Partial Orders, Lattices

Q1 Let $(A, \leq)$ be a partial order defined on a finite set A. Prove that there exists a total order on A that contains the $\leq$ relation. This is essentially topological sorting of a directed acyclic graph. Whether this holds for infinite sets or not is independent of the standard axioms of set theory. It depends on what is called the 'axiom of choice' which is not a standard axiom. Can you think of how this could be proved for an infinite set? The dimension of a partial order is the minimum number of total orders whose intersection is the given partial order. In other words, $a \leq b$ holds in the partial order if and only if it holds in all the total orders. Prove that the dimension of any partial order on a set with n elements is at most $\lfloor n/2 \rfloor + 1$. Give an example of a partial order on n elements whose dimension is $\lfloor n/2 \rfloor + 1$.

Q2 Let $a_1, a_2, \ldots, a_n$ be a sequence of numbers. Prove that for any number $k \geq 1$, either the sequence can be partitioned into k non-decreasing subsequences, or there exists a decreasing subsequence of $k+1$ numbers (but not both). Using this or otherwise, give an efficient algorithm to find the longest subsequence that can be partitioned into k non-decreasing subsequences.

Q3 Let L be a lattice and let $\vee$ and $\wedge$ denote the *lub* and *glb* operations, also called the join and meet. Prove that these operations are commutative, associative and satisfy the absorption laws $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$, for all $a, b \in L$. Conversely, given the two operations satisfying these properties, show that they define a lattice in which $\wedge$ is the *glb* and $\vee$ is the *lub*. Note that it is not necessary to assume the existence of 1 and 0, which may not for infinite lattices. The distributive property is not satisfied by many lattices. In particular, show that the partition lattice does not satisfy distributivity.

Q4 Let $(A, \leq)$ be a partial order on a finite set A. A subset $I \subseteq A$ is called an ideal if for all $a_1, a_2 \in A$, $a_1 \leq a_2$ and $a_2 \in I$ implies $a_1 \in I$. In other words, if an element belongs to I then all elements 'smaller' than it belong to I. Prove that the union and intersection of any two ideals is an ideal. Let C be the collection of all ideals in A with the $\subseteq$ relation defined on it. Then C is a lattice with *glb* as intersection and *lub* as union, and therefore satisfies the distributivity property. Conversely, prove that if L is a finite distributive lattice, that is $\wedge$ and $\vee$ satisfy distributivity apart from the other lattice axioms, then it can be obtained from some finite poset in this way. First show that any such lattice can be represented by a collection of sets that is closed under union and intersection, and then show that any such collection corresponds to ideals in some poset.

Q5 This problem was given to me by a past student, now a faculty member in NUS, (also a visiting faculty here, Kuldeep Meel) and considered one of the 'rising stars' in AI. I don't know the application but the problem is on the Boolean lattice and is interesting by itself. I don't know the answer either (and neither did he,last I checked). Suppose I is an ideal in a Boolean lattice (set of all subsets of a k-element set for some k) with $|I| = n$. What is the minimum number of maximal elements in I? An element in I is maximal if it is not smaller than any other element in I. For example, if $n = 2^k$, then a single maximal element is sufficient. Take any set with k elements, the ideal containing it will have 2^k elements (all subsets of the set), and the set will be the only maximal element. If $n = 2^k + 2^l - 1$ then 2 maximal elements, a subset of size k and a disjoint subset of size l, suffice. The -1 is because the empty set is counted twice. $n = 13$ is the smallest n for which 3 are needed, and a program, if correct, seemed to suggest 419 is the smallest for which 4 are needed. It is not difficult to show that $O(\log n)$ elements suffice, but it seems like $O(\log(\log n))$ are sufficient. Show that there are values of n for which $\Omega(\log(\log n))$ are required. If you can prove the upper bound, you can write to him.