[[x]]$. Suppose we have

$$\mathfrak{m} \subsetneq I \subset A[[x]]$$

for some ideal $I \subset A[[x]]$. Then $I = A[[x]]$ and in particular I contains a unit in $A[[x]]$. Apply contraction to them we have

$$f^{-1}(\mathfrak{m}) \subsetneq f^{-1}(I) \subset A.$$

According to part (i), the ideal I contains unit $g \in A[[x]]$, then $f^{-1}(I)$ contains the constant coefficient $g_0 \in A$, which is a unit. This implies that $f^{-1}(I) = A$, therefore $f^{-1}(\mathfrak{m})$ is a maximal ideal.

2.5.5 (v)

For a given prime ideal $\mathfrak{p} \subset A$, note that

$$(\mathfrak{p} + xA[[x]])^c = \mathfrak{p}$$

under the inclusion map. It suffices to prove that $\mathfrak{p} + xA[[x]]$ is a prime ideal in $A[[x]]$. Firstly of all it's an ideal since it's clearly a subgroup and absorb multiplication from elements of $A[[x]]$. Suppose we have $f, g \in A[[x]]$ where $f = f_0 + f_1 x + \cdots$ and $g = g_0 + g_1 x + \cdots$ such that

$$fg \in \mathfrak{p} + xA[[x]].$$

In particular we have $f_0 g_0 \in \mathfrak{p}$, and then $f_0, g_0 \in \mathfrak{p}$ since it's a prime ideal in A. This actually implies that

$$f = f_0 + f_1 x + \cdots = f_0 + x(f_1 + f_2 x + \cdots) \in \mathfrak{p} + xA[[x]]$$

and similarly for g. So we've proved that both f, g belongs to $\mathfrak{p} + xA[[x]]$, and this confirms that it's a prime ideal.

2.6 Ex 6.

HERE is one solution. HERE is another solution.

Natually we have nilradical is a subset of Jacobson radical.

The non-trivial direction is to prove every maximal ideal is contained in the nilradical. Suppose for the sake of contradiction that there exists a maximal ideal I that's not contained in the nilradical, then by assumption we can find a non-zero idempotent. By Prop 1.9 we know that in particular we have for $y = 1$,

$$1 - xy = 1 - x$$

is a unit. But then we have $(1 - x)x = x - x^2 = 0$, which implies that $1 - x$ is a zero-divisor. This is a contradiction since a unit cannot be a zero-divisor.

2.7 Ex 7.

Fix a prime ideal I. We know A/I is an integral domain. To prove I is a maximal ideal, we only need to check A/I is a field. More precisely, we need to check every nonzero element in A/I has an inverse.

For a nonzero element $a \in A$, there exists $n \geq 2$ such that $a^n = a$.

Note that we have

$$([a]^{n-1} - 1)[a] = 0 \;\Rightarrow\; [a]^{n-1} = 1$$

where $[a]$ is the natural projection from a onto integral domain A/I. Notice that $[a]^{n-2}[a] = 1$, which proves that any non-zero element is a unit as desired. This implies that A/I is a field, hence every prime ideal is a maximal ideal.

2.8 Ex 8.

2.8.1

Nilradical is the minimal element with respect to inclusion. ??✖ This element has to retain to be prime.

Fact: Intersection of prime ideal isn't necessarily prime. For example, in $\mathbb{Z}$, we have two prime ideals $(2), (3)$. Their intersection is precisely (6), which isn't prime.

Fact: How about maximal ideal?
1) As long as we have two maximal ideal with intersection as either one's proper subset, then their intersection mustn't be maximal;
2) since $\mathbb{Z}$ is P.I.D, non-zero prime ideal is maximal ideal. We can take the same example as before.

2.8.2

Although we know that intersection of prime ideals are not necessarily prime ideal, here we only need to consider a special case, which is for applying Zorn's Lemma. We give the order of all ideals $I_1 \leq I_2$ if and only if $I_2 \subset I_1$. This special case is to prove that for all prime ideals that are in one *chain*, their intersection is again a prime ideal. Refers HERE.

??? But for this post, I don't see the reason why it requires existence of prime ideals. Zorn's Lemma only needs "for every chain it has a upper bound..."

Namely if we have a chain of prime ideals $\{I_i\}$ where $i \in A$ for some index set A such that

$$\cdots \subset I_i \subset \cdots .$$

We need to prove that $\bigcap_{i \in A} I_i$ is a prime ideal. We use contrapositive argument here. Suppose we have $a, b \notin \cap_{i \in A} I_i$, then we can find $i_1, i_2 \in A$ such that

$$a \notin I_{i_1} \quad \text{and} \quad b \notin I_{i_2}.$$

Without loss of generality, we can assume that $I_{i_1} \subset I_{i_2}$. So we have $a, b \notin I_{i_1}$, while it's a prime ideal, then we have

$$ab \notin I_{i_1} \quad \Rightarrow \quad ab \notin \cap_{i \in A} I_i.$$

This proves that $\cap_{i \in A} I_i$ is a prime ideal as expected.

2.8.3

See [1] 3.17; minimal prime over an ideal...

2.9 Ex 9.

$\Leftarrow$: Suppose we have some prime ideals $\{P_i\}_{i \in A}$ for some index set A, such that

$$\mathfrak{a} = \bigcap_{i \in A} P_i.$$

In order to prove that $\mathrm{rad}(\mathfrak{a}) = \mathfrak{a}$, it suffices to prove $\mathrm{rad}(\mathfrak{a}) \subset \mathfrak{a}$. For any $x \in \mathrm{rad}(\mathfrak{a})$, this implies that $x^n \in \mathfrak{a}$ for some integer $n > 0$. For any $i \in A$, we have

$$x^n \in P_i.$$

While this is a prime ideal, either x^{n-1} or x belongs to P_i. We can continue in this fashion and conclude that x belongs to P_i. While the index i is arbitrary, we know that

$$x \in \bigcap_{i \in A} P_i = \mathfrak{a}$$

as desired.

$\Rightarrow$: DON'T KNOW HOW TO DO...

!!! This direction is trivial if we recall Prop 1.14. Since radical is intersection of prime ideals which containing $\mathfrak{a}$, the intersection of this set of prime ideals will be $\mathfrak{a}$ given that it's radical.

HERE I present another solution in case you forget Prop 1.14.

Since $\mathfrak{a} \neq (1)$, then we can find a maximal ideal P that containing $\mathfrak{a}$. This ideal P is in particularly prime, and we define

$$B := \{Q \subset R \quad \text{is a prime ideal} \quad | \;\; Q \supset \mathfrak{a}\}$$

as all ideals that containing $\mathfrak{a}$. It's non-empty since we must have $P \in B$.

Now we use non-trivial direction of the inclusion. For any $x \in \operatorname{rad}(\mathfrak{a})$, namely we can fine an integer n such that $x^n \in \mathfrak{a}$, it belongs to $\mathfrak{a}$. So we have $x^n \in P$, which is prime, then we have $x \in P$ by a similar argument as above. This implies that

$$\mathfrak{a} = \operatorname{rad}(\mathfrak{a}) \subset \bigcap_{i \in B} Q_i.$$

It suffices to prove the other direction, in which we intend to use a contrapositive argument. Suppose we have $r \notin \operatorname{rad}(\mathfrak{a})$, this implies that we have

$$r^k \notin \mathfrak{a}$$

for any $k \in \mathbb{N}$. So $S := \{1, r, r^2, ...\}$ is a multiplicative set (this is defined HERE page 5). According to Prop 2.2 Page 5, we know that $R \setminus S$ contains a prime ideal that containing $\mathfrak{a}$. This implies that we have $x \notin \cap_{i \in B} Q_i$ as desired.

2.10 Ex 10.

ii) $\Rightarrow$ iii): We prove that $\mathfrak{R}$ is a maximal ideal. This is because if we have

$$\mathfrak{R} \subsetneq I \subset A$$

for some ideal I, then this implies that we have $i \in I \setminus \mathfrak{R}$. While the nilradical composed of all elements that're nilpotent, then we know that i must be a unit. This implies that $I = A$, which confirms that $\mathfrak{R}$ is prime.

iii) $\Rightarrow$ i): Since it's a field, nilradical $\mathfrak{R}$ is a maximal ideal and is prime. While by definition

$$\mathfrak{R} = \bigcap_{i \in J} P_i$$

for all prime ideals $P_i \subset A$ and for some index set J. By Prop 1.11 (ii), we know that $\mathfrak{R} = P_{i_0}$ for some $i_0 \in J$. This implies that all prime ideals in J is the same, otherwise the equality doesn't hold. So this confirms that A has exactly one prime ideal./.;/'l"";l.l,/.

i) $\Rightarrow$ ii): We have a ring A that only has one prime ideal. While each maximal ideal is prime, then at most we have one maximal ideal. Suppose $A = 0$, the case is trivial. For $A \neq 0$, Theorem 1.3 let us conclude that we have at least one maximal ideal. So in this case we have exactly one maximal ideal, which equals to $\mathfrak{R}$. Each element of A is either a unit or a non-unit. The first case is done, then we consider the case for a non-unit, which is contained in a maximal ideal, namely $\mathfrak{R}$. And it must be nilpotent since it's also in nilradical. So we know that every element of A is either a unit or nilpotent.

2.11 Ex 11.

i) Consider this, for any $x \in A$,

$$x + 1 = (x+1)^2 = x^2 + 2x + 1 = x + 2x + 1 = 3x + 1 \quad \Rightarrow \quad 2x = 0.$$

ii) According to 2.7 and $n = 2 > 1$, we can conclude that every prime ideal is maximal.

According to 2.7, we know that any $x + \mathfrak{p}$ for some non-zero element $x \in A$ is the multiplicative identity since

$$(x + \mathfrak{p})(x + \mathfrak{p}) = x^2 + \mathfrak{p} = x + \mathfrak{p} \quad \Rightarrow \quad x + \mathfrak{p} = 1 + \mathfrak{p}.$$

Note that we cannot have $x^n + \mathfrak{p}$ for some $n \geq 2$ since we're in a Boolean ring. And each $x + \mathfrak{p}$ is equal to $1 + \mathfrak{p}$. Together with $0 + \mathfrak{p}$, we have exactly two elements in the field $A/\mathfrak{p}$.

iii) We consider the ideal $\langle x, y \rangle \subset A$ generated by two distinct elements $x, y \in A$. It suffices to prove that this ideal is principle.

DON'T KNOW HOW TO... HERE

The right candidate for principle ideal isn't xy, but $x + y + xy$.

We need to prove that $\langle x, y \rangle = \langle x + y + xy \rangle$. Clearly we have $\langle x, y \rangle \supset \langle x + y + xy \rangle$. Conversely, we have

$$\begin{aligned} x(x + y + xy) &= x + xy + xy = x + 2(xy) = x, \\ y(x + y + xy) &= xy + y + xy = 2(xy) + y = y. \end{aligned}$$

And this completes the other direction of inclusion.

2.12 Ex 12.

Also see Spring 2023 506 HW 4 Problem 3.

Since it's a local ring, we have exactly one maximal ideal $\mathfrak{m}$. Suppose we have a non-zero idempotent element $x \in A$, then we pass it to the quotient

$$(x+\mathfrak{m})(x+\mathfrak{m}) = x^2 + \mathfrak{m} = x + \mathfrak{m} \quad \Rightarrow \quad x + \mathfrak{m} = 1 + \mathfrak{m}$$

given that $A/\mathfrak{m}$ is a field. We know that $1 - x \in \mathfrak{m}$. Also we know that $x \in \mathfrak{m}$ since it's a non-unit given that $x(x-1) = 0$ (it's a zero-divisor). But this gives us

$$1 = x - (x-1) \in \mathfrak{m},$$

which contradicts the fact that $\mathfrak{m}$ is a maximal ideal.

Solution HERE and HERE.

2.13 Ex 13.

Firstly we prove that the ideal $\mathfrak{a}$ couldn't be the whole ring. Suppose on the contrary that $\mathfrak{a} = (1)$, then we can find a $g \in A := K[x_f]$ such that

$$g \prod_{i \in J} f_i = 1$$

for some index set J. This implies that f is a unit in polynomial ring A. By definition of irreducible polynomial, we know that f couldn't be a constant since K is a field. According to **??**, coefficients other than the constant must be nilpotent, this contradicts the fact that f is a monic polynomial. So we must have $\mathfrak{a} \neq (1)$.

