CS602 Applied Algorithms **Jul-Nov 2023**

Homework (No submission)

Linear Programming Algorithms

Lecture 22 (Oct 27)

Simplex algorithm.

1. Consider the following linear program.

$$\begin{aligned} \max 4x_1 + 3x_2 &\text{ subject to} \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ 2x_1 + x_2 &\leq 6 \\ x_1 + 2x_2 &\leq 6 \\ 2x_1 + 2x_2 &\leq 7 \end{aligned}$$

First write the LP in the standard form $Ax = b, x \geq 0$. Start with the basic feasible solution that has $x_1 = 0, x_2 = 0$. Keep repeating the following iterations of the simplex algorithm.

- Choose one of the non-basic variables to increase. It should be among those whose coefficient in the objective function is positive. If there is no such variable then output the current solution. If there is such a variable then increase it till the maximum possible value while maintaining feasibility.
- This gives you a new basic feasible solution and a new set of basic and non-basic variables.
- Express the objective function in terms of the non-basic variables.

2. Consider an LP where A is a $k \times n$ matrix.

$$\max w^T x \text{ subject to } Ax = b, x \geq 0.$$

Suppose we have a basic feasible solution α, where the basic variables are $\alpha_1, \alpha_2, \ldots, \alpha_k$. That means $\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0$.

- How will you find the values of $\alpha_1, \alpha_2, \ldots, \alpha_k$ using A and b. Note that non-degeneracy assumption means every $k \times k$ submatrix of A is invertible.
- How will you express the objective function in $w^T x$, as a function of $x_{k+1}, x_{k+2}, \ldots, x_n$.
- Prove that α is an optimal solution if and only if the objective function, when expressed as a function of $x_{k+1}, x_{k+2}, \ldots, x_n$, has no positive coefficient.

3. Consider the bipartite maximum matching LP.

$$\begin{aligned} \max \sum x_e &\text{ subject to} \\ x_e \geq 0 &\text{ for } e \in E \\ \sum_{e \in \delta(v)} x_e \leq 1 &\text{ for } v \in V \end{aligned}$$

If we run the simplex algorithm on this LP, what would the successive basic feasible solutions look like? Does it look like the augmenting path algorithm?

Finding the initial feasible solution, convert LP.

Lecture 23, 24 (Oct 31, Nov 3)

Ellipsoid algorithm.

4. Prove Farkas' Lemma from Separating Hyperplane theorem.

5. Prove that for any $0 \leq \alpha < 1/2$, the following ellipsoid

$$\{x \in \mathbb{R}^n : (x_1 - \alpha)^2 + x_2^2(1-2\alpha) + \cdots + x_n^2(1-2\alpha) \leq (1-\alpha)^2\}$$

contains the half ellipsoid given by

$$\{x \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1, x_1 \geq 0\}.$$

6. Minimize $\frac{(1-\alpha)^n}{(1-2\alpha)^{(n-1)/2}}$ for $0 \leq \alpha < 1/2$.

7. Suppose we are given an ellipse

$$E_1 = \{(x_1, x_2) : (27x_1 - 36x_2 - 27)^2 + (64x_1 + 48x_2 - 64)^2 \leq 900\}.$$

Find the smallest ellipse E_2 that contains the intersection of E_1 and the half-space $H_1 = \{(x_1, x_2) : x_1 \geq 1\}$. You also need to explain how you found it.

Hint: You may want to follow these steps: (1) find a transformation T that transforms E_1 into a circle C of radius 1 centered at origin. Apply the same transform on H_1 to get H_1'. (2) Do a rotation R so that H_1' becomes H. (3) First apply the inverse of R on E and then apply inverse of T. The resulting ellipse is the desired one.

8. Consider an LP $Ax = b, x \geq 0$ with n variables and m equality constraints. Let the entries in A and b be integers with at most ℓ bits. Assume that this LP has a basic feasible solution (i.e., solution with $n - m$ zero coordinates). Show that any coordinate in a basic feasible solution is at most $2^{f(n,\ell)}$ for some polynomial function $f(n, \ell)$.

9. Recall the Steiner forest LP we wrote. Let the given pair of terminals be $\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}$.

$$\begin{aligned} min \sum_{e \in E} w_e x_e & \text{ subject to} \\ x_e &\geq 0 \\ \sum_{e \in \delta(S)} x_e &\geq 1 \quad \text{for } S \subseteq V \text{ which separates some terminal pair } (s_i, t_i) \end{aligned}$$

Design a polynomial time separation oracle for this LP.

10. To check if a matrix is positive semidefinite, one can do a two-sided Gaussian elimination, where one does the same row and column transformations on a symmetric matrix and obtains a diagonal matrix at the end. See the example.

