Weekly Reading Summary

November 13, 2023

This document presents a weekly reading summary of the book *Introduction to the Mathematics of Medical Imaging*. The purpose of this summary is to distill the main concepts and insights from the book.

Contents

1 Week 1. The Space of Lines in the Plane

Instead of using a Cartesian coordinate system, we adopt a "point normal" parameterization for an arbitrary line in the plane, introduced in **Section 1.2**. Recall a line in the plane is a set of points that satisfies and equation of the form

$$ax + by = c,$$

where $a^2 + b^2 \neq 0$. We can rewrite this equation as

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y = \frac{c}{\sqrt{a^2+b^2}},$$

which represents the same line. The coefficients $(\frac{a}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}})$ define a point ω on the unit circle S^1 and $\frac{c}{\sqrt{a^2+b^2}}$ can be any number.

This motivates us to parameterize a line in the plane by a pair of a unit vector $\omega(\theta) = (\cos\theta, \sin\theta)$ and a real number t. We call such a line $l_{t,\theta} := l_{t,\omega(\theta)}$ and it is the set of all points (x, y) in $\mathbb{R}^2$ such that

$$x\cos\theta + y\sin\theta = t, \quad \theta \in [0, 2\pi),\ t \in \mathbb{R}.$$

Note that $\omega(\theta)$ is perpendicular to $l_{t,\omega}$. We can verify $l_{t,\omega} = l_{-t,-\omega}$. In fact, the pair (t, ω) specifies an *oriented line.* Define a unit vector

$$\hat{\omega} = (-\sin\theta, \cos\theta)$$

where recall $\omega = (\cos\theta, \sin\theta)$. With such notations, we have a bijection as follows.

Proposition 1. (Proposition 1.2.1.) *There is a bijection between*

$$\{(t,\omega) \ \mid \ t \in \mathbb{R},\ \omega \in \mathbb{S}\} \xrightarrow{\text{bijective}} \{\text{oriented lines in the plane}\}\ .$$

Exercise 1. (Exercise 1.2.1.) *Show that $l_{t,\omega}$ is given parametrically as the set of points*

$$l_{t,\omega} = \{t\omega + s\hat{\omega} : s \in (-\infty, \infty)\}.$$

Answer. Simplify the expression we'll get exactly the same expression. It's a good way to visualise *affine parameter t*. In this equation, it represents precisely the length from origin to the line where the direction is specified by vector ω.

Exercise 2. (Exercise 1.2.2.) *Show that $\hat{\omega}(\theta) = \partial_\theta \omega(\theta)$*

Answer. Differentiate component-wise will work.

2 Week 2. A Basic Model for Tomography

2.1 Beer's Law

We start our exploration of medical imaging, with a mathematical representation of the measurement process used in X-ray tomography. The modeling process commences with a detailed quantitative analysis of how X-rays interact with matter, a phenomenon described by Beer's law, see **Section 3.1**.

Definition 1. In X-ray tomography, we are interested in detecting objects using real-valued funcion defined on $\mathbb{R}^3$. And we define this function as *attenuation coefficient.*

The attenuation coefficient quantifies the tendency of an object to absorb or scatter X-ray of a given energy. For example, bone has a much higher attenuation coefficient than soft tissue.

Definition 2. For radiologists, attenuation coefficient is compared to the attenuation coefficient of water and we define it in terms of a dimensionless quantity, for which we call *Hounsfield unit.* And we define the normalized attenuation coefficient in Hounsfield units as

$$H_{\text{tissue}} := \frac{\mu_{\text{tissue}} - \mu_{\text{water}}}{\mu_{\text{water}}} \times 1000.$$

In practice of measurement, it's difficult to distinguish points where a function is nonzero from points that are "arbitrarily closed" to such points. This motivates us to make a definition as follows:

Definition 3. Let f be a function defined on $\mathbb{R}^n$, a point $\mathbf{x} \in \mathbb{R}^n$ belongs to *support* of f if there's a sequence of points $\langle \mathbf{x}_n \rangle$ such that

$$f(\mathbf{x}_n) \neq 0, \quad \lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}.$$

And we denote the set of all such points as $\operatorname{supp}(f)$.