By Corollary 1.4, there exists a maximal ideal $\mathfrak{m}$ that contains $\mathfrak{a}$. We construct a field $K_1 := A/\mathfrak{m}$, which contains a root for any $f \in \Sigma$ since it's defined as 0 in this field. Repeat this process, and since each polynomial has finite degree, union of infinitely many K_i will contain all of the roots.

The rest is precisely what the problem suggested.

2.14 Ex 14.

Proof is basically Proposition 2.2 of HERE.

We can assume Σ is non-empty since otherwise the proposition is trivially correct. (Do we have to write this?

We wish to apply Zorn's Lemma. Let $i \in J$ be an index set, we consider a chain of ideals

$$\cdots \subset L_i \subset \cdots$$

where $L_i \in \Sigma$. We can define their union as $L = \cup_{i \in J} L_i$. It's a group under addition: for any $x, y \in L$, we can find $x \in L_{i_1}$ and $y \in L_{i_2}$. We can assume

$L_{i_1} \subset L_{i_2}$, then $x - y \in L_{i_2} \subset L$ since L_{i_2} is an ideal. It absorb elements from the ring, fix an element $a \in A$,

$$ax \in L_{i_1} \subset L$$

for arbitrary $x \in L_{i_1}$. So we know that in this chain of inclusion, their union is indeed an ideal. Clearly in every chain, we can form an ideal like this, and it will serve as the upper bound of the chain. Then we can apply Zorn's Lemma and conclude that there exists a maximal element in Σ.

✖For any maximal element $\mathfrak{p} \in \Sigma$, we hope to prove it's a prime ideal. Suppose we have $ab \in \mathfrak{p}$, if $a \notin \mathfrak{p}$, then we have a proper inclusion of ideals

$$\mathfrak{p} \subsetneq \mathfrak{p} + \langle a \rangle.$$

While $\mathfrak{p}$ is assumed to be a maximal element, then $\mathfrak{p} + \langle a \rangle \notin \Sigma$. This implies that we will have an element $p + ax \in \mathfrak{p} + \langle a \rangle$ for some $p \in \mathfrak{p}$ and $x \in A$ such that it's not a zero-divisor. ✖

?How to prove it's prime... HERE, contrapositive is easy ✔

Now we try to prove the maximal element $\mathfrak{p}$ with respect to inclusion is a prime ideal by resorting a contrapositive argument. Suppose we have $a \notin \mathfrak{p}$ and $b \notin \mathfrak{p}$, then we can form two proper inclusion as

$$\mathfrak{p} \subsetneq \mathfrak{p} + \langle a \rangle, \quad \mathfrak{p} \subsetneq \mathfrak{p} + \langle b \rangle.$$

Since we already have $\mathfrak{p} \in \Sigma$ as a maximal element, then both ideals $\mathfrak{p} + \langle a \rangle$ and $\mathfrak{p} + \langle b \rangle$ doesn't belong to Σ. This implies that we can find non-zero-divisors in both ideals. Then we consider their product

$$(\mathfrak{p} + \langle a \rangle)(\mathfrak{p} + \langle b \rangle) \subset \mathfrak{p} + \langle ab \rangle,$$

which will contain at least one non-zero-divisor. So we have $\mathfrak{p} \subsetneq \mathfrak{p} + \langle ab \rangle$ is a proper inclusion, this implies that $xy \notin \mathfrak{p}$ as desired. So we've verified that $\mathfrak{p}$ is a prime ideal.

On page 8 of the book, we know that zero-divisors are union of annihilators

$$D = \bigcup_{x \neq 0} \mathrm{Ann}(x).$$

So zero-divisors are union of ideals (annihilator is a special case for quotient ideal). Each ideal $\mathrm{Ann}(x) \in \Sigma$ since it's composed of all zero-divisors. By the previous part we've build the existence of maximal element with respect to inclusion, so we can cover all zero-divisors D with those maximal elements $\mathfrak{p}$ that are also prime ideals. So we've proved that the set of zero-divisors in A is a union of prime ideals.

2.15 Ex 15.

2.15.1 (i)

We denote the ideal as $\mathfrak{a} = \langle E \rangle \subset A$. Clearly we have $V(E) \supset V(\mathfrak{a})$ since for any prime ideal that contains $\mathfrak{a}$ must contain E in particular. Conversely, give an arbitrary prime ideal $\mathfrak{p} \in V(E)$, this means $\mathfrak{p} \supset E$ by definition. We interpret the ideal generated by E as intersection of all ideals that containing E. Hence we know that

$$\mathfrak{p} \supset \mathfrak{a}.$$

This implies that ideal $\mathfrak{p}$ is a prime ideal that containing $\mathfrak{a}$ so $\mathfrak{p} \in V(\mathfrak{a})$. Therefore we've build the first equality.

For the second equality, again we note that $V(\mathfrak{a}) \supset V(\mathrm{rad}(\mathfrak{a}))$. For any given prime ideal $\mathfrak{q} \supset \mathfrak{a}$ that contains $\mathfrak{a}$, while $\mathrm{rad}(\mathfrak{a})$ is the intersection of the prime ideals which contain $\mathfrak{a}$ by Proposition 1.14, so we have $\mathfrak{q} \supset \mathrm{rad}(\mathfrak{a})$ and $\mathfrak{q} \in V(\mathrm{rad}(\mathfrak{a}))$. In summary we have

$$V(E) = V(\mathfrak{a}) = V(\mathrm{rad}(\mathfrak{a})).$$

2.15.2 (ii)

For $\{0\}$, any prime ideals of A would contain it so $V(0) = X$. Recall in the textbook we defined prime ideal $\neq \langle 1 \rangle$. So we cannot consider the A as a prime ideal of A itself. Hence we have $V(1) = \emptyset$.

2.15.3 (iii)

This is basically unwrap the definition. For a prime ideal $\mathfrak{p}$ that contains $\cup_{i \in I} E_i$, it must contain each E_i. This implies that $\mathfrak{p} \in V(E_i)$ for each index $i \in I$, therefore it belongs to $\cap_{i \in I} V(E_i)$. Conversely, pick any prime ideal $\mathfrak{p}$ such that lives in every $V(E_i)$ where $i \in I$. This just means $\mathfrak{p}$ contains every E_i where $i \in I$, hence we have $\mathfrak{p} \in V(\cup_{i \in I} E_i)$. This completes another direction of inclusion and proved the equality.

2.15.4 (iv)

Now we try to prove the first equality. Clearly we have $V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{a}\mathfrak{b})$ since $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$. Conversely, for any prime ideal $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, it contains product of ideal $\mathfrak{a}\mathfrak{b}$. Since $\mathfrak{p}$ is prime, by Lemma 2.1 on page 4 of HERE, without loss of generality, we have $\mathfrak{p} \supset \mathfrak{a}$. This implies that $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$.

Now we finish the second equality. Clearly we have $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a} \subset \mathfrak{p}$, this implies that $V(\mathfrak{a}\mathfrak{b}) \supset V(\mathfrak{a}) \cup V(\mathfrak{b})$. Now consider any prime ideal $\mathfrak{p} \supset \mathfrak{a}\mathfrak{b}$, by the same approach above, we can assume that this prime ideal contains $\mathfrak{a}$. Hence we have $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. In summary we have

$$V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b}).$$

See [1] Exercise 2.23 on Page 15. ✔

2.16 Ex 16.

2.16.1

For Spec($\mathbb{Z}$), it is composed of prime ideals 0 and $\langle p \rangle$ for all prime number p.

See HERE. See a post HERE. ✔

2.16.2

For Spec($\mathbb{R}$). Since $\mathbb{R}$ is a field, then it's just one point.

See a post HERE. ✔

2.16.3

For Spec($\mathbb{C}[x]$). Since $\mathbb{C}$ is algebraically closed, all prime ideals are of the form $ax + b$ for some $a, b \in \mathbb{C}$.

See a post HERE. See a post HERE for general approach for spectrum of a polynomial ring over a field.

2.16.4

For Spec($\mathbb{R}[x]$).

According to Prop 12 on Page 286 of [3], which states that in *U.F.D.* a nonzero element is prime iff it's irreducible. Hence we need to find all irrducible in $\mathbb{R}[x]$.

All irreducibles in $\mathbb{R}[x]$ are degree 1 polynomials, with quadratic that has no real root. This is because any odd degree real polynomial will necessarily have a real root by intermediate value theory; and any even degree larger than 2 will contradicts the fact that $\mathbb{C}$ is algebraically closed and the degree of extension $[\mathbb{C} : \mathbb{R}] = 2$. See HERE and HERE for a comprehensive explanation. ✔

See a post HERE. See a related post HERE.

2.16.5

For Spec($\mathbb{Z}[x]$), ???

I had no idea...

See a post HERE, HERE, and HERE.

Verified on Page 28, Example 1.8 from [4]. ✔

2.16.6

For irreducibility of $\mathbb{R}[x, y]$, see a post HERE.
See HERE for description for $\operatorname{Spec}(\mathbb{R}[x, y])$.

2.17 Ex 17.

Here we use $D(f) := X_f$. By definition we know Zariski topology is defined by the closed subsets. While $D(\bullet)$ is the complement of closed set, they satisfy the basis for open sets.

2.17.1 (i)

We can compute

$$\begin{aligned}
D(f) \cap D(g) =& V(f)^c \cap V(g)^c \\
=& [V(f) \cup V(g)]^c \\
=& [V(\langle f \rangle) \cup V(\langle g \rangle)]^c \\
=& [V(\langle fg \rangle)]^c \\
=& [V(fg)]^c \\
=& D(fg).
\end{aligned}$$

2.17.2 (ii)

$\Rightarrow$: Assume $D(f) = \emptyset$, then $V(f) = \operatorname{Spec} A$. In particularly, we know $\langle f \rangle \subset \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Spec} A$, which implies

$$\langle f \rangle \subset \bigcap \mathfrak{p} = \operatorname{Nil}(A).$$

And this implies f is nilpotent.

$\Leftarrow$: Assume f is nilpotent, then $f^r = 0$ for some integer $r > 0$. Note that

$$f^r = 0 \in \mathfrak{p} \ \Rightarrow \ f \in \mathfrak{p} \ \Rightarrow \ \langle f \rangle \subset \mathfrak{p}$$

for arbitrary prime ideal $\mathfrak{p}$. While we always have $V(\langle f \rangle) \subset \operatorname{Spec} A$, hence we can conclude

$$V(\langle f \rangle) \subset \operatorname{Spec} A.$$

2.17.3 (iii)

$\Rightarrow$: Note $D(f) = X$ implies $V(f) = V(\langle f \rangle) = \emptyset$. If $\langle f \rangle \subsetneq A$, then it must lies inside some maximal ideal $\mathfrak{m} \in V(\langle f \rangle)$, which contradicts the assumption. Hence we necessarily have $\langle f \rangle = A = \langle 1 \rangle$, which implies f is a unit.

$\Leftarrow$: Say f is a unit, then $V(f) = V(\langle f \rangle) = V(A) = \emptyset$, which forces $D(f) = X$ as expected.

