CS774 : Spectral Graph Theory

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Markov matrix

$M = D^{-1} A$, A = adjacency matrix

D = diagonal s.t. $D_{ii} = \deg(i)$ $\quad p_t^T = p_0^T M^t$

(you can verify this fact easily)

Vague Question

What can I say for a markov matrix M s.t.

$\lim_{t\to\infty} M^t$ exists

e.g. $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ $\quad M^t = \begin{pmatrix} 2^t & 0 \\ 0 & 3^t \end{pmatrix}$

$\lim_{t\to\infty} M^t$ does not exist

Theorem: Let $M \in \text{sym}(\mathbb{R}^{n\times n})$. Then, all the eigenvalues of M are real numbers. Moreover, you can choose an orthonormal eigenbasis of $\mathbb{R}^n$

Suppose $M = D^{-1}A$ is symmetric, then all components are regular (if $(i,j) \in E$, $M_{ij} = \frac{1}{d_i}$, $M_{ji} = \frac{1}{d_j}$)

Theorem: Let $M = M(G) = D^{-1}A$ be the markov matrix for walks on G, if G is regular and connected, $M \in \text{Sym}(\mathbb{R}^{n\times n})$. Then all eigenvals of M lie b/w $+1$ and -1

Proof : We would like to show $I - M$ is p.s.d

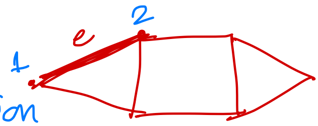
$$I - M = I - D^{-1}A = I - \frac{A}{d}$$

$$= \frac{dI - A}{d} = \frac{L}{d}$$

$\leftarrow$ Laplacian

$L = D - A$

$L = \sum_e L_e$



L_e describes contribution from edge e to the Laplacian

e.g. corresp. to e, $\deg = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and A is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

So, $L = \begin{bmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & 0 \\ 0 & 0 \end{bmatrix}$, and so on for all edges

pattern added with each edge e.

$$(x_1 \; x_2)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 - x_2)^2$$, hence, L_e is p.s.d.

Hence, $\sum_e L_e$ is p.s.d

We would also show $I + M$ is p.s.d, $\parallel^{el}$ pf just $(x_1 + x_2)^2$. $\square$

What if the original matrix was not symmetric? (t.b.d)

$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, eigs $= \pm 1$ $\quad M = U^T \Sigma U$

$$M^t = U^T \Sigma^t U = U^T \begin{pmatrix} 1^t & 0 \\ 0 & (-1)^t \end{pmatrix} U$$

if $U = (u_1 \; u_2)$

$$= u_1 u_1^T + (-1)^t u_2 u_2^T$$

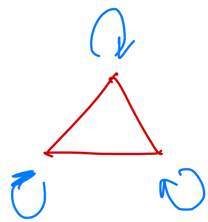
Bad, cuz keeps flip-flopping.

Same thing happens if graph is bipartite!

Idea: force other eigenvalue to never become -1

$I+M$ is p.s.d, $\frac{I+M}{2}$ is p.s.d

and is a markov matrix $W = \frac{I+M}{2}$ (effect of what



$$\frac{I+M}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1/4 & \frac{1}{2} & 4 \\ 1/4 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$ = add loops to every node

{lazy random walk matrix}

would show $\lim_{t \to \infty} \| W^t p - ? \|_2 \Rightarrow 0$

d-regular, connected

$? = \frac{1}{n}$ (uniform dist. ultimately!)

Theorem 1.11. *Let G be a connected d-regular graph. Let $\boldsymbol{M} = \boldsymbol{M}(G)$ denote the Markov Matrix for G. Let $1 = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq -1$ denote the eigenvalues of $\boldsymbol{M}$. Then $\lambda_1 - \lambda_2 \geq 1/n^2$. In other words, λ_2 is strictly smaller than λ_1.*

Proof: TBD

Theorem 1.12. *Let $G = (V, E)$ denote a connected d-regular graph. Let G' denote the graph obtained by adding d loops at every vertex. Let $\boldsymbol{W}$ denote the random walk matrix on G' (recall, $\boldsymbol{W} = \frac{\boldsymbol{I}+\boldsymbol{M}}{2}$ where $\boldsymbol{M}$ is the random walk matrix on G). It holds that*

$$\text{For all distributions } \boldsymbol{p} \text{ on vertices, } \lim_{t\to\infty} \boldsymbol{p}^T \boldsymbol{W}^t = \underbrace{\begin{bmatrix} 1/n & 1/n & 1/n & \cdots & 1/n \end{bmatrix}}_{n \text{ entries}}.$$

Proof : Enough to show $\lim_{t\to\infty} W^t 1_u = \frac{1}{n}$ (since W is symmetric)

(i.e. $1_u^T W^t = [\frac{1}{n} \cdots \frac{1}{n}]$)

$1_u = \sum_{i=1}^{n} c_i v_i$, where v_i are eigenvectors of W, v_1 corresponds to eigenvalue 1, $\lambda_1 - \lambda_2 = 1 - \lambda_2 = \varepsilon > 0$ (thm 1.11)

So, $1_u \cdot v_1 = c_1 = \frac{1}{\sqrt{n}}$ $\left(\because v_1 = \frac{1}{\sqrt{n}}\right)$

$\Rightarrow |W^t 1_u - \frac{1}{n}| = |\sum_{i=2}^{n} c_i \lambda_i v_i| \leq \sum_{i=2}^{n} \|c_i \lambda_i^t v_i\|$

$\leq \sum_{i\geq 2} (1-\varepsilon)^t$

As $t \to \infty$, terms $\to 0$ #

Let stationary distribution $= \pi = [\underbrace{\frac{1}{n} \cdots \frac{1}{n}}_{n \text{ times}}]$, then, $\pi^T W = \pi^T$

Hence, $\pi^T W^t = \pi^T$ $(\because 1 W = 1)$

Definition 1.15. *Let $G = (V, E)$ be a connected (not necessarily regular) graph. Let $\boldsymbol{M} = \boldsymbol{M}(G)$ denote the Markov Matrix for G. Let $\boldsymbol{W} = \frac{\boldsymbol{I}+\boldsymbol{M}}{2}$ denote the lazy random walk matrix for G. Let $\boldsymbol{\pi}$ denote the stationary distribution for lazy random walks on G. The mixing time for lazy random walks on G (or power iterations on $\boldsymbol{W}$) is:*

$$\tau_{mix} = \min\{t \in \mathbb{N} : \|\boldsymbol{1}_u^T \boldsymbol{W}^t - \boldsymbol{\pi}^T\|_1 \leq 1/4 \quad \forall u \in V\}$$

Remark 1.16. *The proof of Theorem 1.12 essentially shows that the mixing time of lazy random walks on a connected d-regular graph is $O(\log n/(1-\lambda_2))$.*

Proof : $\|1_u^T W^t - \pi^t\| \leq \sum_{i\geq 2} (1-\varepsilon)^t$ (connected d-regular graphs)

worst case $= \sqrt{1 - \frac{1}{n}}\, \lambda_2^t = \frac{1}{4}$

$\lambda_2^t = \frac{1}{4}\sqrt{\frac{n}{n-1}}$

$t \log \lambda_2 = \frac{1}{2}(\log n - \log(n-1))$

Basics of Spectral Graph Theory (Warmup theorem)

Theorem 2.1. *Let $G = (V, E)$ be a d-regular graph. Let $\overline{\boldsymbol{L}} = \boldsymbol{I} - \boldsymbol{A}/d$ denote the Laplacian of this graph with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$. Then $\lambda_k = 0$* if and only if *G has at least k connected components.*

Proof: $dI - A$ is p.s.d since $L = \sum_e L_e$ $(\Rightarrow)$

Now, $v^T L v = 0$ (when?) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$$(x_1 \; x_2)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$= (x_1 - x_2)^2 = 0$ iff $x_1 = x_2$ $\Rightarrow$ vector is constant over each connected component

$\lambda_k = 0 \Rightarrow$ k-dimensional subspace for which we get nullspace.

Let vectors be $v_1, \ldots, v_k$

All these are constant over all connected components.

$C_1, C_2, \ldots, C_\ell$ are components,

so, ℓ-dimensional "subspace"-encasing

ℓ.i. vectors $v_1, \ldots, v_k \Rightarrow k \leq \ell$

No other vectors $\Rightarrow k = \ell$

$(\Leftarrow)$ if G has k-connected components, we construct such a set via basis vectors (modified) ∎

Robust version of this result

Definition 2.2. *Let $G = (V, E)$ be a d-regular graph. For a set non-empty S of vertices ($S \neq V$) we define $\varphi(S) = \frac{|E(S,\overline{S})|}{d|S|}$. This quantity can be given a probabilistic interpretation: it measures the probability that a random edge (u, v) incident on a random $u \in S$ has the other end point $v \notin S$.*

Definition 2.3. *For a d-regular graph $G = (V, E)$, we define $\varphi(G) = \min_{0<|S|\leq n/2} \varphi(S)$.*

Motivation $\varphi(G)$ = "connectedness" of a graph.

if $\varphi(G) \geq \alpha$, for any $S \subseteq V$, disconnection required $d|S|\alpha$ edges atleast.

Also, $\varphi(\bar{S}) \leq \varphi(S)$, if $|S| \leq n/2$, so this def^n works

same # cross edges, more size $|S|$

$\varphi(G)$ = expansion of a graph

Cycle: S = contiguous set for # min cross edges

$|\text{edges}| = 2$, $|S| \leq n-1$ so,

$$\varphi(G) = \frac{2}{n-1}$$

Hypercube: (n-dimensional has 2^n vertices, n neighbors for each vertex)

ans

(# ques: expansion of a hypercube)

Homework

1. Detailed balance condition is the unique stationary distribution. (checking using this = trivial).

Note: converges to that too.

$\pi^t W = \pi^t$

if π' enjoys detailed balance, you fail.

#ques: Stationary dist $\Rightarrow$ detailed balance.

i.e.

#ques: For a lazy stochastic matrix, eigenvalue 1 has unique unit eigenvector

Note: for notes about cheeger inequality, refer annotated notes

Thm : [Margulis 1978]
[Lubetzy-Philip-Scrot '84]

depends on α, d

$\forall d \geq 3$ $\forall \alpha \geq 0.001$ $\forall$ suff. large $n \geq n_0$ $\exists$ a d-regular graph $G = (V, E)$ on n-vertices s.t. $\phi(G) \geq \alpha$. (ramanujan graphs).

Thm : [LPS '84]
For any graph G with laplacian $L = I - M$, eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$

$$\underbrace{\frac{\lambda_2}{2} \leq \phi(G)}_{\text{already done.}} \leq \sqrt{2\lambda_2}$$

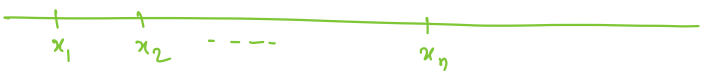
$\phi(G) \leq \sqrt{2\lambda_2}$: It's expansion cannot be very large.

Alon-Milmon Proof of Cheeger's Bound

FindSparseCut(G)

1. Let $\boldsymbol{x}$ be the eigenvector with eigenvalue λ_2. (negative, +ve entries, since $\sum x_i = 0$)
2. Sort $\boldsymbol{x}$ as $\boldsymbol{x}_1 \leq \boldsymbol{x}_2 \leq \cdots \leq \boldsymbol{x}_n$.
3. Assign $S_1 = \{1\}$ and $X = S_1$
4. For $i = 2$ *to* $n - 1$.
   (a) Consider sweep cut $(S_i, \overline{S_i})$ where S_i consists of the first i vertices. → prefix cuts
   (b) Assign $S = \operatorname{argmin}(|S_i|, |\overline{S_i}|)$.
   (c) Assign $X =$ better cut between X *and* S.
5. Return X

Sweep factor based approach



Key idea : People knew you can write a positive vector to obtain cuts in G with lots of interesting properties.

Note : If X, Y are non-negative RVs,

$$\frac{\mathbb{E}[X]}{\mathbb{E}[Y]} \leq \alpha \Rightarrow P\left(\frac{X}{Y} \leq \alpha\right) > 0 \;!$$

Step 1 : We will use one of y and z which gives a better cut

Step 2 : Use the round off thm to obtain a cut. Define $w \in \mathbb{R}^n$ when $w_u = y_u^2$ $\forall v \in V$

Some sanity checks

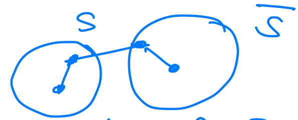
$$R_{\bar{L}}(x) = x^T \bar{L} x = x^T\left(\sum_{e\in E} L_e\right)x = \sum_{e\in E} x^T L_e x$$

$$\left\{ (x_1 \;\; x_2)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 - x_2)^2 \right\}$$

$$= \sum_{(u,v)\in E} (x_u - x_v)^2$$

if $x = \frac{1_S}{|S|} - \frac{1_{\bar{S}}}{|\bar{S}|}$

$x \perp \bar{1}$.

