Extension Complexity for Matroid Union

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Introduction

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2. Small extension complexities have been shown for common matroids such as graphic, transversal, partition matroid.
3. Recently, it was shown that regular matroids have a small extension complexity.
4. Matroid intersection has a known compact formulation, so it is natural to expect such a result for matroid union.
5. Moreover, since matroid union has an efficient algorithm, it is natural to expect a compact formulation

Preliminaries

Extension Complexity

The extension complexity of a polytope $\mathcal{P}$ is the smallest number of facets among convex polytopes $\mathcal{Q}$ that can have $\mathcal{P}$ as a projection

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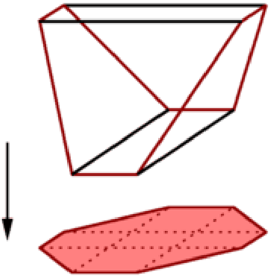

Preliminaries

Matroid

A matroid M is a pair $M = (E, I)$, where E is the finite ground set and $I \subseteq P(E)$ is a nonempty family of subsets of E that satisfies the following two axioms.

1. Closure under subsets. For every $I \in \mathcal{I}$ and $J \subseteq I$ we have $J \in \mathcal{I}$.
2. Augmentation property. For every $I, J \in \mathcal{I}$ where $|I| < |J|$, there is an $j \in J$ such that $I \cup j \in \mathcal{I}$.

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Rank Function

Motivated by Linear Algebra, there is a rank-function of a matroid that is defined for every subset $A \subseteq E$ as the size of the largest independent set that is contained in A,

$$rank(A) = max\{|I| \mid I \in \mathcal{I} \text{ and } I \subseteq A\}$$

Preliminaries

Matroid Polytope

For a set $I \in E$, its characteristic vector $x^I \in R^E$ is defined as

$$x = \begin{cases} 1, & \text{if } e \in I \\ 0, & \text{otherwise} \end{cases}$$

For any collection of sets $A \subseteq P(E)$, the polytope $P(A) \subset R^E$ is defined as the convex hull of the characteristic vectors of the sets in A,

$$P(A) = conv\{x^I \mid I \in A\}$$

For a matroid $M = (E, I)$, its matroid polytope is defined as $P(I) \subset R^E$, i.e., the convex hull of the characteristic vectors of the independent sets. The points $\{x^I \mid I \in \mathcal{I}\}$ are the corners of the matroid polytope $P(I)$

Preliminaries

Edmonds' Characterization

Edmonds gave a simple description of Matroid polytope which uses the rank function of the matroid. For convenience, we define for any $x \in R^E$ and $S \subseteq E$,

$$x(S) = \sum_{e \in S} x_e$$

For a matroid (E, I) with rank function r, a point $x \in R^E$ is in $P(I)$ iff

$$x_e \geq 0 \; \forall \; e \in E \tag{1}$$

$$x(S) \leq r(S) \; \forall \; S \subseteq E \tag{2}$$

Preliminaries

Partition matroid

Partition matroid is a matroid in which E is partitioned into (disjoint) sets $E_1, E_2, .., E_l$ and

$$I = \{X \subseteq E : |X \cap E_i| \ \leq k_i \ \forall \ i = 1, .., l\},$$

for some given parameters $k_1, .., k_l$

Partition Matroid Polytope

For a given partition matroid $P = (E, \mathcal{I})$ and partition $E_1, E_2, .., E_l$, a point $x \in R^E$ is in the polytope $P(I_1 \cap I_2)$ iff

$$x_e \geq 0 \ \forall \ e \in E \tag{3}$$

$$x(E_i) \leq r_1(E_i) \ \forall \ i \tag{4}$$

Matroid Union

Matroid union of $M_1, M_2, M_3, .., M_k$ is defined as

$$M = M_1 \vee M_2 \vee .. \vee M_k = (\cup_{i=1}^{k} S_i, \mathcal{I} = \{\cup_{i=1}^{k} I_i | I_i \in \mathcal{I}\})$$

We can show that M is a matroid!!

Problem Statement

Extension complexity of Matroid Union

Given matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S,\mathcal{I}_2)$ which have a compact formulation, their matroid union polytope has a compact formulation.

Construction of compact formulation

Extension

Let us denote the variables corresponding to elements in ground set of M_1 by x_e for $e \in S$ and similarly for M_2 by z_e for $e \in S$. Let us introduce some new variables v_e for $e \in S$. Let $\mathcal{P}(M_1)$'s compact extended formulation be given by $Q(M_1)$ and similarly $Q(M_2)$ for $\mathcal{P}(M_2)$ with variables of $Q(M_1), Q(M_2)$ being disjoint. Now, consider the polytope $Q(M_1 \vee M_2)$ given by

$$
\begin{aligned}
v_e &\geq 0 \;\forall\; e \in S \\
v_e &\leq 1 \;\forall\; e \in S \\
&Q(M_1) \\
&Q(M_2) \\
v_e &\leq x_e + z_e \;\forall\; e \in S
\end{aligned}
$$

$Q(M_1 \vee M_2)$ is an extended formulation of $\mathcal{P}(M_1 \vee M_2)$.