Usually, X-ray beam is described by a vector valued function $\mathbf{I}(\mathbf{x})$. The direction of $\mathbf{I}$ at $\mathbf{x}$ is the flux at $\mathbf{x}$ and its magnitude, and we denote the intensity of the beam as

$$I(\mathbf{x}) = \|\mathbf{I}(\mathbf{x})\|.$$

The model discussed in the textbook for the interaction of X-rays with matter is phrased in terms of the continuum model and rests on three basic assumptions:

(1) No refraction or diffraction, as X-rays have very high energies.

(2) Monochromatic, waves of X-ray beam are of the same frequency.

(3) **Beer's Law:** the intensity, I of the X-ray beams, satisfies

$$\frac{dI}{ds} = -\mu(x)I$$

where s is the arc-length along the straight-line trajectory of the X-ray beam.

Definition 1. See Example 3.1.5. [**?**] for *isotropic.*

Figure 3.3 gives an example of a failure where we couldn't distinguish objects. However, figure 3.4 gives us a general principle: to distinguish more arrangements of objects we have to make measurements from more directions.

2.2 Shepp-Logan Phantom and its Line Integrals

Understand Figure 3.7 and solve the exercises.

Exercise 3. Think about Exercise 3.2.3.

Exercise 4. Exercise 3.2.3. **Answer:** White bands corresponding to the skull of the highest attenuation. The white band is curved like a wave because the length varies from given different angle θ.

Note: You can skip **Section 3.3** currently!

2.3 Radon Transform

Exercise 5. Think about how to solve 3.4.3.

Definition 2. For simplicity we assume f is a function defined on the plane that is continuous with bounded support. The integral of f along the line $l_{t,\omega}$ is denoted by

$$\mathscr{R}f(t,\omega) = \int_{-\infty}^{\infty} f(s\hat{\omega} + t\omega)ds.$$

The collection of integrals of f along the lines in he plane defines a function on $\mathbb{R} \times \mathbb{S}^1$, called the *Radon transform* of f.

Remark 1. We have following remarks, according to **Section 3.4.**

(1) Properties of Radon transform: linear, monotone, and Rf is an even function.

(2) Radon transform cannot distinguish functions which differ only on a set of measure zero, which is a feature common to any measurement process defined by integrals.

(3) The Radon transform can be defined for a function f whose restriciotn to each line is locally integrable and

$$\int_{-\infty}^{\infty} |f(t\omega + s\hat{\omega})ds| < \infty, \quad \text{for all } (t,\omega) \in \mathbb{R} \times S^1.$$

Functions that satisfy this are in the *natural domain* of the Radon transform.

2.4 Integrable Functions

We briefly recall some definitions in [**?**, Chap 2 & 3]. For a real-valued measurable function f on $\mathbb{R}^n$, we say that f is **Lebesgue integrable** if the non-negative measurable function $|f(x)|$ is integrable, that is, its Lebesgue integral $\int_{\mathbb{R}^n} |f(x)|dx < +\infty$. In fact, the integrable functions form a vector space, where we can define the **norm** of f as

$$\|f\| := \|f\|_{L^1(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |f(x)|dx.$$

The collection of all integrable functions with the above norm gives a (somewhat imprecise) definition of the space $L^1(\mathbb{R}^n)$. For more details, see [**?**, Section 2.2].

We say f is **locally integrable**, if for every ball B in $\mathbb{R}^n$, the function $f(x)\chi_B(x)$ is integrable, where $\chi_B(x)$ is the characteristic function on B. We denote by $L^1_{\text{loc}}(\mathbb{R}^n)$ the space of all locally integrable functions.