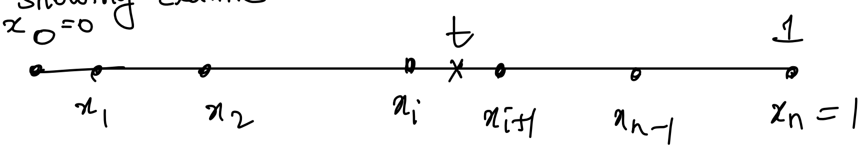


Then, $\dfrac{x^T \bar{L} x}{x^T x} = \dfrac{\sum_{(u,v)\in E_{cut}} \left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right)^2}{\frac{1}{|S|} + \frac{1}{|\bar{S}|}}$

$= E(S,\bar{S}) \cdot \left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right)$

if $(u,v)\in E$
$u,v \in S$ then
$x_u = x_v$
n^y for $u,v\in\bar{S}$
if $u\in S, v\in\bar{S}$
then $x_u = \frac{1}{|S|}$,
$x_v = \frac{-1}{|\bar{S}|}$

showing claims :



$t \sim$ uniform $[0,1]$ $\qquad S_t = \{v\in V \mid x_v \geq t\}$

1) $\mathbb{E}[|S_t|] = \sum x_v$

$$P(|S_t| = k) = P(t \in (x_{n-k}, x_{n-k+1}]) = x_{n-k+1} - x_{n-k}$$

$$\sum k P(|S_t| = k) = \sum_{k=1}^{n} k(x_{n+1-k} - x_{n-k})$$

$$= \sum_{k=1}^{n} (n+1-k)(x_k - x_{k-1})$$

$$= (n+1)(x_n - x_0) - \left(\sum_{k=1}^{n} k x_k - k x_{k-1}\right)$$

$$= (n+1)(1-0) - \left[\sum_{k=1}^{n} k x_k - (k-1)x_{k-1} - x_{k-1}\right]$$

$$= (n+1)(1) - n + \sum_{k=1}^{n} x_{k-1}$$

$$= 1 + \sum_{k=1}^{n} x_{k-1} = \sum_{k=1}^{n} x_k$$

2) $\mathbb{E}[|\#\text{edges crossing } S_t, \overline{S_t}|] = \sum_{k=1}^{n-1} k(n-k)\Pr(S_t = k)$ ← leads nowhere

$$= \sum_{k=1}^{n-1} k(n-k)\,(x_{n+1-k} - x_{n-k}) = \sum_{k=1}^{n-1} k(n-k)(x_{k+1} - x_k)$$

$$= \sum_{k=1}^{n-1} (nk - k^2)(x_{k+1} - x_k) = \sum_{k=1}^{n-1} (nk)(x_{k+1} - x_k) - \sum_{k=1}^{n-1} (k+1)^2 x_{k+1} - k^2 x_k$$

$$+ \sum_{k=1}^{n-1} 2k\,x_{k+1} + x_{k+1}$$

$$= \sum_{k=1}^{n-1} \left((n+2)k\,x_{k+1} - nkx_k\right) - \sum_{k=1}^{n-1} (k+1)^2 x_{k+1} - k^2 x_k$$

$$+ \sum_{k=1}^{n-1} x_{k+1}$$

$$n\left\{\sum_{k=1}^{n-1} (k+1)\,x_{k+1} - k x_k\right\} - \left\{\sum_{k=1}^{n-1} (k+1)^2 x_{k+1} - k^2 x_k\right\}$$

$$+ \left\{\sum_{k=1}^{n-1} x_{k+1}\right\} + \left\{\sum_{k=1}^{n-1} k\,x_{k+1}\right\}$$

$$= n\{n x_n - x_1\} - \{n^2 x_n - x_1\} + \left\{\sum_{k=1}^{n-1} x_{k+1}\right\} +$$

Alternative approach: look at probability of every edge (u,v) contributing to cross edge which is $t \in (x_u, x_v) = |x_u - x_v|$

so, $\mathbb{E}[|\#\text{edges crossing } S_t, \overline{S_t}|] = \sum_{(u,v)\in E} |x_u - x_v|$

Lecture (17/08/23)

Well structured community assumption: Real World Graphs contain $S \subseteq V$ with

$|S| \leq \varepsilon n$

$\phi(S) = \phi << O(1)$

wanted: Runtime $= \tilde{O}(|S|)$

$= |S|\ \text{poly}(\log(n))$

High level: simulate cheeger's sweep on eigenvectors without finding eigenvectors.

eigenvectors of laplacian = eigenvectors of M (random walk matrix)

Key: Eigenvectors of M can be approximated using random walks (general graphs).

$\forall u \in V, \quad \lim_{t\to\infty} M^t \mathbb{1}_u \longrightarrow$ eigenvector

Notation : $G' = (V, E')$ undirected graph

$G'' = (V, E'')$ directed graph (copy of edge running both directions)

$\deg(u) =$ out-degree = in-degree

G = directed + loop.

loops on vertex = degree

So, on G, stationary distribution is well-defined.

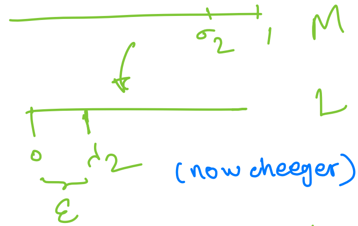
$$\pi(u) = \pi_{G'}(u) \propto \deg(u) \quad \text{(detailed balance)}$$

$$t_{mix}(G) = \min\{ n \text{ s.t. } \forall u \in V(G), \| 1_u M^t - \pi \|_1 < \underbrace{\tfrac{1}{10}}_{} \}$$

(arbitrary, other const also ok)

$$t_{mix}(G) \leq O\left(\frac{\log n}{1-\sigma_2}\right) = O\left(\frac{\log n}{\varepsilon}\right)$$

$$= O\left(\frac{\log n}{\phi^2}\right)$$



(now cheeger)

$$\| M^t 1_v - \tfrac{1}{n} \|_2 \leq \sum_{i=2}^{n} (1-\varepsilon)^t \to 0$$

[ST12] noticed that prev bound on mixing time used $1-\sigma_2$ as "potential f^n"

Don't compare with just a number, instead use a better potential function.

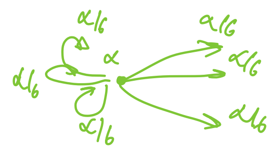
TCS : stochastic analysis, early start = good research career.

[LS95] (it all begins here, giants of analysis)

Definition : Let p denote a distribution supported on $V(G)$. We call the distribution induced on edges an induced distrb from p if

$$\forall e \in \underbrace{\delta^+}_{}(u), \quad \rho(e) = \frac{p(u)}{\deg_G(u)}$$

outgoing edges from u.



Ex : Let $p = \pi$,

$$\rho(\overrightarrow{u,v}) = \frac{\pi(u)}{d_u} \quad \text{(for other than stationary, problematic).}$$

Def^n (Greedy potential) : Let #loops in $G = m$ [i.e. total #edges $= 2m$]

Take some distribution on $V(G)$ say p, consider induced dist ρ on $E(G)$

Sort greedily to obtain $\rho(e_1) \geq \rho(e_2) \geq \dots \geq \rho(e_{2m})$

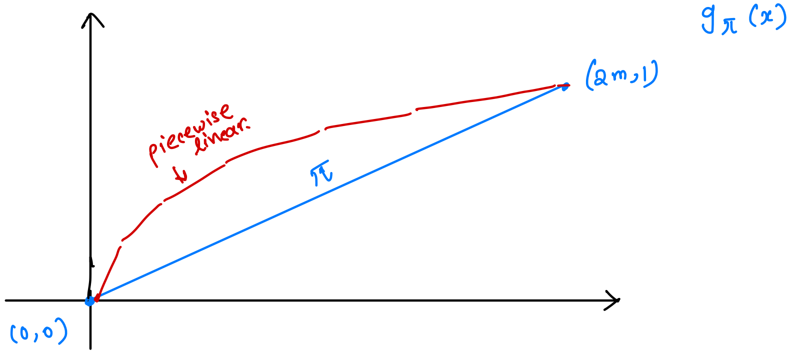
Greedy potential $\quad G_p : \{0, 1, \dots, 2m\} \to [0,1]$

$$G_p(x) = \sum_{i=1}^{x} \rho(e_i)$$

g_p = the continuous LS potential,

$$g_p : [0, 2m] \to [0,1]$$

where $g_p(x)$ = found by linear interpolation on G_p



p is diff from π
$\downarrow$ $\quad$ $\downarrow$
p $\quad$ p'

$\underbrace{p_1 \geq p_2 \geq \dots \geq p_m}_{\text{more sum, as all sum to 1.}} \geq \dots \geq p_{2m}$

Concave? $\to$ diff b/w $g_\pi(x_{i+1}) - g_\pi(x_i) = p_m \leftarrow$ decreases.

Claim: All distributions p satisfy
$$g_p(x) \geq g_{p'}(x) \;\forall\, x \in [0, 2\pi].$$ (p' induced by π) (g_p is increasing and concave)

Let $P_0 = \mathbb{1}_v$ for some $v \in V$. $\quad P_t^T = P_{t-1}^T M$

Is it possible $\boxed{\lim_{t\to\infty} g_t \to g_\pi}$?

prob. dist at step t.

Thm 1: Let $G=(V,E)$ have $\phi(G) \geq \phi$, then for $0 \leq x \leq m$, (LS recurrence)

for $0 \leq x \leq m$ $\quad g_t(x) \leq \dfrac{g_{t-1}(x - 2\phi x) + g_{t+1}(x + 2\phi x)}{2}$

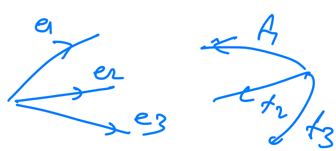
for $x > m$ $\quad g_t(x) \leq \dfrac{g_{t-1}(x - 2\phi(2m-x)) + g_{t-1}(x + 2\phi(2m-x))}{2}$

Thm 2: Let $G=(V,E)$ with $\phi(G) \geq \phi$
Then, $g_t(x) \leq \left(1 - \dfrac{\phi^2}{2}\right)^t \sqrt{x} + \dfrac{x}{2m}$

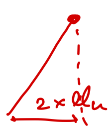
Warmup: g_t's don't rise back up.

Assume: $p(e_1) = p(f_1)$

Then natural order is
$p(e_1) \geq p(e_2) \geq \dots \geq p(e_6) \geq \dots \geq p(f_6)$



Hinge points at step t = (one hinge pt per vertex)
first hinge pts on curve $g \equiv (g(2 \times \deg_G(u)))$
biggest p



we will have n hinge points

Claim : It suffices to show that for all hinges x on g_t, $g_t(x)$ is below $g_{t-1}(x)$

Pf : Prove both hinge points get worse.

Let x be hinge point of g_t, then $\exists S \subseteq V$ s.t.

