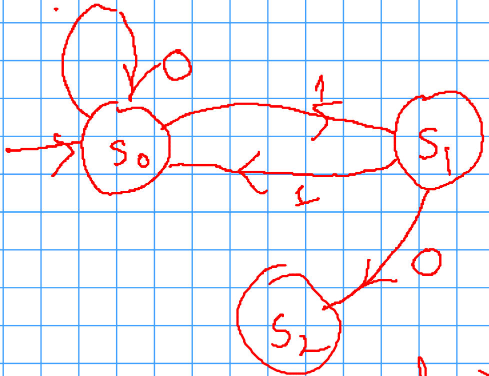
$\{x \in \{0,1\}^* \mid d(x) = 0 \bmod 3\}$ 00110

$d(x) \% 3 == 0$



$n = 0 \bmod 3$

$\Rightarrow$

$2n = 0 \bmod 3$

COMPLETE the automaton!

$S_i = \{x \mid d(x) = i \bmod 3\}$ $\quad 0 \le i < 3$

Prove that $x \in S_i \Rightarrow d(x) = i \bmod 3$

CS310M: Automata Theory (Minor)

Topic 3: Nondeterministic Finite Finite State Automata

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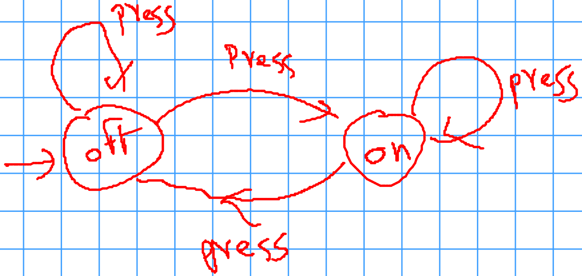
Course URL: https://cse.iitb.ac.in/$\sim$pandya58/CS310M/automata.html

Autumn, 2021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Today's Topics:

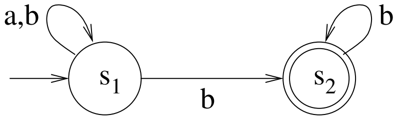
- Nondeterminism in automata
- Syntax, Semantics of NFA, Examples
- Extended Transition Function
- Determinization: Subset Construction

Source: Kozen, Lectures 5 and 6.

Nondeterminism in Automaton

Nondeterministic Finite State Automaton (NFA)

NFA A_2 .

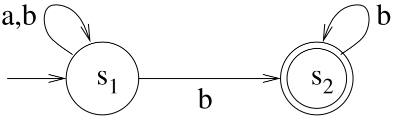


- A run of A_2 over the word *abbb* is:

$$s_1 \xrightarrow{a} s_1 \xrightarrow{b} s_1 \xrightarrow{b} s_2 \xrightarrow{b} s_2.$$

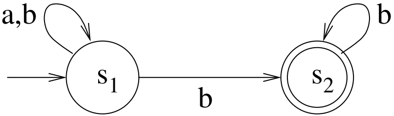
Nondeterministic Finite State Automaton (NFA)

NFA A_2 .



- A run of A_2 over the word *abbb* is:
$$s_1 \xrightarrow{a} s_1 \xrightarrow{b} s_1 \xrightarrow{b} s_2 \xrightarrow{b} s_2.$$
✓ accepting
- Another run over the same word *abbb* is:
$$s_1 \xrightarrow{a} s_1 \xrightarrow{b} s_1 \xrightarrow{b} s_1 \xrightarrow{b} s_1.$$
non-accepting.

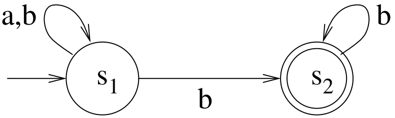
NFA A_2 .



- A run of A_2 over the word *abbb* is:
$$s_1 \xrightarrow{a} s_1 \xrightarrow{b} s_1 \xrightarrow{b} s_2 \xrightarrow{b} s_2.$$
- Another run over the same word *abbb* is:
$$s_1 \xrightarrow{a} s_1 \xrightarrow{b} s_1 \xrightarrow{b} s_1 \xrightarrow{b} s_1.$$
- A_2 has no accepting run over the word *aba* as:
$$s_1 \xrightarrow{a} s_1 \xrightarrow{b} s_2 \xrightarrow{a} \times$$

Nondeterministic Finite State Automaton (NFA)

NFA A_2 .



- A run of A_2 over the word *abbb* is:
  $$s_1 \xrightarrow{a} s_1 \xrightarrow{b} s_1 \xrightarrow{b} s_2 \xrightarrow{b} s_2.$$
- Another run over the same word *abbb* is:
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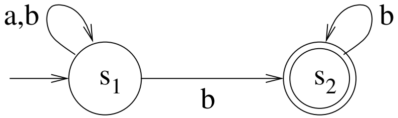
- A_2 has no ~~accepting~~ run over the word *aba* as:
  $$s_1 \xrightarrow{a} s_1 \xrightarrow{b} s_2 \xrightarrow{a} \times$$

NFA can have 0, 1 or more than one runs on a given word.
Run is accepting if it ends in a final state.