The space of square integrable functions on $\mathbb{R}^n$ is denoted by $L^2(\mathbb{R}^n)$. It consists of all mensurable functions f that satisfy

$$\int_{\mathbb{R}^n} |f(x)|^2 dx < +\infty.$$

The resulting $L^2(\mathbb{R}^n)$-norm is defined by

$$\|f\|_{L^2(\mathbb{R}^n)} := (\int_{\mathbb{R}^n} |f(x)|^2 dx)^{1/2}.$$

3 Week 3. Fourier Transform

3.1 The Complex Exponential Function

Definition 3. The logarithm of $z \in \mathbb{C}$ is defined as

$$\log z \ := \ s + i\theta = \log|z| + i\tan^{-1}\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right).$$

One feature of the exponential is that it satisfies an ordinary differential equation

$$\partial_x e^{ix\xi} = i\xi e^{ix\xi}.$$

And we can interpret the formula as saying $e^{ix\xi}$ is an eigenvector with eigenvalue $i\xi$ for the linear operator ∂_x. More details are in **Section 4.1**.

3.2 Fourier Transform for Functions of a Single Variable

Definition 4 (**Fourier Transform**)**.** The Fourier transformation of an L^1-function f that defined on $\mathbb{R}$, is the function $\hat{f}$ defined on $\mathbb{R}$ by the integral

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$

The function f could be reconstructed from $\hat{f}$, by applying the following:

Theorem 1. [Theorem 4.2.1 **Fourier inversion formula**] Suppose that f is an L^1-function such that $\hat{f}$ is also in $L^1(\mathbb{R})$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi. \tag{1}$$

Question. In the proof of 1 the author required an additional assumption that f is continuous. Where did we used such fact during the proof? Is that on the first line of the equality in equation (**4.6**) on Page 95 of [**?**]?

In **Remark 4.2.2.** of [**?**], we introduced some notations.

$$\mathscr{F}(f) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx,$$
$$\mathscr{F}^{-1}(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{ix\xi}d\xi.$$

The proof of Theorem 1. First, we claim that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi = \frac{1}{2\pi}\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\epsilon\xi^2}e^{ix\xi}d\xi.$$

This comes from the dominated convergence theorem, see Theorem 3. Indeed, note that $\hat{f}(\xi)e^{ix\xi}$ converges to $\hat{f}(\xi)e^{-\epsilon\xi^2}e^{ix\xi}$ almost everywhere and

$$|\hat{f}(\xi)e^{-\epsilon\xi^2}e^{ix\xi}| \le |\hat{f}(\xi)|, \quad \text{for } \epsilon \ge 0.$$

By Theorem 3, we have $\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\epsilon\xi^2}e^{ix\xi}d\xi$ converges to $\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi$ as $\epsilon \to 0$.

Next, we plug in the Fourier transform formula for $\hat{f}$ to have

$$\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\epsilon\xi^2}e^{ix\xi}d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{-\epsilon\xi^2}e^{i(x-y)\xi}dyd\xi.$$

This iterated integral is absolutely integrable. Then by Fubini's theorem, see Theorem B.8.1, we can interchange the order of the integration to have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{-\epsilon\xi^2}e^{i(x-y)\xi}dyd\xi = \int_{-\infty}^{\infty} f(y)(\int_{-\infty}^{\infty} e^{-\epsilon\xi^2}e^{i(x-y)\xi}d\xi)dy.$$

By **Example 4.2.4**, we have

$$\mathscr{F}(e^{-\epsilon\xi^2})(x-y) = \sqrt{\frac{\pi}{\epsilon}}e^{-\frac{(x-y)^2}{4\epsilon}}.$$

This implies that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi &= \frac{1}{2\sqrt{\pi\epsilon}}\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} f(y)e^{-\frac{(x-y)^2}{4\epsilon}}dy \\ &= \frac{1}{\sqrt{\pi}}\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} f(x-2\epsilon t)e^{-\frac{-t^2}{4\epsilon}}dt,\end{aligned}$$

where to get the last equality we make substitution $t = (x-y)/(2\sqrt{\epsilon})$. Again by the dominated convergence theorem (note that f is L^1), the last integral converges to

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x)e^{-\frac{-t^2}{4\epsilon}}dt = f(x).$$

$\square$

Here's an important example 4.2.2. in [**?**].