2.17.4 (iv)

$\Rightarrow$: Assume $D(f) = D(g)$, i.e. $V(\langle f\rangle) = V(\langle g\rangle)$. In parcilarly, we try to interpret $V(\langle f\rangle) \subset V(\langle g\rangle)$. This means for any prime ideal $\mathfrak{p} \supset \langle f\rangle$, we have $\mathfrak{p} \supset \langle g\rangle$. Then by Characterisation of Radical, it follows that

$$\langle g\rangle \subset \bigcap_{\mathfrak{p}\supset\langle f\rangle,\ \mathfrak{p}\in\operatorname{Spec} A} \mathfrak{p} = \sqrt{\langle f\rangle}.$$

Similarly, we can get inclusion conversely and

$$\sqrt{\langle g\rangle} \subset \sqrt{\sqrt{\langle f\rangle}} = \sqrt{\langle f\rangle} \subset \sqrt{\sqrt{\langle g\rangle}} = \sqrt{\langle g\rangle} \ \Rightarrow\ \sqrt{\langle f\rangle} = \sqrt{\langle g\rangle}.$$

$\Leftarrow$: Now we assume $\sqrt{\langle f\rangle} = \sqrt{\langle g\rangle}$, which implies

$$\bigcap_{\mathfrak{p}\in\operatorname{Spec} A,\ \mathfrak{p}\supset\langle f\rangle} \mathfrak{p} = \bigcap_{\mathfrak{q}\in\operatorname{Spec} A,\ \mathfrak{q}\supset\langle g\rangle} \mathfrak{q}.$$

For arbitrary prime ideal $\mathfrak{p} \supset \langle f\rangle$, we have

$$\mathfrak{p} \supset \bigcap_{\mathfrak{p}\in\operatorname{Spec} A,\ \mathfrak{p}\supset\langle f\rangle} \mathfrak{p} = \sqrt{\langle g\rangle} \supset \langle g\rangle.$$

Therefore we have $V(\langle f\rangle) \subset V(\langle g\rangle)$; simiarly we get the inclusion conversely. Then it follows that

$$D(f) = V(\langle f\rangle)^c = V(\langle g\rangle)^c = D(g).$$

2.17.5 (v)

We proceed according to the hint. Given an arbitrary open covering for space $X = \bigcup_{i\in I} D(f_i)$ for some index set I. We can further express as

$$X = \bigcup_{i\in I} D(f_i) = \bigcup_{i\in I} V(f_i)^c = \left[\bigcap_{i\in I} V(f_i)\right]^c = \left[V\left(\bigcup_{i\in I}\langle f_i\rangle\right)\right]^c$$
$$\Rightarrow V\left(\bigcup_{i\in I}\langle f_i\rangle\right) = \emptyset \ \Rightarrow\ \bigcup_{i\in I}\langle f_i\rangle = \langle 1\rangle$$

Last implication requires Hilbert's Nullstellensatz?

✔

See a post HERE.

The correct argument is to derive the following equality

$$\emptyset = \bigcap_{i\in I} V(f_i).$$

We claim that $\langle \sum_{i\in I} f_i\rangle = \langle 1\rangle$. Suppose it's not the case, then this implies there's maximal ideal $\mathfrak{m}_0$ will contain $\langle \sum_{i\in I} f_i\rangle$. And this implies $\mathfrak{m}_0 \in V(f_i)$

for each $V(f_i)$, which contradicts the fact that $\emptyset = \bigcap_{i \in I} V(f_i)$. Hence the claim is proved.

The claim told us

$$1 \in \langle \sum_{i \in I} f_i \rangle,$$

which implies we can express 1 as a *finite* linear combination of $\{f_i\}_{i \in J}$ where J is a *finite* index set. Notice that we must have

$$\emptyset = \bigcap_{i \in J} V(\langle f_i \rangle),$$

which is equivalent to say $\{D(f_i)\}_{i \in J}$ is a finite subcover for X.

See a post HERE. Proposition 13.2 on Page 95 [1].

✔

2.17.6 (vi)

One approach is to generalise the proof in previous part (v). But I encountered a problem...

See a post HERE. It used the property proved in (i) to simplify the open cover, which is why my approach didn't work... Given any *basic open set* D_f with an arbitrary open cover

✔

$$D(f) = \bigcup_{i \in I} D(f_i)$$

for some index set I. According to (i) we know $D(f) \cap D(f_i) = D(ff_i)$. Hence we can re-write the open covering as

$$D(f) = \bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} D(f_i) \cap D(f) = \bigcup_{i \in I} D(ff_i).$$

This implies, without loss of generality, we can assume the open covering $\{D(f_i)\}$ has the property that $f_i \in (f)$ (otherwise, take ff_i instead). With such assumption, we take the complement of the above equality gives us

$$V(f) = \bigcap_{i \in I} V(f_i) = V(\bigcup_{i \in I} \langle f_i \rangle) \;\Rightarrow\; \sqrt{\langle f \rangle} = \sqrt{\bigcup_{i \in I} \langle f_i \rangle}.$$

In particularly, we have

$$f \in \sqrt{\bigcup_{i \in I} \langle f_i \rangle} \;\Rightarrow\; f^n = \sum^{<\infty} a_i f_i = \sum_{i \in J} a_i f_i$$

for some $a_i \in A$ and integer $n > 0$. Here it's important that the index set J is *finite* since an element in sum of those ideals are finite sum of such form.

By definitio of radical, it follows that

$$f \in \sqrt{\bigcup_{i \in J} \langle f_i \rangle} \;\Rightarrow\; \langle f \rangle \subset \sqrt{\bigcup_{i \in J} \langle f_i \rangle}.$$

While $\bigcup_{i\in J}\langle f_i\rangle \subset \langle f\rangle$ by our further assumption, we have the converse inclusion of the radicals. Hence we have

$$\sqrt{\langle f\rangle} = \sqrt{\bigcup_{i\in J}\langle f_i\rangle} \Rightarrow Z(\langle f\rangle) = Z(\bigcup_{i\in J}\langle f_i\rangle) \Rightarrow D(f) = \bigcup_{i\in J} D(f_i),$$

which gives a finite subcover of $D(f)$ as expected.

According to the post above, HERE. We can also use a homeomorphism $\operatorname{Spec}(A_f) \simeq X_f$ to finish the proposition.

How to prove this homeo?

2.17.7 (vii)

$\Leftarrow$: This direction is clear by (vi).

$\Rightarrow$: Since all basic open sets form a basis for Zariski topology on A, then for any open quasi-compact set U we can express it as a union of basic open sets

$$U = \bigcup_{i\in I} D(f_i),$$

and it suffices to prove it has a finite subcover...

no idea...

See the prompt of Exercise 13.41, 10.1, on Page 101 of [1].

✔

Hence it suffices to prove the equivalence of the following statements:

An open set U is a union of finitely many basic open sets if and only if $X - U = V(\mathfrak{a})$ where $\mathfrak{a}$ is finitely generated.

Proof. Assume $U = \bigcup_{i=1}^m D(f_i)$, then define $\mathfrak{a} = \langle f_1, ..., f_m\rangle$ should suffice to finish the claim. Because we have

$$X - U = \bigcup_{i=1}^m \mathbf{D}(f_i) = \bigcap_{i=1}^m \mathbf{V}(f_i) = \mathbf{V}(\langle f_1, ..., f_m\rangle) = \mathbf{V}(\mathfrak{a}).$$

Conversely, say $\mathfrak{a} = \langle f_1, ..., f_m\rangle$. Then according the above equality we know

$$\{\mathbf{D}(f_i)\}_{1\le i\le m}$$

will cover U. $\square$

See a lecture note HERE.

2.18 Ex 18.

2.18.1 (i)

Follows immediately according to (ii).

2.18.2 (ii)

By definition we have

$$\overline{\{x\}} = \bigcap_{\mathfrak{p} \subset \mathfrak{p}_x} \mathbf{V}(\mathfrak{p}) \supset \mathbf{V}(\mathfrak{p}_x).$$

Conversely, for any prime ideal $\mathfrak{q} \in V(\mathfrak{p})$ where $\mathfrak{p} \subset \mathfrak{p}_x$ is arbitrary. When $\mathfrak{p} = \mathfrak{p}_x$, we have $\mathfrak{p} \subset \mathfrak{q}$. And this implies $\mathfrak{q} \in \mathbf{V}(\mathfrak{p}_x)$, then

$$\bigcap_{\mathfrak{p} \subset \mathfrak{p}_x} \mathbf{V}(\mathfrak{p}) = \mathbf{V}(\mathfrak{p}_x) \;\Rightarrow\; \overline{\{x\}} = \mathbf{V}(\mathfrak{p}_x)$$

See Exercise 13.16 [1] on Page 98.

???

2.18.3 (iii)

We use contrapositive argument. We wish to prove that if y lies in any neighborhood of x and vice versa, then $x = y$.

Wrong contrapositive ???

A potentially wrong approach... Negation of the later part gives us for any open set O, we have either $a, b \in O$ or $a, b \notin O$. Take $O := \mathbf{D}(a)$ in the second case will forces $b \in \mathbf{V}(a)$. Similarly we know the other inclusion, which gives us $a = b$.

Is this correct?

See HERE for a solution, which used a middle claim... See Kolmogorov space from Wiki.

✔

2.19 Ex 19.

2.19.1 Equivalent Definition for Irreducible

Take X a topological space. T.F.A.E.

- (a) Space X cannot be expressed as a union of two closed proper subset.
- (b) Every pair of non-empty open sets in X intersect.
- (c) Every non-empty open set is dense in X.

(a) $\Leftrightarrow$ (b): Contrapositive. Suppose there exist U_1, U_2 that do not intersect in X, then X is reducible given

$$X = (X \setminus U_1) \cup (X \setminus U_2)$$

where both $X \setminus U_1, X \setminus U_2$ are non-empty and closed. Conversely, suppose X is reducible. Then $X = V_1 \cup V_2$ where V_1, V_2 are non-empty closed subsets of X. Then both $X \setminus V_1, X \setminus V_2$ will be non-empty proper open subsets of X. But they won't have intersection given

$$(X \setminus V_1) \cap (X \setminus V_2) = X \setminus [V_1 \cup V_2] = \emptyset.$$

(a) ⇔ (c): Contrapositive. Assume a non-empty open set U isn't dense in X, i.e. $\overline{U} \subsetneq X$. Therefore we have

$$X = \overline{U} \cup (X \setminus U),$$

which contradicts the assumption that X is irreducible. Conversely, suppose X is reducible. Then $X = V_1 \cup V_2$ where V_1, V_2 are non-empty proper closed subsets of X. But immediately we have $(X \setminus V_1)$ is a non-empty open set in X, and it's not dense in X since

$$\overline{X \setminus V_1} \subset V_2 \subsetneq X.$$

2.19.2

Proof. Assume $\mathfrak{q} := \mathrm{Nil}(A)$ is a prime ideal, we wish to show that $\mathrm{Spec}(A)$ is irreducible. By one characterisation of nilradical, we have

$$\mathfrak{q} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, A} \mathfrak{p} \;\Rightarrow\; \mathrm{Spec}\, A \subset \mathbf{V}(\mathfrak{q}) \;\Rightarrow\; \mathrm{Spec}\, A = \mathbf{V}(\mathfrak{q}).$$

Furthermore, recall in previous exercise 2.18 we can express the variety as a closure of a point.

$$\mathrm{Spec}\, A = \mathbf{V}(\mathfrak{q}) = \overline{\{\mathfrak{q}\}}.$$

While a single point $\mathfrak{q}$ is irreducible , then we can apply a result from 2.20 to conclude that $\mathrm{Spec}\, A$ is irrducibe in itself, hence irreducible.

See HERE in case you're uncomfortable about the previous claim...

Now we have the deal with the other direction. We proceed by contrapositve. Assume $\mathfrak{b} := \mathrm{Nil}(A)$ is not a prime ideal, though it's an ideal anyway. Hence there exist some $xy \in \mathfrak{b}$ but $x \notin \mathfrak{b}$ and $y \notin \mathfrak{b}$. Now we can use results from previous exercise 17 2.17. While xy is nilpotent, then $\mathbf{V}(xy) = \mathrm{Spec}\, A$; while x, y aren't nilpotent, then $\mathbf{V}(x), \mathbf{V}(y) \subsetneq \mathrm{Spec}\, A$. Notice we have two proper closed subsets

$$\mathbf{V}(x) \cup \mathbf{V}(y) = \mathbf{V}(xy) = \mathrm{Spec}\, A,$$

which proves $\mathrm{Spec}\, A$ is reducible as expected. □

See a post HERE. See a blog HERE.
See a Lemma on Page 2 of a note HERE.

✔

2.20 Ex 20.

2.20.1 (i)

Both Y and $\overline{Y}$ are topological space with subspace topology given by X. For any pair of non-empty open subset $U_1, U_2 \in \overline{Y}$, there exist W_1, W_2 open in X such that

$$U_1 = W_1 \cap \overline{Y},\; U_2 = W_2 \cap \overline{Y}.$$

While Y is irreducible, then $(W_1 \cap Y) \cap (W_2 \cap Y)$ will intersect in Y. By definition of subspace topology in Y, we know both $(W_1 \cap Y), (W_2 \cap Y)$ are open. Furthermore, we note

$$\emptyset \subsetneq [(W_1 \cap Y) \cap (W_2 \cap Y)] \cap Y \subset (U_1 \cap U_2) \cap \overline{Y}.$$

Thus it follows that any pair of two open sets in $\overline{Y}$ will intersection in $\overline{Y}$, which means $\overline{Y}$ is irreducible.

