CS310M: Automata Theory (Minor)

Topic 4: Regular Expressions

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Course URL: https://cse.iitb.ac.in/∼pandya58/CS310M/automata.html

Autumn, 2021

Specifying Patterns of Occurrence of Letters

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- E.g. $L(a^* \cdot b + b \cdot a^*) = \{a^n \cdot b,\ b \cdot a^n \mid n \geq 0\}$

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- @ Matches any word from Σ. $L(@) = \Sigma^*$

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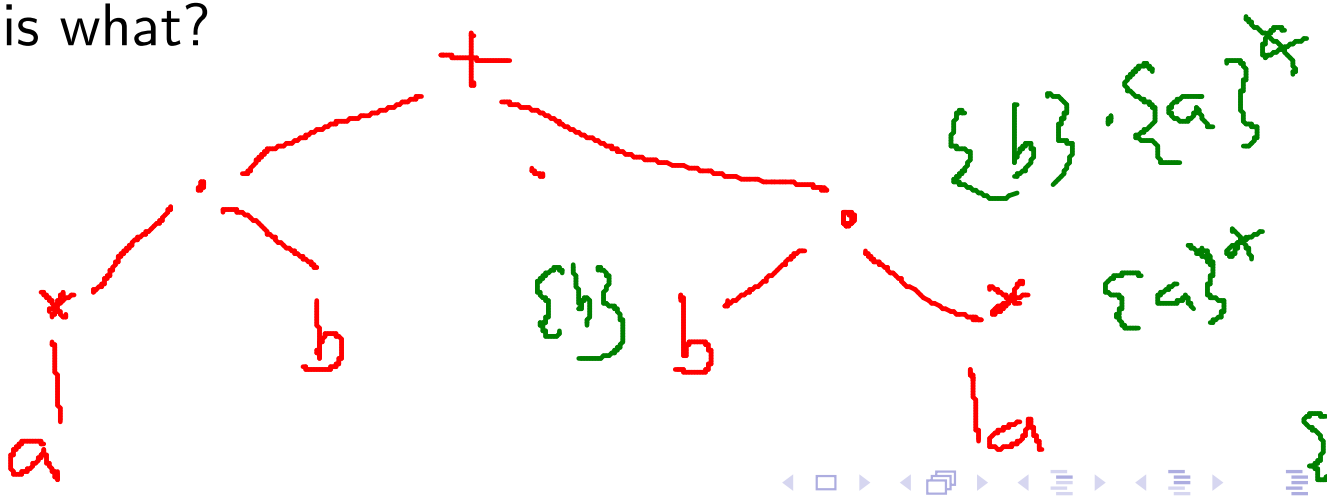

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- $(\alpha \cap \beta)$. Intersection. $L((\alpha \cap \beta)) = L(\alpha) \cap L(\beta)$.
- $(\sim \alpha)$. Complement.
  $L((\sim \alpha)) = \sim L(\alpha) = \Sigma^* - L(\alpha)$.

Example of Pattern expression and its langauge

Let α be the pattern $(((a^*) \cdot b) + (b \cdot (a^*)))$. Then $L(\alpha)$ is what?

Inductive Definition

- Every pattern expression is uniquely constructed from atomic patterns by successive application of compound operators.
- The parse tree makes clear the structure of the expression.
- THe language of pattern expression can be computed in bottom up fashion by applying rules for each node.

This method is quite general. Also used to define programming languages.

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- Example $(((a^*) \cdot b) + (b \cdot (a^*)))$ can be written as $a^*b + ba^*$.

Examples

- $L(@) = \Sigma^* = L(\#^*)$
- a^*b^*
- $@a@a@a@$ gives words with three occurrences of a
- $(\#\cap a)^*$ strings without letter a.
- Sentences having "cat" and "nation" in it.
  $(@cat@) \cap (@nation@)$.
  E.g. The nation of cats. and The catenation of two words is a word.

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Example $(a+b)c \equiv ac + bc$.

Redundant Operators

- Let $\Sigma = \{a_1, a_2, \ldots, a_n\}$. Then $\# \equiv (a_1 + a_2 + \ldots + a_n)$.
- $@ \equiv \#^*$.
- $\alpha \cap \beta \equiv \sim(\sim \alpha \cup \sim \beta)$.
- $\sim$ is also redundant (difficult proof).