Example 1. Define the function

Example 2 (Example 4.2.4)**.** The Gaussian, e^{-x^2}, is a function of considerable importance in image processing and mathematics. Its Fourier transform is still a Gaussian function

$$\mathcal{F}(e^{-x^2})(\xi) = \int_{-\infty}^{\infty} e^{-x^2}e^{-ix\xi}dx = \sqrt{\pi}e^{-\xi^2/4}.$$

The proof is based on complex contour integral, see Section 4.2.3.

4 Week 4 Dominated Convergence Theorem

In this week we read some definitions and results in Stein's textbook [**?**].

4.1 Measurable Functions

Theorem 2 ([**?**, Chapter 1, Theorem 1.4 & Chapter 2, Theorem 1.13])**.** Every open subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable union of *almost disjoint* closed cubes.

Exterior measure m_*, which attempts to describe the volume of a set E by approximating it from the outside, assigns to any subset of $\mathbb{R}^d$ a first notion of size.

Properties of Exterior Measure

4.2 Convergence Theorem

Theorem 3 ([**?**, Chapter 2, Theorem 1.13])**.** Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. x, as n tends to infinity. If $|f_n(x)| \leq g(x)$, where g is absolutely integrable, then

$$\int |f_n - f| \to 0 \quad \text{as} \quad n \to \infty,$$

and consequently

$$\int f_n \to \int f \quad \text{as} \quad n \to \infty.$$

In this case, we can first integrate and then take the limit, which will give us $\int f$.

4.3 Some Notations

The limit inferior and limit superior of a sequence x_n are defined by

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} (\inf_{m\geq n} x_m), \qquad \limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{m\geq n} x_m).$$

If $\liminf_{n\to\infty} x_n$ exists, then it is the largest real number a such that for any $\epsilon > 0$, there exists an integer N satisfying $x_n > a - \epsilon$ for any $n > N$. Only finitely many elements of the sequence are less than $a - \epsilon$. Similarly, if $\limsup_{n\to\infty} x_n$ exists, then it is the smallest real number b such that for any $\epsilon > 0$, there exists an integer N satisfying $x_n < b + \epsilon$ for any $n > N$. Only finitely many elements of the sequence are greater than $b + \epsilon$.

5 Week 5. Properties of Fourier Transform

As is discussed below **Remark 4.2.2,** the operation performed to recover f from $\hat{f}$ is almost the same as the operation performed to obtain $\hat{f}$ from f, if we compare the Fourier transform with the inverse Fourier transform. Indeed, if we define

$$f_r(x) := f(-x),$$

then we have

$$\mathcal{F}^{-1}(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{ix\cdot\xi}d\xi = \frac{1}{2\pi}\hat{f}(-x) = \frac{1}{2\pi}\mathcal{F}(f_r).$$

This illustrates some "symmetry" between Fourier transform and its inverse and accounts for many of the Fourier transform's properties.

5.1 Regularity and Decay

It is a general principle that the regularity properties of a function f on $\mathbb{R}^n$ are reflected in the decay properties of its Fourier transform $\hat{f}$ and similarly, the regularity of $\hat{f}$ is a reflection of the decay properties of f.