See Stacks Project HERE.

2.20.2 (ii)

I have no idea what is needed to prove? Isn't this equivalent to some sort of Axiom of Choice or Zorn?

See [1] Lemma 16.50 on Page 125, *"irreducible component"*. Also see a post HERE.

Use Zorn's Lemma... Order all irreducible spaces by inclusion. In any *chain* of irreducible spaces, we claim the union of irreducible spaces are again irreducible. Then any chain is bounded above and apply Zorn's Lemma gives the existence of a maximal irreducible space. Therefore any irreducible space is contained in a maximal one, for which we call *irreducible component* in (iii) later on.

2.20.3 (iii)

Say $A \subset X$ is an irreducible component, which means maximal with respect to inclusion. It follows immediately A is closed given

$$A \subset \overline{A} \Rightarrow A = \overline{A}.$$

For any point $x \in X$, note that $\{x\}$ is irreducible. Hence it lies in some irreducible component. This means all irreducible components will cover X.

For a Hausdorff space X, irreducible compoent is just a singleton $\{x\}$ for any $x \in X$.

See [1] Lemma 16.50 on Page 125; See an official solution HERE.

2.20.4 (iv)

According to 2.18, we know $\mathbf{V}(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ is closed for being a closure of a single point by (ii). It's maximal given $\mathfrak{p}$ is minimal.

See a post HERE.

2.21 Ex 21.

3 Chapter 2

3.1 Ex 2.2 Page 20

i) For any element $a \in \text{Ann}(M+N)$, namely for any element such that

$$a(M+N) = 0.$$

This implies that in particularly, $aM = 0$ and $aN = 0$ given $M, N \subset M+N$ by definition of sum of modules. Hence we have $a \in \text{Ann}(M) \cap \text{Ann}(N)$.

Conversely, if we have $a \in \text{Ann}(M)$ and $a \in \text{Ann}(N)$. This implies that for any finite sum

$$a(m_1 + \cdots + m_i + n_1 + \cdots + n_j) = 0,$$

so we have $a(M+N) = 0$ for any $a \in A$. This completes the other direction and therefore we have the equality

$$\text{Ann}(M+N) = \text{Ann}(M) \cap \text{Ann}(N).$$

ii) For any element $x \in (N : P)$, by definition $xP \subset N$. Now we try to interpret annihilator of $(N+P)/N$, they're all ring elements $x \in A$ such that

$$x(N+P)/N = 0 \;\Rightarrow\; x(N+P) \subset N.$$

If we have $xP \subset N$, then naturally we have $x(N+P) \subset N$, which completes one direction of inclusion. Conversely, if we're given $x \in \text{Ann}((N+P)/N)$, equivalently we know that $x(N+P) \subset N$, while $xN \subset N$ given that N is module. We must have $xP \subset N$, hence $x \in (N : P)$. So in summary we have

$$(N : P) = \text{Ann}((N+P)/N).$$

3.2 Prop 2.3 Remark

It works well without assumption of finitely generated? See this post discuss why every module is a quotient of free module HERE.

3.3 Remark ii) Page 25

✎ *This is an exercise 2 in the section of tensor product on Dummit and Foote Page 375. It's meant to illustrate the fact that tensor product notation $x \otimes y$ is ambiguous unless we specify which tensor product it lives in.*

On the book we've already seen that $2 \otimes 1 = 0$ in $\mathbb{Z}$-module $\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$.
Now we wish to prove it's non-zero in $2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$.

3.4 Proposition 2.4

Generalized Version of the Cayley–Hamilton Theorem

HERE is a useful paraphrase and some errata.

See an excellent explanation HERE.

➻ A counterexample of Nakayama's Lemma HERE. Also see HERE, explaining why $(\mathbb{Q}, +)$ is not finitely generated; and HERE, explaining $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.

Proposition 2.4 Proof

Let $x_1, ..., x_n$ be a set of generators of M. For each element in the ideal $a \in \mathfrak{a}$, we have to consider it as an endormorphism, i.e., we consider $a := \psi(a) \in \mathrm{Hom}_A(M, M)$. Then for each x_i where $1 \leq i \leq n$, we have (here all $a_{ij} := \psi(a_{ij})$)

$$\phi(x_i) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = \sum_{j=1}^{n} a_{ij}x_j$$
$$\Rightarrow \ (-a_{i1})x_1 + \cdots + (\phi - a_{ii})x_i + \cdots + (-a_{in})x_n = 0$$
$$\Rightarrow \ \sum_{j=1}^{n} (\delta_{ij}\phi - a_{ij})x_j = 0.$$

for all $a_{ij} \in \mathfrak{a}$. And we can write them in the form of a matrix as

$$\begin{pmatrix} \phi - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \phi - a_{22} & \cdots & -a_{2n} \\ \cdots & & & \\ -a_{n1} & -a_{n2} & \cdots & \phi - a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = 0.$$

Moreover, we denote the first matrix as A and the later column vector as v. Now we can apply adjugate matrix on that and have

$$\mathrm{adj}(A)A = \det AI = 0$$
$$\Rightarrow \mathrm{adj}\, Av = \det AIv = 0.$$

Note that this implies the matrix $\det AI$ annihilates all generators, then it must be zero endormorphism. So we have $\det AI = 0 \in \mathrm{Hom}_A(M, M)$. This implies $\det A = 0$ as a scalar, so we can expand the determinant to get the desired result

$$\phi^n + \psi(b_1)\phi^{n-1} + \cdots + \psi(b_n) = 0$$

where $b_i \in \mathfrak{a}$ since it's the product of some a_i.

3.5 Corollary 2.13

HERE is a post discussing this.

3.6 Corollary 2.7

Recall that element in product of an ideal and the module is finite sum of ... See the post HERE.

3.7 Proposition 2.8

See this post HERE and HERE.

3.8 Exercise 2.20

Fix an exact sequence of $B-$module

$0 \longrightarrow N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \longrightarrow 0.$

Let's consider this after tensor with M_B. Note that we have

$$0 \longrightarrow N_1 \otimes M_B \xrightarrow{g_1 \otimes 1} N_2 \otimes M_B \xrightarrow{g_2 \otimes 1} N_3 \otimes M_B \longrightarrow 0$$

$$\Rightarrow 0 \longrightarrow N_1 \otimes_B B \otimes_A M \xrightarrow{g_1 \otimes 1} N_2 \otimes_B B \otimes_A M \xrightarrow{g_2 \otimes 1} N_3 \otimes_B B \otimes_A M_B \longrightarrow 0$$

$$\Rightarrow 0 \longrightarrow N_1 \otimes_A M \xrightarrow{g_1 \otimes 1} N_2 \otimes_A M \xrightarrow{g_2 \otimes 1} N_3 \otimes_A M_B \longrightarrow 0,$$

which is exact since N_1, N_2, N_3 inherits $A-$module structure and M is flat as an $A-$module. And this proves that M_B is flat as $A-$module.

3.9 Ex 1.

According to Qing Liu's book Remark 1.3, that the tensor product of modules is generated by all elements of the form $a \otimes b$ and any element in the tensor product can be written as $\sum_{\text{finite}} a_i \otimes b_j$ a finite sum of tensor products. It suffices to prove for any $a \in \mathbb{Z}/m\mathbb{Z}$ and $b \in \mathbb{Z}/n\mathbb{Z}$, their tensor product is zero. Since m, n are coprime, then we have $mx + ny = 1$ for some integers $x, y \in ZZ$. In $\mathbb{Z}$ we have 1, then this gives us

$$\begin{aligned} a \otimes b &= (mx + ny)a \otimes b \\ &= mxa \otimes b + (ny)a \otimes b \\ &= mxa \otimes b + a \otimes nyb \\ &= 0 \otimes b + a \otimes 0 \\ &= 0. \end{aligned}$$

Since for every element in the tensor product we have proved it's zero, then the module is zero.

3.10 Ex 2.

For the exact sequence we can tensor it with M as

$$
\begin{aligned}
0 \;\to\; \mathfrak{a} \;\to\; A \;\to\; A/\mathfrak{a} \;\to\; 0, \quad & \quad (1)\\
\Rightarrow \mathfrak{a}\otimes M \;\to\; A\otimes M \;\to\; (A/\mathfrak{a})\otimes M \;\to\; 0, \quad & \quad (2)
\end{aligned}
$$

which is exact by Proposition 2.18. The first map in (2) is $\iota\otimes 1$ where $\iota : \mathfrak{a}\to A$ is inclusion. We have to figure out the image of the first map in (2),

$$\iota\otimes 1(\mathfrak{a}\otimes M) = \mathfrak{a}\otimes M.$$

✍ Now we claim that $\mathfrak{a}\otimes_A M \cong \mathfrak{a}M$. We'll prove this by using universal property. ???

We wish to prove this by universal property. Let N be any $A-$module with an $A-$module homomorphism $f : \mathfrak{a}\times M \to N$.

$$
\begin{tikzcd}
\mathfrak{a}\times M \arrow[d, "T"'] \arrow[dr, "f"] & \\
\mathfrak{a}M \arrow[r, dashed, "\bar{f}"'] & N
\end{tikzcd}
$$

We need to verify that for any f, there exist A-linear map $\bar{f}$ that factors through $\mathfrak{a}M$. And by universal property of tensor product we'll know that

$$\mathfrak{a}M \cong \mathfrak{a}\otimes_A M.$$

And this gives us

$$A\otimes_A M/\mathfrak{a}\otimes_A M \cong M/\mathfrak{a}M \cong (A/\mathfrak{a})\otimes_A M.$$

HERE is a post explaining an isomorphism.

3.11 Ex 3.

HERE is one solution.
HERE is a post, discussing an important isomorphism.

We follow the hint. Let $\mathfrak{m}\subset A$ be the maximal ideal of local ring A, then we can define $k := A/\mathfrak{m}$ as its residue field. According to 3.10, we can define a module as

$$M_k := k\otimes_A M \cong M/\mathfrak{m}M.$$

Notice that M is finitely generated and $\mathfrak{m}$ lies inside Jacobson Radical $\mathfrak{R}$, so assumption of Nakayama's Lemma are verified. If we have $M/\mathfrak{m}M = 0$, then we can conclude that

$$M = 0.$$

Now we back to the problem itself, observe that

$$\begin{aligned}
&M \otimes_A N = 0 \\
\Rightarrow\ & (k \otimes_k k) \otimes_A M \otimes_A N = 0 \\
\Rightarrow\ & [k \otimes_k (k \otimes_A M)] \otimes_A N = 0 \quad \text{by Qing Liu Prop 1.10 on Page 3} \\
\Rightarrow\ & (k \otimes_A M) \otimes_k k \otimes_A N = 0 \\
\Rightarrow\ \Rightarrow\ & (k \otimes_A M) \otimes_k (k \otimes_A N) = 0 \\
\Rightarrow\ & M_k \otimes_k N_k = 0.
\end{aligned}$$

Notice that since M_k is annihilated by $\mathfrak{m}$, then it's inherits $A/\mathfrak{m}-$module structure, whihc implies it's a vector field. While M, N finitely generated, so we have $M_k = M/\mathfrak{m}M$ is a finite dimensional vector space. Since all vector spaces are free, so we can write them as

$$M_k \otimes_k N_k = (A)^{\operatorname{rank} M \cdot \operatorname{rank} N}.$$

The fact that this free module equals to 0 will force each component to be 0, i.e. either $M_k = 0$ or $N_k = 0$, then by the previous argument we know we must have either $M = 0$ or $N = 0$ as expected.

3.12 Ex 4.

Fix an injective $A-$module $f : N_1 \to N_2$. Since $M = \oplus_{i\in I} M_i$ is flat, so we have

$$\begin{aligned}
& f \otimes 1 : N_1 \otimes M \to N_2 \otimes M \\
\Rightarrow\ & f \otimes 1 : \oplus_{i\in I}(N_1 \otimes M_i) \to \oplus_{i\in I}(N_2 \otimes M_i)
\end{aligned}$$

is injective. We define its component function as

$$(f \otimes 1)_i := N_1 \otimes M_i \to N_2 \otimes M_i$$

for arbitrary $i \in I$. Equivalently we can express the function as

$$f \otimes 1 = \oplus_{i\in I}(f \otimes 1)_i.$$

It suffices to prove the following claim: we claim that $f \otimes 1$ is injective if and only if every $(f \otimes 1)_i$ is injective for all $i \in I$.

