Weekly Reading Summary

November 14, 2023

This document presents a weekly reading summary of the book *Introduction to the Mathematics of Medical Imaging*. The purpose of this summary is to distill the main concepts and insights from the book.

Contents

1 Week 1. The Space of Lines in the Plane

Instead of using a Cartesian coordinate system, we adopt a "point normal" parameterization for an arbitrary line in the plane, introduced in **Section 1.2**. Recall a line in the plane is a set of points that satisfies and equation of the form

$$ax + by = c,$$

where $a^2 + b^2 \neq 0$. We can rewrite this equation as

$$\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y = \frac{c}{\sqrt{a^2+b^2}},$$

which represents the same line. The coefficients $(\frac{a}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}})$ define a point ω on the unit circle S^1 and $\frac{c}{\sqrt{a^2+b^2}}$ can be any number.

This motivates us to parameterize a line in the plane by a pair of a unit vector $\omega(\theta) = (\cos\theta, \sin\theta)$ and a real number t. We call such a line $l_{t,\theta} := l_{t,\omega(\theta)}$ and it is the set of all points (x, y) in $\mathbb{R}^2$ such that

$$x\cos\theta + y\sin\theta = t, \quad \theta \in [0, 2\pi),\ t \in \mathbb{R}.$$

Note that $\omega(\theta)$ is perpendicular to $l_{t,\omega}$. We can verify $l_{t,\omega} = l_{-t,-\omega}$. In fact, the pair (t, ω) specifies an *oriented line.* Define a unit vector

$$\hat{\omega} = (-\sin\theta, \cos\theta)$$

where recall $\omega = (\cos\theta, \sin\theta)$. With such notations, we have a bijection as follows.

Proposition 1. (Proposition 1.2.1.) *There is a bijection between*

$$\{(t, \omega) \ \mid\ t \in \mathbb{R},\ \omega \in \mathbb{S}\} \xrightarrow{\text{bijective}} \{\text{oriented lines in the plane}\}\ .$$

Exercise 1. (Exercise 1.2.1.) *Show that $l_{t,\omega}$ is given parametrically as the set of points*

$$l_{t,\omega} = \{t\omega + s\hat{\omega} : s \in (-\infty, \infty)\}.$$

Answer. Simplify the expression we'll get exactly the same expression. It's a good way to visualise *affine parameter t*. In this equation, it represents precisely the length from origin to the line where the direction is specified by vector ω.

Exercise 2. (Exercise 1.2.2.) *Show that $\hat{\omega}(\theta) = \partial_\theta \omega(\theta)$*

Answer. Differentiate component-wise will work.

2 Week 2. A Basic Model for Tomography

2.1 Beer's Law

We start our exploration of medical imaging, with a mathematical representation of the measurement process used in X-ray tomography. The modeling process commences with a detailed quantitative analysis of how X-rays interact with matter, a phenomenon described by Beer's law, see **Section 3.1**.

Definition 1. In X-ray tomography, we are interested in detecting objects using real-valued funcion defined on $\mathbb{R}^3$. And we define this function as *attenuation coefficient.*

The attenuation coefficient quantifies the tendency of an object to absorb or scatter X-ray of a given energy. For example, bone has a much higher attenuation coefficient than soft tissue.

Definition 2. For radiologists, attenuation coefficient is compared to the attenuation coefficient of water and we define it in terms of a dimensionless quantity, for which we call *Hounsfield unit.* And we define the normalized attenuation coefficient in Hounsfield units as

$$H_{\text{tissue}} := \frac{\mu_{\text{tissue}} - \mu_{\text{water}}}{\mu_{\text{water}}} \times 1000.$$

In practice of measurement, it's difficult to distinguish points where a function is nonzero from points that are "arbitrarily closed" to such points. This motivates us to make a definition as follows:

Definition 3. Let f be a function defined on $\mathbb{R}^n$, a point $\mathbf{x} \in \mathbb{R}^n$ belongs to *support* of f if there's a sequence of points $\langle \mathbf{x}_n \rangle$ such that

$$f(\mathbf{x}_n) \neq 0, \quad \lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}.$$

And we denote the set of all such points as $\operatorname{supp}(f)$.