Theorem 4 (Theorem 4.4.2 **The Riemann-Lebesgue Lemma**). *If f is an L^1-function, then its Fourier transform $\hat{f}$ is a continuous function that goes to zero at infinity. That is, for $\eta \in \mathbb{R}$,*

$$\lim_{\xi\to\eta}\hat{f}(\xi) = \hat{f}(\eta) \quad \textit{and} \quad \lim_{\xi\to\pm\infty}\hat{f}(\xi) = 0.$$

Proof. First, we prove $\hat{f}(\xi)$ is continuously (actually it is uniformly continuous). For this purpose, let $h > 0$ be small and we compute

$$|\hat{f}(\xi + h) - \hat{f}(\xi)| =$$

□

Definition 5 (Definition 4.2.2.). For $k \in \mathbb{N}\cup\{0\}$, the set of functions on $\mathbb{R}$ with k continuous derivatives is denoted by $\mathscr{C}(\mathbb{R})$. The set of infinitely differentiable functions is denoted by $\mathscr{C}^\infty(\mathbb{R})$.

Definition 6 (Definition 4.2.3.). A function, f, defined on $\mathbb{R}^d$, decays like $\|\mathbf{x}\|^{-a}$ if there are constant C and R so that

$$|\ f(\mathbf{x})\ | \leq \frac{C}{\|\mathbf{a}\|} \quad \text{for} \quad \|\mathbf{x}\| > R.$$

And we use the notation "$f = \mathcal{O}(\|\mathbf{x}\|^{-a})$ as $\|\mathbf{x}\|$ tends to infinity."

Question. Notation for $f^{[j]}$? **Answer:** Here $f^{[j]}(x)$ denotes the jth derivation of $f(x)$.

5.2 Quantitative Measures of Regularity and Decay

Recall the integration by parts formula: for differentiable function f and g on the interval $[a, b]$, we have

$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx.$$

To use integration by parts in Fourier analysis, we need to consider this formula when $a = -\infty$ and $b = +\infty$. For our purpose, if we assume $fg, f'g, fg'$ are absolutely integrable, then we have

$$\lim_{x\to\pm\infty} fg(x) = 0,$$

and therefore we have

$$\int_{-\infty}^{\infty} f'(x)g(x)dx = -\int_{-\infty}^{\infty} f(x)g'(x)dx. \tag{2}$$

Suppose f has j integrable derivatives, for $j \geq 1$. Then for any $\xi \neq 0$, we can use (2) to obtain a formula that relates $\mathcal{F}(f)$ with $\mathcal{F}(f^{[j]})$:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx = \int_{-\infty}^{\infty} f'(x)\frac{e^{-ix\xi}}{i\xi}dx = \ldots = \int_{-\infty}^{\infty} f^{[j]}(x)\frac{e^{-ix\xi}}{(i\xi)^j}dx.$$

Note that the last equality can be regarded as $\frac{1}{(i\xi)^j}$ multiplied by the Fourier transform of $f^{[j]}(x)$. Thus, we conclude that

$$\mathcal{F}(f) = \frac{1}{(i\xi)^j}\mathcal{F}(f^{[j]})$$

when f has j integrable derivatives.

Proposition 2 (Proposition 4.2.1.)**.** *Let j be a positive integer. If f has j integrable derivatives, then there is a constant C so $\hat{f}$ satisfies the estimate*

$$|\hat{f}(\xi)| \leq \frac{C}{(1+ \mid \xi \mid)^j}.$$

Moreover, for $1 \leq l \leq j$, the Fourier transform of $f^{[l]}$ is given by

$$\widehat{f^{[l]}}(\xi) = (i\xi)^l \hat{f}(\xi).$$

The rate of decay in $\hat{f}$ is also reflected in the smoothness of f.

Example 1 (Example 4.2.2. (**Sinc Function**))**.** See a post HERE computing the last integral.

Example 2 (Example 4.2.3.)**.**

5.3 The Parseval Formula

Definition 7 (Definition 4.2.4.)**.** A complex-valued function f, defined on $\mathbb{R}^n$, is L^2 or *square integrable* if

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^n} \mid f(\mathbf{x}) \mid^2 d\mathbf{x} < \infty.$$

Denote the set of all such functions, with norm defined by $\|\cdot\|_{L^2}$, by $L^2(\mathbb{R}^n)$. And with such norm $L^2(\mathbb{R}^n)$ is a complete, normed vector space. Here the norm on $L^2(\mathbb{R}^n)$ is defined by an inner product,

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}^n} f(\mathbf{x})\overline{g(\mathbf{x})} d\mathbf{x}.$$

And this inner product satisfies Cauchy-Schwarz inequality.