#cut edges from $S = X$

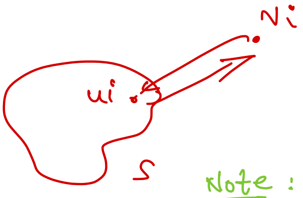
$$g_t(x) = \sum_{i=1}^{X} \rho_t(e_i) = \sum_{i=1}^{X} \underbrace{\rho_t(\overrightarrow{u_i, v_i})}_{\text{mass was sitting on vertices in prev timestep}} = \sum_{u \in S} P_t(u)$$

reverse edges (i.e. incident on S)

$$= \sum_{i=1}^{X} \rho_{t-1}(\overleftarrow{v_i, u_i})$$

$$\le \underbrace{g_{t-1}(X)}_{X \text{ heaviest edges}}$$

some set of X edges



Note : self loops are counted in $\overrightarrow{u_i, v_i}$ as well as $\overrightarrow{v_i, u_i}$

Lecture (2/18/23)

ILP formulation

$$\rho_t(2\cdot 3) = \max \vec{c} \cdot \vec{e} = (e_1, \dots, e_{2m}) \quad \{\text{fractional knapsack}\}$$

$$\text{s.t. } \sum c_i = 2\cdot 3$$

$$0 \le c_i \le 1$$

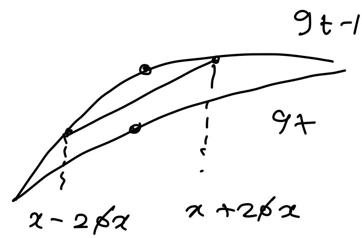
(curves fall royally)

Thm 1 : For any distribution p_0 on $V(G)$ any time step t and any hinge point of g_t $\quad 0 \le x \le m$,

$$g_t(x) \le \frac{g_{t-1}(x - 2\phi x) + g_{t-1}(x + 2\phi x)}{2}$$

(hinges) $\quad m < x \le 2m$

$$g_t(x) \le \frac{g_{t-1}(x - 2\phi(2m-x)) + g_{t-1}(x + 2\phi(2m-x))}{2}$$



Thm 2 : For any prob. p_0 any t, any x

$$g_t \le \frac{X}{2m} + \left(1 - \frac{\phi^2}{2}\right)^t \sqrt{X} \quad \text{(falls rapidly)}$$

Expansion for irregular graphs

for $S \subseteq V$, $\mathrm{vol}(S) = \sum_{u \in S} d_u$

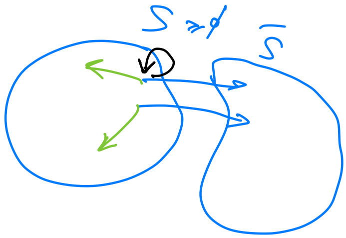
$$\phi(S) = \frac{|E(S, \bar{S})|}{\mathrm{vol}(S)}, \quad \phi(G) = \min_{\substack{S \subseteq V \\ \mathrm{vol}(S) \le \frac{\mathrm{vol}(V)}{2}}} \phi(S)$$

Proof : Fix a hinge point x, call corresponding ~~cut~~ outgoing edges X.

$$g_t(x) = \sum p_t(e_i) = \sum_{i=1}^{x} p_t(\overrightarrow{u_i,v_i}) = \sum_{u\in S} p_t(u)$$

$$= \sum_{i=1}^{x} p_{t-1}(\overleftarrow{u_i,v_i})$$

$$= \sum_{e\in W_1} p_{t-1}(e) + \sum_{e\in W_2} p_{t-1}(e)$$



W_1 = all edges internal to S

W_2 = all edges escaping S + loops

$|W_1| + |W_2| = X$ (#outgoing edges)

#loops $= \frac{X}{2}$ (by construction)

$|E(S,\bar{S})| \geq \phi \cdot \mathrm{vol}(S)$
$= \phi \cdot X$

$|W_2| = |E(S,\bar{S})| + \#\text{loops}$
$= \phi X + X/2$

$|W_1| \leq \frac{X}{2} - \phi X$

$$\boxed{g_{t-1}(|W_1|) \leq g_{t-1}\left(\frac{X}{2} - \phi X\right)}$$

(since g is monotonic ↑)

Hope : $g_{t-1}(|W_2|) \leq g_{t-1}\left(\frac{X}{2} + \phi X\right)$?

(imp. as g is monotone ↑)

Idea : ① $g_{t-1}(|W_1|) \leq \frac{1}{2} g_{t-1}(X - 2\phi X)$

② $g_{t-1}(|W_2|) \leq \frac{1}{2} g_{t-1}(X + 2\phi X)$

mass-reshuffling.

$$W_1' = \bigcup_{e\in W_1,\ e' \text{ is companion loop}} \{e, e'\}$$

$p(e) = p(e')$ since same vertex.

W_1' has twice many edges as W_1

$p_{t-1}(e) = p_{t-1}(e')$.

$p_{t-1}(e) = \frac{1}{2}\left(p_{t-1}(e) + p_{t-1}(e')\right)$

$$\sum_{e\in W_1} p_{t-1}(e) = \frac{1}{2}\left(\sum_{e\in W_1} p_{t-1}(e) + p_{t-1}(e')\right) = \frac{1}{2}\left(\sum_{e\in W_1'} p_{t-1}(e)\right)$$

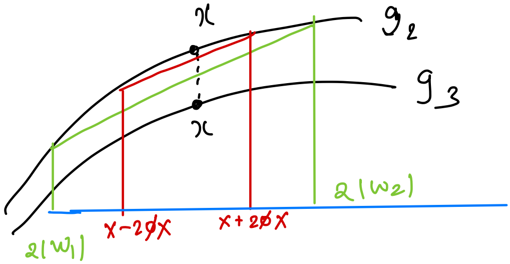
$$\leq \frac{1}{2} g_{t-1}(|W_1'|)$$

not quite there → $= \frac{1}{2} g_{t-1}(2|W_1|) \leq \frac{1}{2} g_{t-1}(X - 2\phi X)$

$W_2' = \bigcup_{e \in W_2} \{e, e'\}$ # loops = # out-edge, so pair an edge appropriately (bijection)

Fix $e \in W_2$ $\rho_{t-1}(e) = \rho_{t-1}(e')$

$$\sum_{e \in W_2} \rho_{t-1}(e) = \frac{1}{2} \sum_{e \in W_2} \rho_{t-1}(e) + \rho_{t-1}(e') = \frac{1}{2} \sum_{e \in W_2'} \rho_{t-1}(e)$$
$$\leq \frac{1}{2} g_{t-1}(|W_2'|)$$
$$= \frac{1}{2} g_{t-1}(2|W_2|)$$



Suppose you know $g_3(x) \leq$ val x at green chord

Suppose x is mid point of green chord

$\Rightarrow g_3(x) \leq$ val of x on red chord

Recall: LS have shown $g_t(x) \leq \frac{1}{2} g_{t-1}(2|W_1|) + \frac{1}{2} g_{t-1}(2|W_2|)$ (green) (i.e. doesn't intx $g(n)$)

from earlier

$\Rightarrow g_t(x) \leq \frac{1}{2} g_{t-1}(\phi - 2\phi x) + \frac{1}{2} g_{t-1}(x + 2\phi x)$ (red)

$\sum_{e \in W_1} \rho_{t-1}(e) + \sum_{e \in W_2} \rho_{t-1}(e) \leq (\quad)$

Now, just need to show to intersections with x.

Q. How to get algorithmic mileage from LS toolkit

A. As pioneered [ST12], you can use LS curve technique to find sparse cuts in graphs.

You know $S \subseteq V$, $|S| = \varepsilon m$

You use LS to conclude $g_t(x) \leq \frac{x}{2m} + \left(1 - \frac{\phi^2}{2}\right)^t \sqrt{x}$ holds

if $S_1, S_2, \ldots, S_t$ all have conductance $\phi(S_i) \geq \phi$

random walks, non-zero probs

Suppose you know $g_t(x) > \varepsilon$, $x = \varepsilon m$

$\phi = \sqrt{\frac{\phi(S)}{\alpha}}$, $t = \frac{\log n}{\phi(S)} \alpha$, then $g_t(x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{6} = \frac{2\varepsilon}{3}$

But we know $g_t(x) > \varepsilon$. $\Rightarrow$ contradiction, i.e. we encounter sparse cuts!

Lecture UGC and a subexponential time alg. (24/8/23)

Invent a new coloring problem "between" 2-col & 3-col.

$G = (V, E)$ where each edge $e = (u,v) \in E$ carries a permutation.

$\pi_{uv} : [k] \to [k]$

$u \xrightarrow{\pi_{uv}} v$

$\ell : V \to [k]$

Colors accepted by edge e of the form $(\ell_u, \pi_{uv}(\ell_u))$

ULP (unique labelling problem)

$(G = (V, E), \{\pi_{uv}\}_{(u,v) \in E})$

Warm-up problem of deciding if ULP instance is satisfiable is in P.
Algorithm is very similar to 2-coloring (one vertex fixes coloring)

$\mathrm{Val}(G, \ell)$ = fraction of constraints satisfied by ℓ.

$$\mathrm{Val}(G) = \max_{\ell \in \text{labellings}} \mathrm{Val}(G, \ell)$$

Q: Given a ULP instance G with the promise that $\mathrm{Val}(G) \geq 0.99$
Can you recover a coloring ℓ s.t. $\mathrm{Val}(G, \ell) \geq 0.01$

[Kho '02] 1. Hardness of general ULP $\Rightarrow$ Hardness of ULP on regular instances.
2. Problem remains hard when all permutations are cyclic.

Consider the following lin system.

$x_i + x_j = c_{ij} \mod k$ (cyclic shift)

there is a hard instance.

Unique games conjecture : For every $\varepsilon > 0$ $\exists k = k(\varepsilon)$ s.t. $\mathrm{ULP}(G, \varepsilon)$ is NP-hard

Def ($\mathrm{ULP}(G, \varepsilon, k)$) : Is $\mathrm{val}(G) \geq 1 - \varepsilon$ or $\mathrm{Val}(G) \leq \varepsilon$ (given one of them is true)

(Vizing's thm) Take a graph $G = (V, E)$ with $\Delta(G) \leq d$. Then you can properly edge color G with $\leq d+1$ colors $\chi_{\text{edge}} \in \{d, d+1\}$.

Note: deciding χ_{edge} is NP-hard.

Vizing's $\Rightarrow$ $\exists$ matching of size $\dfrac{|E|}{d+1}$, and we just satisfy these edges

$1/d+1$ $\quad \varepsilon$

$\varepsilon > 1/d+1$
$d+1 > 1/\varepsilon$
$d > 1/\varepsilon - 1$

$$\Rightarrow \mathrm{Val}(G) \geq \frac{1}{d+1}$$

Remark : UGC is asserted only for d-regular graphs with $d > 2/\varepsilon$

Thm [ABS'10] : ULP(G, ε) ∈ RTIME$\left(2^{n^{f(\varepsilon)}}\right)$ $\overset{o(n)}{\uparrow}$ RP $\overset{?}{=}$ P (believed)

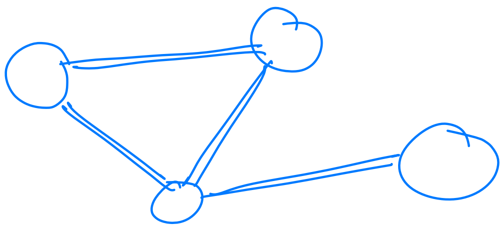
ETH (exp time hypothesis) : 3-sat needs $2^{\Omega(n)}$ time

But, we know ULP ∈ TIME$(2^{n^{0.01}})$ (breaks ETH?)

(polynomial time reduction)

↳ there might not be 3-SAT $\leq_p$ ULP , so doesn't disprove ETH.

ABS idea : [use expander decomposition]



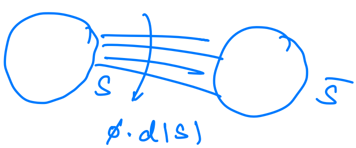
Solve ULP on all the "expanding pieces". Give up on cross edges. Show that there are not too many crossedges.

<u>Thm</u> : The vertex set of any d-regular graph can be partitioned into some subsets where ϕ(G[each subset]) $\geq \frac{1}{\log^3 n}$

cross-edges $\leq \frac{dn}{\sqrt{\log n}}$

<u>Proof</u> : If $\phi(G) \geq \phi$ (target) then done. (ϕ is adjustable)

If not,

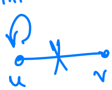


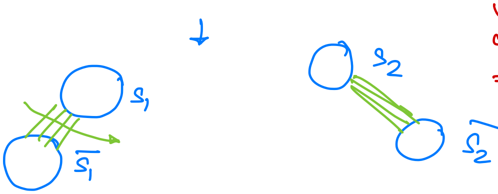
Have everybody in S pay up ϕd money)

money = #cross-edges

pays = when $\phi(S) \leq \phi$ pays ϕd.

introduce.





Whenever a vertex pays up, size of component it is assigned to shrinks by 1/2.

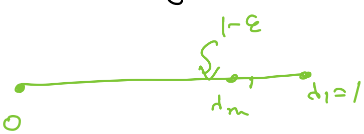
⇒ A vertex pays $\leq \log_2 n$ times.

Total money per vertex $\leq \phi d \log n$.

All money collected $\leq \phi n d \log n$

<u>Def</u> : [Threshold rank of graph] $G = (V, E)$

Let W = lazy random walk matrix = $\frac{I + A/d}{2}$



$\mathrm{rank}_{1-\varepsilon}(G) = |\{ i \in [n] : \lambda_i \geq 1-\varepsilon \}|$

<u>Note</u> : If good expanders, λ_2 is away from 1. (spectral gap is large ← expander)

$\text{rank}_{1-\varepsilon}$ (Good expander) $= 1$

Thm: The vertex set of any d-regular graph can be partitioned into subsets $V_1, V_2, \ldots, V_b$ s.t. $\text{rank}_{1-\varepsilon}(G[V_i]) \leq n^{O(\varepsilon^{1/3})}$ $\forall i \in [b]$ (not too many eigs above $1-\varepsilon$)

$\#$ cross-edges $\leq 0.01 \, |E(G)|$

This partition can be found in time poly(n)

Lemma: Fix $\varepsilon > 0$. Then $\exists$ an algorithm which can input a graph H with $n_H \leq n$ vertices and $\text{rank}_{1-\varepsilon}(H) \geq n^{O(1/3)}$ and returns a set $S \subseteq V(H)$ s.t.