Usually, X-ray beam is described by a vector valued function $\mathbf{I}(\mathbf{x})$. The direction of $\mathbf{I}$ at $\mathbf{x}$ is the flux at $\mathbf{x}$ and its magnitude, and we denote the intensity of the beam as

$$I(\mathbf{x}) = \|\mathbf{I}(\mathbf{x})\|.$$

The model discussed in the textbook for the interaction of X-rays with matter is phrased in terms of the continuum model and rests on three basic assumptions:

(1) No refraction or diffraction, as X-rays have very high energies.

(2) Monochromatic, waves of X-ray beam are of the same frequency.

(3) **Beer's Law:** the intensity, I of the X-ray beams, satisfies

$$\frac{dI}{ds} = -\mu(x)I$$

where s is the arc-length along the straight-line trajectory of the X-ray beam.

Definition 1. See Example 3.1.5. [1] for *isotropic.*

Figure 3.3 gives an example of a failure where we couldn't distinguish objects. However, figure 3.4 gives us a general principle: to distinguish more arrangements of objects we have to make measurements from more directions.

2.2 Shepp-Logan Phantom and its Line Integrals

Understand Figure 3.7 and solve the exercises.

Exercise 3. Think about Exercise 3.2.3.

Exercise 4. Exercise 3.2.3. **Answer:** White bands corresponding to the skull of the highest attenuation. The white band is curved like a wave because the length varies from given different angle θ.

Note: You can skip **Section 3.3** currently!

2.3 Radon Transform

Exercise 5. Think about how to solve 3.4.3.

Definition 2. For simplicity we assume f is a function defined on the plane that is continuous with bounded support. The integral of f along the line $l_{t,\omega}$ is denoted by

$$\mathscr{R}f(t,\omega) = \int_{-\infty}^{\infty} f(s\hat{\omega} + t\omega)ds.$$

The collection of integrals of f along the lines in he plane defines a function on $\mathbb{R} \times \mathbb{S}^1$, called the *Radon transform* of f.

Remark 1. We have following remarks, according to **Section 3.4.**

(1) Properties of Radon transform: linear, monotone, and Rf is an even function.

(2) Radon transform cannot distinguish functions which differ only on a set of measure zero, which is a feature common to any measurement process defined by integrals.

(3) The Radon transform can be defined for a function f whose restriciotn to each line is locally integrable and

$$\int_{-\infty}^{\infty} |f(t\omega + s\hat{\omega})ds| < \infty, \quad \text{for all } (t,\omega) \in \mathbb{R} \times S^1.$$

Functions that satisfy this are in the *natural domain* of the Radon transform.

2.4 Integrable Functions

We briefly recall some definitions in [3, Chap 2 & 3]. For a real-valued measurable function f on $\mathbb{R}^n$, we say that f is **Lebesgue integrable** if the non-negative measurable function $|f(x)|$ is integrable, that is, its Lebesgue integral $\int_{\mathbb{R}^n} |f(x)|dx < +\infty$. In fact, the integrable functions form a vector space, where we can define the **norm** of f as

$$\|f\| \coloneqq \|f\|_{L^1(\mathbb{R}^n)} \coloneqq \int_{\mathbb{R}^n} |f(x)|dx.$$

The collection of all integrable functions with the above norm gives a (somewhat imprecise) definition of the space $L^1(\mathbb{R}^n)$. For more details, see [3, Section 2.2].

We say f is **locally integrable**, if for every ball B in $\mathbb{R}^n$, the function $f(x)\chi_B(x)$ is integrable, where $\chi_B(x)$ is the characteristic function on B. We denote by $L^1_{\text{loc}}(\mathbb{R}^n)$ the space of all locally integrable functions.

The space of square integrable functions on $\mathbb{R}^n$ is denoted by $L^2(\mathbb{R}^n)$. It consists of all mensurable functions f that satisfy

$$\int_{\mathbb{R}^n} |f(x)|^2 dx < +\infty.$$

The resulting $L^2(\mathbb{R}^n)$-norm is defined by

$$\|f\|_{L^2(\mathbb{R}^n)} \coloneqq (\int_{\mathbb{R}^n} |f(x)|^2 dx)^{1/2}.$$

3 Week 3. Fourier Transform

3.1 The Complex Exponential Function

Definition 3. The logarithm of $z \in \mathbb{C}$ is defined as

$$\log z \ := \ s + i\theta = \log|z| + i \tan^{-1}\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right).$$

One feature of the exponential is that it satisfies an ordinary differential equation

$$\partial_x e^{ix\xi} = i\xi e^{ix\xi}.$$

And we can interpret the formula as saying $e^{ix\xi}$ is an eigenvector with eigenvalue $i\xi$ for the linear operator ∂_x. More details are in **Section 4.1**.