Proposition 3 (Proposition 4.2.4.)**.** *If $f, g \in L^2(\mathbb{R}^n)$, then*

$$\mid \langle f, g \rangle_{L^2} \mid \leq \|f\|_{L^2} \|g\|_{L^2}.$$

The relationship between absolutely integrable functions and square integrable functions is complicated in the sense that neither of them contains the other one. Some pathological examples are provided below.

Example 3 (Example 4.2.7.)**.** The function

$$f(x) = (1+ \mid x \mid)^{-\frac{3}{4}}$$

is not absolutely integrable, but it is square integrable. On the other hand, the funciton

$$g(x) = \frac{\chi_{[-1,1]}(x)}{\sqrt{\mid x \mid}}$$

is absolutely integralble but not square integrable.

See a post HERE.

Proof. It's not absolutely integrable because

$$\int_{\mathbb{R}} |f(x)|\, dx \geq \int_{0<x<\infty} |f(x)|\, dx = \int_{0<x<\infty} \left(\frac{1}{(1+x)^{3/4}} \right) , dx = \infty.$$

While we can compute L^2-norm as

$$\int_{\mathbb{R}} f(x) = \int_{\mathbb{R}} \mid (1+|x|)^{-\frac{3}{4}} \mid^2 = \int_{-\infty}^{0} (1-x)^{-\frac{3}{2}}\, dx + \cdots = 2+2 = 4.$$

□

Exercise 1 (Exercise 4.2.9.)**.** Let f be an L^1-function. Show that $\hat{f}$ is a continuous function. Extra credit: Show that $\hat{f}$ is uniformly continuous on the whole real line.

6 Week 6.

6.1 Fourier Transformation on $L^2(\mathbb{R})$

According to this pathological example 3, we know being absolutely integrable and square integrable won't imply between themselves. They're "parallel", and the present of both conditions will give us a stronger statement, namely Parseval formula. This formula tells us when will the Fourier transformation of a function $\hat{f}$ be square integrable, and it gave an explicit description for L^2-norm of $\hat{f}$.

Question. How about L^1 function? Fourier inversion formula assumed a priori $\hat{f} \in L^1(\mathbb{R})$. Only tool we have seems to be Riemann-Lebesgue Lemma, which states certain nice "decay" property for L^1 functions. But all we can conclude, under condition of Riemann-Lebesgue is that function is continuous and will go to zero at infinity. One naïve hope is that this will be enough to deduce it's in L^1, but it appears to be false? See a post HERE, which redirects us to Chapter 8 of [**?**].

Example 4. There are something in L^2, after Fourier transform, not in $1/x$.

We defined Fourier transformation for L^1- functions. In fact, we can extend Fourier transformation to $L^2(\mathbb{R})$ as follows:

Let $f \in L^2(\mathbb{R})$ and for each real nubmer $R > 0$ define

$$\hat{f}_R(\xi) = \int_{-R}^{R} f(x)e^{-ix\xi}\, dx.$$

Parseval's formula gives us, for $R_1 < R_2$,

$$\|\hat{f}_{R_1} - \hat{f}_{R_2}\|^2_{L^2} = 2\pi \int_{R_1 \le |x| \le R_2} |f(x)|^2\, dx \tag{3}$$

Notice that f being L^2 forces RHS of 3 goes to 0 when $R_1 \to \infty$ and $R_2 \to \infty$. Since otherwise the integral will be infinity, contradicts the definition.

