Computational Complexity

30/7/24

Ref : course webpage

Grading :
- Scribe (5%)
- 2 or 3 Assignments (20%)
- Presentations (20%)
- Midsem (20%)
- Endsem (35%)

Pre-req : Basic Algorithm Design,
* Basic Linear Algebra,
Basic Graph Theory,
NP, NP-C,
Turing Machines

Can we show that 3-coloring problem requires atleast $\Omega(2^n)$? (1 million dollars :))

Naive : Try all colors $\rightarrow 3^n$ time

Resources : Memory (Working Memory)

Given a graph n vertices, 2 vertices u, v whether $u \sim v$

Input is in separate memory
[RAM too small for graph size]

BFS : $\Omega(n)$ memory (maintain a queue of vertices)
↳ too much if graph is big

n vertices $\Rightarrow$ $\log n$ bits to store index of vertex.

2005 Reingold det-algo. [Ambitious goal : $O(\log n)$ bits of memory randomization
using random walk $O(n^3)$ time with high probability
$\rightarrow 1$ with $n \uparrow$

Resource : Randomness

Expensive resource : seed (systime / sound) $\xrightarrow{f}$ f : complicated fn, looks random.

Not predictable $\Rightarrow$ random [view of complexity theory]

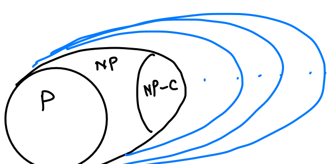
Resource : Communication

Computation b/w many parties [distributed computing].

Want to limit comm. (local computation is cheap)

Course : Connections between different concepts of complexity

Connect problems that are hard, hardness assumptions.



PSPACE

Proving pseudo randomness $\equiv$ Proving ckt lower bounds

Hard to compute f^n $\longrightarrow$ generate pseudo-random
↓
Make algos deterministic.

Does use of randomness give you more power ? [Open for 30 years]

E.g. Using random $O(\log n)$ bits memory w.h.p.
without random show $> \log n$

will show ~~randomized~~ algo more powerful than deterministic.

→ Belief : Equal power

Interactive Proofs, Zero-Knowledge, Prob. checkable Pf

Graph : Is there a path from u to v of length ≤ 100

Yes → give path

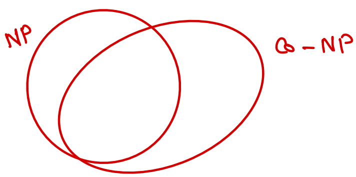
No → give BFS pf, run algo and show

Graph : Is there a 3-coloring

Yes → give coloring scheme (easy proof)

No → Maybe some way ? (e.g. existence of 4-clique)

If a logical statement is true, then there is always a small proof ? (Open)



(belief, Co-NP $\neq$ NP)

Allowing interaction makes it possible ! (Q = f (Ans) : Proof w.h.p.)

↓ Poly # rounds, prover unlimited power.

3-coloring has unlimited power. (90s)

Blockchains : Certain nodes compute, convince other parties of that computation,
so interaction is useful here.

Probabilistic checkable proofs ← useful for blockchains : verifier reads small part of pf.

Give a proof , see 3-bits and that will convince w.h.p. (reading 3 bits of pf is enough)

Probabilistic Checkable Proofs $\Longleftrightarrow$ Hardness of Approximation

Zero-Knowledge Proofs [Goldwasser]

Want to convince I know , without giving away information

Basic Lower Bound on Sorting

Claim : Sorting n numbers requires $\Omega(n \log n)$ comparisons

Adversarial argument :

Initially $\boxed{n!} = \boxed{A_i > A_j}\ \boxed{A_i < A_j}$ [First query $A_i > A_j$]

Adversary picks answer with bigger set from the two. (atleast half of initial size)

$n! \xrightarrow{1^{st}\text{ query}} n!/2 \xrightarrow{2^{nd}} n!/4$ } $\log_2(n!)$ queries needed to obtain permutation

$\Rightarrow$ needed for sorting

$= \Omega(n \log n)$

Above is an information theoretic lower bound $\Rightarrow$ Need x queries to obtain enough info

- Complexity Lower Bound : Given all info, how much computation needed to get answer
- Can you write a program for any given problem ?
  - Lack of understanding information
  - Lack of computational power

Puzzle : n numbers, exactly two of them are equal

Queries $A_i, A_j \longrightarrow \{<, >, =\}$

Goal : Find the pair that is equal

Computational Task :

Input $\in \{0,1\}^*$

Output $\in \{0,1\}^*$

1. Search Problem : $R \subseteq \{0,1\}^* \times \{0,1\}^*$

(OR) $f : \{0,1\}^* \to (2^{\{0,1\}^*} - \phi)$

2. Decision Problem : $f : \{0,1\}^* \to \{0,1\}$

For every search problem there is a natural decision problem s.t. solving the latter solves the former and vice versa [e.g. in poly time]

1. SAT : ϕ , output : a satisfying assignment ,

decision : given ϕ , is there a satisfying assignment ?

search $\xrightarrow[\text{to}]{\text{reduces}}$ decision

(self reduction)

2. Input : integer n (input size = $\log n$)

Output : prime factors of n

Decision : is there a factor of $n < k$? $\to$ Binary search.