1. $|S| \leq |V| / n^{O(\varepsilon^{1/3})}$
2. $\phi(S) \leq O(\varepsilon^{1/3})$

$\|^{el}$ to previous division, instead of $1/2$, we get fewer steps, so easier bounds using lemma to prove thm & we'll get appropriate # cross-edges.

Lecture

31/08/2023

Announcements:

1. Second session of probabilistic methods on Saturday 02 Sep @ 11.00 am
2. would you like in-class exercise sessions?
3. Pizzas today!

ABS algorithm [an almost wrap-up]

$\text{rank}_{1-\varepsilon}(G) = |\{ i \in [n] \mid \lambda_i(W) \geq 1-\varepsilon \}|$

Dense graph: $|E| \geq \Omega(n^2)$

For such graphs, $\text{rank}_{1-\varepsilon}(G) \leq O(1)$

- Almost all graphs are dense graphs. (if a random graph is chosen, deg vertex is $n/2$) {similar to chernoff bound}

claim: G dense $\Rightarrow$ $\underbrace{\text{rank}_{1-\varepsilon}(G) \leq O(1)}_{\text{"almost expander"}}$ (10,000)

e.g. $\frac{n}{10}$-regular

{Packing arguement}

$$\|A\|_F^2 = \|A_{\text{flatten}}\|_2^2 = \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} A_{ij}^2 = tr(A^T A)$$
(n² coord.)

$$= tr(A^2), \quad A \text{ symmetric}$$

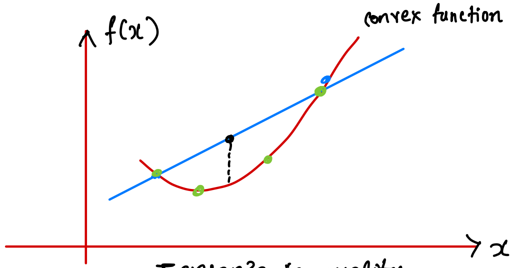
$$= \sum_{i=1}^{n} \lambda_i^2, \quad \lambda_i \text{ is an eigenvalue of } \underset{\text{adjacency}}{A}$$

$$= \frac{n}{10} \cdot n = n^2/10$$

Threshold eig of $A \geq \frac{n}{10}(1-\varepsilon)$. so, very few eigs of that order

Q: Suppose $G = (V, E)$ with $|E| = m$, $|V| = n$.

$\#\text{triangles}(G) \leq m^{3/2}$



Jensen's inequality

Ans. i,j entry of A^n = n length paths from i to j

3 length paths from i to i, but same triangle counted 6 times (perms)

So, Ans $= \dfrac{tr(A^3)}{6} = \dfrac{\sum \lambda_i^3}{6}$

$tr(A^2) = \sum_i \lambda_i^2 = \#\text{edges} = m$

$(tr(A^2))^3 \geq (tr(A^3))^2$ (Jensen)

$n = \#\text{vertices}$

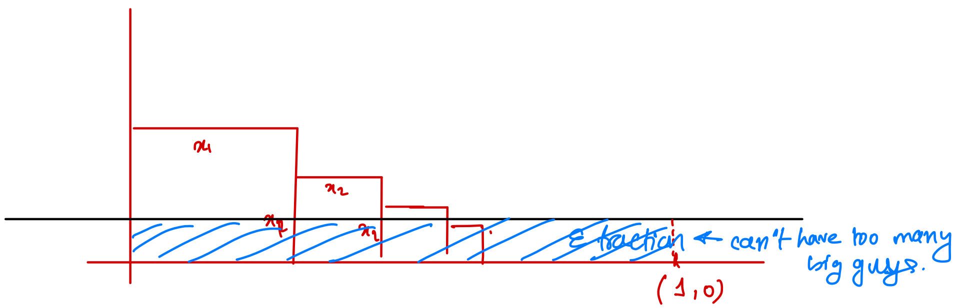
$\beta = O(\varepsilon^{1/3})$

$m = n^{O(\beta)}$

Claim: Fix $\varepsilon > 0$ and let H denote a graph with $n_H \leq n$.

let $t = \dfrac{\beta \log n}{\varepsilon}$. then $\exists\, u \in V(H)$ s.t. $\| W_H^t \mathbb{1}_u \|_2^2 \geq \dfrac{n^\beta}{n_H}$

Analytic sparsity: $X \in \mathbb{R}^n$ and $\|X\|_1 \leq \sqrt{\delta_n}\, \|X\|_2$



$\langle x, Wx \rangle \geq \langle x, W^2 x \rangle$

$\langle x, (W - W^2)x \rangle \geq 0$ $\quad \because \lambda_W \geq \lambda_W^2$.

Intuition: correlation b/w initial, final state reduces with increasing number of steps.

Almost expander: $\text{rank}_{1-\varepsilon}[\,L(\text{each subset})\,] \leq m$

Thm: $\exists$ an algo which on almost expanders, runs in time $2^{n \cdot O(\varepsilon^{1/3})}$ and return an assignment val $\geq 1 - \varepsilon^{1/6}$

- In $\hat{G}$, atmost k cycles. ($K = \#$ colors) for each cycle in G. (Spectral connection)

Lec-03 notes review

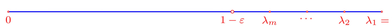
(Parameters)

Idea [ABS]: divide, solve ULP on subgraphs, combine sol^ns.

- $n \in \mathbb{N}$, the number of vertices in the original graph.
- $\varepsilon > 0$.
- $\beta = O(\varepsilon^{1/3})$.
- $m = n^{O(\beta)}$.

UniqueLabelingProblem(G, ε) (problem)

1. There exists $\ell \in [k]^n$ so that $\geq (1-\varepsilon)$ fraction of the edges are satisfied.
2. For all $\ell \in [k]^n$, ℓ satisfies at most ε fraction of the edges.



Definition 3.1. *Fix some sufficiently small $\varepsilon > 0$. Take a d-regular graph $G = (V, E)$ with the lazy random walk matrix being $\boldsymbol{W}$. We say that the $(1-\varepsilon)$ threshold rank of G is at most m if the number of eigenvalues of $\boldsymbol{W}$ which exceed $(1-\varepsilon)$ is at most m. This is written as,* {almost expander}

$$\mathsf{rank}_{1-\varepsilon}(G) = |\{\lambda \colon \lambda \text{ is an eigenvalue of } \boldsymbol{W}, \lambda \geq 1-\varepsilon\}| \leq m.$$ i.e. $\lambda_m(G) \geq 1-\varepsilon$

Flow:

1. Lemma to break d-reg graph into subsets with $\phi \geq \frac{1}{\log^3 n}$ (almost expanders), removing $o(n)$ edges (cheeger-tree)

cheeger sweep to find cuts S s.t. $\frac{E(S,\bar{S})}{d|S|} < \phi$

2. Thm, to break d-reg graph into $V_1, \ldots, V_b$, $\text{rank}_{1-\varepsilon}[G(V_i)] \leq m$, removing 1% edges (cheeger-tree with the red subroutine instead of cheeger cuts)

- $\varepsilon > 0$, $\exists$ alg with input H subgraph G, $\text{rank}_{1-\varepsilon} H \geq m$ which efficiently gives cut $S \subseteq V(H)$ with $|S| \leq |V|/m$, $\phi(S) \leq O(\sqrt{\varepsilon/\beta})$

$\to$ (Subroutine algo)

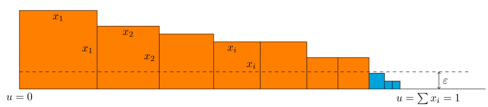
* Analytical sparsity: S has small support

Definition 3.5. *A vector $\boldsymbol{x} \in \mathbb{R}^n$ is called δ-analytically sparse if*

$$\|\boldsymbol{x}\|_1 \leq \sqrt{\delta n} \cdot \|\boldsymbol{x}\|_2.$$

- Motivation: majority of entries are small

$\|x\|_1 = 1$ then $\|x\|_2^2 \geq \frac{1}{\delta n^2}$



* $\varepsilon > 0$, H is subgraph, $|V(H)| = n_H$, $\text{rank}_{1-\varepsilon}(H) \geq m$. Let $t = \frac{\beta}{\varepsilon} \log n$,

$\exists u \in V(H)$ s.t. $\|W_H^t \mathbb{1}_u\|_2^2 \geq \frac{n^\beta}{n_H}$ $\left\{ \underbrace{\operatorname{Tr}(W^{2t})}_{\text{sum of eig}^2} = \sum_u \mathbb{1}_u W^{2t} \mathbb{1}_u = \sum_u \|W \mathbb{1}_u\|_2^2 \geq n^\beta / n_H \right\}$

* Further idea: random walks not mixing after t steps, $\Rightarrow$ probab. dist. is analytically sparse, it finds a low conductance cut.

large norm

- find some step where mixing is bad $\|W^{s+1} \mathbb{1}_u\| \geq \left(1 - \frac{\varepsilon}{\beta}\right) \|W^s \mathbb{1}_u\|$. If never then $\|W^t \mathbb{1}_u\| \leq \left(1 - \frac{\varepsilon}{\beta}\right)^t$

$W^s \mathbb{1}_u = x$. $\langle Wx, Wx \rangle \geq 1 - \frac{\varepsilon}{\beta} \langle x, x \rangle$

$\langle W^2 x, x \rangle \geq 1 - \varepsilon/\beta \langle x, x \rangle$ x is almost eigenvec. of W^2.

$x = W_H^t \mathbb{1}_u \Rightarrow$ sparse as $\|x\|_1 = 1$, 2-norm is lower bounded.

* Hence, $\exists x$ s.t. $\frac{\|Wx\|_2^2}{\|x\|_2^2} \geq \left(1 - \frac{\varepsilon}{\beta}\right)$. Now we want to use this to find sparse cut.
{sparse}
(rayleigh quotient small but large support for x)

* H is regular subgraph, $W_H = W$, x is analytically-sparse with $\frac{\|Wx\|_2^2}{\|x\|_2^2} \geq 1-\alpha$
for suff. small $\alpha > 0$, $\exists S \subseteq V(H)$ with $|S| \leq \delta n$ s.t. $\phi_H(S) \leq 8\sqrt{\alpha}$

Plan: obtain y from x which is sparse with better rayleigh quotient & use cheeger.

- $y_u = \begin{cases} x_u - 1/4, & x_u \geq 1/4 \\ 0, & \text{otherwise} \end{cases}$, $\|x\|_1 \leq \delta n$, $\|x\|_2^2 = \delta n$ {scaling}

$|\text{support}\{y\}| \leq 4\delta n$: #(elements $\geq 1/4$) $\leq 4\delta n$.

$$R_L(y) = \frac{\sum_{(i,j)\in E(H)} (y_i - y_j)^2}{d \sum y_i^2}$$

$(y_i - y_j)^2 \leq (x_i - x_j)^2$,
$\sum y_i^2 \geq (x_i - 1/4)^2 \geq \sum x_i^2 - \frac{1}{2}\sum x_i \geq \frac{\delta n}{2} = \frac{\|x\|_2^2}{2}$

$\Rightarrow R_L(y) \leq 2R_L(x)$

$\Rightarrow \frac{\langle y, Wy\rangle}{\|y\|_2^2} \geq 1 - 2\alpha$

* Do cheeger rounding with this y to get the required cut, i.e. satisfying cheeger ineq.

3. Solving ULP on each of these almost expanders : ULP(G, ε, k)

(parameters)

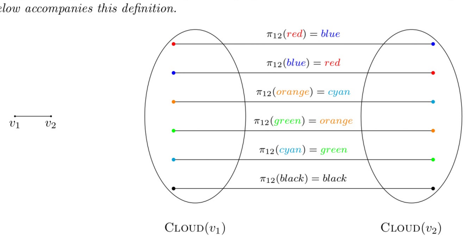
- $n \in \mathbb{N}$, the number of vertices in the original graph.
- $0 < \gamma \ll 0.01$.
- $\varepsilon = \gamma^6$.
- $\beta = O(\varepsilon^{1/3}) = O(\gamma^2)$.
- $m = n^{O(\beta)}$.