Regular Expressions

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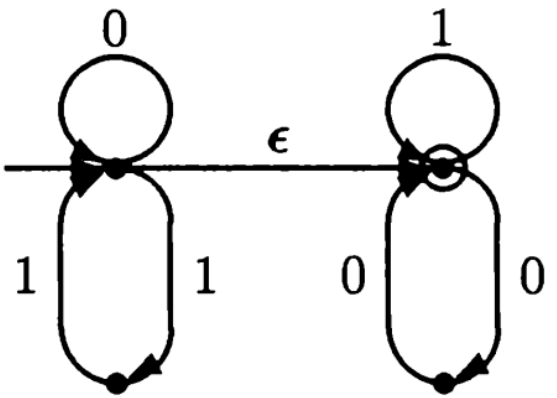
Regular Expressions

Pattern expressins without $@, \#, \cap, \sim$. Abstract syntax: let α, β range over *RegExp*.

$$\alpha ::= a \mid \phi \mid \epsilon \mid \alpha \cdot \beta \mid \alpha + \beta \mid \alpha^*$$

Question What kind of languages are definable using regular expressions? Can they be accepted by a DFA?

$$(11+0)^*(00+1)^*$$



In general?

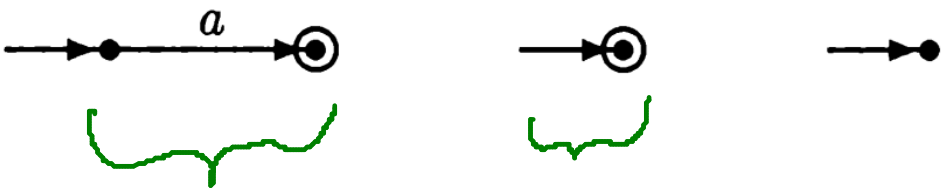
From RegExp to ϵ-NFA

Theorem

For every regular expression α we can construct a normalized ϵ-NFA $A(\alpha)$ such that $L(\alpha) = L(A(\alpha))$.

Proof (By stuctural induction). We construct the automaton bottom up.

- Normalized ϵ-NFA for a, ϵ, ϕ respectively.



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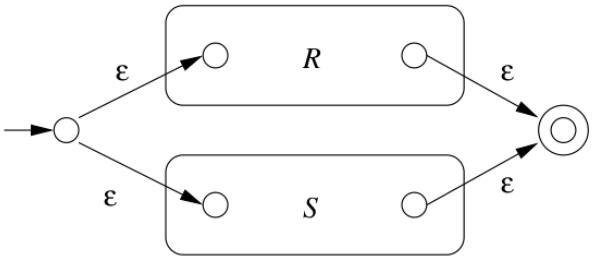
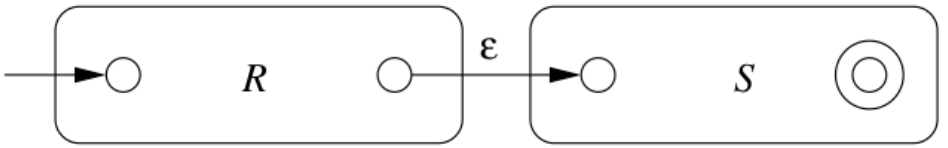
- For $\alpha + \beta$, inductively assume that we have normalized ϵ-NFA $A(\alpha)$ and $A(\beta)$. Construct union automaton recognizing $L(A(\alpha)) \cup L(A(\beta))$ as before.

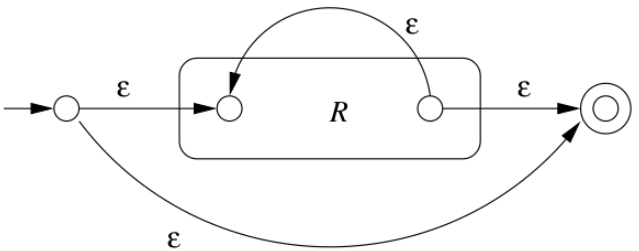


Proof (cont)

- For $\alpha \cdot \beta$, inductively assume that we have normalized ϵ-NFA $A(\alpha)$ and $A(\beta)$. Construct catenation automaton recognizing $L(A(\alpha)) \cdot L(A(\beta))$ as before.



- For α^*, inductively assume that we have normalized ϵ-NFA $A(\alpha)$. Construct Kleene closure automaton recognizing $L(A(\alpha))^*$ as before.



Size of constructed automaton

- size of regular expression α counts number of atomic expressions and number of composition operators in α. This is denoted $|\alpha|$.
  E.g. $|(a+b)^*aaa| = 11$.
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  - Similar induction steps for $\beta \cdot \gamma$ and β^*

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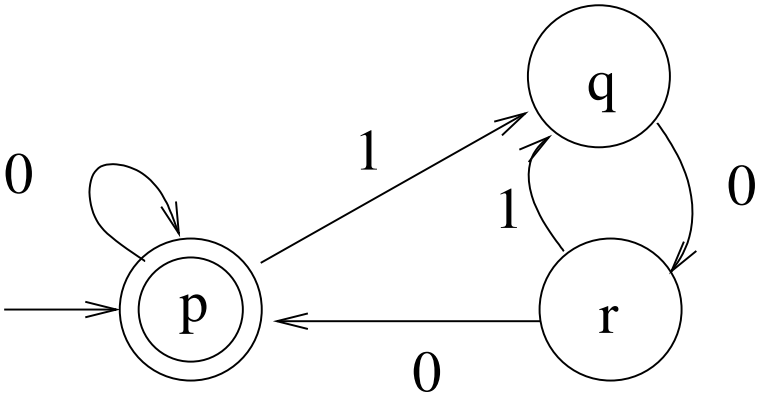