$\Leftarrow$: For distinct $\mathbf{x}, \mathbf{y} \in \oplus_{i\in I}(N_1 \otimes M_i)$, then we can find at least one $i_0 \in I$ such that $\mathbf{x}_{i_0} \neq \mathbf{y}_{i_0}$, then we must have $(f \otimes 1)_{i_0}(\mathbf{x}_{i_0}) \neq (f \otimes 1)_{i_0}(\mathbf{y}_{i_0})$ give each component function is injective. And this will force $(f \otimes 1)(\mathbf{x}) \neq (f \otimes 1)(\mathbf{y})$ as expected.

$\Rightarrow$: Given an index $j \in I$. For distinct $\mathbf{x}_j, \mathbf{y}_j$, we can embed it into $\mathbf{x} = \mathbb{1}_{\{j\}}\mathbf{x}_k$ (we adopt the notatio nof indicator function HERE, this just means an element of $N_1 \otimes M$ that are nonzero only in j-th coordinate) and similarly for $\mathbf{y}_j$. Given that $f \otimes 1$ is injective, so we have

$$\begin{aligned}
& (f \otimes 1)(\mathbf{x}) \neq (f \otimes 1)(\mathbf{y}) \\
\Rightarrow\ & (f \otimes 1)_j(\mathbf{x}_j) \neq (f \otimes 1)_j(\mathbf{y}_j),
\end{aligned}$$

which proves that $(f \otimes 1)_j$ is injective.

3.13 Ex 5.

HERE is a post discussing the structure of polynomial ring, and write it into direct sum of modules.

For a polynomial ring, we can write it as direct sum of distinct powers of indeterminate

$$A[x] \cong A \oplus A\{x\} \oplus A\{x^2\} \oplus \cdots \cong \oplus_{i \in \mathbb{N}} A\{x^i\} \cong \oplus_{i \in \mathbb{N}} A.$$

According to 3.12, it remains to check each module A is flat for $i \in \mathbb{N}$. Give an injective $A-$module homomorphism $f : N_1 \to N_2$, we have

$$f : N_1 \otimes_A A \to N_1 \otimes_A A \ \Rightarrow f : N_1 \to N_2$$

is injective, which proves that $A[x]$ is flat as desired.

3.14 Ex 6.

✍ Potential typo: here I assume M is an $A-$module.
Notice that polynomial ring $A[x] \cong \oplus_{i \in \mathbb{N}} A_i$ where $A_i \cong A$. So we have

$$(\oplus_{i \in \mathbb{N}} A_i) \otimes_A M \cong \oplus_{i \in \mathbb{N}} M \cong M[x].$$

3.15 Ex 7.

Let $\mathfrak{p}$ be a prime ideal in A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, is $\mathfrak{m}[x]$ a maximal ideal in $A[x]$?

Refers to Dummit and Foote , Section 9.1 of Polynomial Ring, on Page 296. "However, the ideal generated by $\mathfrak{m}$ and x is maximal..."

➻ Proposition 4 of Chapter 7.2 on Page 235 of Dummit and Foote;
➽ Proposition 2 of Chapter 9.1 on Page 296 of Dummit and Foote.
Since we have, by Prop 2 above

$$A[x]/\mathfrak{p}[x] \cong A/\mathfrak{p}[x].$$

Note that $A/\mathfrak{p}$ is an integral domain given $\mathfrak{p}$ is a prime ideal in A. According to Prop 4 above, we know that $A/\mathfrak{p}[x]$ is an integral domain. This proves that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.

When $I \subset A$ is a maximal ideal. See contents after Prop 2 above.

$$\mathbb{Z}[x]/\langle 2\mathbb{Z}[x]\rangle \cong \mathbb{Z}/2\mathbb{Z}[x].$$

It's not a field, so it's never a maximal ideal.
➽ See HERE, HERE, and HERE.
Note that $\langle 2, x\rangle$ is maximal ideal in $\mathbb{Z}[x]$ since we have

$$\mathbb{Z}[x]/\langle 2, x\rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

We can define a map explicitly as

$$\begin{gathered} \varphi : \mathbb{Z}[x] \to \mathbb{Z}/2\mathbb{Z} \\ a_0 + a_1 x + \cdots + a_n x^n \mapsto [a_0] + [a_1] \cdot 0 + \cdots + [a_n] \cdot 0^n. \end{gathered}$$

And we can verify that the kernel is exactly $\langle 2, x\rangle$.

Or we can interpret this isomorphism by applying Third Isomorphism. See HERE, and HERE.

3.16 Ex 8.

i) Use Prop 2.19 characterisation (iii) twice, together with the associativity of tensor product.

We start with an injetive A-module homomorphism $f : L' \to L$. Since M is flat as A-module, so we have

$$f \otimes_A 1_M : L' \otimes_A M \to L \otimes_A M$$

is injective. Again, we can apply the assumption that N is flat as an A-module, which gives us

$$\begin{aligned} & f \otimes_A 1_M \otimes_A 1_N : L' \otimes_A M \otimes_A N \to L \otimes_A M \otimes_A N \\ \Rightarrow\ & f \otimes_A (1_{M \otimes_A N}) : L' \otimes_A (M \otimes_A N) \to L \otimes_A (M \otimes_A N), \end{aligned}$$

which confirms that $M \otimes_A N$ is flat.

ii) Needs change the order when we applying flatness.

Let's start with an injective B-module homomorphism $g : L' \to L$. Since N is flat B-module, then we know that

$$g \otimes_B 1_N : L' \otimes_B N \to L \otimes_B N$$

is injective. And since B is A-algebra, then we can regard this homomorphism as A-module homomorphism, together with the fact that B is flat A-algebra, we have injective maps

$$\begin{aligned} F := g \otimes_B 1_N \otimes_A 1_B : L' \otimes_B N \otimes_A B \to L \otimes_B N \otimes_A B \quad & \text{this induces injective map} \\ \Rightarrow \;\; F' : L' \otimes_B B \otimes_A N \to L \otimes_B B \otimes_A N \quad & \text{this induces injective map} \\ \Rightarrow \;\; F'' : L' \otimes_A N \to L \otimes_A N, \quad & \end{aligned}$$

which gives us N is a flat A-module. Notice that the reason we have apply associativity of Exercise 2.15 from book is because both N, B are (A, B)-bimodule.

3.17 Ex 9.

Notice that we have $M/M' \cong M''$. And we have a fact from Dummit and Foote, Chapter 10.3 Exercise 7, Page 356.

3.18 Ex 10.

✦ *Surjectivity could be interpreted as cokernel is trivial.*

⚮ *I had some misunderstanding about the proposition initially. Need counterexample for help.*

♣ Elegant solution, just see the post HERE, which relies on the fact that tensoring preserve cokernel by right exactness, a more detailed version is HERE; also see the post HERE for the solution.

Solution 1 According to Corollary 2.7, since N is finitely generated and $\mathfrak{a} \subset \mathfrak{R}$ in Jacobson radical, we have

$$u(M) + \mathfrak{a}N = N \;\Rightarrow\; u(M) = N.$$

Clearly we have $u(M) + \mathfrak{a}N \subset N$, it suffices to check the other direction of inclusion. For arbitrary $x \in N$, we can find $n_0 \in N$ such that

$$x \in n_0 + \mathfrak{a}N.$$

Notice that since $M \to M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective and we denote the compostion map as h, then we know that there exists m_0 such that $h(m_0) = n_0 + \mathfrak{a}N$. This gives us

$$x \in n_0 + \mathfrak{a}N = u(m_0) + \mathfrak{a}N \subset u(M) + \mathfrak{a}N.$$

I couldn't proceed with this proof, because I don't know if the diagram is commute or not???

Solution 2 In general for cokernel, we have an exact sequence as

$$M \;\to\; N \;\to\; \operatorname{coker}(u) := N/u(M) \;\to\; 0$$

According to Proposition 2.18, we know that we can tensor this sequence with $(A/\mathfrak{a})$ as

$$\begin{aligned} M \otimes_A (A/\mathfrak{a}) \;&\to\; N \otimes_A (A/\mathfrak{a}) \;\to\; \operatorname{coker}(u) \otimes_A (A/\mathfrak{a}) \;\to\; 0 \\ \Rightarrow\; M \otimes_A (A/\mathfrak{a}) \;&\to\; N \otimes_A (A/\mathfrak{a}) \;\to\; \operatorname{coker}(u) \otimes_A (A/\mathfrak{a}) \;\to\; 0 \\ \Rightarrow\; M/\mathfrak{a}M \;&\to\; N/\mathfrak{a}N \;\to\; \operatorname{coker}(u)/\mathfrak{a}\operatorname{coker}(u) \;\to\; 0. \end{aligned}$$

Notice that since the induced map $u' : M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, so $\operatorname{coker}(u) = \mathfrak{a}\operatorname{coker}(u)$, it's finitely generated since it's a quotient of finitely generated module N. While $\mathfrak{a}$ lives inside Jacobson radical, by Nakayama's Lemma, we can can conclude $\operatorname{coker}(u) = 0$, which is equivalently to say that $u : M \to N$ is surjective.

See a solution HERE, and HERE.

Counterexample Converse is true in general. Why does the ideal $\mathfrak{a}$ have to lie in Jacobson radical?

Take a non-local ring $\mathbb{Z}$ and it acts on itself. Consider this diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, two heads] & \mathbb{Z} \arrow[d, two heads] \\ \mathbb{Z}/3\mathbb{Z} \arrow[r, dashed, "\cong"'] & \mathbb{Z}/3\mathbb{Z} \end{tikzcd}$$

There's another problem of similar taste.

Let I be a nilpotent ideal in a commutative ring R. Let M, N be R-modules with $\varphi : M \to N$ an R-module homomorphism. Show that if the induced map $\overline{\varphi} : M/IM \to N/IN$ is surjective, then φ is surjective.

See this post HERE.

But here we don't have the fin.gen. assumption, which forbids us to use Nakayama.

3.19 Ex 11.

3.19.1 i)

According to the hint, we have induced isomorphisms as

$$\begin{aligned} 1\otimes_A : (A/\mathfrak{m}) \otimes_A A^m &\to (A/\mathfrak{m}) \otimes_A A^n \\ \Rightarrow\ (1\otimes_A)' : (A/\mathfrak{m} \otimes_A A)^m &\to (A/\mathfrak{m} \otimes_A A)^n \\ \Rightarrow\ (1\otimes_A)'' : (A/\mathfrak{m})^m &\to (A/\mathfrak{m})^n. \end{aligned}$$

Notice that we have the last isomorphism between vector space, while in vector space of finite dimension, they're isomorphic if and only if dimension is the same. And this case we can deduce that $m = n$ as expected.

3.19.2 ii)

3.20 Ex 12.

We follow the hint, wishing to prove that $\mathrm{Ker}(\phi)$ is a direct summand of M. While M is finitely generated, then so will be its direct summand.

Let $e_1, ..., e_n$ be basis of A^n, while ϕ is surjective, we define

$$u_i := \phi^{-1}(e_i) \in M$$

for all integers $1 \leq i \leq n$. And we define the module all such elements generated inside M as

$$U = \langle u_1, ..., u_n \rangle.$$

Clearly we have $U + \mathrm{Ker}(\phi) \subset M$.

And we must have $U \cap \mathrm{Ker}(\phi) = 0$ since otherwise we'll have a nontrivial element $m = m_1u_1 + \cdots + m_nu_n \in M$. Here nontriviality of m enforces that $m_1, ..., m_n$ are not all 0. Hence we know

$$0 = \phi(m) = \phi(m_1u_1 + \cdots + m_nu_n) = m_1e_1 + \cdots + m_ne_n.$$

But $m_1e_1 + \cdots + m_ne_n \neq 0$ since $m_1, ..., m_n$ are not all trivial and $\{e_1, ..., e_n\}$ is a basis for A^n.

To prove that $M = U \oplus \mathrm{Ker}(\phi)$, it suffices to check $M \subset U + \mathrm{Ker}(\phi)$. For any element $m \in M$, its image is $a := (a_1, ..., a_n) := \phi(m) \in A^n$. If we have $\phi(m) = 0$, then we can write it as $m = 0 + m \in U + \mathrm{Ker}(\phi)$ since $m \in \mathrm{Ker}(\phi)$.