Definition 4.1. *Let* UniqueLabeling(G, ε, k) *denote an instance of the unique labeling problem where $G = (V, E)$ is the underlying graph. We define the graph $\widehat{G} = (\widehat{V}, \widehat{E})$ where the vertices of $\widehat{G}$ have the form (v, i) where $v \in V(G)$ and $i \in [k]$. This is to say, for $v \in V(G)$, we let*

$$\textsc{Cloud}(v) = \{(v,1), (v,2), \cdots (v,k)\}$$

which is to say $\textsc{Cloud}(v)$ *contains k copies of the vertex v, one in each color. Note that $|\widehat{V}| = kn$. Now, we define $\widehat{E}$. An edge in $\widehat{G}$ is of the form $((u,a),(v,b))$ where $u, v \in V(G)$ and the edge rule π_{uv} satisfies $\pi_{uv}(a) = b$. In other words, given an edge $(u,v) \in E(G)$, we include k edges in $\widehat{E}$.*

These edges form a perfect matching running between vertices in $\textsc{Cloud}(u)$ *and* $\textsc{Cloud}(v)$ *and they satisfy the edge rule π_{uv} in the "natural way". This graph is called the* Label-Extended Graph. *The picture below accompanies this definition.*



Edges between cloud(v_1) and cloud(v_2) connect vertices of the form $(1,a)$ with $(2,b)$ iff $\pi_{12}(a) = b$.

- Define label-extended graphs
  * for a satisfiable instance,
  $S = \{u, \sigma(u)\}_{u \in V(G)}$
  is disconnected from rest of G.

- Plan: unique labelling then corresponding low-conductance cut.
  * Consider ULP on d-regular graphs with $d \geq 2/\alpha$, having an assignment satisfying $(1-\alpha)$ fraction then,
  $\phi(\widehat{G}) \leq 2\alpha$
  - Consider vertex set = assignment.
    if (u,v) violates σ, pair with two edges satisfying
    $\Rightarrow$ both crossing cut, so # cut-edges $\leq 2 \cdot \alpha \cdot \frac{d|S|}{\text{total edges}}$.

- Plan : low-conductance cut $\Rightarrow$ unique labelling ?
  * can pick more than one vertex in a cloud, so not straightforward.

* observation : corresponding to every cycle in G there are k-cycles in $\hat{G}$
(when labelling satisfied we get k-cycles. if not, lesser)
and cycle in $\hat{G}$ $\Rightarrow$ cycle in G.

* Idea : large eigenvalue eigenvectors to find labellings

* Top eigenspace : **Definition 4.5.** *Let $\widehat{W}$ denote the lazy random walk matrix for the label extended graph $\widehat{G}$. The $(1-\gamma^5)$-top eigenspace of $\widehat{W}$ is written as:*

$$\mathcal{U} = \mathcal{U}(\widehat{W}, \gamma^5) = \mathsf{SPAN}\left(\{x : x \text{ is an eigenvector of } \widehat{W} \text{ with eigenvalue } \geq (1-\gamma^5)\}\right).$$

- Bound dimensionality of this space. (almost expander so bounded)
- $\hat{G}$ has kn nodes, corresponding $\hat{W}$ has $1 = \hat{\lambda}_1 \geq \dots \geq \hat{\lambda}_{kn} \geq 0$

then for any $0 < \delta < 1$, $\hat{\lambda}_{km \cdot n^{O(\beta)}} < 1 - \delta \cdot a$

$$r(W) = r_{1-\alpha}(W) = \inf_{k \in \mathbb{N}} \frac{\mathrm{Tr}(W^{2k})}{(1-\alpha)^{2k}}.$$

$\mathrm{Tr}(W^{2k}) \geq (1-\alpha)^{2k} \mathrm{rank}_{1-\alpha}(W)$ {packing arguement?}

* if ULP has assign of $1-\varepsilon$ fraction, $r_{1-\alpha}(W) \leq m$ then $\hat{r}_{1-\alpha}(W) \leq km$

Pf : $r_{1-\alpha}(\hat{W}) \leq \dfrac{\mathrm{Tr}(\hat{W}^{2t})}{(1-\alpha)^{2t}} = \dfrac{\mathrm{Tr}\left(\left(\frac{I+\hat{A}}{2}\right)^{2t}\right)}{(\ )} = \sum \dfrac{\binom{n}{i}}{2^i} \underbrace{\mathrm{Tr}(\hat{A}^i)}_{\text{cycles of length } i} \leq \sum \dfrac{\binom{n}{i}}{2^i} k^i \cdot \mathrm{Tr}(A^i) \leq \dfrac{k^{2t}\,\mathrm{Tr}(W^{2t})}{(1-\alpha)^{2t}}$

Now, choose t which "achieves" infimum

$$r_{1-\alpha}(\hat{W}) \leq r_{1-\alpha}(W)\, k^{2t} \underset{?}{\leq} km$$

#ques : How to we this result is unclear.

* Finding good assignment / vector in $\mathcal{U}$.

- Let projector to $\mathcal{U}$ be P_u $\qquad \varphi(S) \leq 2\varepsilon$
- Claim : $\|P_u x_S\|_2^2 \geq 0.99 \|x_S\|_2^2$, where $x_S = \frac{1_S}{\sqrt{S}}$ (pf elementary bounding $x_S^T \hat{W} x_S$)
- Gives x_S corresp to sparse cut in label extended graph, does not loose mass in $\ell 2$-norm, but looses lot of ℓ_1 mass

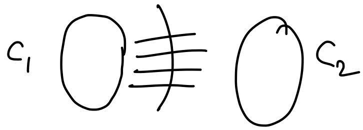
1. Come up with sparse cut $\tilde{S}$: subspace-enumeration, $\varphi(\tilde{S}) = O(\varepsilon)$
2. Come up with assignment corresponding to a sparse cut : TBD

Lecture

07/09/23

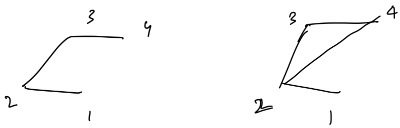
$A \succeq B$ if $A - B$ is p.s.d.

Q5.



supergraph G

subgraph G'



$L \qquad\qquad L'$

$L_{G'} \precsim L_G$ (all eigenvalues of $G' \le G$)

(don't need $\varepsilon \ll \phi^2$)

$L \preceq L'$

History:

[AKKTV '08] UG easy on good expanders, wanted ϕ(each piece) $\geq \frac{1}{\log\log\log n}$

P1 : Let P_n denote the path on n-vertices, $\lambda_2(P_n) \geq 4/n^2$

P2 : Let B_n denote complete binary tree on n vertices,

$$\lambda_2(B_n) \geq \Omega\left(\frac{1}{n \log n}\right)$$

P3 : In any graph (d-regular connected) $\lambda_2 \geq n^{-c}$, $\lambda_n \leq n^{-c}$ ($c > 1$ is a large constant)

A1 :

A3 : $0 \leq \lambda_i \leq 2$ $\quad I - \frac{A}{d}$

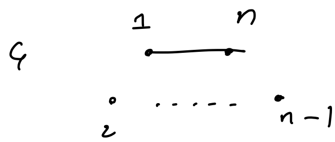
mult. by d

$$\boxed{\lambda_n \leq 2d}$$

M2 : $\lambda_n \leq \sum \lambda_i^2 = tr(A^2)$
$\leq nd^2$

$$\lambda_2 \geq \frac{\cancel{\phi}^2}{2} \geq \frac{2}{(nd)^2} \qquad (c=2)$$

Q. Show that $nL(P_n) \geq L(G)$



$n\left((y_1-y_2)^2 + \dots + (y_{n-1}-y_n)^2\right)$

$- (y_1 - y_n)^2$

$$\frac{\sum x_i^2}{n} \geq \left(\frac{\sum x_i}{n}\right)^2$$

$$(y_1-y_2)^2 + \dots + (y_{n-1}-y_n)^2 \geq \frac{(y_1-y_n)^2}{n}$$

Now, $L(G) = \begin{pmatrix} 1 & -1 & & 0 \\ -1 & 1 & & \\ & 0 & & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ -x_1 + x_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\quad x_3 = \cdots = x_n = 0$

$\lambda = 0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda x_1 = (x_1 - x_2)$

$\lambda x_2 = (-x_1 + x_2)$

$x_1 + x_2 = 0.$

$\lambda (x_1 - x_2) = 2(x_1 - x_2)$

$x_1 \neq x_2$

$\lambda = 0$ if $x_1 = x_2$

$\lambda_1, \lambda_2, \cdots, \lambda_{n-2} = 0$

$\lambda_{n-1} = 2 = \lambda_n$

A1.

Hint for P_n:

① Show $\displaystyle\sum_{1 \le u \le v \le n} \underbrace{(v-u) \sum_{i=u}^{v-1} L_{i,i+1}}_{} \succeq L_{K_n}$

$$\sum_{1 \le u \le v \le n} (v-u) \sum_{i=u}^{v-1} (y_i - y_{i+1})^2$$

$$\geq \sum_{1 \le u \le v \le n} (y_u - y_v)^2 = y^T L_{K_n} y.$$

② Show $\dfrac{n^3}{4} L_{P_n} \succeq (*)$ $\quad L_{P_n} = \displaystyle\sum_{1 \le u \le n} L_{i,i+1}$. $\displaystyle\sum_{1 \le u \le v \le 1} (v-u^*) \sum_{i=u}^{v-1} L_{i,i+1}$

$v-u$ for $u \le i \le v$

$\displaystyle\sum_{u \le i} \sum_{i \le v} (v-u) L_{i,i+1} \le n^3/4$

$\dfrac{n^3}{4} L_{P_n}$ $\quad \lambda_2$ compare with K_n

$$\begin{pmatrix} n-1-\lambda & & -1 & -1 \\ -1 & \ddots & & -1 \\ -1 & -1 & & \end{pmatrix}$$

$\sum d_i = n(n-1) \geq \lambda_2 (n-1)$

$J = n, 0, \cdots$

$nI - J = 0, n, \cdots, n$

$\boxed{\lambda_2 = n}$

$\Rightarrow \quad \dfrac{n^3}{4} \lambda_2 \geq n \Rightarrow \boxed{\lambda_2 \geq \dfrac{4}{n^2}}$

P4: $G=(V,E)$ d-regular and W is lazy random walk matrix, let

$$x = \begin{pmatrix} N(0,1) \\ \cdots \\ N(0,1) \end{pmatrix}$$

$$Y = X - \frac{\mathbb{1}}{\sqrt{n}} \langle X, \frac{\mathbb{1}}{\sqrt{n}} \rangle = X - \frac{\mathbb{1}}{n} \langle X, \mathbb{1} \rangle$$

Fix $\varepsilon > 0$

initial random (anti of Lovasz-Simonovits)

P1 for $k >> 1/\varepsilon \log n/\varepsilon$, letting $z = W^k y$

$$z^T L z \leq (\lambda_k + \varepsilon) z^T z \quad w/ \text{ prob.} \geq 2/3$$

{ dies out very fast in random walks }

Pf: L, W are p.s.d.

eigenbasis $[v_1, v_2, \ldots, v_n]$

evalues (L) $\quad 0 = \lambda_1 \leq \cdots \leq \lambda_n$

(W) $\quad 1 = \mu_1 \geq \cdots \geq \mu_n$

$$\mu_i' = 1 - \frac{\lambda_i'}{2} \qquad \begin{cases} \lambda_i : L \\ \mu_i : W \end{cases}$$

You know, $v_1 = \frac{\mathbb{1}}{\sqrt{n}}$, $\quad \alpha_i = \langle y, v_i \rangle = \langle x, v_i \rangle \; \forall i \geq 2$

$$\Rightarrow y = \sum_{i \geq 2} \alpha_i v_i$$

$$z = W^k \sum_{i \geq 2} \alpha_i v_i = \sum_{i \geq 2} \alpha_i \mu_i^k v_i$$

$$\frac{z^T L z}{z^T z} = \frac{(\sum \alpha_i \mu_i^k v_i)^T L (\sum \alpha_i \mu_i^k v_i)}{\sum \alpha_i^2 \mu_i^{2k}}$$

$$= \frac{\sum_{i \geq 2} \alpha_i^2 \mu_i^{2k} \lambda_i}{\sum_{i \geq 2} \alpha_i^2 \mu_i^{2k}}$$

Let $I = \{ i \geq 2 : \lambda_i \leq \lambda_2 + \frac{\varepsilon}{2} \}$

$|I| = i^*$

$$(*) = \left(\sum_{i=2}^{i^*} \alpha_i^2 \mu_i^{2k} \lambda_i' + \sum_{i=i^*+1}^{n} \alpha_i^2 \mu_i'^{2k} \lambda_i' \right) \Big/ \sum_{i \geq 2} \alpha_i^2 \mu_i'^{2k}$$

$$= T_1 + T_2$$

$$T_1 \leq \frac{\sum_{i=2}^{i^*} \alpha_i^2 \mu_i^{2k} \lambda_i}{\sum_{i\geq 2}^{i^*} \alpha_i^2 \mu_i^{2k}} \leq \lambda_{i^*} \frac{\cancel{\Sigma}}{\cancel{\Sigma}} = \lambda_{i^*}$$

$\mu_i - \mu_2 \geq \varepsilon$ (by prop of i^*)

$$T_2 = \frac{\sum_{i=i^*+1}^{n} \alpha_i^2 \mu_i^{2k} \lambda_i}{\sum_{i\geq 2} \alpha_i^2 \mu_i^{2k}} \leq \frac{\Sigma}{\alpha_2^2 \mu_2^{2k}} = \frac{\sum_{i=i^*+1}^{n} \alpha_i^2 \left(\frac{\mu_i}{\mu_2}\right)^{2k} \lambda_i}{\alpha_2^2} \leq \frac{\sum \alpha_i^2 \left(1 - \frac{\mu_i - \mu_2}{\mu_2}\right)^{2k} \lambda_i}{\alpha_2^2}$$

$$\leq \frac{\sum \alpha_i^2 (1-\varepsilon)^{2k} \lambda_i}{\alpha_2^2}$$

$$\leq \frac{\sum \alpha_i^2 e^{-\varepsilon k/2} \lambda_i}{\alpha_2^2}$$

will show $\left.\begin{array}{l} \sum_{i=i^*+1}^{n} \alpha_i^2 \leq O(n \log n) \\ \alpha_2^2 \geq \Omega\left(\frac{k}{n}\right) \end{array}\right\}$ finishes proof.