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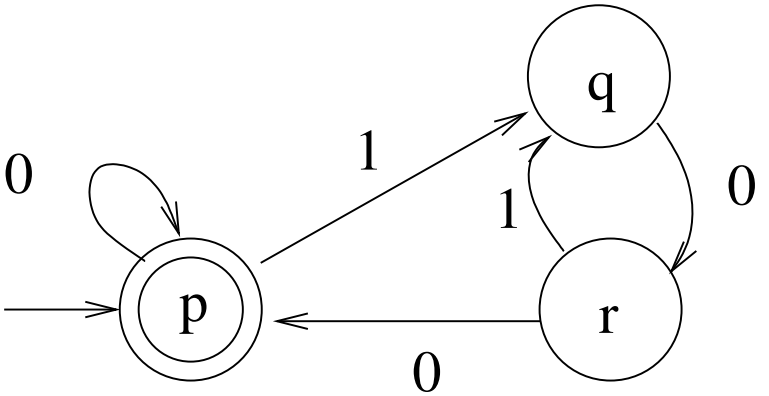

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(Mcnaughton - Yamada Lemma)

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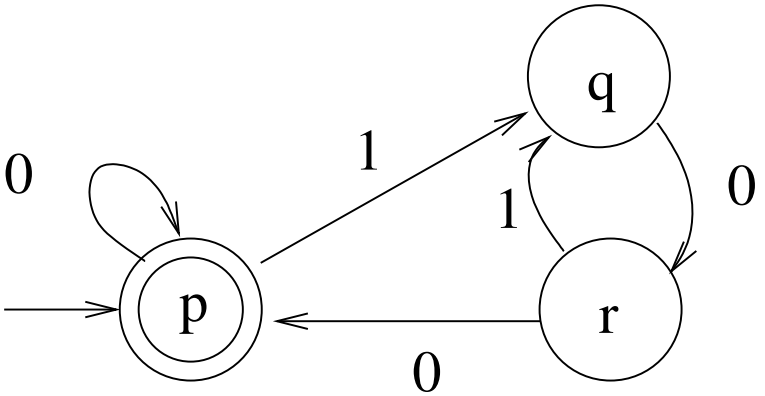
(via some sort of DP)



Example:

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Aim: To compute $\alpha^{p,q,r}_{p,p}$

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- For any $q \in X$,

$$\alpha^X_{p,r} = \alpha^{(X-\{q\})}_{p,r} + \alpha^{(X-\{q\})}_{p,q} \cdot \left(\alpha^{(X-\{q\})}_{q,q}\right)^* \cdot \alpha^{(X-\{q\})}_{q,r}$$

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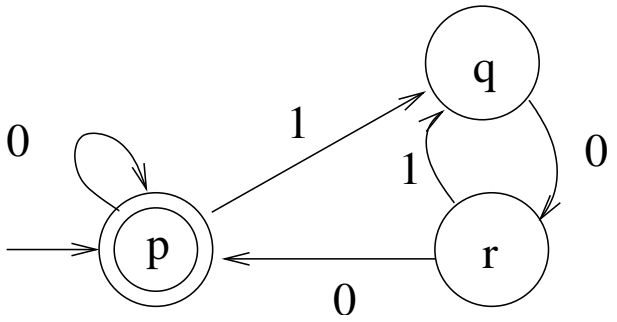
The desired regular expression is given as follows:

$$\sum_{p \in I} \sum_{q \in F} \alpha^Q_{p,q}.$$

Example

automaton → regex $O(4^n)$



Compute $\alpha_{p,p}^{p,q,r}$ as

$$\alpha_{p,p}^{p,q,r} = \alpha_{p,p}^{p,r} + \alpha_{p,q}^{p,r} \cdot (\alpha_{q,q}^{p,r})^* \cdot \alpha_{q,p}^{p,r}.$$

It is easy to see that

$$\alpha_{p,p}^{p,r} = 0^*, \qquad \alpha_{p,q}^{p,r} = 0^* \cdot 1$$
$$\alpha_{q,p}^{p,r} = 00(0^*), \qquad \alpha_{q,q}^{p,r} = \epsilon + 01 + 00(0^*)1$$

Hence $\alpha_{p,p}^{p,q,r}$

$$= 0^* + 0^*1 \cdot (\epsilon + 01 + 00(0^*)1)^* \cdot 00(0^*)$$

} exponential blowup.

every exp → a tuple exp

Kleene Algebra (summary)

simplification of regular expression (language equivalent regex)

$\alpha = \beta$ iff $L(\alpha) = L(\beta)$

$\alpha \leq \beta$ iff $L(\alpha) \subseteq L(\beta)$

Algebraic Laws for equivalence of regular expressions. Useful in simplifying regexp.