Nondeterministic Finite State Automaton (NFA)

NFA A_2 recognises words which end in b.



- A run of A_2 over the word *abbb* is:
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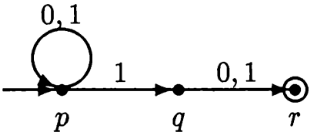
- Another run over the same word *abbb* is:
  $$s_1 \xrightarrow{a} s_1 \xrightarrow{b} s_1 \xrightarrow{b} s_1 \xrightarrow{b} s_1.$$
- A_2 has no accepting run over the word *aba* as:
  $$s_1 \xrightarrow{a} s_1 \xrightarrow{b} s_2 \xrightarrow{a} \times$$

NFA can have 0, 1 or more than one runs on a given word.
Run is accepting if it ends in a final state.

NFA Example

$$A = \{x \in \{0,1\}^* \mid \text{the second symbol from the right is } 1\}.$$

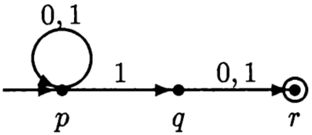


Some possible runs on word 01011

- $p \xrightarrow{0} p \xrightarrow{1} p \xrightarrow{0} p \xrightarrow{1} p \xrightarrow{1} p$

NFA Example

$$A = \{x \in \{0,1\}^* \mid \text{the second symbol from the right is } 1\}.$$



Some possible runs on word 01011

- $p \xrightarrow{0} p \xrightarrow{1} p \xrightarrow{0} p \xrightarrow{1} p \xrightarrow{1} p$
- $p \xrightarrow{0} p \xrightarrow{1} p \xrightarrow{0} p \xrightarrow{1} q \xrightarrow{1} r$

NFA Example

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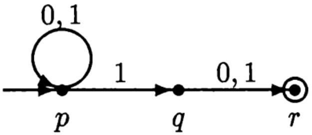

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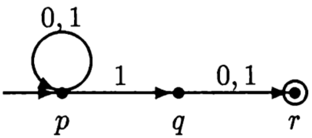
NFA Example

$$A = \{x \in \{0,1\}^* \mid \text{the second symbol from the right is 1}\}.$$



$\Delta(p,1) = \{p,q\}$

$\Delta(p,0) = \{p\}$

$\Delta(q,0) = \phi$

Some possible runs on word 01011

- $p \xrightarrow{0} p \xrightarrow{1} p \xrightarrow{0} p \xrightarrow{1} p \xrightarrow{1} p$
- $p \xrightarrow{0} p \xrightarrow{1} p \xrightarrow{0} p \xrightarrow{1} q \xrightarrow{1} r$
- $p \xrightarrow{0} p \xrightarrow{1} p \xrightarrow{0} p \xrightarrow{1} p \xrightarrow{1} q$
- $p \xrightarrow{0} p \xrightarrow{1} q \xrightarrow{0} r \xrightarrow{1} \times$

Angelic Non-determinism

When will an automaton accept a word? (There can be several runs on the word)

- $\forall$ Nondeterminism Word is accepted if every run is accepting.
- $\exists$ Nondeterminism Word is accepted if there exists a run which is accepting.

In this course we will use $\exists$ nondeterminism throughout. It will just be called nondeterminism.

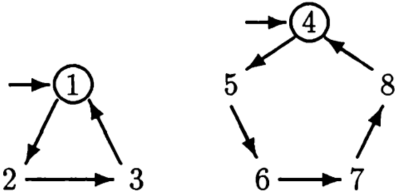
Example: Union

NNFA

$Q = \{1, 2, \ldots, 8\}$

Nondeterministic choice of start state

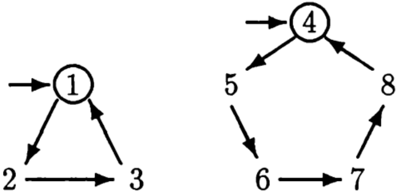
$S = \{1, 4\}$



Some runs on word 11111

- $1 \xrightarrow{1} 2 \xrightarrow{1} 3 \xrightarrow{1} 1 \xrightarrow{1} 2 \xrightarrow{1} 3$ X

Nondeterministic choice of start state



Some runs on word 11111

- $1 \xrightarrow{1} 2 \xrightarrow{1} 3 \xrightarrow{1} 1 \xrightarrow{1} 2 \xrightarrow{1} 3$
- $4 \xrightarrow{1} 5 \xrightarrow{1} 6 \xrightarrow{1} 7 \xrightarrow{1} 8 \xrightarrow{1} 4$

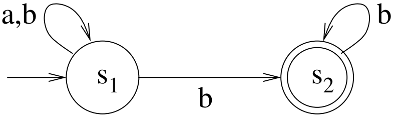
NFA Definition

An NFA is $(Q, \Sigma, \Delta, S, F)$

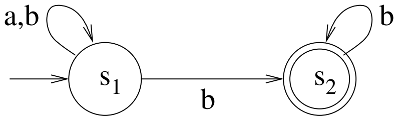
- Q Finite set of states.
- Σ Alphabet – finite set of letters.
- $S \subseteq Q$ set of initial states.
- $F \subseteq Q$ set of final states.
- Nondeterminstic transition function $\Delta : Q \times \Sigma \rightarrow 2^Q$
  Notation $2^Q \; = \; \{A \mid A \subseteq Q\}$.
  We use $p \xrightarrow{a} q$ to denote $q \in \Delta(p, a)$.