$\alpha_i \sim \langle v_i, y \rangle = \underbrace{\langle v_i, X \rangle}_{= \text{gaussian}}$

$x \quad y$

$f_x \quad d_y$

$e^{-x^2} \cdot e^{-y^2} = e^{-(x^2+y^2)}$

Lecture

(11/09/23)

Announcements

1. Dud Problem : P3
2. Project list out this week
3. Planning visit to TIFR

Bourgain's Embedding [A foray into discrete geometry]

E.g. Shortest path problem

Find s-t shortest paths takes $\Omega(m)$ time. Naive precomputing: ${}^nC_2$ entries

Metric embedding approach: ① Suppose you can attach to a vertex v, coord. (x_v, y_v)

② Also suppose $d_G(u,v) \cong \sqrt{(x_u - x_v)^2 + (y_u - y_v)^2}$. for $\log n$ coord., $n \log n$ space.

Preliminaries on discrete geometry

A metric space captures the notion of distance.

X, $|X| = n$. (X, d) is metric space if

① $d(x,x) = 0 \quad \forall x \in X$

② $d(x,y) \geq 0 \quad \forall x, y \in X$

③ $d(x,y) = d(y,x)$ ④ $d(x,y) + d(y,z) \geq d(x,z) \quad \forall x,y,z \in X$.

Think of d as a vector $\mathbb{R}^{{}^nC_2}$

Eg : ① Line metric

A metric space (X, d) is called a line metric if $\exists f : X \to \mathbb{R}$ s.t.

$$d(x,y) = |f(x) - f(y)|$$

② Cut metric

(X, d) is a cut-metric if $\exists S \subseteq X$ s.t.

$$d(x,y) = \begin{cases} 1 & x \in S, y \notin S \\ 0 & \text{otherwise} \end{cases}$$

③ L_1 - metric

(X, d) is ℓ_1 - metric if $\exists f : X \to \mathbb{R}^n$ s.t. $d(x,y) = \| f(x) - f(y) \|_1$

④ Graph metric

(X, d) is a graph metric if $\exists$ graph G d can be realised as shortest path metric on some on G

$$\forall x, y \in X, \quad d(x,y) = dist_G(u,v)$$

Q. If $G \setminus \{e\}$ preserves metric, good. Critical graph (no more edges can be dropped) depends on sequence of droppings. Given metric d, X is finite, can you find the critical graph with fewest # edges?

For general metric d, take K_n weighed s.t. $w\{u,v\} = d(u,v)$, works because of Δ inequality

Def [Cut Cone]

Definition 2.3. ***Cut-Cone*** *We define cut-cone to be the set of all metrics $d \in \mathbb{R}^{\binom{n}{2}}$. That is, we say*

$$CutCone = \{d \in \mathbb{R}^{\binom{n}{2}} : d = \sum_{S \subseteq [n]} \alpha_S d_S \text{ where } \alpha_S \geq 0 \text{ and } d_S \text{ is a cut-metric}\}.$$

Sps $d \in$ Cut Cone, d is legit metric. (check properties)

Thm : ℓ_1 metric $\in$ Cut - CONE.

Claim : Take a line metric where $\forall u, v \in X$, $d(u,v) \in [0,1]$ (unit interval line metric)

Then $d \in$ CUT-CONE.

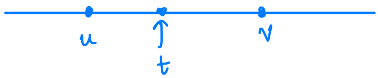
Proof : let f denote the function $f : X \to [0,1]$ s.t. $d(x,y) = |f(x) - f(y)|$

Pick a number $t \sim_{unif} [0,1]$

Define $S_t = \{ x \in X \mid f(x) \geq t \}$

We show $\mathbb{E}_t[d_{S_t}(u,v)] = d(u,v)$

Pf.



$$\mathbb{E}[d_{S_t}(u,v)] = \Pr(f(u) \le t \le f(v)) = |f(v) - f(u)| = d(u,v) \blacksquare$$

Define $\alpha_i = f(x_{i+1}) - f(x_i)$

Pf : ℓ_1-metric over $\mathbb{R}^k \in$ cut-cone, go coordinate wise!

Fact : ℓ_2 metrics are computationally nice.

metric version of min-cut : sps min-cut is $(S, \bar{S})$

$$\min_{S} \sum_{(i,j)\in E} d_S(i,j)$$

Ribe Program (ICM 2014, Asaf Naor) {like langlands program}

vector spaces $\leftrightarrow$ metric spaces

Metric version of sparsest cut problem

Sparsest cut problem seeks to find $S \subseteq V$; $|S| \le \frac{|V|}{2}$ s.t.

$\frac{|E(S,\bar{S})|}{|S|}$ is minimized.

$$\min \frac{\sum_{(i,j)\in E} d_S(i,j)}{\sum_{i<j} d_S(i,j)} = \frac{\#\text{cut-edges}}{|S| + |\bar{S}|} \le \frac{2E(S,\bar{S})}{|S|} \longleftarrow \text{NP-hard (finding min conductance cut)}$$

[Def] [Distortion of embedding]

Let (X_1, d_1), (X_2, d_2) be metric spaces, $f : X_1 \to X_2$

we say distortion of f is atmost α $(\alpha > 1)$ if $\exists$ a number $r > 0$ s.t.

$$\forall x, y \in X_1;\ r\, d_1(x,y) \le d_2(f(x), f(y)) \le \alpha \cdot r \cdot d_1(x,y)$$

In practice, we only consider host spaces X_2, where X_2 is a normed-space.

{vector space & metric}

Thm : [Bourgain 1985]

Every metric space (X, d) embeds into ℓ_1 with distortion $\le O(\log n)$

↳ $(\mathbb{R}^k, \ell_1)$

Thm : [Folklore]

Fix a metric (X, d), α_X = min distortion of all embeddings from X to X_2 can be found in poly(n) time.

Thm : [Linial/London/Rabinovich]

Given any finite metric space (X, d) with $|X| = n$, there exists an embedding of (X, d) into $(\mathbb{R}^k, \ell_1)$ where

1. $k = O(\log^2 n)$
2. distortion $= O(\log n)$

Bourgain Embedding (X, d)

1. For $i = 1, \dots, \log n$
   - for $j = 1, \dots, c \log n$
     - * add each $x \in X$ to $S_{i,j}$ with prob. $1/2^i$

For $x \in X$

$$f(x) = [\, d(x, S_{1,1}) \;\; d(x, S_{1,2}) \; \cdots \; d(x, S_{\log n, c\log n}) \,]^T$$

$$d(x, S) = \min_{y \in S} d(x, y)$$

Fix $i \in \{1, 2, \dots, \log n\}$

$i = 1$. $\quad S_{1,1}, \dots \dots, S_{1, c\log n}$ $\qquad$ Each $S_{i,j}$ has $\approx n/2$ pts.

$x \in X \to S_{i,j}$ prob $= 1/2$

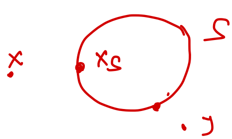
$S_{2,1} \; \dots \; S_{2, c\log n}$ $\qquad i = 2$ $\qquad n/4$ pts

$S_{\log n, 1} \; \dots \; S_{\log n, c\log n}$ $\qquad i = \log n$ $\qquad \approx 1$ pt.

Claim : Fix $S \subseteq X$. define $f_S(x) = d(x, S)$ then, this embedding is non-expanding [Fuchat embedding]

i.e. $|f_S(x) - f_S(y)| \le d(x, y)$

Pf :



$f_S(y) - f_S(x) \overset{?}{\le} d(x, y)$

$f_S(y) - f_S(x) \le d(y, x_S) - d(x, x_S)$
$\le d(x, y)$

$\text{``}\,\, f_S(x) - f_S(y) \le d(x, y)$

Up next: Lower bound on $|f_S(x) - f_S(y)|$.

Lecture

Announcements:

1. Bourgain's thm notes out
2. Delay in H/w grading
3. Project list out tomorrow
4. Lecture 3 on prob. methods Saturday @ 11.00am

$(X, d) \longrightarrow (\mathbb{R}^k, \ell_1/\ell_2)$

Guest/source — Host/Target

Given map $f : X \to \mathbb{R}^k$ (non-contractive)

distances don't reduce (via scaling e.g.)

distortion$(f) = \alpha \Rightarrow$

$\forall x, y \in X$

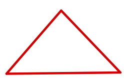
$$d(x,y) \le \|f(x) - f(y)\| \le \alpha\, d(x,y)$$

Consider a 3-pt metric space,

$d(a,b) = 1$

$d(b,c) = 1$

$d(c,a) = 1$



Consider a 4-pt metric space:

$d(a,b) = d(b,c) = d(c,d) = d(d,a) = 1$ {no embedding in $\mathbb{R}^2$. i.e. distortion > 1 }

$d(a,c) = d(b,d) = 2$

Claim: Embedding 4-pt metric above in ℓ_2 (any $\mathbb{R}^k$), necessarily suffers $\sqrt{2}$ distortion.

Idea: find two functions $w_1, w_2 : X \times X \to \mathbb{R}_{\ge 0}$ s.t. $\forall f$ on $X \to \mathbb{R}^k$ for some k

$$\text{s.t.}\quad \frac{\sum w_1(x,y)\, d(x,y)^2}{\sum w_2(x,y)\, d(x,y)^2} \ge k \cdot \frac{\sum w_1(x,y)\, \|f(x) - f(y)\|^2}{\sum w_2(x,y)\, \|f(x) - f(y)\|^2}$$

Suppose distortion $\le \alpha$

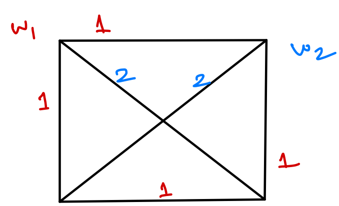
$$\text{LHS} \le \frac{\sum w_1(x,y)\, \alpha^2\, \|f(x) - f(y)\|^2}{\sum w_2(x,y)\, \|f(x) - f(y)\|^2}$$

if $\alpha^2 < k$ then we get a contradiction.

So, if $k = 2$, we get $\alpha \ge \sqrt{2}$.

Find w_1, w_2. $\quad w_2(b,d) = w_2(a,c) = 1$

$w_2(x,y) = 0$ otherwise



ques: Work out proof from here

[Folklore] For $S \subseteq X$, S non-empty, (X, d) is metric space.

let $f_S(x) = d(x, S)$.

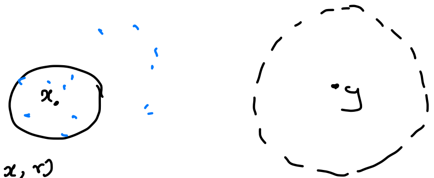
This mapping is non-expanding. i.e. $|f_S(x) - f_S(y)| \le d(x, y)$

Pf: Trivial (take x_S, apply Δ inequality)

Now, need to show not too-shrinking.

Idea: Force a situation where lots of $S_{i,j}$'s contribute a large value to $\|f(x) - f(y)\|_1$

Submodule:



$|B(x,r)| \ge 2^i$

$|B(y,R)| \le 2^{i+1}$

$B(x,r) = \{z : d(z,x) \le r\}$ $\quad$ $B(y,R) = \{z : d(z,y) \le R\}$

$d(x, S) \le r$ $\quad$ $d(y, S) \ge R$

$f(x) = [d(x, S_{11}) \ldots \ldots d(x, S)$

$f(y) = [d(y, S_{11} \ldots \ldots d(y, S)$.