3.2 Fourier Transform for Functions of a Single Variable

Definition 4 (Fourier Transform). The Fourier transformation of an L^1-function f that defined on $\mathbb{R}$, is the function $\hat{f}$ defined on $\mathbb{R}$ by the integral

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$

The function f could be reconstructed from $\hat{f}$, by applying the following:

Theorem 1. [Theorem 4.2.1 **Fourier inversion formula**] Suppose that f is an L^1-function such that $\hat{f}$ is also in $L^1(\mathbb{R})$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi. \tag{1}$$

Question. In the proof of 1 the author required an additional assumption that f is continuous. Where did we used such fact during the proof? Is that on the first line of the equality in equation (**4.6**) on Page 95 of [1]?

In **Remark 4.2.2.** of [1], we introduced some notations.

$$\mathscr{F}(f) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx,$$
$$\mathscr{F}^{-1}(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{ix\xi}d\xi.$$

The proof of Theorem 1. First, we claim that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi = \frac{1}{2\pi}\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\epsilon\xi^2}e^{ix\xi}d\xi.$$

This comes from the dominated convergence theorem, see Theorem 3. Indeed, note that $\hat{f}(\xi)e^{ix\xi}$ converges to $\hat{f}(\xi)e^{-\epsilon\xi^2}e^{ix\xi}$ almost everywhere and

$$|\hat{f}(\xi)e^{-\epsilon\xi^2}e^{ix\xi}| \le |\hat{f}(\xi)|, \quad \text{for } \epsilon \ge 0.$$

By Theorem 3, we have $\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\epsilon\xi^2}e^{ix\xi}d\xi$ converges to $\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi$ as $\epsilon \to 0$.

Next, we plug in the Fourier transform formula for $\hat{f}$ to have

$$\int_{-\infty}^{\infty} \hat{f}(\xi)e^{-\epsilon\xi^2}e^{ix\xi}d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{-\epsilon\xi^2}e^{i(x-y)\xi}dyd\xi.$$

This iterated integral is absolutely integrable. Then by Fubini's theorem, see Theorem B.8.1, we can interchange the order of the integration to have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{-\epsilon\xi^2}e^{i(x-y)\xi}dyd\xi = \int_{-\infty}^{\infty} f(y)(\int_{-\infty}^{\infty} e^{-\epsilon\xi^2}e^{i(x-y)\xi}d\xi)dy.$$

By **Example 4.2.4**, we have

$$\mathscr{F}(e^{-\epsilon\xi^2})(x-y) = \sqrt{\frac{\pi}{\epsilon}}e^{-\frac{(x-y)^2}{4\epsilon}}.$$

This implies that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi &= \frac{1}{2\sqrt{\pi\epsilon}}\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} f(y)e^{-\frac{(x-y)^2}{4\epsilon}}dy \\ &= \frac{1}{\sqrt{\pi}}\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} f(x-2\epsilon t)e^{-\frac{-t^2}{4\epsilon}}dt,\end{aligned}$$

where to get the last equality we make substitution $t = (x-y)/(2\sqrt{\epsilon})$. Again by the dominated convergence theorem (note that f is L^1), the last integral converges to

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x)e^{-\frac{-t^2}{4\epsilon}}dt = f(x).$$

□

Here's an important example 4.2.2. in [1].

Example 1. Define the function

Example 2 (Example 4.2.4)**.** The Gaussian, e^{-x^2}, is a function of considerable importance in image processing and mathematics. Its Fourier transform is still a Gaussian function

$$\mathcal{F}(e^{-x^2})(\xi) = \int_{-\infty}^{\infty} e^{-x^2}e^{-ix\xi}dx = \sqrt{\pi}e^{-\xi^2/4}.$$

The proof is based on complex contour integral, see Section 4.2.3.

4 Week 4 Dominated Convergence Theorem

In this week we read some definitions and results in Stein's textbook [3].

4.1 Measurable Functions

Theorem 2 ([3, Chapter 1, Theorem 1.4 & Chapter 2, Theorem 1.13])**.** Every open subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable union of *almost disjoint* closed cubes.