Note that every DFA is an NFA.

Three Representations of NFA



Three Representations of NFA

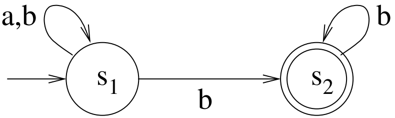


Five Tuple

$(\{s_1, s_2\},\ \{a, b\}, \Delta, \{s_1\}, \{s_2\})$ where

$$\begin{array}{lllllll} \Delta(s_1, a) & = & \{s_1\}, & & \Delta(s_1, b) & = & \{s_1, s_2\} \\ \Delta(s_2, a) & = & \emptyset, & & \Delta(s_2, b) & = & \{s_2\} \end{array}$$

Three Representations of NFA



Transition diagram

Five Tuple

Q Σ fn st st. final.

5 tuple

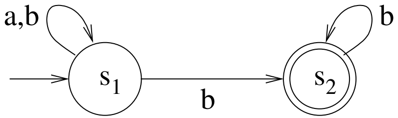
$(\{s_1, s_2\}, \{a, b\}, \Delta, \{s_1\}, \{s_2\})$ where

$$\begin{aligned}
\Delta(s_1, a) &= \{s_1\}, & \Delta(s_1, b) &= \{s_1, s_2\} \\
\Delta(s_2, a) &= \emptyset, & \Delta(s_2, b) &= \{s_2\}
\end{aligned}$$

Transition Table

	a	b
s_1	$\{s_1\}$	$\{s_1, s_1\}$ 2
s_2	$\emptyset$	$\{s_2\}$

Transition table



Five Tuple

$(\{s_1, s_2\},\ \{a, b\}, \Delta, \{s_1\}, \{s_2\})$ where

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\Delta(s_1, a) & = & \{s_1\}, & \Delta(s_1, b) & = & \{s_1, s_2\} \\
\Delta(s_2, a) & = & \emptyset, & \Delta(s_2, b) & = & \{s_2\}
\end{array}
$$

Transition Table

			a	b
$\rightarrow$	s_1		$\{s_1\}$	$\{s_1, s_1\}$
	s_2	F	$\emptyset$	$\{s_2\}$

NFA semantics

- A run of NFA N on $x = a_0, a_1, \ldots, a_{n-1}$ is a sequence of states $q_0, q_1, \ldots, q_n$ s.t. $q_i \xrightarrow{a_i} q_{i+1}$ for $0 \leq i < n$ and $q_0 \in I$.

NFA semantics

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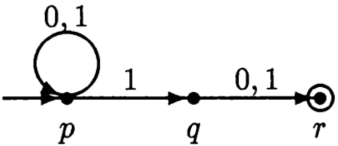
- For a given word x the NFA can have zero, one or more runs.
- A run accepting if last state $q_n \in F$.
- Language accepted by N is $L(N) = \{x \in \Sigma^* \mid N \text{ has an accepting run on } x\}$.

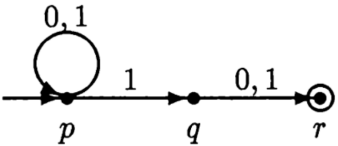
Extended Transition Function

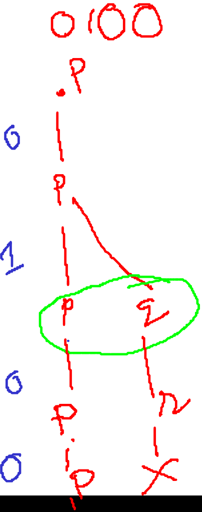
Let $\hat{\Delta}(\{q\}, x)$ give all the states which can be reached at the end of some run starting from q on word x.



Extended Transition Function (2)

For $A \subseteq Q$, $\hat{\Delta}(A, x)$ give all the states which can be reached at the end of some run starting from some $q \in A$ on word x.