$d(y,S) - d(x,S) \ge R - r$.

Claim: Fix two arbit points $x, y \in X$ and fix an index $i \in [\log n]$ and radii $r < R$

Let S be picked in ith iteration.

Assume $|B(x,r)| \ge 2^i$, $|B(y,R)| \le 2^{i+1}$, $B(x,r) \cap B(y,R) = \phi$

Then with prob. $\ge \frac{1}{32}$, $\quad \|f(x) - f(y)\| = \sum |d(x, S_{ij}) - d(y, S_{ij})|$

$|d(y,S) - d(x,S)| \ge R - r$ (i.e. $\mathbb{E}_S[|d(y,S) - d(x,S)|] \ge \Omega(1) \cdot R - r$)

since $\frac{1}{32}$ times $R - r$ atleast

Pf: $\mathcal{E}_1$ = event that S hits $B(x,r)$ $\Big\}$ indep (due to point picking being i.i.d)

$\mathcal{E}_2$ = event that S misses $B(y,R)$

$B(x,r) \ge 2^i$

$$p(\mathcal{E}_1) = 1 - \underbrace{\left(1 - \frac{1}{2^i}\right)}_{\text{not in } S}^{\underbrace{|B(x,r)|}_{\text{pts in ball.}}} \ge 1 - \left(1 - \frac{1}{2^i}\right)^{2^i} \ge 1/4$$

||ly $P(\mathcal{E}_2) \ge 1/8$

$\Pr(\mathcal{E}_1 \wedge \mathcal{E}_2) \ge \frac{1}{32}$ $\quad$ {since indep events}

$\mathcal{E}_1 \wedge \mathcal{E}_2 \Rightarrow \quad d(y,S) - d(x,S) \ge R - r.$

Bourgain defines sequence of radii

$$0 < \delta_1 \le \delta_2 \le \cdots \le \delta_t$$

δ_1 is chosen $\le \frac{d(x,y)}{3}$ (so that both balls $|B(x,\delta_i)|, |B(y,\delta_i)| \ge 2^i$ pts)

Consider radius δ_{i+1}, pretend $|B(y,\delta_{i+1})| \le 2^{i+1}$

You get w. prob. $\frac{1}{32}$, each of $c\log n$ sets contributes $\delta_{i+1} - \delta_i$ to
(iteration over j) $\|f(x) - f(y)\|_1$

Now,

fix x, y, iteration i (Chernoff)

w. prob. $\ge 1 - \frac{1}{n^n}$, in i^{th} iteration,

$$\sum_{i=1}^{\log n} \sum_{j=1}^{c\log n} |d(x, S_{ij}) - d(y, S_{ij})| \underset{??}{\ge} \sum_{i=1}^{\log n} c\frac{\log n}{64}(\delta_{i+1} - \delta_i) \ge \frac{c\log n \cdot \delta_t}{64}$$

$$= \frac{k}{\log n} \frac{\delta_t}{64}$$

$$= \frac{k}{\log n} \cdot \frac{d(x,y)}{3 \cdot 64}$$

Lecture

25/09/23

Announcements

1. TIFR trip postponed to October (Date TBD)
2. HW2 out this week
3. Talk by Nikhil Srivastava on many nodal domains
4. Decide project by Oct 11.

We study a map $f: X \longrightarrow \mathbb{R}^k$ has distortion $\le \alpha$ if $\quad k = O(\log^2 n)$, $\alpha = O(\log n)$

$\exists\ 0 < c \le 1$ s.t. $\forall x, y \in X$,

$$d(x,y) \le \|f(x) - f(y)\|_1 \le \alpha\, d(x,y)$$

(scale invariant)
{can multiply f by constant}

Recall, we showed $\|f_S(x) - f_S(y)\| \le d(x,y)$ $\{f_S(x) = d(x,S)\}$

Can we embed in a higher dim sp. s.t. "not too shrinking"

Thm [Leighton – Rao 1989] : $\exists$ algo which on input d-regular graph G runs in time poly(n) which returns a cut $(S, \bar{S})$ s.t. $\phi(S) \leq O(\log n)\, \phi(G)$

{use of bourgain for sparse cut}

Cheeger: $\phi(S) \leq O(\sqrt{\phi(G)})$

Say $\phi_G = \frac{1}{n}$. Cheeger: $\frac{1}{\sqrt{n}}$, LR : $\frac{\log n}{n}$, so LR is much better.

Lesson 1: Any metric $\underbrace{\;}_{O(\log n)}$ some l_1 metric

Lesson 2 : Any l_1 metric lies in the cut cone i.e. it is a convex combination of cut metric

Def [Sparsest cut] $sc(S) = \dfrac{E(S,\bar{S})}{|S|\,|\bar{S}|} \dfrac{n}{d} \leq 2\phi(S)$

we will ignore this (normalisation)

$sc(\bar{S}) = sc(S)$

$$\Omega(G) = \min_{\substack{S \subseteq V \\ S \neq \emptyset, V}} sc(S)$$

Sparsity for indicators.

$$sc(S) = \frac{\sum_{(u,v)\in E} |\mathbb{1}_S(u) - \mathbb{1}_S(v)|}{\sum_{u,v \in V} |\mathbb{1}_S(u) - \mathbb{1}_S(v)|}$$

P1 : $$\min_{\substack{S \subseteq V \\ S\neq \emptyset, V}} sc(S) \geq \min_{\substack{d:\, d \text{ is a} \\ \text{dist } f^n}} \frac{\sum_{(u,v)\in E} d(u,v)}{\sum_{u,v\in V} d(u,v)}$$

$\mathbb{1}_S$ is a distance f^n !
(cut-metric)

i.e.

$d(u,u) = 0$

$d(u,v) = d(v,u)$

$d(u,v) + d(v,w) \geq d(u,w)$

$\sum_{u,v\in V} d(u,v) = 1$ ← because of rescaling

$\min \sum_{(u,v)\in E} d(u,v)$

gives us an LP. for RHS (P1)

(let sol^n be d^*)

Using Bourgain, we obtain

$F : V \to \mathbb{R}^k$ s.t.

$d^*(u,v) \leq \|F(u) - F(v)\| \leq O(\log n)\, d^*(u,v)$

Consider

$$sc(G) \geq \frac{\sum_{(u,v)\in E} d^*(u,v)}{\sum_{u,v\in V} d^*(u,v)} \geq \frac{1}{O(\log n)} \frac{\sum_{(u,v)\in E} \|F(u)-F(v)\|_1}{\sum_{(u,v)\in V} \|F(u) - F(v)\|_1}$$

$l_1 = \sum \alpha_t S_t$

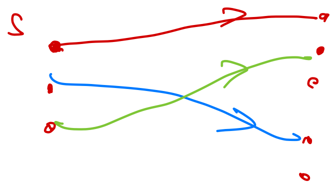
$$= \frac{1}{O(\log n)} \frac{\sum_E \sum_{S\in S_f} \alpha_S |1_S(u) - 1_S(v)|}{\sum_V \sum_{S\in S_f} \alpha_S |1_S(u) - 1_S(v)|}$$

$$= \frac{\sum_{S\in S_f} \alpha_S \sum_E |1_S(u)-1_S(v)|}{\sum_{S\in S_f}\alpha_S \sum_V |1_S(u)-1_S(v)|}\cdot\frac{1}{O(\log n)} \geq \min_S \frac{\sum_E |1_S(u)-1_S(v)|}{\sum_V |1_S(u)-1_S(v)|}\cdot \frac{1}{O(\log n)}$$

Now, $\dfrac{\sum\alpha_i}{\sum\beta_i} \geq \min_i \dfrac{\alpha_i}{\beta_i}$

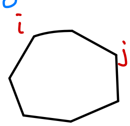
Hence, $\dfrac{sc(G)}{O(\log n)} \le 2\phi(G) \Rightarrow \boxed{sc(S)\le O(\log n)\, sc(G)}$

$$sc(S) = \frac{\sum_{(u,v)\in E} |\mathbb{1}_S(u)-\mathbb{1}_S(v)|}{\sum_{(u,v)\in V} |\mathbb{1}_S(u)-\mathbb{1}_S(v)|} = \frac{|E(S,\bar{S})|}{|S|\,|\bar{S}|}$$



Some generalisation of max-flow min-cut (Multi-commodity flow problem)

Intuition for Bourgain's result

Metric	Embedding Algo	$\mathbb{E}$[distortion]	Report card
cycle metric 	Pick $r\sim V$ $\mathbb{E}\,|d(r,u)-d(r,v)|$	$O(1)$.	✓
Shortest path from i to j	Pick $r\sim V$ $\mathbb{E}[d(r,u)-d(r,v)] = \frac{2}{n}$	$O(1)$ after rescaling.	✓
1 - metric	($r=u,v$ then 1, else 0)	$O(\frac{1}{n})$ dist = 1	✗
1-2 metric		$1+\frac{2}{n}$ dist = 2	
1 - 2 metric	$A \subseteq_{rand} V$ $(Pr = 1/2)$ $\mathbb{E}[d(x,A)-d(y,A)]$	$O(1)$	✓

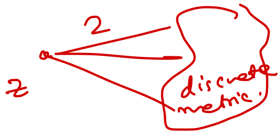
SP metric on any planar graph $\xrightarrow[O(1)]{} \ell_1$ [Ref: Anupam Gupta]

Best: $O(\sqrt{\log n})$

e.g. of 1 - 2 metric

$(X, \frac{1}{2})$ here $\forall x, y \in X \setminus \{z\}$ $d(x,y) = 1$

$\forall x \neq z,$ $d(x, z) = 2$



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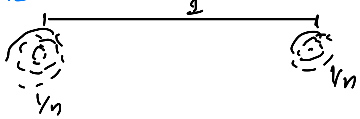
far clusters



If we pick 1 pt, far pts messed up,
$A \subseteq V$, then close pts messed up.
(mixing of distribution idea)

Lecture

05/10/23

Announcements

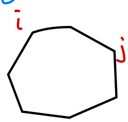
1. Would the class prefer one single question as both HW3, HW4
   [Difficult problem : Do not google]

2. which one do you prefer
   → Presenting your project (or)
   → Submitting a written report {final exam week}

Randomized map

$f: X \to \mathbb{R}^m$ is RV. $\forall u,v \in X$ $d(u,v) \le \mathbb{E}\|f(u)-f(v)\|_1 \le \alpha \cdot d(u,v)$

Intuition for Bourgain's result

Metric	Embedding Algo	$\mathbb{E}$[distortion]	Report card
cycle metric  Shortest path from i to j	Pick $r \sim V$ $\mathbb{E}\,|d(r,u)-d(r,v)|$ $f(u)=d(r,u)$.	$O(1)$.	✓
1 - metric	Pick $r \sim V$ $\mathbb{E}[d(r,u)-d(r,v)]$ $= \frac{2}{n}$ ($r=u,v$ then 1, else 0)	$O(1)$ after rescaling.	✓
1-2 metric	Pick $r \sim V$ $\mathbb{E}[d(r,u)-d(r,v)]$	$\begin{cases} \frac{2}{n} & \text{if } d(u,v)=1 \\ 1+\frac{2}{n} & \text{if } d(u,v)=2 \end{cases}$	✗ ← distortion $=\Omega(n)$
1 -2 metric	$A \subseteq_{rand} V$ $(Pr = 1/2)$ $\mathbb{E}[d(x,A)-d(y,A)]$	$O(1)$	✓

1-2 metric.

$A \subseteq V$ w.p. $1/2$
w/p $1/4$, we get actual distance

$d(r, Hub) - d(r, v)$

$n-2$ pts. 1

2 pts. 2.

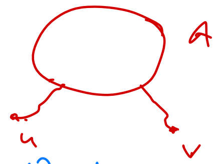
$\frac{n-2}{n} + \frac{2}{n}(2) = 1 + 2/n$

$\mathbb{E}_A \, |d(A,u) - d(A,v)| \geq d(u,v)/2$ {1-2 metric}

$p = \frac{1}{2}$ one in, other out.

dist is +ve.