Exterior measure m_*, which attempts to describe the volume of a set E by approximating it from the outside, assigns to any subset of $\mathbb{R}^d$ a first notion of size.

Properties of Exterior Measure

4.2 Convergence Theorem

Theorem 3 ([3, Chapter 2, Theorem 1.13])**.** Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. x, as n tends to infinity. If $|f_n(x)| \leq g(x)$, where g is absolutely integrable, then

$$\int |f_n - f| \to 0 \quad \text{as} \quad n \to \infty,$$

and consequently

$$\int f_n \to \int f \quad \text{as} \quad n \to \infty.$$

In this case, we can first integrate and then take the limit, which will give us $\int f$.

4.3 Some Notations

The limit inferior and limit superior of a sequence x_n are defined by

$$\liminf_{n\to\infty} x_n \coloneqq \lim_{n\to\infty} (\inf_{m\geq n} x_m), \qquad \limsup_{n\to\infty} x_n \coloneqq \lim_{n\to\infty} (\sup_{m\geq n} x_m).$$

If $\liminf_{n\to\infty} x_n$ exists, then it is the largest real number a such that for any $\epsilon > 0$, there exists an integer N satisfying $x_n > a - \epsilon$ for any $n > N$. Only finitely many elements of the sequence are less than $a - \epsilon$. Similarly, if $\limsup_{n\to\infty} x_n$ exists, then it is the smallest real number b such that for any $\epsilon > 0$, there exists an integer N satisfying $x_n < b + \epsilon$ for any $n > N$. Only finitely many elements of the sequence are greater than $b + \epsilon$.

5 Week 5. Properties of Fourier Transform

As is discussed below **Remark 4.2.2,** the operation performed to recover f from $\hat{f}$ is almost the same as the operation performed to obtain $\hat{f}$ from f, if we compare the Fourier transform with the inverse Fourier transform. Indeed, if we define

$$f_r(x) \coloneqq f(-x),$$

then we have

$$\mathcal{F}^{-1}(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{ix\cdot\xi}d\xi = \frac{1}{2\pi}\hat{f}(-x) = \frac{1}{2\pi}\mathcal{F}(f_r).$$

This illustrates some "symmetry" between Fourier transform and its inverse and accounts for many of the Fourier transform's properties.

5.1 Regularity and Decay

It is a general principle that the regularity properties of a function f on $\mathbb{R}^n$ are reflected in the decay properties of its Fourier transform $\hat{f}$ and similarly, the regularity of $\hat{f}$ is a reflection of the decay properties of f.

Theorem 4 (Theorem 4.4.2 **The Riemann-Lebesgue Lemma**). *If f is an L^1-function, then its Fourier transform $\hat{f}$ is a continuous function that goes to zero at infinity. That is, for $\eta \in \mathbb{R}$,*

$$\lim_{\xi\to\eta}\hat{f}(\xi) = \hat{f}(\eta) \quad \textit{and} \quad \lim_{\xi\to\pm\infty}\hat{f}(\xi) = 0.$$

Proof. First, we prove $\hat{f}(\xi)$ is continuously (actually it is uniformly continuous). For this purpose, let $h > 0$ be small and we compute

$$|\hat{f}(\xi+h) - \hat{f}(\xi)| =$$

□

Definition 5 (Definition 4.2.2.). For $k \in \mathbb{N}\cup\{0\}$, the set of functions on $\mathbb{R}$ with k continuous derivatives is denoted by $\mathscr{C}(\mathbb{R})$. The set of infinitely differentiable functions is denoted by $\mathscr{C}^\infty(\mathbb{R})$.

Definition 6 (Definition 4.2.3.). A function, f, defined on $\mathbb{R}^d$, decays like $\|\mathbf{x}\|^{-a}$ if there are constant C and R so that

$$| \ f(\mathbf{x}) \ | \leq \frac{C}{\|\mathbf{a}\|} \quad \text{for} \quad \|\mathbf{x}\| > R.$$

And we use the notation "$f = \mathcal{O}(\|\mathbf{x}\|^{-a})$ as $\|\mathbf{x}\|$ tends to infinity."

Question. Notation for $f^{[j]}$? **Answer:** Here $f^{[j]}(x)$ denotes the jth derivation of $f(x)$.