$\hat{\Delta}(\{p\}, 01) = \{p, q\}$

$\hat{\Delta}(\{p\}, 0100) = \{p\}$

Properties

- $\hat{\Delta}(A, x) \quad = \quad \bigcup_{q \in A} \hat{\Delta}(\{q\}, x)$

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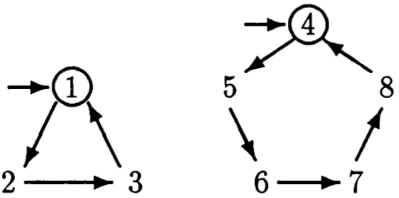
- $\hat{\Delta}(A, a) \quad = \quad \bigcup_{q \in A} \Delta(q, a)$

Properties

- $\hat{\Delta}(A, x) \ = \ \bigcup_{q \in A} \hat{\Delta}(\{q\}, x)$
- $\hat{\Delta}(A, \epsilon) \ = \ A$
- $\hat{\Delta}(A, a) \ = \ \bigcup_{q \in A} \Delta(q, a)$
- $\hat{\Delta}(A, xa) \ = \ \bigcup_{q \in \hat{\Delta}(A, x)} \Delta(q, a)$

Example 2



$$\hat{\Delta}(\{1,5\},11) = \{3,7\}$$

Formal Definition of Extended Transition Function

Definition (Extended Nondeterministic Transition Function)

Define $\hat{\Delta} : 2^Q \times \Sigma^* \rightarrow 2^Q$ using structural induction:

$$
\begin{aligned}
\hat{\Delta}(A, \epsilon) &= A \\
\hat{\Delta}(A, xa) &= \bigcup_{q \in \hat{\Delta}(A,x)} \Delta(q, a)
\end{aligned}
$$

Definition (Extended Nondeterministic Transition Function)

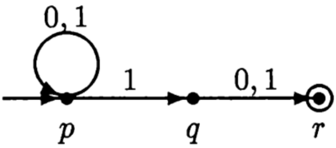
Define $\hat{\Delta} : 2^Q \times \Sigma^* \to 2^Q$ using structural induction:

$$\begin{aligned} \hat{\Delta}(A, \epsilon) &= A \\ \hat{\Delta}(A, xa) &= \bigcup_{q \in \hat{\Delta}(A,x)} \Delta(q, a) \end{aligned}$$

Note that

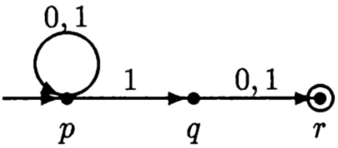
$$\left.\begin{aligned} \hat{\Delta}(A, a) &= \bigcup_{q \in \hat{\Delta}(A,\epsilon)} \Delta(q, a) \\ &= \underline{\bigcup_{q \in A} \Delta(q, a)} \end{aligned}\right\}$$

Extended Transition Function Computation



Calculate $\hat{\Delta}(\{p\}, 010)$

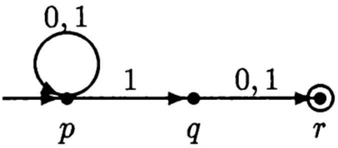
Extended Transition Function Computation



Calculate $\hat{\Delta}(\{p\}, 010)$

- $\hat{\Delta}(\{p\}, \epsilon) \;=\; \{p\}$

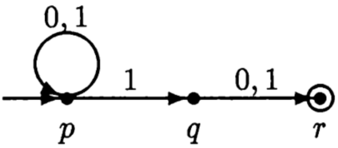
Extended Transition Function Computation



Calculate $\hat{\Delta}(\{p\}, 010)$

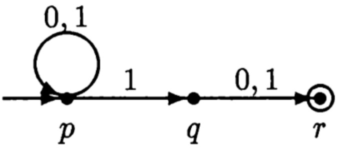
- $\hat{\Delta}(\{p\}, \epsilon) \;=\; \{p\}$
- $\hat{\Delta}(\{p\}, 0) \;=\; \bigcup_{s \in \{p\}} \; \Delta(s, a)$
  $= \; \{p\}$

Extended Transition Function Computation



Calculate $\hat{\Delta}(\{p\}, 010)$

- $\hat{\Delta}(\{p\}, \epsilon) \;=\; \{p\}$
- $\hat{\Delta}(\{p\}, 0) \;=\; \bigcup_{s \in \{p\}} \Delta(s, a)$
  $= \{p\}$
- $\hat{\Delta}(\{p\}, 01) \;=\; \bigcup_{s \in \hat{\Delta}(\{p\},0)} \Delta(s, 1)$
  $= \bigcup_{s \in \{p\}} \Delta(s, 1) = \Delta(p, 1)$
  $= \{p, q\}$