If it's not 0, i.e. we know $\phi(m) = a_1e_1 + \cdots + a_ne_n \neq 0$ for some nontrivial coefficients $a_1, ..., a_n \in A$. Furthermore, we'll get $m \in U$ by definition of U, which means we can write $m = m + 0 \in U + \mathrm{Ker}(\phi)$.

In summary, we've proved that $M \subset U + \mathrm{Ker}(\phi)$ as desired.

➽ See a post HERE and HERER, which elegantly uses projectiveness of free module A^n in exact sequence.

See a proof HERE.

"Homomorphism preserve finitely generated", see this post HERE.

3.21 Ex 13.

Clearly we can verify that

$$p \circ g = \mathrm{id}_N,$$

which implies that g is inejctvie. So we can consider the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "g"'] & N_B \arrow[l, bend right, dashed, "p"'] \arrow[r] & g(N) \arrow[r] & 0 \end{tikzcd}$$

We know that this sequence split, so we know $N_B = N \oplus g(N) \cong$.

See this solution HERE.
See the post HERE, and HERE.

3.22 Ex 14.

Direct limits

3.23 Ex 15.

Since C is direct sum of all modules and μ_i is the restriction of surjective map $\mu : C \to C/D$. For any element $x \in M := C/D$, we can express its preimage

$$\mu^{-1}(x) = \bigoplus_{I_0} x_i \in C = \bigoplus_{i \in I} M_i$$

where $x_i \in M_i$ and I_0 is a *finite* index set.

For the first two distinct components x_1, x_2, by definition of the direct system we can find $k \in I$ such that $1 \le k$ and $2 \le k$. Furthermore, we have

$$x_1 - \mu_{1k}(x_1) \in D, \; x_2 - \mu_{2k}(x_2) \in D.$$

But this implies we can replace x_1, x_2 in $\mu^{-1}(x)$ by 0 and write $\mu_{1k}(x_1)+\mu_{2k}(x_2)$ at k-th coordinate, i.e. we define an element

$$y := 0 \oplus 0 \oplus \left(\oplus_{I_0 \setminus \{1,2\}} x_i\right) + \underbrace{(0, ..., 0, \mu_{1k}(x_1) + \mu_{2k}(x_2), 0..., 0)}_{k\text{-th coordinate}}.$$

See HERE for the horizontal curly braces.

We claim that $\mu(y) = \mu(x)$ by construction.

The difference is that y is direct sum whose nonzero components are fewer than $|I_0|$. And we can inductively do this process, given I_0 is finite, and reach to one point that there's only one nonzero compoent as expected.

See HERE for a discussion post on MathStackExchange.

If we have $\mu_i(x_i) = 0$, more precisely we have

$$\mu_i(x_i) \in D = \langle x_i - \mu_{ij}(x_i) \rangle_{i,j \in I}.$$

Also see "A Term..." Corollary 7.5 on Page 53.

3.24 Ex 16.

Given the construction and verify the universal property. See Theorem 7.4 of A Term of Commutative Algebra...

3.25 Ex 17.

In "A Term of Commutative Algebra", Ex 7.2 on Page 52 proved that

$$\varinjlim M_i = \bigcup M_i.$$

The third equality holds, since the inclusion...

See a post HERE.

3.26 Ex 18.

$$\begin{tikzcd}
M_i \arrow[r, "\mu_{ij}"] \arrow[d, "\phi_i"'] & M_j \arrow[r, "\mu_j"] \arrow[d, "\phi_j"'] & M \arrow[d, dashed, "\exists!\phi"] \\
N_i \arrow[r, "v_{ij}"'] & N_j \arrow[r, "v_j"'] & N
\end{tikzcd}$$

Notice that we're given another cone under $\{M_i\}$ with nadir N by composing $v_i \circ \phi_i : M_i \to N$. By universal property of M there exists a unique map $\phi : M \to N$ as expected.

3.27 Ex 19.

$$\begin{tikzcd}
0 \arrow[r] & M_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_{ij}"'] & N_i \arrow[r, "\psi_i"] \arrow[d, "\beta_{ij}"'] & P_i \arrow[r] \arrow[d, "\gamma_{ij}"'] & 0 \\
0 \arrow[r] & M_j \arrow[r, "\phi_j"] \arrow[d, "\alpha_j"'] & N_j \arrow[r, "\psi_j"] \arrow[d, "\beta_j"'] & P_j \arrow[r] \arrow[d, "\gamma_j"'] & 0 \\
0 \arrow[r] & \varinjlim M \arrow[r, "\phi"] & \varinjlim N \arrow[r, "\psi"] & \varinjlim P \arrow[r] & 0
\end{tikzcd}$$

Note that we'll decide the index i, j along the proof.

According to the assumption, the first two rows are exact, we wish to show the last row is exact. Here ϕ, ψ are given by applying 3.26. Furthermore, 3.26 tells us the diagram is commute.

We claim that $\operatorname{Im}\phi \subset \operatorname{Ker}\psi$. Pick an element $x \in \varinjlim M$, by 3.23 there exists an index i such that $\alpha(x_i) = x$ where $x_i \in M_i$. While the diagram commute, we can chase the x_i under the map in either ways

$$\begin{aligned}&\psi \circ \phi \circ \alpha_i(x_i) = \gamma_i \circ \psi_i \circ \phi_i(x_i) = \gamma_i(0) = 0\\ &\Rightarrow\ \psi \circ \phi(x) = 0,\end{aligned}$$

which proves the claim.

Conversely, we need to show $\operatorname{Im}\phi \supset \operatorname{Ker}\psi$. Start by picking an element $y \in \operatorname{Ker}\psi$, then apply 3.23. Hence there exists j such that $\beta_j(y_j) = y$ for some $y_j \in N_j$. Given the diagram is commute, $y_i \in \operatorname{Ker}\psi_j = \operatorname{Im}\phi_j$. This implies there exists some $y' \in M_j$ such that $\phi_j(y') = y_j$. While the diagram involving $M_j, N_j, \varinjlim M, \varinjlim N$ commute, we know $\phi(\alpha_j(y')) = y$, which proves the claim.

3.28 Ex 20.

Follow the hint, using universal property...

3.29 Ex 21.

The colimit $\varinjlim A$ is a $\mathbb{Z}-$module. To be defined as a ring, we need to define multiplication, unit.

Firstly, we define the multiplication of $a, b \in \varinjlim A$. Pick two index such that $\alpha_i(a_i) = a,\ \alpha_j(b_j) = b$ by invoking 3.23. According to the definition of filtered set, we can pick another index k such that $i, j \leq k$. While the diagram is commute, we also have $\alpha_k \circ \alpha_{ik}(a_i) = a,\ \alpha_k \circ \alpha_{jk}(b_j) = b$. And we define

$$a \cdot b \ := \ \alpha_{ik}(a_i) \cdot_{A_k} \alpha_{jk}(b_j)$$

where the multiplication is taken from A_k. It is well-defined because the diagram is commute, the value is independent of choice of index. Also we can define $1 := \alpha_k(1_k)$, which is also well-defined for the diagram is commute. See a post HERE.

3.30 Ex 27. Absolutely Flatness

3.30.1

i) $\Rightarrow$ ii): I have no idea what the hint was...
See a post HERE, and a solution HERE.

Let's start with the exact sequence

$$0 \longrightarrow \langle x\rangle \longrightarrow A \longrightarrow A/\langle x\rangle \longrightarrow 0$$

Since A is absolutely flat, we know the functor $-\otimes_A A/\langle x\rangle$ is an exact functor. Apply this exact functor gives us

$$0 \longrightarrow \langle x\rangle \otimes_A A/\langle x\rangle \longrightarrow A \otimes_A A/\langle x\rangle \longrightarrow A/\langle x\rangle \otimes_A A/\langle x\rangle \longrightarrow 0$$

Now we can apply quotient isomorphism we proved in 3.10, which gives us

$$\langle x\rangle \otimes_A A/\langle x\rangle \simeq \langle x\rangle/\langle x\rangle^2 = \langle x\rangle/\langle x^2\rangle.$$
$$A/\langle x\rangle \otimes_A A/\langle x\rangle \simeq A/\langle x\rangle/\langle x\rangle A/\langle x\rangle \simeq A/\langle x\rangle.$$

More precisely, we get a short exact sequence as follows

$$0 \longrightarrow \langle x\rangle/\langle x\rangle^2 \longrightarrow A/\langle x\rangle \longrightarrow A/\langle x\rangle \longrightarrow 0$$

Then we can apply first isomorphism theorem to enforce $\langle x\rangle/\langle x\rangle^2 = 0$.

??? *What are the map induced from $A/\langle x\rangle \to A/\langle x\rangle$, it should be identity but I don't know...*

3.30.2

ii) $\Rightarrow$ iii):

3.30.3

iii) $\Rightarrow$ i): See a post HERE without using Tor functor...

3.31 Ex 28.

3.31.1 Boolean Ring

Clearly a Boolean ring is *absolutely flat* by applying characterisation of absolutely flatness in 3.30.

3.31.2 Ring from Chapter 1 Exercise 7

The requirement of Exercise 7 in Chapter 1 2.7 is $x = x^n$ for any $x \in A$ with an integer $n \geq 2$.

??? Didn't complete the proof. See HERE, and HERE.

Note that in general we have $\langle x\rangle \supset \langle x^2\rangle$. Conversely, we observe $ax = ax^n = ax^{n-2}x^2 \in \langle x^2\rangle$, which proves $\langle x\rangle = \langle x^2\rangle$.

3.31.3 Homomorphic image

Let $f : A \to B$ be a ring homomorphism where A is *absolutely flat*, then $f(A)$ is *absolutely flat*.

We use characterisation from 3.30. For any principal ideal $\mathfrak{I} \subset f(A)$, we can express it as $\mathfrak{I} = \langle f(a) \rangle \subset f(A)$ where $a \in A$. Note that the surjective map $A \to f(A)$ will map an ideal in A to an ideal in $f(A)$, therefore we have

$$\langle f(a) \rangle = f(\langle a \rangle) = f(\langle a^2 \rangle) = \langle f(a)f(a) \rangle,$$

which proves that $f(A)$ is idempotent.

One crutial step is the first and the last equality (without surjectivity assumption we'll only get $\langle f(a) \rangle \supset f(\langle a \rangle)$ and similarly for the other one). See a post HERE, be careful see HERE!

3.31.4 Absolutely flat local ring, non-unit...

Assume local ring $(A, \mathfrak{m}, k)$ is absolutely flat. Didn't work out... See "A Term of Commutative Algebra" Ex 10.26. Take a non-unit $x \in A$. Since $\langle x \rangle = \langle x^2 \rangle$ we have $x = ax^2$ for some $a \in A$. This implies $x(ax - 1) = 0$, while x is a non-unit so $ax - 1 \neq 0$. So we know a non-unit must be a zero-divisor.

Now let's consider the ideal $\langle x \rangle$, it's not the whole ring given x is assumed to be a non-unit. So it must lies inside the maximal ideal of the local ring, i.e.

$$\langle x \rangle \subset \mathfrak{m}$$

Recall the characterisation of Jacobson radical, we know that $ax - 1$ is a unit in A. Therefore we have

$$x = (ax - 1)^{-1}(ax - 1)x = 0 \ \Rightarrow \ \mathfrak{m} = 0.$$

And this proves that A is a field.

??? In fact, we can say more about the converse. Any field is a local ring and absolutely flat.

3.31.5

Apart from this problem, we found a post HERE, which is about "nilradical of an absolutely flat ring is trivial".

4 Chapter 3

4.1 Example

Warning: localisation might not produce a local ring, for example see HERE. If we localise at a prime ideal $\mathfrak{p}$ then it's local. See HERE.

4.2 Proposition 3.7

Localisation commutes with tensor product. See HERE for a pure tensor manipulation. See HERE for a post, using universal property of tensor product.

4.3 Proposition 3.11

For i), I have some doubts in the last step of HERE.

For iv), in the proof we have *"*$S^{-1}A/S^{-1}\mathfrak{p} \simeq \overline{S}^{-1}(A/\mathfrak{p})$*"*. In fact we if we apply exact functor $S^{-1}(-)$ to first isomorphism exact sequence we'll get it, with S^{-1} but not $\overline{S}^{-1}$???

See a post discussing this HERE. In the second answer, it used Universal Property of Localisation to prove the isomorphism...