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{row_2 \leftarrow row_2 - 2 \times row_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{col_2 \leftarrow col_2 - 2 \times col_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{row_3 \leftarrow row_3 + (2/3) \times row_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 7/3 \end{pmatrix} \xrightarrow{col_3 \leftarrow col_3 + (2/3) \times col_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7/3 \end{pmatrix}$$

Show that a symmetric matrix is PSD if and only if the diagonal matrix at the end of two-sided Gaussian elimination has all non-negative entries. Essentially, you need to show that PSD property is preserved when we do same row and column transformations.

11. Recall the below semidefinite program for the MAXCUT problem. Design a polynomial time separation oracle for this SDP. First solving the above problem will be helpful.

$$\max \sum_{i<j} w_{i,j} x_{i,j} \text{ subject to}$$
$$\begin{pmatrix} 1 & 1-2x_{1,2} & \cdots & 1-2x_{1,n} \\ 1-2x_{1,2} & 1 & \cdots & 1-2x_{2,n} \\ & \vdots & & \\ 1-2x_{1,n} & 1-2x_{2,n} & \cdots & 1 \end{pmatrix} \succeq 0.$$

Lecture 25, 26 (Nov 7, 10) Interior point method

12. Consider the following linear program.

$$\begin{aligned} \min -4x_1 + 3x_2 &\text{ subject to} \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_1 + x_2 &\leq 3 \end{aligned}$$

We want to convert this an unconstrained optimization problem with parameter η and a barrier function $B(x_1, x_2)$.

$$\min \eta \times (-4x_1 + 3x_2) + B(x_1, x_2).$$

What will be the barrier function corresponding to the above constraints. Let $x_\eta^* \in \mathbb{R}^2$ be the optimal point for the above function with parameter η. Plot x_η^* as a function of η, from $\eta = 0$ to $\eta = \infty$. For example, you can try `https://www.wolframalpha.com/` to plot this.

1. Consider the following linear program.

$$\begin{aligned} \max 4x_1 + 3x_2 \text{ subject to} \\ x_1 \geq 0 \\ x_2 \geq 0 \\ 2x_1 + x_2 \leq 6 \\ x_1 + 2x_2 \leq 6 \\ 2x_1 + 2x_2 \leq 7 \end{aligned}$$

First write the LP in the standard form $Ax = b, x \geq 0$. Start with the basic feasible solution that has $x_1 = 0, x_2 = 0$. Keep repeating the following iterations of the simplex algorithm.

- Choose one of the non-basic variables to increase. It should be among those whose coefficient in the objective function is positive. If there is no such variable then output the current solution. If there is such a variable then increase it till the maximum possible value while maintaining feasibility.
- This gives you a new basic feasible solution and a new set of basic and non-basic variables.
- Express the objective function in terms of the non-basic variables.

max $4x_1 + 3x_2$
$x_1 \geq 0$
$x_2 \geq 0$
$2x_1 + x_2 \leq 6$
$x_1 + 2x_2 \leq 6$
$2x_1 + 2x_2 \leq 7$

$x_3 = 6 - 2x_1 - x_2$
$x_4 = 6 - 2x_2 - x_1$
$x_5 = 7 - 2(x_1 + x_2)$
$x \geq 0$
max $4x_1 + 3x_2$

Note: non-basic variable = appears in ≥ 2 constraint/objective. (i.e. x_1, x_2)

$x_1 = 3,\ x_2 = 0,\ x_3 = 0,\ x_4 = 3,\ x_5 = 1$

relaxing x_2

$x_1 = 3 - \frac{\alpha}{2},\ x_2 = \alpha,\ x_3 = 0,\ x_4 = 6 - 2(\alpha) - 3 + \alpha/2 = 3 - \frac{3\alpha}{2},\ x_5 = 7 - 2(3 + \frac{\alpha}{2}) = 1 - \alpha$

$4x_1 + 3x_2 = 4(3 - \frac{\alpha}{2}) + 3\alpha = 12 + \alpha$

Put $\underline{\alpha = 1}$.

$x_1 = 5/2,\ x_2 = 1,\ x_3 = 0,\ x_4 = 3/2,\ x_5 = 0$ $\quad$ $\underline{\text{obj}} = 4 \times 5/2 + 3 \times 1 = 13$

Now, relaxing $x_3 = \alpha, x_5 = 0$ $\quad$ $\alpha = 6 - 2x_1 - x_2$, $2(x_1 + x_2) = 7$ $\Rightarrow 4x_1 + 3x_2 = 7 + 6 - \alpha = 13 - \alpha$

Hence, doesn't work.