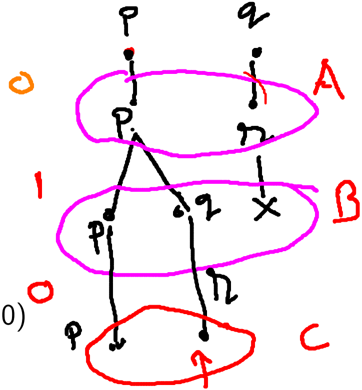


$= \{p, r\}$

$\hat{\Delta}(\{p, r\}, 010)$

Calculate $\hat{\Delta}(\{p\}, 010)$

- $\hat{\Delta}(\{p\}, \epsilon) \;=\; \{p\}$
- $\hat{\Delta}(\{p\}, 0) \;=\; \bigcup_{s \in \{p\}} \Delta(s, a)$
  $= \{p\}$
- $\hat{\Delta}(\{p\}, 01) \;=\; \bigcup_{s \in \hat{\Delta}(\{p\}, 0)} \Delta(s, 1)$
  $= \bigcup_{s \in \{p\}} \Delta(s, 1) = \Delta(p, 1)$
  $= \{p, q\}$
- $\hat{\Delta}(\{p\}, 010) \;=\; \bigcup_{s \in \hat{\Delta}(\{p\}, 01)} \Delta(s, 0)$
  $= \bigcup_{s \in \{p, q\}} \Delta(s, 0)$
  $= \Delta(p, 0) \cup \Delta(q, 0)$
  $= \{p\} \cup \{r\}$



Some Properties

Theorem

$\hat{\Delta}(A, xy) \ = \ \hat{\Delta}(\ \hat{\Delta}(A, x),\ \ y)$ *Proof: see Kozen Lemma 6.1*

The proof is exactly identical to the similar proof for DFA given in the last class.

Some Properties

$N = (Q, \Sigma, \Delta, S, F)$

Theorem

$\hat{\Delta}(A, xy) \;=\; \hat{\Delta}(\;\hat{\Delta}(A, x),\; y)$ *Proof: see Kozen Lemma 6.1*

The proof is exactly identical to the similar proof for DFA given in the last class.

Theorem

If $\hat{\Delta}(A, x) = B$ *then for every* $q \in B$ *there exists* $p \in A$ *s.t. there is a run from* p *to* q *on word* x.

Suppose $\hat{\Delta}(S, x) = B$

When will $x \in L(N)$?

Some Properties

$N = (Q, \Sigma, \Delta, S, F)$

Theorem

$\hat{\Delta}(A, xy) \; = \; \hat{\Delta}(\; \hat{\Delta}(A, x), \; y)$ *Proof: see Kozen Lemma 6.1*

The proof is exactly identical to the similar proof for DFA given in the last class.

Theorem

If $\hat{\Delta}(A, x) = B$ then for every $q \in B$ there exists $p \in A$ s.t. there is a run from p to q on word x.

Theorem

$x \in L(N)$ if and only if $\hat{\Delta}(S, x) \cap F \neq \emptyset$

Definition

Language of NFA Given NFA $N = (Q, \Sigma, \Delta, S, F)$

- N accepts word x iff $(\hat{\Delta}(S, x) \cap F) \neq \emptyset$
- Language recognized by N is
  $L(N) = \{x \in \Sigma^* \mid N \text{ accepts } x\}$

Is NFA a machine?

NFA   specification or Model

DFA   implementation

Is given NFA implementable?

Theorem (Rabin-Scott)

For every NFA $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$ we can construct DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ such that $L(N) = L(M)$.

Construction of M is called the subset construction. Each subset of Q_N becomes a state of M.

NFA N $\xmapsto{\text{subset cons.}}$ DFA M

Q_N

Q_M

Theorem (Rabin-Scott)

For every NFA $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$ we can construct DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ such that $L(N) = L(M)$.

Construction of M is called the subset construction. Each subset of Q_N becomes a state of M.

$$Q_M \stackrel{\text{def}}{=} 2^{Q_N},$$

$$\delta_M(A,a) \stackrel{\text{def}}{=} \underline{\widehat{\Delta}_N(A,a)},$$

$$s_M \stackrel{\text{def}}{=} S_N,$$

$$F_M \stackrel{\text{def}}{=} \{A \subseteq Q_N \mid A \cap F_N \neq \varnothing\}.$$