Hence we know sequence $\{\hat{f}_R\}_{R\in\mathbb{R}}$ is Cauchy. While $L^2(\mathbb{R})$ is complete and normed vector space, we can define it's limit inside $L^2(\mathbb{R})$. In summary, we define $\hat{f}$ as this limit. One convention: we call limit of a sequence in L^2-norm a *limit in the mean* and denote it by LIM.

Definition 8 (Defintion 4.2.5.)**.** If f is a function in $L^2(\mathbb{R})$, then its Fourier transform is defined to be

$$\hat{f} = \mathrm{LIM}_{R\to\infty}\, \hat{f}_R.$$

Proposition 4 (Proposition 4.2.5.)**.** *The Fourier transform extends to define a continuous map from $L^2(\mathbb{R})$ to itself. If $f \in L^2(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$$

Question. Why Parseval implies continuity of f?

A consequence of Parseval's formula is uniqueness statement. The slogan is *"A function in L^2 is determined by its Fourier transform."*

One application is to determine whether two functions in $L^2(\mathbb{R})$ are equal, we can compute Fourier transformation of their subtraction. More precisely, it's the following corollary

Corollary 1. If $f \in L^2(\mathbb{R})$ and $\hat{f} = 0$, then $f \equiv 0$.

Proposition 1 (Fourier inversion for $L^2(\mathbb{R})$).

Here we give a summary of basic properties of Fourier transform that hold for integrable function or L^2-functions.

Question. Why not absolutely integrable? Connections between integrable and absolutely integrable. I saw a post HERE.

- `LINEARITY`:
- `SCALING`:
- `TRANSLATION`: Let f_t be the function f shifted by t [i.e., $f_t(x) = f(x-t)$]. The Fourier transform of f_t is given by

$$\begin{aligned}\hat{f}_t(\xi) &= \int_{-\infty}^{\infty} f(x-t)e^{-i\xi x}\,dx \\ &= \int f(y)e^{-i\xi(y+t)}\,dy \\ &= e^{-i\xi t}\hat{f}(\xi).\end{aligned}$$

- `REALITY`: If f is a real-valued function, then its Fourier transform satisfies $\hat{f}(\xi) = \overline{\hat{f}(-\xi)}$. This shows that the Fourier transform of a real-valued function is completely determined by its values for positive (or negative) frequencies. ?
- `EVENNESS`:

Figure 4.6.
Figure 4.7.

6.2 A General Principle in Functional Analysis

As we know *completeness* is such an important property for a normed linear space, we recall defintion of *dense* with a theorem.

Question. Is this saying every normed linear space is complete or just saying it's a property might or might not have?

Whenever we have a dense subspace $S \subset V$ where $(V, \|\cdot\|)$ is a normed linear space, we can use a sequence of points from V to approach the point in S.

Definition 9 (Defintion 4.2.7.)**.** Let $(V, \|\cdot\|)$ be a normed linear space. A subspace S of V is *dense* if for every $\mathbf{v} \in V$ there is a sequence $\{\mathbf{v}_k\}_{k\in\mathbb{N}} \subset S$ such that

$$\lim_{k\to\infty} \|\mathbf{v} - \mathbf{v}_k\| = 0.$$

Here we'll introduce a theorem that gives us a general principle: a bounded linear map, defined on a dense subset, extends to the whole space.

Theorem 5 (Theorem 4.2.4.)**.** *Let $(V_1, \|\cdot\|)$ and $(V_2, \|\cdot\|)$ be normed, linear spaces and assume that V_2 is complete. Suppose that S_1 is a dense subspace of V_1 and A is a linear map from S_1 to V_2. If there exists a constant M such that*

$$\|A\mathbf{v}\|_2 \leq M\|\mathbf{v}\|_1,$$

for all $\mathbf{v}$ in S_1, then A extends to define a linear map from V_1 to V_2, satisfying the same esimate.

Question. How did we define linear space? "Satisfying the same estimate" means we have the above inequality, but $\mathbf{v}$ could be chosen from the whole space V_1?