5.2 Quantitative Measures of Regularity and Decay

Recall the integration by parts formula: for differentiable function f and g on the interval $[a,b]$, we have

$$\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)dx.$$

To use integration by parts in Fourier analysis, we need to consider this formula when $a = -\infty$ and $b = +\infty$. For our purpose, if we assume $fg, f'g, fg'$ are absolutely integrable, then we have

$$\lim_{x\to\pm\infty} fg(x) = 0,$$

and therefore we have

$$\int_{-\infty}^{\infty} f'(x)g(x)dx = -\int_{-\infty}^{\infty} f(x)g'(x)dx. \tag{2}$$

Suppose f has j integrable derivatives, for $j \geq 1$. Then for any $\xi \neq 0$, we can use (2) to obtain a formula that relates $\mathcal{F}(f)$ with $\mathcal{F}(f^{[j]})$:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx = \int_{-\infty}^{\infty} f'(x)\frac{e^{-ix\xi}}{i\xi}dx = \ldots = \int_{-\infty}^{\infty} f^{[j]}(x)\frac{e^{-ix\xi}}{(i\xi)^j}dx.$$

Note that the last equality can be regarded as $\frac{1}{(i\xi)^j}$ multiplied by the Fourier transform of $f^{[j]}(x)$. Thus, we conclude that

$$\mathcal{F}(f) = \frac{1}{(i\xi)^j}\mathcal{F}(f^{[j]})$$

when f has j integrable derivatives.

Proposition 2 (Proposition 4.2.1.)**.** *Let j be a positive integer. If f has j integrable derivatives, then there is a constant C so $\hat{f}$ satisfies the estimate*

$$|\hat{f}(\xi)| \leq \frac{C}{(1+ \mid \xi \mid)^j}.$$

Moreover, for $1 \leq l \leq j$, the Fourier transform of $f^{[l]}$ is given by

$$\widehat{f^{[l]}}(\xi) = (i\xi)^l \hat{f}(\xi).$$

The rate of decay in $\hat{f}$ is also reflected in the smoothness of f.

Example 1 (Example 4.2.2. (**Sinc Function**))**.** See a post HERE computing the last integral.

Example 2 (Example 4.2.3.)**.**

5.3 The Parseval Formula

Definition 7 (Definition 4.2.4.)**.** A complex-valued function f, defined on $\mathbb{R}^n$, is L^2 or *square integrable* if

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^n} \mid f(\mathbf{x}) \mid^2 d\mathbf{x} < \infty.$$

Denote the set of all such functions, with norm defined by $\|\cdot\|_{L^2}$, by $L^2(\mathbb{R}^n)$. And with such norm $L^2(\mathbb{R}^n)$ is a complete, normed vector space. Here the norm on $L^2(\mathbb{R}^n)$ is defined by an inner product,

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}^n} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x}.$$

And this inner product satisfies Cauchy-Schwarz inequality.

Proposition 3 (Proposition 4.2.4.)**.** *If $f, g \in L^2(\mathbb{R}^n)$, then*

$$\mid \langle f, g\rangle_{L^2} \mid \leq \|f\|_{L^2}\|g\|_{L^2}.$$

The relationship between absolutely integrable functions and square integrable functions is complicated in the sense that neither of them contains the other one. Some pathological examples are provided below.

Example 3 (Example 4.2.7.)**.** The function

$$f(x) = (1+ \mid x \mid)^{-\frac{3}{4}}$$

is not absolutely integrable, but it is square integrable. On the other hand, the funciton

$$g(x) = \frac{\chi_{[-1,1]}(x)}{\sqrt{\mid x \mid}}$$

is absolutely integralble but not square integrable.

See a post HERE.

Proof. It's not absolutely integrable because

$$\int_{\mathbb{R}} |f(x)|\, dx \geq \int_{0<x<\infty} |f(x)|\, dx = \int_{0<x<\infty} \left(\frac{1}{(1+x)^{3/4}}\right) , dx = \infty.$$

While we can compute L^2-norm as

$$\int_{\mathbb{R}} f(x) = \int_{\mathbb{R}} \mid (1+|x|)^{-\frac{3}{4}} \mid^2 = \int_{-\infty}^{0} (1-x)^{-\frac{3}{2}}\, dx + \cdots = 2+2 = 4.$$

$\square$

Exercise 1 (Exercise 4.2.9.)**.** Let f be an L^1-function. Show that $\hat{f}$ is a continuous function. Extra credit: Show that $\hat{f}$ is uniformly continuous on the whole real line.

