Core : All possible ways of distributing total worth s.t. no sub-coalition has incentive to secede

1. Worth of assignment game = OPT of primal LP
2. Core imputation ; distribution = optimal solution to dual LP

Degeneracy ≡ optimal assignment is not unique

Shapley and Shubik : "in the most common case", optimal assignment is unique

Earlier approach : perturb weights to make assignment unique (this is improved without perturbation).

Setting (captures issues) :

U = set of women players, V = set of men players

$|U|=m$, $|V|=n$, $G=(U,V,E)$ is bipartite,

$(i,j)\in E$ if (i,j) can participate as a mixed doubles team
$i\in U, j\in V$

$w_{i,j}>0$ represents expected earning if (i,j) participate in tournament

Worth of game = maximum weight bipartite matching

Ques : How to distribute profits (among strong, weak, unmatched), so that no one seceeds ?

Definition 1. The set of all players, $U \cup V$, is called the *grand coalition*. A subset of the players, $(S_u \cup S_v)$, with $S_u \subseteq U$ and $S_v \subseteq V$, is called a *coalition* or a *sub-coalition*.

Definition 2. The *worth* of a coalition $(S_u \cup S_v)$ is defined to be the maximum profit that can be generated by teams within $(S_u \cup S_v)$ and is denoted by $p(S_u \cup S_v)$. Formally, $p(S_u \cup S_v)$ is the weight of a maximum weight matching in the graph G restricted to vertices in $(S_u \cup S_v)$ only. $p(U \cup V)$ is called the *worth of the game*. The *characteristic function* of the game is defined to be $p : 2^{U \cup V} \rightarrow \mathcal{R}_+$.

Definition 3. An *imputation*[2] gives a way of dividing the worth of the game, $p(U \cup V)$, among the agents. It consists of two functions $u : U \rightarrow \mathcal{R}_+$ and $v : V \rightarrow \mathcal{R}_+$ such that $\sum_{i \in U} u(i) + \sum_{j \in V} v(j) = p(U \cup V)$.

Definition 4. An imputation (u, v) is said to be in the *core of the assignment game* if for any coalition $(S_u \cup S_v)$, the total worth allocated to agents in the coalition is at least as large as the worth that they can generate by themselves, i.e., $\sum_{i \in S_u} u(i) + \sum_{j \in S_v} v(j) \geq p(S)$.

$$\begin{aligned}
\max \quad & \sum_{(i,j)\in E} w_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{(i,j)\in E} x_{ij} \leq 1 \quad \forall i \in U, \\
& \sum_{(i,j)\in E} x_{ij} \leq 1 \quad \forall j \in V, \\
& x_{ij} \geq 0 \quad \forall (i,j) \in E
\end{aligned}$$

← LP relaxation

dual LP →

$$\begin{aligned}
\min \quad & \sum_{i\in U} u_i + \sum_{j\in V} v_j \\
\text{s.t.} \quad & u_i + v_j \geq w_{ij} \quad \forall (i,j) \in E, \\
& u_i \geq 0 \quad \forall i \in U, \\
& v_j \geq 0 \quad \forall j \in V
\end{aligned}$$

Theorem 1. *(Shapley and Shubik [SS71]) The imputation (u, v) is in the core of the assignment game if and only if it is an optimal solution to the dual LP, (2).*

Related Work

1. Stable matching : coalition of 2 ppl should be happy (Gale-Shapley algorithm '62)
2. Generalise to general graph matching game. $G = (V, E)$ and $w : E \to \mathbb{R}^+$ where any 2 agents can trade if $(i, j) \in E$
   1. Assume (K_n, E).

      Worth of $S \subseteq V$, $p(S)$ = max weight matching in $G[S]$

      Core is empty then, $v(S) < p(S)$ for some imputation v.
   2. Least core : $\max_{v} \min_{S \subseteq V} v(S) - p(S)$

      $v(\phi) = 0$

      $v(V) = p(V)$
   3. Nucleolus (history given in paper)

Definition 5. For an imputation $v : V \to \mathcal{R}_+$, let $\theta(v)$ be the vector obtained by sorting the $2^{|V|} - 2$ values $v(S) - p(S)$ for each $\emptyset \subset S \subset V$ in non-decreasing order. Then the unique imputation, v, that lexicographically maximizes $\theta(v)$ is called the *nucleolus* and is denoted $\nu(G)$.

Open : 1. Combinatorial PTIME algorithm for general graph.
2. No upper bound for $v(S) - p(S)$

Solved : 1. Ellipsoid base PTIME algorithm [KPT '20]

4. Approximate Core [Vaz '22]

found $\frac{2}{3}$-approximate core (each sub-coalition gets $\frac{2}{3}$ profit atleast),

best approx ratio since it is integrality gap of LP.

