Equivalence relations, Countability

Q1 Let f be a one-to-one function from a set A to itself. Prove formally by induction that if A is finite then f is also an onto function and hence a bijection. Give an example of a set A and function f which is one-to-one but not onto. Prove that if A is not finite there exists a one-to-one function from A to itself which is not onto.

Q2 Let $A_1, A_2, \ldots$ be an infinite sequence of sets such that the intersection of any finite number of sets in the sequence is not empty. Is it true that there exists an element x such that $x \in A_i$ for all i? If so prove it, else give an example for which it is false. Suppose $a_1, a_2, \ldots$ is an infinite sequence of numbers such that the *gcd* of any finite set of numbers in the sequence is greater than 1. Prove that there exists a prime p such that p divides a_i for all i.

Q3 Let R be any arbitrary relation defined on a set A. Let I denote the identity relation, and R^{-1} the converse of R. Let R^k be the relation defined by $R^0 = I$ and $R^k = R.R^{k-1}$ for $k > 0$. Write down an expression for the smallest equivalence relation E containing R using these and other standard set operations. E is smallest in the sense that any equivalence relation that contains R must also contain E. Suppose E is an equivalence relation on a finite set A with n elements and k equivalence classes. What is the minimum number of elements in a relation R on A such that the smallest equivalence relation containing R is the given relation E? Prove your answer using induction.

Q4 Prove that the intersection of two equivalence relations on a set A (considered as a collection of ordered pairs) is also an equivalence relation. For any relation R on A, the smallest equivalence relation containing R, called the closure of R and denoted R^c is the intersection of all equivalence relations containing R. Show that this is equivalent to the definition in problem 3. Let $\mathcal{E}$ be the set of all equivalence relations on a set A. Define two operations . and + on $\mathcal{E}$ as: $E_1.E_2 = E_1 \cap E_2$ and $E_1 + E_2 = (E_1 \cup E_2)^c$. Which of the axioms of Boolean algebra do these operations satisfy? If they do not, give examples to show it, otherwise prove it. Prove that for all E_1, E_2, $E_1.(E_1 + E_2) = E_1$ and $E_1 + E_1.E_2 = E_1$. These operations define a lattice called the lattice of partitions of A. An equivalence relation E_2 covers the relation E_1 if $E_1 \subset E_2$ and there is no relation E such that $E_1 \subset E \subset E_2$. Prove that if E_1 covers $E_1.E_2$ then E_2 is covered by $E_1 + E_2$. Is the converse of this statement true?

Q5 Let f be function from a set A to itself. Suppose there exists a number k such that f^k is the identity function. Prove that f is a bijection. Prove that the converse is true if A is finite but is not true if A is infinite. For a finite set A with n elements, what is the smallest number k such that for every bijection f from A to A, $f^k = I$. Prove your answer.