6 Week 6.

6.1 Fourier Transformation on $L^2(\mathbb{R})$

According to this pathological example 3, we know being absolutely integrable and square integrable won't imply between themselves. They're "parallel", and the present of both conditions will give us a stronger statement, namely Parseval formula. This formula tells us when will the Fourier transformation of a function $\hat{f}$ be square integrable, and it gave an explicit description for L^2-norm of $\hat{f}$.

Question. How about L^1 function? Fourier inversion formula assumed a priori $\hat{f} \in L^1(\mathbb{R})$. Only tool we have seems to be Riemann-Lebesgue Lemma, which states certain nice "decay" property for L^1 functions. But all we can conclude, under condition of Riemann-Lebesgue is that function is continuous and will go to zero at infinity. One naïve hope is that this will be enough to deduce it's in L^1, but it appears to be false? See a post HERE, which redirects us to Chapter 8 of [2].

Example 4. There are something in L^2, after Fourier transform, not in $1/x$.

We defined Fourier transformation for L^1- functions. In fact, we can extend Fourier transformation to $L^2(\mathbb{R})$ as follows:

Let $f \in L^2(\mathbb{R})$ and for each real nubmer $R > 0$ define

$$\hat{f}_R(\xi) = \int_{-R}^{R} f(x)e^{-ix\xi}\, dx.$$

Parseval's formula gives us, for $R_1 < R_2$,

$$\|\hat{f}_{R_1} - \hat{f}_{R_2}\|_{L^2}^2 = 2\pi \int_{R_1 \le |x| \le R_2} |f(x)|^2\, dx \tag{3}$$

Notice that f being L^2 forces RHS of 3 goes to 0 when $R_1 \to \infty$ and $R_2 \to \infty$. Since otherwise the integral will be infinity, contradicts the definition.

Hence we know sequence $\{\hat{f}_R\}_{R\in\mathbb{R}}$ is Cauchy. While $L^2(\mathbb{R})$ is complete and normed vector space, we can define it's limit inside $L^2(\mathbb{R})$. In summary, we define $\hat{f}$ as this limit. One convention: we call limit of a sequence in L^2-norm a *limit in the mean* and denote it by LIM.

Definition 8 (Defintion 4.2.5.)**.** If f is a function in $L^2(\mathbb{R})$, then its Fourier transform is defined to be

$$\hat{f} = \mathrm{LIM}_{R\to\infty}\, \hat{f}_R.$$

Proposition 4 (Proposition 4.2.5.)**.** *The Fourier transform extends to define a continuous map from $L^2(\mathbb{R})$ to itself. If $f \in L^2(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$$

Question. Why Parseval implies continuity of f?

A consequence of Parseval's formula is uniqueness statement. The slogan is *"A function in L^2 is determined by its Fourier transform."*

One application is to determine whether two functions in $L^2(\mathbb{R})$ are equal, we can compute Fourier transformation of their subtraction. More precisely, it's the following corollary

Corollary 1. If $f \in L^2(\mathbb{R})$ and $\hat{f} = 0$, then $f \equiv 0$.

Proposition 1 (**Fourier inversion for** $L^2(\mathbb{R})$)**.**

Here we give a summary of basic properties of Fourier transform that hold for integrable function or L^2-functions.

Question. Why not absolutely integrable? Connections between integrable and absolutely integrable. I saw a post HERE.

- `LINEARITY`:
- `SCALING`:
- `TRANSLATION`: Let f_t be the function f shifted by t [i.e., $f_t(x) = f(x-t)$]. The Fourier transform of f_t is given by

$$\begin{aligned}\hat{f}_t(\xi) &= \int_{-\infty}^{\infty} f(x-t)e^{-i\xi x}\,dx \\ &= \int f(y)e^{-i\xi(y+t)}\,dy \\ &= e^{-i\xi t}\hat{f}(\xi).\end{aligned}$$

- `REALITY`: If f is a real-valued function, then its Fourier transform satisfies $\hat{f}(\xi) = \overline{\hat{f}(-\xi)}$. This shows that the Fourier transform of a real-valued function is completely determined by its values for positive (or negative) frequencies. ?
- `EVENNESS`:

Figure 4.6.
Figure 4.7.