7 Week 7.

7.1 Tools from Appendix B: Basic Analysis

Theorem 1 (Theorem B.8.1 (Fubini's Theorem))**.** Let f be a function defined on $\mathbb{R}^n$ and let $n = k + l$ for positive integers k and l. If either of the iterated integrals

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^l} |f(\mathbf{w},\mathbf{y})|\, d\mathbf{w}\, d\mathbf{y} \quad \text{or} \quad \int_{\mathbb{R}^l}\int_{\mathbb{R}^k} |f(\mathbf{w},\mathbf{y})|\, d\mathbf{y}\, d\mathbf{w}$$

is finite, then the other is as well. In this case f is integrable over $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^l} |f(\mathbf{w},\mathbf{y})|\, d\mathbf{w}\, d\mathbf{y} = \int_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^l}\int_{\mathbb{R}^k} |f(\mathbf{w},\mathbf{y})|\, d\mathbf{y}\, d\mathbf{w}.$$

Remark 2. Informally speaking, the order of the integrations can be interchanged under the assumption that f is *absolutely* integrable. Because there are examples of functions defined on $\mathbb{R}^2$ so that both iterated integrals

$$\int\int f(x,y)dxdy, \quad \int\int f(x,y)dydx$$

exist but are not equal, with f not integrable on $\mathbb{R}^2$.

7.2 The Heisenberg Uncertainty Principle

Here we study relationships between the $\operatorname{supp} f$ and $\operatorname{supp} \hat{f}$. The simplest such result states that if a function has bounded support, then its Fourier transform cannot.

Proposition 1 (Proposition 4.4.1.)**.** Suppose $\operatorname{supp} f$ is contained in the bounded interval $(-R, R)$. If $\hat{f}$ also has bounded support then $f \equiv 0$.

7.3 The Fourier Transformation for Functions of Several Variables

We make some notation conventions that will be useful for multi-variable functions. We often use lowercase Roman bold letter to denote

$$\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n.$$

And it's customary to use lowercase Greek bold letters for points on the Fourier transform space such as

$$\xi = (\xi_1, ..., \xi_n).$$

Again, we start by defining for functions from $L^1(\mathbb{R}^n)$.

Definition 1 (Definition 4.5.1.)**.** If f belongs to $L^1(\mathbb{R}^n)$, then the Fourier transform, $\hat{f}$ of f, is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\langle \xi, \mathbf{x}\rangle}\, d\mathbf{x} \quad \text{for} \quad \xi \in \mathbb{R}^n.$$

Furthermore, we can express this integral as

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, ..., x_n) e^{-ix_1\xi_1}\, dx_1 \cdots e^{-ix_n\xi_n}\, dx_n.$$

While f is assumed to be absolutely integrable, then Fubini's theorem ensures that we can interchange the order of integration.

Theorem 2 (Theorem 4.5.1 (Fourier Inversion Formula))**.** Suppose that f is an L^1-function defined on $\mathbb{R}^n$. If $\hat{f}$ also belongs to $L^1(\mathbb{R}^n)$, then

$$f(\mathbf{x}) = \frac{1}{[2\pi]^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\mathbf{x}\cdot\xi}\, d\xi.$$

8 Week 8.

8.1 Convolution

Our interest is to study a smooth function of one variable with data contaminated by noise. And we let n be the function that "*models*" the noise. This Noise function n, is typically represented by a rapidly varying function that is *locally of mean zero*.

Definition 2. We say a rapidly varying function is *locally of mean zero* if for any x, and a large enought δ, the average

$$\frac{1}{\delta}\int_{x}^{x+\delta} n(y)\,dy$$

is small compared to the size of n.

Question. How do we define size of n?

Definition 3. The *moving average* of f is defined to be

$$\mathcal{M}_\delta(f)(x) = \frac{1}{\delta}\int_{x}^{x+\delta} f(y)\,dy.$$

???