For v), about extension of ideal, which really should be discussed in Chap 1. HERE is a post, discussion the definition.

For a more detailed treatment, see The Rising Sea 1.3.F. EXERCISE.

4.4 Proposition 3.14

One might wonder if the condition of finitely-generated is omitted what will happen?

Let $A = k[x]$ where k is a field, and define

$$M = \bigoplus_{i=2}^{\infty} \frac{k[x]}{x^i} = \frac{k[x]}{x^2} \oplus \frac{k[x]}{x^3} \oplus \cdots .$$

It's not finitely-generated as $k[x]-$module.

Let $S = \{1, x, x^2, ...\}$.

Notice that by definition $S^{-1}M = 0$ given that for any $m \in M$ we can always choose a large enough $s \in S$ such that $sm = 0$.

On the other hand we have $\mathrm{Ann}(M) = 0$, which gives us

$$0 = S^{-1}(\mathrm{Ann}(M)) \neq \mathrm{Ann}(S^{-1}M) = \mathrm{Ann}(0) = k[x]$$

4.5 Proposition 3.16

Question on the last equality. Purely based on the contents we knew

$$\mathfrak{q}^c \supset \mathfrak{p}^{ec} = \mathfrak{p}.$$

Denote the ring homomorphism from $A \to B$ as f. Conversely, we proceed by a contrapositive arguement. Suppose we have $x \in \mathfrak{q}^c \setminus \mathfrak{p}$, then $f(x) \in \mathfrak{q}$ and $f(x) \in f(A - \mathfrak{p}) = S$ by definition. But it was ensured that $\mathfrak{q} \cap S = \emptyset$, contradiction.

4.6 Side Notes

There are some examples from Reid's Undergraduate C.A, see P41.

4.7 Ex 1.

$\Leftarrow$: Assume the existence of an element $s \in S$ such that $sM = 0$. For any element $m_0/s_0 \in S^{-1}M$, we claim

$$m_0/s_0 = 0/1 \ \text{ given } \ s(m_0 \cdot 1 - s_0 \cdot 0) = sm_0 = 0.$$

Since every element is zero, hence $S^{-1}M = 0$.

$\Rightarrow$: Since M is finitely-generated, let $m_1, ..., m_n$ be its generators for some $n \in \mathbb{Z}$. For each m_i, there's some $s_i \in S$ such that $s_i m_i = 0$ by assumption. Now we define

$$s = \prod_{i=1}^{n} s_i$$

and claim that $sM = 0$. This because for any $m \in M$, we can express it as $a_1 m_1 + \cdots + a_n m_n = m \in M$ for some coefficients $a_i \in A$. Furthermore, we notice

$$sm = s(a_1 m_1 + \cdots + a_n m_n) = a_1 s m_1 + \cdots a_n s m_n = 0,$$

which proves $sM = 0$.

✍ "finitely-generated A-module M" is important? Any counterexample???

4.8 Ex 2.

We recall the charactersation of Jacobson radical in a ring. For any element $x \in \text{Jac}()$

4.9 Ex 3.

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[dr, "g_1"'] & S^{-1}A \arrow[r, "f_2"] \arrow[d, "f_3"'] & U^{-1}(S^{-1}A) \arrow[dl, dashed, "\exists!\phi"', bend left] \\
& (ST)^{-1}A \arrow[ur, dashed, "\exists!\psi"', bend right] &
\end{tikzcd}$$

Firstly, we claim that $f_2 \circ f_1$ will send every element of ST to a unit in $U^{-1}(S^{-1}A)$. Hence by universal property there's a unique map

$$\psi : (ST)^{-1}A \to U^{-1}(S^{-1}A).$$

Proof for the first claim. Element of ST is like st for some $s \in S$ and $t \in T$. Under f_1, $s/1$ is a unit in $S^{-1}A$ and $t/1 \in U$. Furthermore, under f_2, element $f_2(t/1)$ will be a unit in $U^{-1}(S^{-1}A)$. And we have $f_2 \circ f_1(st) = \frac{st/1}{1}$ is a unit for there exists

$$\frac{1/s}{t/1} \in U^{-1}(S^{-1}A) \text{ such that } \frac{st/1}{1} \cdot \frac{1/s}{t/1} = 1.$$

Secondly, we claim that f_3 will send every element of U to unit in $(ST)^{-1}A$, which give rise to the unique existence of

$$\phi : U^{-1}(S^{-1}A) \to (ST)^{-1}A.$$

Proof for the second claim. Note that the map

$$f_3 : S^{-1}A \to (ST)^{-1}A$$

is a ring inclusion. Clearly $t/1 \in U$ will be a unit under f_3 in $(ST)^{-1}A$ given $1/t \in (ST)^{-1}A$ and $t/1 \cdot 1/t = 1$.

Both maps are unique and making the diagram commute, hence they're isomorphisms.

See Stack Project Prop 10.9.10 HERE. See this page, for a discussion between isomorphisms of localisations as either ring or module...

4.10 Ex 4.

My approach was to produce two unique morphisms by universal property of $S^{-1}B$ and $T^{-1}B$, respectively.

$$\begin{tikzcd} & S^{-1}B \arrow[dd, dashed, bend left, "\exists!"] \\ B \arrow[ur, "b\mapsto b/1"] \arrow[dr, "b\mapsto b/f(1)"'] & \\ & T^{-1}B \arrow[uu, dashed, bend left, "\exists!"] \end{tikzcd}$$

But it seems to be difficult to write in valid proof....

See a post discuss a more straight-forward approach HERE, and HERE. In the first post, there's an obvious map to define and module homomorphism together with surjectivity easy to check. For the injectivity, we can check...

4.11 Ex 5.

I encountered a problem, wishing to prove $x/1 \neq 0/1$ in the localised ring.

A potentially incorrect solution

We wish to prove that

$$\mathrm{Nil}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, A} \mathfrak{p} = 0.$$

For the localised ring $A_{\mathfrak{p}}$, by Prop 3.11 (i) and Coro 3.13, we know every ideals are extended ideals and

$$0 = \mathrm{Nil}(A_{\mathfrak{p}}) = \bigcap_{\mathfrak{q}\in\mathrm{Spec}\,A,\ \mathfrak{q}\subset\mathfrak{p}} \mathfrak{q}^e \supset \left(\bigcap_{\mathfrak{q}\in\mathrm{Spec}\,A,\ \mathfrak{q}\subset\mathfrak{p}} \mathfrak{q}\right)^e$$

where the last inclusion is given by Exercise 1.18 on Page 10 of Atiyah's textbook. But the only way that an extended ideal to be zero is the ideal

$$\bigcap_{\mathfrak{q}\in\mathrm{Spec}\,A,\ \mathfrak{q}\subset\mathfrak{p}} \mathfrak{q} = 0.$$

Therefore we have

$$\mathrm{Nil}(A) = \bigcap_{\mathfrak{p}\in\mathrm{Spec}\,A} \mathfrak{p} \subset \bigcap_{\mathfrak{q}\in\mathrm{Spec}\,A,\ \mathfrak{q}\subset\mathfrak{p}} \mathfrak{q} = 0.$$

See a post HERE. For examples...

See a post HERE. It used Corollary 3.12.

"Being an integral domain is not a local property." See a post HERE.

Possible counterexamples including $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{C}[x,y]/\langle xy\rangle$, etc. But I'm not sure about the what will be the localisation... One very simple way to construct a counterexample to take product of two integral domains. Say $R = \mathbb{Q}\times\mathbb{Q}$. According to Milne's Notes on Algebraic Number Theory Page 15, "Ideals in product of rings", we know that prime ideals in R is of the form $\mathfrak{q} = \mathfrak{p}\times\mathbb{Q}$ or $\mathbb{Q}\times\mathfrak{p}$ where $\mathfrak{p}\subset\mathbb{Q}$ is a prime ideal. We can see that $R_{\mathfrak{q}}$ is always an integral domain. However, R is not an integral domain for $(1,0),(0,1)$ are nontrivial zero-divisors.

4.12 Ex 6.

Apply Zorn's Lemma. Since $A\neq 0$, hence it has identity. Then $\{1\}$ would be a multiplicatively closed subset. Hence $\sum$ is nonempty. Each *chain* is bounded above by $A\setminus\{\text{zerodivisors}\}$. ✖!!! This is a wrong way to apply Zorn. Because I was confused how to find the correct upper bound?...

The set $\sum$ is nonempty and ordered by inclusion as illustrated above. Fix a chain, the upper bound is the union of all multiplicatively close subsets

$$S = \bigcup S_i$$

where S_i are all multiplicatively closed subsets in the chain. Clearly, union of multiplicatively closed subsets S is again multiplicatively closed and crutially we know $0\notin S$. Therefore each chain is bounded above, then we can apply Zorn's Lemma to conclude the existence of a maximal element $S\in\sum$. *This proof is exactly the same as proper ideal lies in some maximal ideal...*

$\Rightarrow$: Use contrapositive for "minimal"(Is this legal??? In fact we can contradicts either three properties including minimal, prime, or ideal...). Suppose on the contrary that we have a prime ideal $\mathfrak{p} \subsetneq A - S$, then $A - \mathfrak{p} \in \sum$ is a multiplicatively closed subset such that $0 \notin A - \mathfrak{p}$. Clearly we have $A - \mathfrak{p} \supsetneq S$, and this contradicts the assumption that S is the maximal element in $\sum$.

$\Leftarrow$: Assume $A - S$ is a minimal prime ideal. Hence S is a multiplicatively closed set such that $0 \notin S$. Furthermore, multiplicatively closed set S is maximal given

4.12.1

One concern was about contrapositive, is that legal? Another problem I encountered was taking complement of multiplicatively closed subset. It will not necessarily give us an ideal. For example, consider $\mathbb{Q}[x]$ with a multiplicativly closed subset $S_0 = \{1, x, x^2, ...\}$, the complement isn't even closed under addition for we have $x + 1/2, x - 1/2$. Or another even simpler example, take $S_1 = \{1\}$ in $\mathbb{Z}$.

However, "complement of maximal multiplicatively closed subset *is* a prime ideal." See this post HERE.

✔

4.12.2

A post HERE. A post HERE, see this post "conversely".

4.12.3

Note that on "A Term..." there's another approach, requiring saturation...

4.13 Ex 7.

4.13.1 (i)

$\Leftarrow$: Assume $A - S$ is a union of prime ideals. Then S is a intersection of multiplicatively closed subset, which is again a multiplicativly closed subset. We only need to prove that

$$xy \in S \;\Rightarrow\; x \in S \text{ and } y \in S.$$

This is true since $xy \notin \mathfrak{p}_i$ for each prime ideals implies $x \notin \mathfrak{p}_i$ and $y \notin \mathfrak{p}_i$. While $\mathfrak{p}_i$ is arbitrary, we can conclude that $x \in S$ and $y \in S$.

Verified at [1] Exercise 3.24.

✔

$\Rightarrow$: Assume S is saturated.

We note that $A - S$ contains no unit, so we can cover it by $\{\mathfrak{p}_i\}_{i \in I}$ for some index I in which every $\mathfrak{p}_i$ is a maximal ideal. Now we need to prove $\mathfrak{p}_i \cap (A - S)$ is an ideal and prime.

The above approach is potentially wrong... I couldn't work out. Because I couldn't prove *is* an additive Abelian group. See a post HERE.

4.13.2

Potentially wrong... It turned out I choose the wrong covering. See a post HERE. It has some slight problems...

Start with fact that any $x \in A - S$ is a non-unit, under the ring homomorphism $f : A \to S^{-1}A$ the element $f(x)$ will still be a non-unit (???). Hence we can find a maximal ideal $\mathfrak{q} \subset S^{-1}A$ contains $f(x)$. In particularly, it's prime. By correspondence theorem we know its preimage $f^{-1}(\mathfrak{q})$ will have trivial intersection with S. Hence

$$x \in f^{-1}(\mathfrak{q}) \subset A - S$$

and this proves that $A - S$ is a union of prime ideals.

I have some doubts towards the above argument. Ring homo will map unit to unit. But ring homo won't necessarily map non-unit to non-unit, for example take inclusion map $\mathbb{Z} \to \mathbb{Q}$. But is the argument right in this specific case?

4.13.3 a verified approach

According to a post HERE. One important claim is that we can find a maximal ideal containing $\langle x \rangle$ and disjoint from S. This is possible, see Eisenbud's "Commutative Algebra...", Page 70, construction of a prime ideal.