6.2 A General Principle in Functional Analysis

As we know *completeness* is such an important property for a normed linear space, we recall defintion of *dense* with a theorem.

Question. Is this saying every normed linear space is complete or just saying it's a property might or might not have?

Whenever we have a dense subspace $S \subset V$ where $(V, \|\cdot\|)$ is a normed linear space, we can use a sequence of points from V to approach the point in S.

Definition 9 (Defintion 4.2.7.)**.** Let $(V, \|\cdot\|)$ be a normed linear space. A subspace S of V is *dense* if for every $\mathbf{v} \in V$ there is a sequence $\{\mathbf{v}_k\}_{k\in\mathbb{N}} \subset S$ such that

$$\lim_{k\to\infty} \|\mathbf{v} - \mathbf{v}_k\| = 0.$$

Here we'll introduce a theorem that gives us a general principle: a bounded linear map, defined on a dense subset, extends to the whole space.

Theorem 5 (Theorem 4.2.4.)**.** *Let $(V_1, \|\cdot\|)$ and $(V_2, \|\cdot\|)$ be normed, linear spaces and assume that V_2 is complete. Suppose that S_1 is a dense subspace of V_1 and A is a linear map from S_1 to V_2. If there exists a constant M such that*

$$\|A\mathbf{v}\|_2 \leq M\|\mathbf{v}\|_1,$$

for all $\mathbf{v}$ in S_1, then A extends to define a linear map from V_1 to V_2, satisfying the same esimate.

Question. How did we define linear space? "Satisfying the same estimate" means we have the above inequality, but $\mathbf{v}$ could be chosen from the whole space V_1?

7 Week 7.

7.1 Tools from Appendix B: Basic Analysis

Theorem 1 (Theorem B.8.1 (Fubini's Theorem))**.** Let f be a function defined on $\mathbb{R}^n$ and let $n = k + l$ for positive integers k and l. If either of the iterated integrals

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^l} |f(\mathbf{w},\mathbf{y})|\, d\mathbf{w}\, d\mathbf{y} \quad \text{or} \quad \int_{\mathbb{R}^l}\int_{\mathbb{R}^k} |f(\mathbf{w},\mathbf{y})|\, d\mathbf{y}\, d\mathbf{w}$$

is finite, then the other is as well. In this case f is integrable over $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^l} |f(\mathbf{w},\mathbf{y})|\, d\mathbf{w}\, d\mathbf{y} = \int_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^l}\int_{\mathbb{R}^k} |f(\mathbf{w},\mathbf{y})|\, d\mathbf{y}\, d\mathbf{w}.$$

Remark 2. Informally speaking, the order of the integrations can be interchanged under the assumption that f is *absolutely* integrable. Because there are examples of functions defined on $\mathbb{R}^2$ so that both iterated integrals

$$\int\int f(x,y)dxdy, \quad \int\int f(x,y)dydx$$

exist but are not equal, with f not integrable on $\mathbb{R}^2$.

7.2 The Heisenberg Uncertainty Principle

Here we study relationships between the $\operatorname{supp} f$ and $\operatorname{supp} \hat{f}$. The simplest such result states that if a function has bounded support, then its Fourier transform cannot.

Proposition 1 (Proposition 4.4.1.)**.** Suppose $\operatorname{supp} f$ is contained in the bounded interval $(-R, R)$. If $\hat{f}$ also has bounded support then $f \equiv 0$.

7.3 The Fourier Transformation for Functions of Several Variables

We make some notation conventions that will be useful for multi-variable functions. We often use lowercase Roman bold letter to denote

$$\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n.$$

And it's customary to use lowercase Greek bold letters for points on the Fourier transform space such as

$$\xi = (\xi_1, ..., \xi_n).$$

Again, we start by defining for functions from $L^1(\mathbb{R}^n)$.

Definition 1 (Definition 4.5.1.)**.** If f belongs to $L^1(\mathbb{R}^n)$, then the Fourier transform, $\hat{f}$ of f, is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\langle \xi, \mathbf{x}\rangle}\, d\mathbf{x} \quad \text{for} \quad \xi \in \mathbb{R}^n.$$

Furthermore, we can express this integral as

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, ..., x_n) e^{-ix_1\xi_1}\, dx_1 \cdots e^{-ix_n\xi_n}\, dx_n.$$

While f is assumed to be absolutely integrable, then Fubini's theorem ensures that we can interchange the order of integration.