$x_5 = \alpha, x_3 = 0$.

$7 - 2(x_1 + x_2) = \alpha$, $2x_1 + x_2 = 6$

$4x_1 + 3x_2 = 13 - \alpha$, hence, 13 is the optimal value ■

2. Consider an LP where A is a $k \times n$ matrix.

$$\max w^T x \text{ subject to } Ax = b, x \geq 0.$$

Suppose we have a basic feasible solution α, where the basic variables are $\alpha_1, \alpha_2, \ldots, \alpha_k$. That means $\alpha_{k+1} = \alpha_{k+2} = \cdots = \alpha_n = 0$.

- How will you find the values of $\alpha_1, \alpha_2, \ldots, \alpha_k$ using A and b. Note that non-degeneracy assumption means every $k \times k$ submatrix of A is invertible.
- How will you express the objective function in $w^T x$, as a function of $x_{k+1}, x_{k+2}, \ldots, x_n$.
- Prove that α is an optimal solution if and only if the objective function, when expressed as a function of $x_{k+1}, x_{k+2}, \ldots, x_n$, has no positive coefficient.

A is $k \times n$ matrix $\quad$ let $A = [A'_{k\times k}\ \ A'']$ $\quad$ Now, $A'x' + A''x'' = b$

$x' = (A')^{-1}(b - A''x'')$

let $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix} = \alpha'$, $\begin{bmatrix} \alpha_{k+1} \\ \vdots \\ \alpha_n \end{bmatrix} = \alpha''$

$x'' = \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \end{bmatrix}$

$Ax = A'\alpha' + A''\cancel{\alpha''}^{0} = b$

$\boxed{\alpha' = (A')^{-1} b}$

$w = \begin{bmatrix} w' \\ w'' \end{bmatrix}$, $w^T x = (w')^T (A')^{-1}(b - A''x'') + (w'')^T x''$

Now, when $w^T x$ is expressed as a function of $x_{k+1}, \ldots, x_n$

$w^T x = c_{k+1}x_{k+1} + \cdots + c_n x_n + c$. Now, $\alpha_{k+1} = \cdots = \alpha_n = 0$

($\Rightarrow$) $\alpha_{k+1} = \cdots = \alpha_n = 0$ and have no +ve coeff
$\Rightarrow$ none of them can be increased
$\Rightarrow$ opt

($\Leftarrow$) opt $\Rightarrow$ none can be inc $\Rightarrow$ no +ve coeff.

3. Consider the bipartite maximum matching LP.

$$\max \sum x_e \text{ subject to}$$
$$x_e \geq 0 \text{ for } e \in E$$
$$\sum_{e\in\delta(v)} x_e \leq 1 \text{ for } v \in V$$

If we run the simplex algorithm on this LP, what would the successive basic feasible solutions look like? Does it look like the augmenting path algorithm?

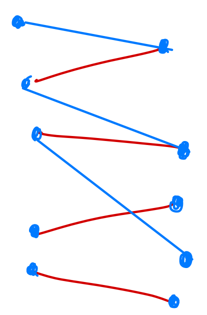
Finding the initial feasible solution, convert LP.

$$\sum_{e\in\delta(v)} x_e + s_v = 1 \quad \forall v \in V$$

Initial feasible solution $x_e = 0$, $s_v = 1$ $\forall e, v$

$x_e = \alpha \to s_v = 1-\alpha$ for both endpts

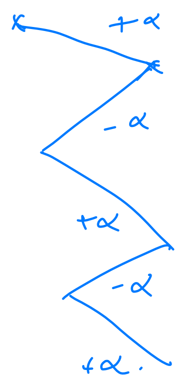
$x_e = 1$. (1^{st} generate matching)



$x_e = 0$

successive sol^n = growing MM.

$x_e \to \alpha$



4. Prove Farkas' Lemma from Separating Hyperplane theorem.

$K \subseteq \mathbb{R}^n$ is closed, convex, bounded (compact) and $\alpha \in \mathbb{R}^n$. If $\alpha \notin K$ then $\exists a, b$ s.t.
$a^T x \leq b \ \forall x \in K$, $a^T \alpha > b$

Farkas lemma

$A = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = b,\ x_1 \geq 0, \ldots, x_n \geq 0\}$ is not feasible iff

$\exists y \in \mathbb{R}^k$, $y^T \alpha_i \geq 0 \ \forall i$, $y^T b < 0$.

Consider K' = ball bounding $\|b\|_2$, $K = A \cap K'$

($\Rightarrow$)
if $\exists y \in \mathbb{R}^k$, $y^T \alpha_i \geq 0 \ \forall i$, $y^T b < 0 \Rightarrow y^T(\sum x_i \alpha_i) \geq 0$, $y^T b < 0$
$\Rightarrow b \notin A$.