* $f : \{0,1\}^* \to \{0,1\}$

Counting argument,

$\# f^n$ = uncountable

no bijection ($\exists f^n$ not computable by programs)

Program = $\{0,1\}^*$ is countable

Diagonalisation

Let G have a program.

Well defined f^n : 1. G : input natural no. i

output = $\begin{cases} 1, & \text{if } i^{th} \text{ program on input } i \text{ halts} \\ 0, & \text{otherwise} \end{cases}$

Consider program $P \to$ input $i \in \mathbb{N}$

run program G on input i

If output is $1 \to$ loop

$0 \to$ return 0

Let j = index of program P

If $G(j) = 1$: Program P halts on input j } contradiction
but P loops

$= 0$: Program P doesn't halt on j } contradiction.
but P returns 0

$\Rightarrow$ G doesn't have a program.

H : input i, x

output 1 if i^{th} program halts on input x

0 otherwise

Halting problem

HW Problem : Given two C++ programs, do they have same behaviour?

[Show undecidable]

P : input x, y

output $\begin{cases} 1 & \text{if } x, y \text{ have same behaviour} \\ 0 & \text{otherwise} \end{cases}$

Let P have a program.

Hilbert's 10^{th} problem

Input : Poly multivariable eq w/ integer coeff
Output : Is there an integer sol^n

} undecidable

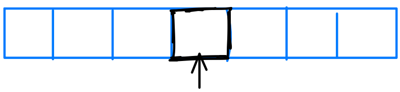
- Any C++ program can be converted into a polynomial eq^n s.t. program on this input halts iff equation has integer solution

→ Similar approach for undecidability of game of life

Turing Machine

$Q \leftarrow$ states , $\Sigma = \{0, 1\}$

- Infinite Tape , head



Transition function

$$\delta : Q \times \Sigma \longrightarrow \underset{\substack{\uparrow \\ \text{write symbol}}}{Q \times \Sigma} \times \underset{\substack{\uparrow \\ \text{movement of head}}}{\{\text{left}, \text{right}, \text{stay}\}}$$

Church - Turing Thesis : Every function computable by "natural", "reasonable" model of computation can be computed by a turing machine.

Abstract RAM Machine

(unbounded)

Infinite memory cells ← int
(finite) registers ← int
program counter ← int

Instructions

reset (r) (makes it zero)
dec (r)
inc (r)
load (r_1, r_2)
store (r_1, r_2)
cond goto (r, l)

Universal Turing Machine (2 state, 3 symbol) ← can simulate the TM
Which takes description of any TM with input and it can simulate it

Time Complexity

$A \leftarrow$ TM which always halts

$$t_A : \{0,1\}^* \to \mathbb{N}$$

$$t_A(n) = \max_{x \in \{0,1\}^n} t_A(x) \leftarrow \text{Time complexity}$$

Efficient Computation

Cobham 1964
Edmonds 1965

DTIME$(t(n))$ = class of problems with TM running time atmost $t(n)$ $\leftarrow$ deterministic time

$$P = \bigcup_{c \geq 1} \text{DTIME}(n^c)$$

Cobham - Edmonds Thesis : Any function computable by any reasonable model can be by TM with polynomial time overhead.

Palindrome : 2 tape $\leftarrow O(n)$

1 tape $\leftarrow O(n^2)$

needs $\Omega(n^2)$ time.

a b c b a

reason why polytime is a universal measure.

Multitape TM $t(n)$

$\downarrow$

Single tape $t(n)^2$

(try!)

Eg of problems not in P

Input : description of TM and an input x for it

Output : whether it stops in $2^{|x|}$ time

Obvious : Simulate program $2^{|x|}$ time algo.

Can show : No faster algorithm $\leftarrow$ diagonalization argument

Input : A boolean ckt with 2ℓ variables.

(Defines a graph on 2^{ℓ} vertices)

output : whether $s \sim t$ in this graph

Trivial algo : $2^{\ell} = O(?$ size of formula$)$

Exp-time needed in this.