- $\equiv$ is reflexive, symmetric and transitive.
- $\equiv$ is a congruence.
- $(Regexp, +, \phi)$ is a commutative monoid with $+$ idempotent.
- $(Regexp, \cdot, \epsilon)$ is a monoid.
- Distributivity: $(\alpha + \beta)\gamma \equiv (\alpha\gamma + \beta\gamma)$ and $\gamma(\alpha + \beta) \equiv (\gamma\alpha + \gamma\beta)$
- Rules for Kleene closure α^*. See Kozen.

Kleene Algebra (1)

$$\alpha + (\beta + \gamma) \equiv (\alpha + \beta) + \gamma$$

$$\alpha + \beta \equiv \beta + \alpha$$

$$\alpha + \emptyset \equiv \alpha$$

$$\alpha + \alpha \equiv \alpha$$

$$\alpha(\beta\gamma) \equiv (\alpha\beta)\gamma$$

$$\epsilon\alpha \equiv \alpha\epsilon \equiv \alpha$$

$$\alpha(\beta + \gamma) \equiv \alpha\beta + \alpha\gamma$$

$$(\alpha + \beta)\gamma \equiv \alpha\gamma + \beta\gamma$$

$$\emptyset\alpha \equiv \alpha\emptyset \equiv \emptyset$$

Kleene Algebra Example

$$
\begin{aligned}
(1+01+001)^*(\epsilon+0+00) &\equiv ((\epsilon+0+00)1)^*(\epsilon+0+00) \\
&\equiv ((\epsilon+0)(\epsilon+0)1)^*(\epsilon+0)(\epsilon+0).
\end{aligned}
$$

Kleene Algebra (2)

$$\epsilon + \alpha\alpha^* \equiv \alpha^*$$
$$\epsilon + \alpha^*\alpha \equiv \alpha^*$$
$$\beta + \alpha\gamma \leq \gamma \Rightarrow \alpha^*\beta \leq \gamma$$
$$\beta + \gamma\alpha \leq \gamma \Rightarrow \beta\alpha^* \leq \gamma$$

laws are sound and complete

Soundness : $\alpha \equiv_A \beta \Rightarrow L(\alpha) = L(\beta)$

Completeness : $L(\alpha) = L(\beta) \Rightarrow \alpha \equiv_A \beta$

Useful Derived Identities

$$(\alpha\beta)^*\alpha \equiv \alpha(\beta\alpha)^*$$
$$(\alpha^*\beta)^*\alpha^* \equiv (\alpha + \beta)^*$$
$$\alpha^*(\beta\alpha^*)^* \equiv (\alpha + \beta)^*$$
$$(\epsilon + \alpha)^* \equiv \alpha^*$$
$$\alpha\alpha^* \equiv \alpha^*\alpha$$

Kleene Algebra Example (2)

$|RE| \to O(n)$

$|A(\alpha)| = ?$

$\varepsilon RE \mapsto O(n)$

$|A(\alpha)| = O(2^{2^{\cdots^{2}}} n)$ ← lower bound.

$$
\begin{aligned}
& 0^* + 0^*1(\epsilon + 00^*1)^*000^* \\
& \equiv 0^* + 0^*1(00^*1)^*000^* && \text{by (9.17)} \\
& \equiv \epsilon + 00^* + 0^*10(0^*10)^*00^* && \text{by (9.10) and (9.14)} \\
& \equiv \epsilon + (\epsilon + 0^*10(0^*10)^*)00^* && \text{by (9.8)} \\
& \equiv \epsilon + (0^*10)^*00^* && \text{by (9.10)} \\
& \equiv \epsilon + (0^*10)^*0^*0 && \text{by (9.18)} \\
& \equiv \epsilon + (0 + 10)^*0 && \text{by (9.15).}
\end{aligned}
$$

(recursion)

naive method.

Extended regular expression (ERE) ← doesn't add power (Reg → DFA → operations → Reg)

$\emptyset \mid \varepsilon \mid a \mid r_1 + r_2 \mid r_1 \cdot r_2 \mid r^* \mid \underbrace{r_1 \cap r_2 \mid \sim r}_{\text{(extended re)}}$

"succintness" : descriptional complexity

sere (semi-extended re)

(ERE → RE : $n \mapsto 2^{2^{\cdots^{2}}}$ } height $\sim n$)