($\Leftarrow$)
if $b \notin A$ iff $b \notin K \Rightarrow \exists \alpha, \beta$ s.t. $\alpha^T b > \beta$, $\alpha^T x \leq \beta \ \forall x \in K$

$\Rightarrow \alpha^T(\sum \alpha_i x_i) \leq \beta$, $x_i \geq 0 \ \forall i$

$\Leftrightarrow \alpha^T \alpha_i \leq 0 \ \forall i \Rightarrow \beta \geq 0 \Rightarrow \alpha^T b > 0$.

5. Prove that for any $0 \leq \alpha < 1/2$, the following ellipsoid

$$\{x \in \mathbb{R}^n : (x_1 - \alpha)^2 + x_2^2(1-2\alpha) + \cdots + x_n^2(1-2\alpha) \leq (1-\alpha)^2\}$$

contains the half ellipsoid given by

$$\{x \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1, x_1 \geq 0\}.$$

Let $x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1$, $x_1 \geq 0$

Then, $(x_1-\alpha)^2 + (1-2\alpha)\sum x_i^2 \leq (x_1-\alpha)^2 + (1-2\alpha)(1-x_1^2)$

$f(x_1, \alpha) = (x_1-\alpha)^2 + (1-2\alpha)(1-x_1^2)$

$\frac{\partial f}{\partial x_1} = 2(x_1-\alpha) + (1-2\alpha)(-2x_1) = \cancel{2x_1} - 2\alpha - \cancel{2x_1} + 4x_1\alpha$

$= 2\alpha(1-2x_1)$

$x_1 = 1/2$.

So, $(\frac{1}{2}-\alpha)^2 + (1-2\alpha)(\frac{3}{4}) = \cancel{\frac{1}{4}} - \alpha + \alpha^2 + \cancel{\frac{3}{4}} - \frac{3}{2}\alpha = 1 - \frac{5}{2}\alpha + \alpha^2$

$f(0) = 1$, $f(\frac{1}{2}) = 1 - \frac{5}{4} + \frac{1}{4} = 0$.

6. Minimize $\frac{(1-\alpha)^n}{(1-2\alpha)^{(n-1)/2}}$ for $0 \leq \alpha < 1/2$.

$$\frac{df}{d\alpha} = \frac{(1-2\alpha)^{\frac{n-1}{2}}\, n(1-\alpha)^{n-1}(-1) + (1-\alpha)^n \frac{n-1}{2}(1-2\alpha)^{\frac{n-3}{2}}(-2)}{(1-2\alpha)^{n-1}}$$

$$= (n-1)(1-\alpha)^n(1-2\alpha)^{\frac{n-3}{2}} - n(1-\alpha)^{n-1}(1-2\alpha)^{\frac{n-1}{2}}$$

$$= \frac{(1-\alpha)^{n-1}(1-2\alpha)^{\frac{n-3}{2}}}{(1-2\alpha)^{n-1}}\left((n-1)(1-\alpha) - n(1-2\alpha)\right)$$

$$= (\quad)(\cancel{n}-1-\alpha n+\alpha-\cancel{n}+2n\alpha)$$

$$= (\quad)(n\alpha+\alpha-1) = 0$$

$$\Rightarrow \boxed{\alpha = \frac{1}{n+1}}$$

7. Suppose we are given an ellipse

$$E_1 = \{(x_1, x_2) : (27x_1 - 36x_2 - 27)^2 + (64x_1 + 48x_2 - 64)^2 \leq 900\}.$$

Find the smallest ellipse E_2 that contains the intersection of E_1 and the half-space $H_1 = \{(x_1, x_2) : x_1 \geq 1\}$. You also need to explain how you found it.

Hint: You may want to follow these steps: (1) find a transformation T that transforms E_1 into a circle C of radius 1 centered at origin. Apply the same transform on H_1 to get H_1'. (2) Do a rotation R so that H_1' becomes H. (3) First apply the inverse of R on E and then apply inverse of T. The resulting ellipse is the desired one.