Theorem 2 (Theorem 4.5.1 (Fourier Inversion Formula))**.** Suppose that f is an L^1-function defined on $\mathbb{R}^n$. If $\hat{f}$ also belongs to $L^1(\mathbb{R}^n)$, then

$$f(\mathbf{x}) = \frac{1}{[2\pi]^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\mathbf{x}\cdot\xi}\, d\xi.$$

8 Week 8.

Here are some notes for Stein's textbook in Fourier anlysis [4].

8.1 Chapter 5.1

Definition 2 (Chapter 5, Page 131 Stein)**.** We say that a funciton defined on $\mathbb{R}$ to be of moderate decrease if f cotinuous and there exists a constant $A > 0$ so that

$$|f(x)| \leq \frac{A}{1+x^2} \quad \text{for all} \quad x \in \mathbb{R}.$$

We shall denote by $\mathcal{M}(\mathbb{R})$ the set of functions of moderate decrease on $\mathbb{R}$.

The inequality condition forces f to be bounded. Also that it decays at infinity at least as fast as $1/x^2$.

Question. Connection between moderate decrease with L^1, L^2, etc?

Under the usual point-wise addition and multiplication, $\mathcal{M}(\mathbb{R})$ is a vector space over $\mathbb{C}$. why?

Here we give some examples.

Example 3. Both $1/(1 + |x|^n)$ and $e^{-a|x|}$ are of moderate decrease for $n \geq 2$.

The reason we introduce $\mathcal{M}(\mathbb{R})$ is because we can therefore define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_{N}^{N} f(x)\,dx.$$

The limit exists, and it could be proved by Cauchy sequence argument. This argument is essentially the same as the process where we tried to define Fourier transform for function in $L^2(\mathbb{R})$.

Here we summarise some properties of moderate decrease functions defined on $\mathbb{R}$.

Proposition 2 (Chapter 5 Proposition 1.1.)**.** The integral of a function of moderate decrease defined as

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_{N}^{N} f(x)\,dx$$

satisfies the following properties:

- *Linearity*
- *Translation invariance*
- *Scaling under dilations*
- *Continuity*

TODO: Give a proof for 1......

Back to Epstein's Book.

8.2 Convolution

Our interest is to study a smooth function of one variable with data contaminated by noise. And we let n be the function that "*models*" the noise. This Noise function n, is typically represented by a rapidly varying function that is *locally of mean zero.*

Definition 3. We say a rapidly varying function is *locally of mean zero* if for any x, and a large enought δ, the average

$$\frac{1}{\delta}\int_{x}^{x+\delta} n(y)\,dy$$

is small compared to the size of n.

Question. How do we define size of n?

Definition 4. The *moving average* of f is defined to be

$$\mathcal{M}_{\delta}(f)(x) = \frac{1}{\delta}\int_{x}^{x+\delta} f(y)\,dy.$$

In practice of medical image we only consider convolution in lower dimension, but we can actually define it generally.

Definition 5 (Definition 5.1.1.)**.** If f is an L^1-function defined on $\mathbb{R}^n$ and g is a bounded, locally integrable function then, the *convolution product* of f and g iis the function on $\mathbb{R}^n$ defined by the integral

$$f * g(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})g(\mathbf{y})\,d\mathbf{y}.$$

This definition is crafted specifically for Theorem 5.1.1. **??** see Exercise 5.1.8. **??**

Exercise 6 (5.1.8.)**.** Suppose that the convolution product were defined by

$$f * g(\mathbf{x}) = \int f(\mathbf{y})g(\mathbf{y}-\mathbf{x})\,d\mathbf{y}.$$

Show that equation of Theorem 5.1.1. would not hold. What would replace it?

Remark 3 (Remark 5.1.1.)**.**

References

[1] Charles L Epstein. *Introduction to the mathematics of medical imaging.* SIAM, 2007.

[2] Gerald B Folland. *Real analysis: modern techniques and their applications*, volume 40. John Wiley & Sons, 1999.

[3] Elias M Stein and Rami Shakarchi. *Real analysis: measure theory, integration, and Hilbert spaces.* Princeton University Press, 2009.

[4] Elias M Stein and Rami Shakarchi. *Fourier analysis: an introduction*, volume 1. Princeton University Press, 2011.