Or see "A Term..." Proposition 3.9, Exercise 3.24. See HERE.

4.13.4 Examples

Assumption of S being saturated is necessary. Note that $\{1\} \subset \mathbb{Z}$ is a multiplicatively closed subset that's not *saturated*, whereas $\{-1, 1\}$ is saturated. And clearly $\mathbb{Z} \setminus \{1\}$ isn't a union of prime ideals for we have $1, -1$ inside that could never be covered by prime ideals. Indeed, we note $\mathbb{Z} \setminus \{-1, 1\}$ is a union of primes. (?)

4.13.5 (ii)

Let $\sum$ denotes the set of all *saturated* multiplicatively closed subset in A that containing S. Note that in each *chain* $\{S_i\}_{i \in I}$ where I is some index set, the intersection

$$A := \bigcap_{i \in I} S_i$$

is again saturated multiplicatively closed subset: clearly it contains 1; multiplicatively closed for each S_i is so; and saturated for the similar reason. Also it must contain S, so $A \in \sum$.

Hence each chain in $\sum$ is bounded below, Zorn's Lemma gives us the existence of minimal element. Suppose we have two distinct minimal element A_1, A_2, i.e. we have $A_1 - A_2 \neq \emptyset, A_2 - A_1 \neq \emptyset$. Pick $x \in A_1 - A_2$ and $y \in A_2 - A_1$. Then $xy \in A_1 \cap A_2$, while A_1 is saturated then $x \in A_2$, contradiction! Therefore there's only one minimal element, i.e. the smallest one, in $\sum$ and we can denote it as $\overline{S}$.

Let the set $\{\mathfrak{Q}_i\}_{i\in J}$ denotes the set of all prime ideals in A that do not meet S for some index set J. According to (i),

$$A - \bigcup_{i\in J} \mathfrak{Q}_i$$

is a saturated multiplicatively closed subset. Clearly we have

$$S \subset A - \bigcup_{i\in J} \mathfrak{Q}_i,$$

while $\overline{S}$ is smallest saturated one we know

$$\overline{S} \subset A - \bigcup_{i\in J} \mathfrak{Q}_i.$$

Also we must have the inclusion in converse direction. Otherwise, say $A - \bigcup_{i\in J} \mathfrak{Q}_i \subsetneq \overline{S}$, we'll get a strictly larger set of prime ideals that don't meet S indexed by J_0. This is because $\overline{S}$ is saturated we can apply (i) above. And this is a contradiction for definition of J. Hence we can conclude

$$\overline{S} = A - \bigcup_{i\in J} \mathfrak{Q}_i.$$

See a verified post HERE. ✔

4.13.6 Compute a specifc example, wrong

For the case where $\mathfrak{a} = A$, we have $1 + \mathfrak{a} = A$ and $\overline{S} = A$.

Now we can assume $1 \notin \mathfrak{a}$. Note that $\langle 1 + \mathfrak{a},\ \mathfrak{a}\rangle = \langle 1\rangle = A$. So the set of all prime ideals in A that do not meet S lies strictly inside $\mathfrak{a}$. × Wrong

For any prime ideal $\mathfrak{p} \subsetneq \mathfrak{a}$, we claim that $\mathfrak{p} \cap (1 + \mathfrak{a}) = \emptyset$. Suppose we have

$$1 + a = p$$

for $a \in \mathfrak{a}$ and $p \in \mathfrak{p} \subset \mathfrak{a}$, then $1 \in \mathfrak{a}$, contradiction. Therefore we know that any prime ideals lies strictly inside $\mathfrak{a}$ are exactly the set of prime ideals that do not meet S. Hence we can conclude

$$\overline{S} = \bigcup_{\mathfrak{p}\in \operatorname{Spec} A,\ \mathfrak{p}\subsetneq \mathfrak{a}} \mathfrak{p}.$$

Above argument is wrong! I made a mistake in the original proof ×

??? Recall the localisation, we have

$$\overline{S} = A - \left(\bigcup_{\mathfrak{q}\in \operatorname{Spec}(S^{-1}A)} f^{-1}(\mathfrak{q}) \right)$$

where $f : A \to S^{-1}A$.

4.13.7 Compute a specifc example, verified

See HERE. And [1] Exercise 3.25.
The correct approach is to consider all primes that containing $\mathfrak{a}$.

$$\overline{S} = A - \left(\bigcup_{\mathfrak{p} \in \operatorname{Spec}(A),\ \mathfrak{a} \subset \mathfrak{p}} \right) \mathfrak{p}.$$

5 Chapter 4

5.1 Example 3) Page 51

My question was about that quotient ring...
See a post HERE.
See a post dicussing the quotient ring $k[x,y,z]/\langle xy - z^2 \rangle$ HERE.

5.2 Prop 4.2

For the original proof. Note that $A/\mathfrak{a}$ is a local ring, then elements in $A/\mathfrak{a}$ is either a unit or a nonunit. In case of a nonunit, it lies in the nilradical (intersection of all prime ideals).

Another proof is HERE.

6 Chapter 5

6.1 Example

7 Chapter 6

7.1 Corollary 6.4

One might wonder if $\bigoplus_{i\in I} M_i$ is also Noetherian given the index set I is not necessarily finite with each M_i being Noetherian. Consider this example 3) on Page 76

$$k[x_1, x_2, ...] = \langle x_1\rangle \oplus \langle x_2\rangle \oplus \cdots .$$

Not sure if the above example works well as a *counterexample*???

See a post HERE. Once we allow infinte index set, at least countably infinite, we'll get $M_1 \subsetneq M_1 \oplus M_2 \subsetneq ...$

7.2 Proposition 6.7

We cannot interpret this inclusion set-theorically? Any element $n_0 + N_i$ must also be a.... I cannot resolve the coset $+N_i$ problem...

The inclusion $N_{i-1}/N_i \subset M_{i-1}/M_i$ is given by noticing N_i is *exactly* kernel of the map (by definition the kernel is exactly $M_i \cap N = N_i$)

$$N_{i-1} \hookrightarrow M_{i-1} \xrightarrow{\pi} M_{i-1}/M_i$$

And this induces an injective from $N_{i-1}/N_i \to M_{i-1}/M_i$, for which we can interpret it as set inclusion.

7.3 Ex 1.

7.3.1 (i)

Since M is Noetherian, then it's finitely generated in particular. Then we can apply one form of Nakayama's Lemma. One approach is to use induction on number of generators of M ???

I didn't get the approach suggested by the hint HERE. One crutial observation is the ascending chain of modules

$$\mathrm{Ker}(u) \subset \mathrm{Ker}(u^2) \subset \mathrm{Ker}(u^3) \subset \cdots$$

7.3.2 (ii)

Couldn't understand hint from both the textbook and the post HERE.

A post HERE.

According to Artinian assumption there exists $n \in \mathbb{Z}$ such that

$$\mathrm{Im}(u) \supset \mathrm{Im}(u^2) \supset \cdots \supset \mathrm{Im}(u^n) = \mathrm{Im}(u^{n+1}) = \cdots$$

And we only need to prove the claim $M = \mathrm{Im}(u^n) + \mathrm{Ker}(u^n)$...

Another cute argument is to consider any $m \in M$. There exists $m' \in M$ such that

$$u^n(m) = u^{n+1}(m').$$

By injectivity of u we must have $u^{n-1}(m) = u^n(m')$. Inductively, we know $m = u(m')$. Hence $\mathrm{Im}(u) = M$.

7.3.3

For (ii), assumption of Artinian is necessary. Since for non-Artinian $\mathbb{Z}$-module $\mathbb{Z}$, we have $\times 2 : \mathbb{Z} \to \mathbb{Z}$ injective but not surjective.

7.4 Ex 2.

Apply argument of Prop 6.2 in Atiyah's [2]. Verified at [1] Exercise 16.32. ✔

8 Chapter 10

8.1 Remarks on Page 101

The reason why G is Hausdorff. One equivalent condition for G being Hausdorff is the diagonal element in $G \times G$ is closed. Because for point (x, y) that's off-diagonal, i.e. $x \neq y$, we can always find $U \times V$ (by applying Hausdorff in G) that's open in $G \times G$ with trivial intersection to the diagonal. See a post HERE.

8.2 Lemma 10.1

8.2.1 Details of (i)

could be found HERE and HERE.

8.2.2 Details of (ii)

Details could be found at the same link HERE, the first answer.

8.2.3 Details for (iii)

For each element $g + H \in G/H$, it's the preimage of the continuous map

$$g + H \xrightarrow{-g} H$$

While H is closed, hence $g + H$ is closed as expected. In fact, this translation map is a homeomorphism, see Atiyah's comment before Lemma 10.1.

8.2.4 Details for (iii), without incurring (ii)

???potentially *incorrect*!

Conversely, assume $H = 0$. Pick distinct points $x_1, x_2 \in G$, then we have $x_1 - x_2 \neq 0$ in G. While $H = \cap_{i \in I} U_i$ where each U_i are all 0-neighborhood. Hence we can find a 0-neighborhood U_1 such that

$$x_1 - x_2 \notin U \;\Rightarrow\; x_1 \notin x_2 + U_1.$$

Similarly, we can find another U_2 such that

$$x_2 - x_1 \notin U \;\Rightarrow\; x_2 \notin x_1 + U_2.$$

Therefore $x_1 + U_2 \setminus U_1$ and $x_2 + U_1 \setminus U_2$ are two disjoint neighborhoods that contains x_1, x_2 respectively, which proves G is Hausdorff.

8.2.5 Details for (iv)

$\Rightarrow$: Assume G is Hausdorff but $H \neq 0$... Pick an element $y \in H$... See a post HERE.

Part II

A Term of Commutative Algebra

9 Chapter 3

9.1 Definition 3.17

We can easily generalise the argument in 2.8 to conclude the existence of minimal prime over a given ideal.

9.2 Ex 3.16

Notice that in general intersection of two prime ideals are not necessarily prime. For example, take intersection of $\langle 2\rangle, \langle 3\rangle \subset \mathbb{Z}$ will give us $\langle 6\rangle$, which isn't a prime ideal. However, taking intersection in a *chain* of prime ideals will give us, again, a prime ideal.

10 Chapter 13

10.1 Ex 13.41

Let R be a ring, $X := \operatorname{Spec}(R)$, and U an open subset. Show U is quasi-compact if and only if $X - U = V(\mathfrak{a})$ where $\mathfrak{a}$ is finitely generated.

One equivalent statement for the latter part is "U is a finite union of basic open sets". See Atiyah's Exercise 17,2.17, of Chapter 1.

Proof. Assume U is quasi-compact. Let $\{D(f_i)\}_{i\in I}$ be an open cover for U where I is an index set. Given U is quasi-compact, there exists a *finite* index set $J \subset I$ such that $\{D(f_i)\}_{i\in J}$ is a subcover. Since U is open, then there exists a closed subset

$$V(f) = X - U$$

for some $f \in R$ and we have

$$U = \bigcup_{i\in I} D(f_i) = \bigcup_{i\in J} D(f_i) \ \Rightarrow\ V(f) = \bigcap_{i\in J} V(f_i) = V\left(\bigcup_{i\in J}\langle f_i\rangle\right).$$

Now we define $\mathfrak{a} = \bigcup_{i\in J}\langle f_i\rangle$, which is finitely-generated as desired.

Conversely, assume we have $U = X - \mathbf{V}(\mathfrak{a})$ for a finitely-generated ideal $\mathfrak{a} = \langle f_1, ..., f_m\rangle \subset R$ where $m \in \mathbb{Z}_+$. And this implies

$$U = X - \mathbf{V}(\mathfrak{a}) = \bigcup_{1\le i\le m} D(f_i).$$

While each $D(f_i)$ is quasi-compact by (vi) of 2.17, we know U is quasi-compact. □

11 References

[1] Allen Altman and Steven Kleiman. *A Term of Commutative Algebra*. 04 2021.

[2] M.F. Atiyah and I.G. MacDonald. *Introduction To Commutative Algebra*. Addison-Wesley series in mathematics. Avalon Publishing, 1994.

[3] David Steven Dummit and Richard M Foote. *Abstract algebra*, volume 3. Wiley Hoboken, 2004.

[4] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6. Oxford Graduate Texts in Mathe, 2002.