Proof sketch

Equivalent Result

Given a set $I \subseteq S$, I is independent in $M_1 \vee M_2$ if and only if there is a feasible point $p \in Q(M_1 \vee M_2)$ such that $v(p) = \chi_I$, where χ_I denotes the indicator vector corresponding to I in $R^{|S|}$. We can show that the above result is equivalent to this result, $Q(M_1 \vee M_2)$ is an extended formulation of $\mathcal{P}(M_1 \vee M_2)$

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Integral Corners

For all corners p_i of $Q(M_1 \vee M_2)$, $v(p_i)$ are integral. We can show this by arguments on no of tight constraints, as no of tight constraints for a polytope are maximized at the corners.

Proof sketch

The forward direction of the lemma stated above (equivalent result) is easy to show.
In order to show the backward direction, say $v(p)$ is integral, then we need to show that the indicator vector described by v corresponds to an independent set.

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In order to show the backward direction, say $v(p)$ is integral, then we need to show that the indicator vector described by v corresponds to an independent set.

Backward direction (equivalent)

Given $I \subseteq S$, $p \in Q(M_1 \vee M_2)$ such that $v(p) = \chi_I$, then there exists $p' \in Q(M_1 \vee M_2)$ such that $x(p'), z(p')$ are integral and $v(p) = v(p') = \chi_I$.

Partition Matroids

First, let's look at the case of partition matroids. Let's consider matroids,

$$
\begin{aligned}
x_2 + x_3 &\leq 1 \\
x_1 + x_2 + x_3 &\leq 2 \\
x &\geq 0 \\
x &\leq 1
\end{aligned}
$$

and

$$
\begin{aligned}
z &\leq 1 \\
z &\geq 0 \\
z_1 + z_2 &\leq 1 \\
z_2 + z_3 &\leq 1 \\
z_3 + z_1 &\leq 1 \\
z_1 + z_2 + z_3 &\leq 1
\end{aligned}
$$

with $I = \{1, 2, 3\}$.

Partition Matroids

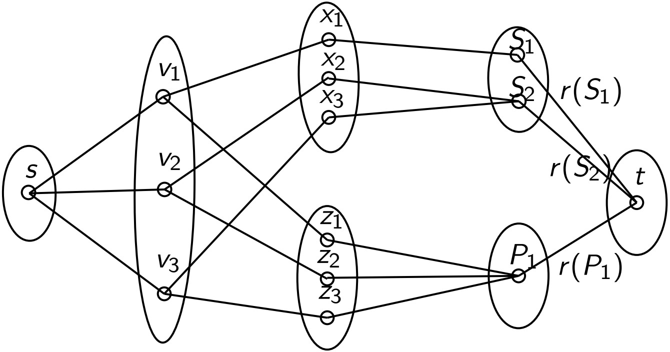

Figure: Flow network obtained from partition matroid

Moving to general matroids

Hence, union of partition matroids with a compact formulation has a compact formulation ! But what about general matroids?

Tight sets

Definition (Tight sets)

Let $\mathcal{M}$ be a matroid with the rank function r. Let $x \in \mathcal{P}(\mathcal{M})$ (matroid polytope). We call a set S a tight set of x with respect to r if $x(S) = r(S)$

Tight sets

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Let $\mathcal{M}$ be a matroid with the rank function r. Let $x \in \mathcal{P}(\mathcal{M})$ (matroid polytope). We call a set S a tight set of x with respect to r if $x(S) = r(S)$

Lemma (Maximal Chain of Tight Sets)

Let $\mathcal{M}$ be a matroid with rank function r, and let $x \in \mathcal{P}(\mathcal{M})$. Let $\mathcal{C} = \{C_1, \cdots, C_k\}$ with $\phi \subset C_1 \subset \cdots \subset C_k$ be an inclusion-wise maximal chain of tight sets of x with respect to r. Then every tight set T of x with respect to r must satisfy $\chi_T \in span\{\chi_C : C \in \mathcal{C}\}$

In comes network flow $\cdots$

Given a feasible solution $p \in Q(M_1 \vee M_2)$, with integral $v(p)$, we need to construct another solution p' with integral x, z, v co-ordinates. So, given such a solution, we exploit network flow yet again to partition the variables occuring in tight constraints!