(1) $X = 27x_1 - 36x_2 - 27$

$Y = 64x_1 + 48x_2 - 64$

$$\begin{bmatrix} 27 & -36 \\ 64 & 48 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -27 \\ -64 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Let $T(x) = (Ax+b)/30$

$E = \{x \mid \|Ax+b\|_2^2 \leq 900\}$

$T(E) = \{T(x) \mid \|Ax+b\|_2^2 \leq 900\}$

$= \{T(x) \mid \|T(x)\|_2^2 \leq 1\}$

$H_1 = \{(x_1, x_2) \mid x_1 \geq 1\}$

$T(H_1) = \{T(x_1, x_2) \mid x_1 \geq 1\}$

$= \{Ax+b \mid x_1 \geq 1\}$

$\{x_1 \geq 1 \Rightarrow Ax+b = \begin{pmatrix} 27(x_1-1) - 36x_2 \\ 64(x_1-1) + 48x_2 \end{pmatrix}\}$

$= \{(x_1, x_2) \mid 4x_1 + 3x_2 \geq 0\}$

$R \rightarrow \quad x_1 \geq 0$

8. Consider an LP $Ax = b, x \geq 0$ with n variables and m equality constraints. Let the entries in A and b be integers with at most ℓ bits. Assume that this LP has a basic feasible solution (i.e., solution with $n - m$ zero coordinates). Show that any coordinate in a basic feasible solution is at most $2^{f(n,\ell)}$ for some polynomial function $f(n, \ell)$.

$Ax = b$ } m constraints

$x \geq 0$ } n constraints

entries in A, b have atmost ℓ bits, for basic feasible solⁿ.

for feasible solⁿ, $x = A^{-1}b$

$\frac{1}{\det A}[\mathrm{cof}(A)]\, b$

9. Recall the Steiner forest LP we wrote. Let the given pair of terminals be $\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}$.

$$min \sum_{e \in E} w_e x_e \text{ subject to}$$

$$x_e \geq 0$$

$$\sum_{e \in \delta(S)} x_e \geq 1 \quad \text{for } S \subseteq V \text{ which separates some terminal pair } (s_i, t_i)$$

Design a polynomial time separation oracle for this LP.

Solⁿ: Check minimum x-capacity $s_i - t_i$ cut is ≥ 1.

min s-t cut = max s-t flow. (polytime: ford-fulkerson)

If min-cut < 1, we have a violating constraint.

10. To check if a matrix is positive semidefinite, one can do a two-sided Gaussian elimination, where one does the same row and column transformations on a symmetric matrix and obtains a diagonal matrix at the end. See the example.

$$\begin{pmatrix}1 & 2 & 0\\ 2 & 1 & 2\\ 0 & 2 & 1\end{pmatrix} \xrightarrow{row_2 \leftarrow row_2 - 2\times row_1} \begin{pmatrix}1 & 2 & 0\\ 0 & -3 & 2\\ 0 & 2 & 1\end{pmatrix} \xrightarrow{col_2 \leftarrow col_2 - 2\times col_1} \begin{pmatrix}1 & 0 & 0\\ 0 & -3 & 2\\ 0 & 2 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & -3 & 2\\ 0 & 2 & 1\end{pmatrix} \xrightarrow{row_3 \leftarrow row_3 + (2/3)\times row_2} \begin{pmatrix}1 & 0 & 0\\ 0 & -3 & 2\\ 0 & 0 & 7/3\end{pmatrix} \xrightarrow{col_3 \leftarrow col_3 + (2/3)\times col_2} \begin{pmatrix}1 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 7/3\end{pmatrix}$$

Show that a symmetric matrix is PSD if and only if the diagonal matrix at the end of two-sided Gaussian elimination has all non-negative entries. Essentially, you need to show that PSD property is preserved when we do same row and column transformations.

Symmetric operations corresp to unitary matrices M, M^T, each M is invertible

$M^T U M = D$

M is invertible $\Rightarrow$ it's full span

$u^T M^T U M u = u^T D u \geq 0 \;\forall u$

$\Leftrightarrow$ D is non-neg entries

$\parallel^y$ converse follows.

11. Recall the below semidefinite program for the MAXCUT problem. Design a polynomial time separation oracle for this SDP. First solving the above problem will be helpful.

$$\max \sum_{i<j} w_{i,j} x_{i,j} \text{ subject to}$$

$$\begin{pmatrix} 1 & 1-2x_{1,2} & \cdots & 1-2x_{1,n}\\ 1-2x_{1,2} & 1 & \cdots & 1-2x_{2,n}\\ & \vdots & & \\ 1-2x_{1,n} & 1-2x_{2,n} & \cdots & 1 \end{pmatrix} \succeq 0.$$

Gaussian elimination gives M s.t. $M^T X M = D$. If $D \geq 0$, return $x \in K$

If $D < 0$, for some diagonal element

$m_i^T X m_i = -d_i$

SDP: $\sum z_i z_j p_{i,j} \geq 0 \quad \forall z \in \mathbb{R}^n$

putting this value of $z = m_i$ gives a violating constraint and hence a sep hyperplane.