$$\widehat{\Delta}(A,a) = \bigcup_{q \in A} \Delta(q,a)$$

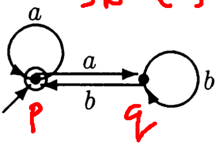
Subset Construction by Example

NFA

$\{p\}$

$S_N = \{p\}$

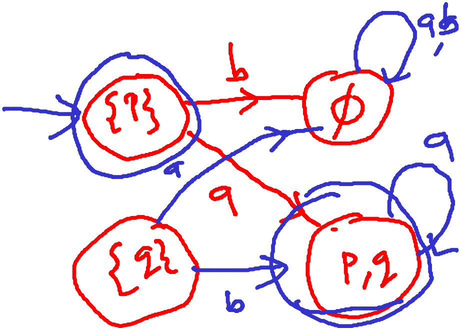
$Q_N = \{p, q\}$



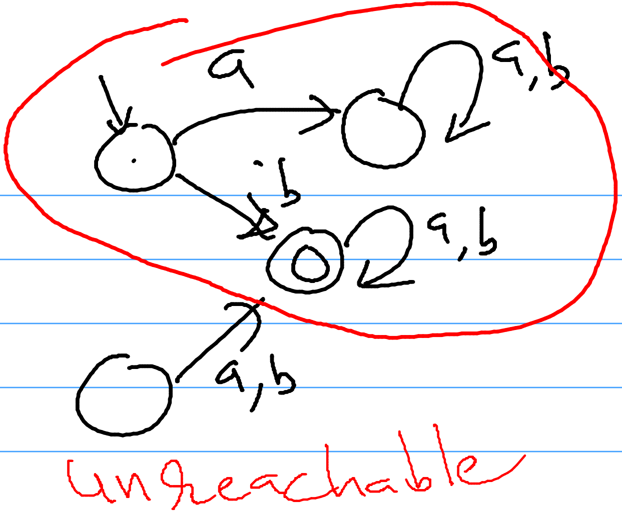
$Q_M = \{\phi, \{p\}, \{q\}, \{p,q\}\}$

Q_M DFA

	a	b	
S_0 ϕ	ϕ	ϕ	
$S_1 \rightarrow \{p\}$	$\{p,q\}$	ϕ	F
S_2 $\{q\}$	ϕ	$\{p,q\}$	
S_3 $\{p,q\}$	$\{p,q\}$	$\{p,q\}^F$	



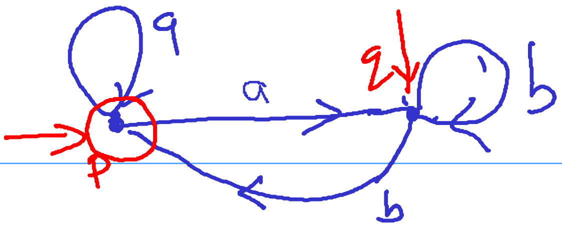
$$\hat{\Delta}_N(\{p,q\}, a) = \{p,q\}$$


a
a,b
b
a,b
a,b
unreachable

$\Sigma = \{a, b\}$

NFA

N



$Q_N = \{p, q\}$ $\quad S_N = \{p, q\}$

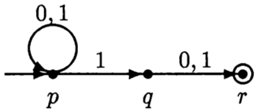
$F_N = \{p\}$

DFA

	a	b
$\emptyset$	$\emptyset$	$\emptyset$
$\{p\}$ F	$\{p, q\}$	$\emptyset$
$\{q\}$	$\emptyset$	$\{p, q\}$
$\rightarrow$ $\{p, q\}$ F	$\{p, q\}$	$\{p, q\}$

Subset construction Example 2

NFA N



$Q_N = \{p, q, r\}$

$S_N = \{p\}$

$F_N = \{r\}$

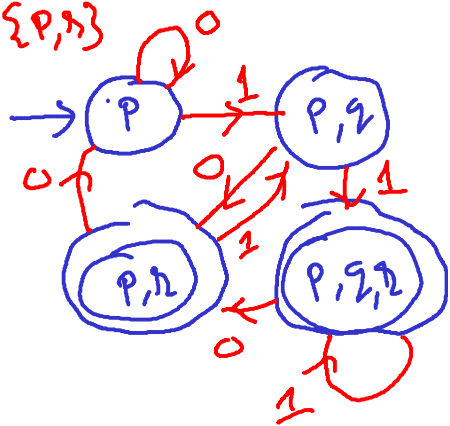
p p p q r

0110

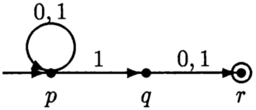
DFA

$\{p\}\{p\}\{p,q\}\{p,q,r\}\{p,r\}$

	0	1
→ $\{p\}$	$\{p\}$	$\{p,q\}$
$\{p,q\}$	$\{p,r\}$	$\{p,q,r\}$
$\{p,r\}$ F	$\{p\}$	$\{p,q\}$
$\{p,q,r\}$ F	$\{p,r\}$	$\{p,q,r\}$



N



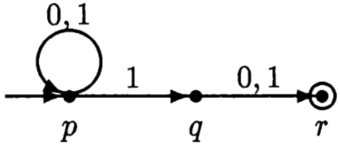
$S_N = \{p\}$
$F_N = \{r\}$

DFA
M

	0	1
$\varnothing$	$\varnothing$	$\varnothing$
$\rightarrow \{p\}$	$\{p\}$	$\{p,q\}$
$\{q\}$	$\{r\}$	$\{r\}$
$\{r\}F$	$\varnothing$	$\varnothing$
$\{p,q\}$	$\{p,r\}$	$\{p,q,r\}$
$\{p,r\}F$	$\{p\}$	$\{p,q\}$
$\{q,r\}F$	$\{r\}$	$\{r\}$
$\{p,q,r\}F$	$\{p,r\}$	$\{p,q,r\}$

$\hat{\Delta}(\{p,q\},0)$
$\hat{\Delta}(\{p,q\},1)$

Keep only reachable subsets



	0	1
$\rightarrow \{p\}$	$\{p\}$	$\{p,q\}$
$\{p,q\}$	$\{p,r\}$	$\{p,q,r\}$
$\{p,r\}F$	$\{p\}$	$\{p,q\}$
$\{p,q,r\}F$	$\{p,r\}$	$\{p,q,r\}$

Properties

- $\hat{\Delta}(A, x) \;=\; \bigcup_{q \in A} \hat{\Delta}(q, x)$
- $\hat{\Delta}(A, \epsilon) \;=\; A$
- $\hat{\Delta}(A, a) \;=\; \bigcup_{q \in A} \Delta(q, a)$

Theorem

If $\hat{\Delta}(A, x) = B$ then for every $q \in B$ there exists $p \in A$ s.t. there is a run from p to q on word x.