In comes network flow $\cdots$

Consider the matroids $M_1 = (S, I_1), M_2 = (S, I_2)$ with $S = \{1, 2, 3\}$. Consider $I \subseteq S$ as $I = \{1, 2, 3\}$.Let feasible point $p \in Q(M_1 \vee M_2)$ be such that $x(p) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ and $z(p) = (\frac{3}{4}, \frac{1}{2}, \frac{3}{4})$. Then, we have the tight constraints,

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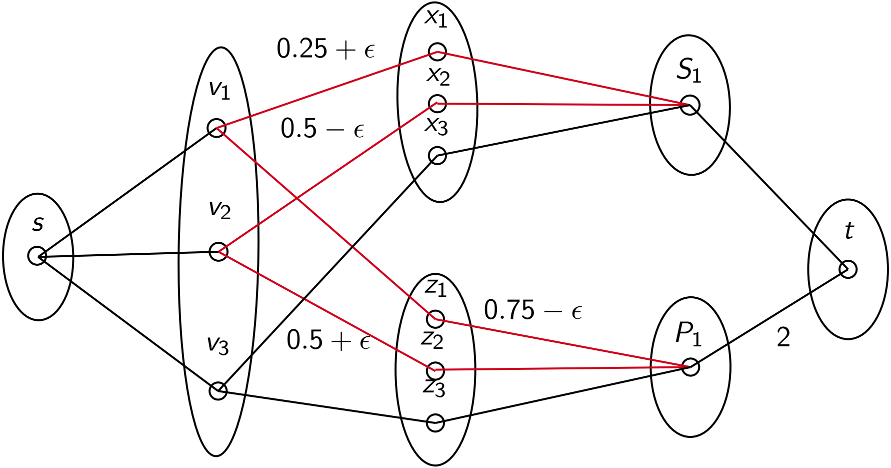
$$x_1 + x_2 + x_3 \leq 1$$
$$z_1 + z_2 + z_3 \leq 1$$

In comes network flow $\cdots$



In comes network flow $\cdots$

We see that in order to augment the flow, while respecting rank constraints, we can set $\epsilon = 0.5$ to get $x = (0.75, 0, 0.25)$ and $z = (0.25, 1, 0.75)$. Hence, we get the new tight constraints as,

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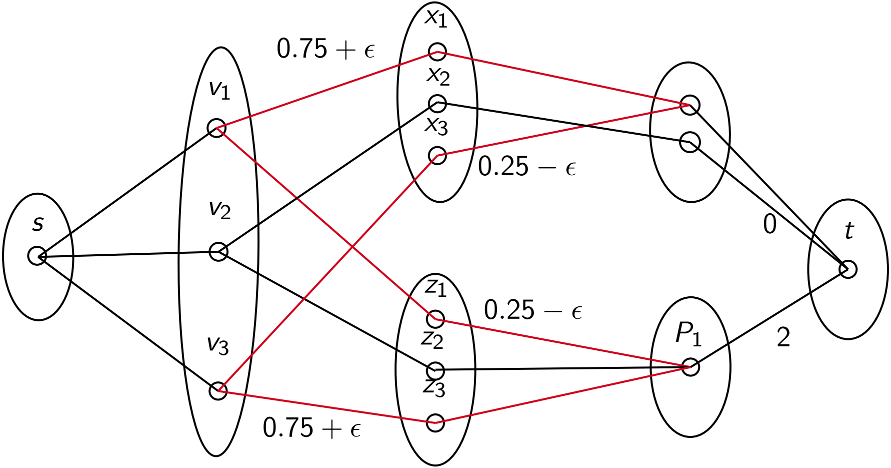
$$\begin{aligned} x_1 + x_3 &\leq 1 \\ x_1 + x_2 + x_3 &\leq 1 \\ z_1 + z_2 + z_3 &\leq 2 \end{aligned}$$

In comes network flow · · ·



In comes network flow $\cdots$

Here, choosing $\epsilon = 0.25$ does the trick, giving us the required integral solutions as $x = (1, 0, 0)$ and $z = (0, 1, 1)$. Observe that the union of the sets corresponding to these indicators indeed gives us I.

Proof sketch for the general case

Given a feasible solution with integral v. Consider the below *augmenting algorithm*.

1. Find the tight constraints, and find their corresponding chaining
2. Construct a network flow using partition of the chain of tight constraints, into equalities among disjoint sets of variables (similar to partition matroid)
3. Augment ϵ to the flow, such that some variables become integral or some constraints become tight
4. Repeat above process until we get an integral feasible solution

Future Work

In the direction of finding small extension complexities for independent set polytopes, one of the major open questions is finding a small extension for linear matroids. We are also interested in conjectures given by Tony Huynh in his blog post. Both these open questions can serve as directions for future work.