5, 6. NR08, CE15 (ref : paper)

Definition 6. By a *mixed doubles team* we mean an edge in G; a generic one will be denoted as $e = (u, v)$. We will say that e is:

1. *essential* if e is matched in every maximum weight matching in G.
2. *viable* if there is a maximum weight matching M such that $e \in M$, and another, M' such that $e \notin M'$.
3. *subpar* if for every maximum weight matching M in G, $e \notin M$.

Definition 7. Let y be an imputation in the core of the game. We will say that e is *fairly paid in* y if $y_u + y_v = w_e$ and it is *overpaid* if $y_u + y_v > w_e$[4]. Finally, we will say that e is *always paid fairly* if it is fairly paid in every imputation in the core.

Definition 8. A generic player in $U \cup V$ will be denoted by q. We will say that q is:

1. *essential* if q is matched in every maximum weight matching in G.
2. *viable* if there is a maximum weight matching M such that q is matched in M and another, M' such that q is not matched in M'.
3. *subpar* if for every maximum weight matching M in G, q is not matched in M.

Definition 9. Let y be an imputation in the core. We will say that *q gets paid in* y if $y_q > 0$ and *does not get paid* otherwise. Furthermore, *q is paid sometimes* if there is at least one imputation in the core under which q gets paid, and it is *never paid* if it is not paid under every imputation.

Theorem 2. *The following hold:*

1. *For every team $e \in E$:*

$$e \text{ is always paid fairly} \iff e \text{ is viable or essential}$$

Proof uses strict CS (not covered in 602, so explain)

2. *For every player $q \in (U \cup V)$:*

$$q \text{ is paid sometimes} \iff q \text{ is essential}$$

Consequences of thm 2

1). Negating both sides of the first statement proved in Theorem 2 we get the following double-implication. For every team $e \in E$:

$$e \text{ is subpar} \iff e \text{ is sometimes overpaid}$$

Counter-intuitive fact: *Whereas viable and essential teams are always paid fairly, subpar teams are sometimes overpaid.*

Justification (ref: paper)

2). The second statement of Theorem 2 is equivalent to the following. For every player $q \in (U \cup V)$:

$$q \text{ is never paid} \iff q \text{ is not essential}$$

Corollary 1. *In the assignment game, the set of essential players is non-empty.* (some players in every MWM!)

To do: Make comments on bipartite polytope integrality gap.

3). Clearly the worth of the game is generated by teams that do play. Assume that (u, v) is such a team in an optimal assignment. Since $x_{uv} > 0$, by complementary slackness we get that $y_u + y_v = w_{uv}$, where y is a core imputation. Thus core imputations distribute the worth generated by a team among its players only. In contrast, the $\frac{2}{3}$-approximate core imputation for the general graph matching game given in [Vaz22] distributes the worth generated by teams which play to non-playing agents as well, thereby making a more thorough use of the TU aspect.

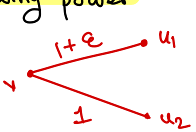
4). Next we use Theorem 2 to get insights into degeneracy. Clearly, if an assignment game is non-degenerate, then every team and every player is either always matched or always unmatched in the set of maximum weight matchings in G, i.e., there are no viable teams or players.

Corollary 2. *Imputations in the core of an assignment game treat viable and essential teams in the same way. Additionally, they treat viable and subpar players in the same way.* } thm 2

Note: For this page construct some examples (mixed doubles) which illustrate each point and why it makes sense.

Negotiating power

ex 1

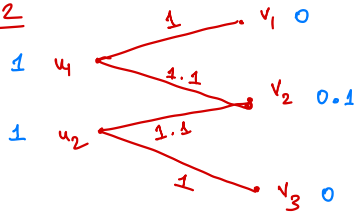


(v, u_1) chosen $v \leftarrow 1$, $u_1 \leftarrow \epsilon$

if u_1 asks for more, v shifts to u_2. so v only has negotiating power

Thus core imputations reward only those agents who always play. This raises the following question: Can't a non-essential player, say q, team up with another player, say p, and secede, by promising p almost all of the resulting profit? The answer is "No", because the dual (2) has the constraint $y_q + y_p \geq w_{qp}$. Therefore, if $y_q = 0$, $y_p \geq w_{qp}$, i.e., p will not gain by seceding together with q.

Ex 2



worth = 2.1

award = (1, 1, 0, 0.1, 0)

Why can't v_2 be allocated more than 0.1?

if asks for more, one will switch out to v_1

u_1, u_2 are stronger because they have ready partner available.