Theorem

$x \in L(M)$ if and only if $\hat{\Delta}(S, x) \cap F \neq \emptyset$

NFA Determinization

Theorem (Rabin-Scott)

For every NFA $N = (Q_N, \Sigma, \Delta_N, S_N, F_N)$ we can construct DFA $M = (Q_M, \Sigma, \delta_M, s_M, F_M)$ such that $L(N) = L(M)$.

Construction of M is called the subset construction. Each subset of Q_N becomes a state of M.

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$$
\begin{aligned}
Q_M &\stackrel{\text{def}}{=} 2^{Q_N}, \\
\delta_M(A, a) &\stackrel{\text{def}}{=} \widehat{\Delta}_N(A, a), \\
s_M &\stackrel{\text{def}}{=} S_N, \\
F_M &\stackrel{\text{def}}{=} \{A \subseteq Q_N \mid A \cap F_N \neq \varnothing\}.
\end{aligned}
$$

$\hat{\Delta}$ = operates on sets instead of elements of S

Lemma 6.9

Defn of M

For any $A \subseteq Q_N$ and $x \in \Sigma^*$,

$$\boxed{\widehat{\delta}_M(A, x) = \widehat{\Delta}_N(A, x).}$$

$\delta_M(A, a) = \widehat{\Delta}_N(A, a)$

Proof. Induction on $|x|$.

Basis

For $x = \epsilon$, we want to show

$$\widehat{\delta}_M(A, \epsilon) = \widehat{\Delta}_N(A, \epsilon).$$

$\widehat{\delta}(A, \epsilon) = A$

$\widehat{\Delta}(A, \epsilon) = A$

But both of these are A, by definition of $\widehat{\delta}_M$ and $\widehat{\Delta}_N$.

Proof Of Determization Theorem (2)

$\delta(q, a) = r$.

Induction Step:

Assume that

$$\widehat{\delta}_M(A, x) = \widehat{\Delta}_N(A, x).$$

We want to show the same is true for xa, $a \in \Sigma$.

$$
\begin{aligned}
\widehat{\delta}_M(A, xa) &= \delta_M(\widehat{\delta}_M(A, x), a) && \text{definition of } \widehat{\delta}_M \\
&= \delta_M(\widehat{\Delta}_N(A, x), a) && \text{induction hypothesis} \\
&= \widehat{\Delta}_N(\widehat{\Delta}_N(A, x), a) && \text{definition of } \delta_M \\
&= \widehat{\Delta}_N(A, xa) && \text{Lemma 6.1.}
\end{aligned}
$$

Proof Of Determization Theorem (3)

Theorem

For NFA N and constructed DFA M we have L(M) = L(N)

Proof. For any $x \in \Sigma^*$,

DFA

$$
\begin{array}{lll}
& x \in L(M) & \\
\iff & \widehat{\delta}_M(s_M, x) \in F_M & \text{definition of acceptance for } M \\
\iff & \widehat{\Delta}_N(S_N, x) \cap F_N \neq \varnothing & \text{definition of } s_M \text{ and } F_M\text{, Lemma (} \\
\iff & x \in L(N) & \text{definition of acceptance for } N.
\end{array}
$$

Size of the Constructed DFA

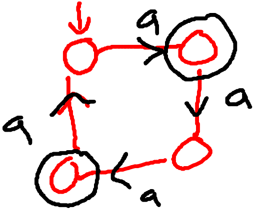
Given NFA N with n states what is the maximum number of states in the DFA M constructed by the subset construction?

DFA M $\leq 2^n$ state.

Two ways

→ Unreachable states.

→



(Same language)

Size of the Constructed DFA

Given NFA N with n states what is the maximum number of states in the DFA M constructed by the subset construction?

$\underbrace{\text{Is there a better construction giving smaller size DFA?}}_{}$

Lower bound for # states in DFA (of NFA)

2^N

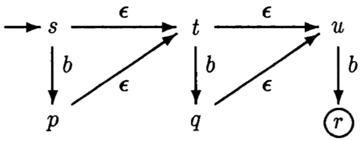
Size of the Constructed DFA

Given NFA N with n states what is the maximum number of states in the DFA M constructed by the subset construction?

Is there a better construction giving smaller size DFA?

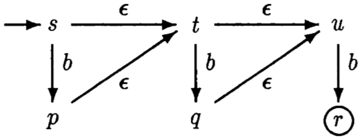
Answer: No in worst case.

Automata with Silent transitions



An ϵ-NFA is like an NFA $M = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$ where $\Delta : Q \times (\Sigma \cup \{\epsilon\}) \to 2^Q$. Symbol $\epsilon \notin \Sigma$ denotes silent action.

Automata with Silent transitions



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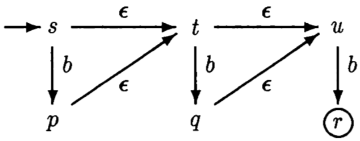
- Silent transtions $p \xrightarrow{\epsilon} q$.
  A silent transition can be spontaneously made without consuming any letter.



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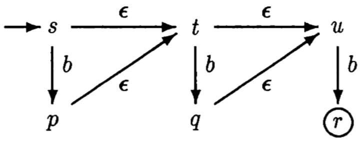
Automata with Silent transitions



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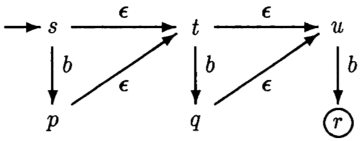
- Note that $\epsilon b \epsilon b$ is internally accepted by above N.
  We say that bb is externally accepted by N.

Automata with Silent transitions



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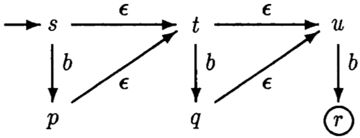
Automata with Silent transitions



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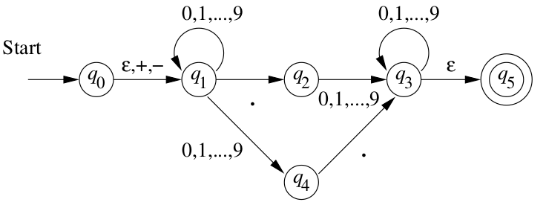
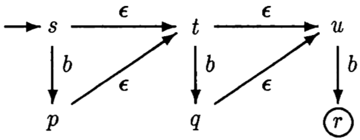
- The language
  $L(M) = \{w \Downarrow \; \mid \; w \text{ is internally accepted by } N\}$.
  Language $\{\epsilon, \; b, \; bb, \; bbb\}$

Example: Decimal numbers



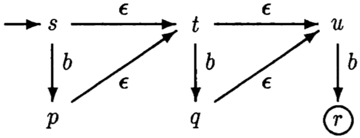
Automata with silent transitions



An ϵ-NFA is like an NFA $M = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$.

- Let $p \xRightarrow{\epsilon^*} q$ denote a finite seqeunce of epsilon moves from p to q. .

Automata with silent transitions



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- Let $p \stackrel{\epsilon^*}{\Longrightarrow} q$ denote a finite seqeunce of epsilon moves from p to q. E.g. $s \stackrel{\epsilon^*}{\Longrightarrow} u$.
- Epsilon-closure: For $A \subseteq Q$, let $C_\epsilon(A) = \{q \mid \exists p \in A.\ p \stackrel{\epsilon^*}{\Longrightarrow} q\}$. It denotes all states reachable by epsilon paths from states in A.

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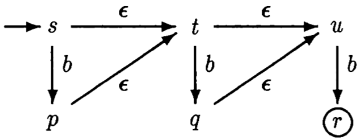
Automata with silent transitions



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  E.g. $C_\epsilon(\{p, r\}) = \{p, t, u, r\}$.

Equivalence

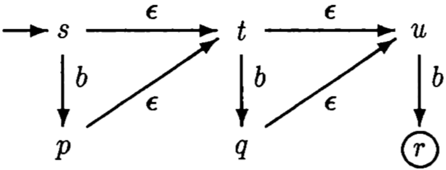
Theorem

For every ϵ-NFA $N = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$ we can construct a DFA $M = (Q_M, \Sigma, \Delta_M, s_M, F_M)$ such that $L(M) = L(N)$.

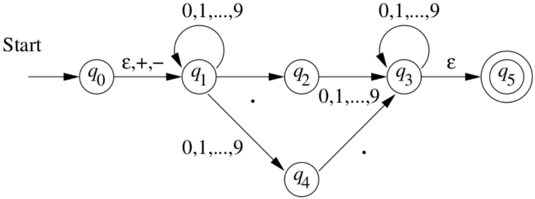
Modified Subset Construction of M

- $Q_M \;=\; \{X \subseteq Q \;\mid\; C_\epsilon(X) = X\}$ epsilon closed subsets.
- $s_M \;=\; C_\epsilon(S)$.
- $F_M \;=\; \{X \in Q_M \;\mid\; X \cap F \neq \emptyset\}$
- $\Delta_M(X, a) \;=\; C_\epsilon(\hat{\Delta}(X, a))$.

Equivalent NFA construction Example



Example: Decimal numbers



Constructed DFA for Decimal NFA