$\geq 1/2$



w.p. 1/4

$d(u,v) = 1$ or 2

so, $\geq d(u,v)/4$

[Def] For clusters

$d(\text{across}) = 1$

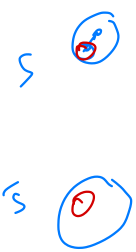
$d(\text{within}) = 1/n$

if $u, v \in S$ $\quad \mathbb{E}_r \, |d(u,r) - d(v,r)| = \frac{2}{n^2}$ if $r = u$ or $r = v$ then $\frac{1}{n}$ w.p. $2/n$

if $u \in S, v \in \bar{S}$ $\quad \mathbb{E}_r \, | \qquad | = 1.$

Single pt doesn't work, try picking A random set with den $(A) = \frac{1}{2}$

$\mathbb{E}_A \, |d(u,A) - d(v,A)|$



w/p 1/2 one in, one out

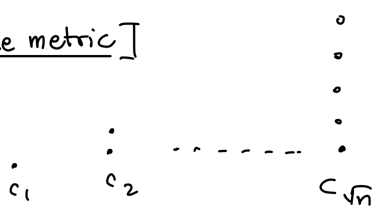
w/p 1/2 one in, one out

$u, v \in S$ then $d = \frac{1}{n} \Rightarrow \mathbb{E} = 1/2n$

$u \in S, v \in \bar{S}$ then $d = \frac{1}{n} + \underbrace{\exp(-n)}_{\text{if } \bar{S} \text{ is missed completely}}$ } messed up

Idea : Pick w.p. 1/2 one of two sizes $\{1, \frac{n}{2}\}$ ← density parameter

Def [Tree metric]



Pick $t \sim \{1, 2, \ldots, \frac{n}{2}\}$

Idea : Hit all cluster w/p size of cluster

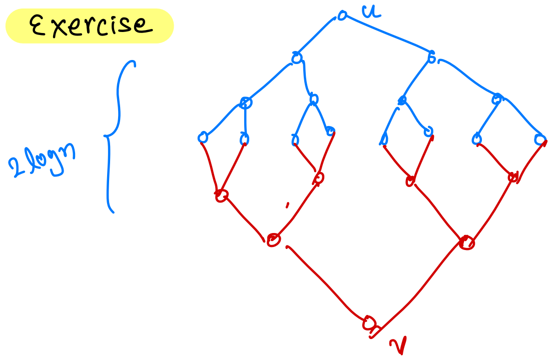
Pick $t \sim (1, 2, 4, \ldots, 2^{\lfloor \log n \rfloor})$

A picks pt w/p $1/t$. We witness one correct value.

Correct choice w/p $\frac{1}{\log n}$.

Ref : Luca Trevisan (Metric Embeddings)

Exercise



for any t,

$$\mathbb{E}[d(u,A) - d(v,A)] = \frac{d(u,v)}{\log n}.$$

[Kirchoff 1847]:

1. Lays down foundations of electrical network n/w theory
2. Gives KCL + KVL [electric flow] KVL $\Longleftrightarrow$ ohm's law
3. Among all "flows" electric flow minimizes energy.
4. Presents an algo to count # of spanning trees.
5. Algo to generate random spanning tree.

Theorem [Kirchoff] let $G = (V, E)$ be a graph, let $L = D - A$

Then #spanning trees of $G = K(G)$
$= \det(L^{(i)})$ $\forall i \in [n]$

delete ith row, ith column

Thm [Bernardi '12] : {Candidate for paper presentation}

Let H_d denote the d-dimensional hypercube $\{e(u,v)$ if $d_H(u,v) = 1\}$.

Then, $K(H_d) = \frac{1}{2^d} \cdot \prod_{i=1}^{d} (2i)^{\binom{d}{i}}$

The eigenvalues of $L(H_d)$

$\rightarrow$ $(2i)$ multiplicity $\binom{d}{i}$ $\forall i \in [d]$ (conn: power of edges d times)

Note : For K_n, #Spanning trees = n^{n-2}

M1 : Proofer code.

Consider set of strings with $\{1, \dots, n\}$ alphabet of length $n-2$.

For any tree

1. Take smallest number which is a leaf, add parent to string
2. Delete leaf and repeat.

for any tree $\exists$ string.

Now are all strings computed? Yes (can be shown by)

1. Take min idx not in string, take 1^{st} letter as parent of that
2. Delete 1^{st} letter and repeat.

M2 : Graph laplacians.

#spanning trees = $\det(L^{(i)})$

$$L^{(i)} = \begin{matrix} n-1 & & & \\ -1 & n-1 & & \\ \cdots & & \ddots & \\ -1 & & & \end{matrix} = nI - J$$

, where dim $= (n-1)$ for both

for this eigs of $J = (n-1), 0, \dots, 0$

so, eigs $= 1, \underbrace{n, n, \dots, n}_{n-2 \text{ values}}$ } $n-1$ values

So, #spanning trees $= n^{n-2}$

Lecture

Announcements

1. Decide Projects Soon
2. Talk by Nikhil Srivastava
3. HW2 discussion?
4. Decide Project type (Research or Reading)

Generate Random Spanning Tree

[Thm] Bernardi '12

Let H_d denote d-dimensional hypercube, then #spanning trees

Then $K(H_d) = \frac{1}{2^d} \prod_{i=1}^{d} (2i)^{\binom{d}{i}}$

K: number of spanning trees

Remark: Bernardi's pf is combinatorial (find a bijection of some sort)

1. $V(H_d) = \{0,1\}^n$
   $E(H_d) = \{u, u \oplus e_i\}_{i=1}^{n}$

   $\}$ matching b/w $n-1$ dim H.C.

Fact: Eigenvalue of $L(H_d)$ are $2k$ with mult. $\binom{d}{k}$ for $k=0$ to d.

Matrix Tree Theorem

$L^{(i)}$: laplacian with i^{th} row, i^{th} column deleted.

Thm: For any n-vertex graph $G=(V,E)$ with laplacian L and for any $i \in [n]$
You have $\det(L^{(i)}) = K(G)$

Claim: $A \in \mathbb{R}^{n \times n}$, E = diagonal matrix where $E \in \mathbb{R}^{n \times n}$ with $E(i,i)=1$, 0 o/w

then, $\det(A+E) = \det(A) + \det(A^{(i)})$

write $X = A + E$

$$\det(X) = \sum_{\sigma \in S_n} \text{sgn}(\sigma)\, X_{1,\sigma(1)} \cdots X_{n\sigma(n)}$$

$X_{i,i} = A_{i,i} + 1$

$$\det(X) - \det(A) = \sum_{\substack{\sigma \in S_n \\ \sigma(i)=i}} \text{sgn}(\sigma) \; (\underbrace{(A_{i,i})}_{-A_{i,i}} + 1) \prod_{\substack{j=1 \\ j \neq i}}^{n} A_{j,\sigma(j)}$$

$$= \det(A^{(i)})$$

Pf [of MTT]

fix some edge $e \in E = (i,j)$

$$K(G) = K(G \setminus e) + K(G/e)$$

e delete — T doesn't contain e

e contracted — T contains e

Base case : $L = 0,\ L^{(i)} = 0,\ K(G) = 0$

We will show

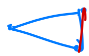
① $\det(L^{(i)}_{G \setminus e}) = K(G-e)$

and $\det(L^{(i,j)}) = K(G/e)$ ← delete 2 row, 2 col

product of edge weights || include weight of spanning tree

② $\det L^{(i)} = \det L^{(i)}_{G \setminus e} + \det L^{(i,j)}$

① Trivial by induction ← nope :



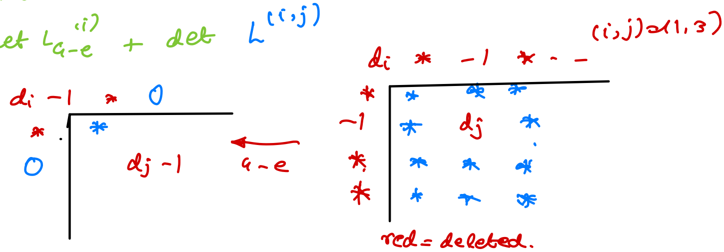
e contract.

no longer a simple graph.

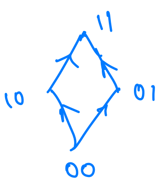
Towards item 2

$L^{(i)} = L^{(i)}_{G-e} + E$ where $E = 1$ at $(j-1, j-1)$, 0 o/w.

$\det L^{(i)} = \det L^{(i)}_{G-e} + \det L^{(i,j)}$



Pf : $D_{out} \in \mathbb{R}^{2^d \times 2^d}$ (diagonal matrix)



orientation : acc. to sum.

$D_{out}(x,x) = 2\ \#\ 1\text{'s in } x$
$= 2\ \mathrm{Ham}(x)$

we will show L is similar to D_{out}

(similarity preserves spectrum)

$G \in \mathbb{R}^{2^d \times 2^d}$, the similarity transform.

$$G(a,b) = g_{a,b} = (-1)^{a \circ b} = (-1)^{\Sigma a_i b_i}$$

Step 1 : G has orthogonal rows

Step 2 : $G^2 = 2^d I$

Step 3 : $L = G^{-1} D_{out} G$

Pf : $2 \Rightarrow 3$

$G^{-1} D_{out} G = \frac{G}{2^d} D_{out} G = Y$

$$Y_{a,b} = \frac{1}{2^d} \sum (-1)^{a\cdot x} D_{out}(x,x) (-1)^{b\cdot x}$$

① $a=b$, $(-1)^{a\cdot x}(-1)^{b\cdot x} = 1$

$$Y_{a,a} = \frac{1}{2^d}\sum D_{out}(x,x) = \frac{1}{2^d} \times 2 \times \sum_x \#\text{ones}(x)$$

(2^d nos, $\frac{1}{2}$ ones)

$$= \frac{1}{2^d}\, 2\sum k\binom{d}{k}$$

$$= \frac{1}{2^d}\, 2\cdot d\cdot 2^{d-1}$$

$$= d.$$

② $a\neq b$, a,b neighbors

$$Y = 2\left(\sum_{x\in M_1} Ham(x) - \sum_{x\in M_2} Ham(x)\right)\Big/2^d$$

$$= -2\ \#\text{extra ones}\ /2^d$$

$$= \frac{-2\times 2^{d-1}}{2^d} = \frac{-2^d}{2^d} = -1.$$

$a\cdot x = a_1x_1 + a_2x_2 + \dots + a_nx_n$

$b\cdot x = b_1x_1 + \dots$

only differ here

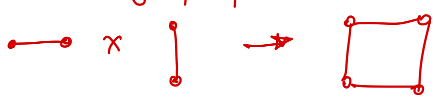
↓

break into p.m with $i=0$, $i=1$

③ $a\neq b$, a,b not neighbors.

Show $Y_{a,b} = 0$.

Next up: Define graph products



$\pm 1 \quad \pm 1 \quad \pm 1 \quad \pm 1$

$$V(G \,\square\, H) = V(G) \times V(H)$$

$$E(G \,\square\, H) = \{(u,v),(u,v')\} \text{ iff } (v,v') \in E(H)$$
$$+ \{(u,v),(u',v)\} \text{ iff } (u,u') \in E(G)$$

Lecture 23/10/23

Recall : $\tau(H_d) = \frac{1}{2^d} \prod_{i=1}^{d} (2i)^{\binom{d}{i}}$ {Bernardi's result}

$J =$ all ones matrix.

Thm [MTT++]

$$\tau(G) = \det(L^{(i)}) = \frac{1}{n} \prod_{i \geq 2} \lambda_i(L)$$

Thm : Let $A \in \mathrm{Sym}(\mathbb{R}^{n\times n})$ with row sum $= 0$
Then, all cofactors of A are equal.

Proof: $A^{(i,j)} = A$ minus i^{th} row, j^{th} column.

$C \in \mathbb{R}^{n\times n}$ is a cofactor matrix i.e. $C_{i,j} = (-1)^{i+j} \det A^{(i,j)}$

Def : [Adjugate] $\mathrm{adj}(A) = C^T$

Fact : $A\,\mathrm{adj}(A) = \det(A)\, I$

Now, let $C =$ co-factor matrix.

wanna show, all rows of C lie in span of $\mathbb{1}$ (enough for result)

Since all cols of A add to 0,
$\mathrm{rank}(A) \leq n-1$

$\Rightarrow$ $A \cdot \mathrm{adj}(A) = A\,C^T = 0$ (since $\det = 0$)

If $\mathrm{rank}(A) \leq n-2 \Rightarrow C = 0$ (removing one row, col gives LD vectors again).

Now, if $\mathrm{rank}\, A = n-1$ (only non-trivial case)

Recall $\mathbb{1} \in \mathrm{Nullsp}(A)$

and $A\,C^T = 0$

$\Rightarrow$ all columns of $C^T \in \lambda \mathbb{1}$

$\Rightarrow$ all elements in a col are equal

II^{ly} for $A^T \Rightarrow$ all el in row are equal

$\Rightarrow$ All cofactors are equal ■

Pf (of MTT++) : Want to produce a symmetric matrix whose eigenvalues are all non-zero eigenvalues of laplacian upto rescaling.

Consider matrix $L+J$ [J = all ones matrix]
$= \mathbb{1}\,\mathbb{1}^T$

eigenvalues of $L+J$ = $\{n, \lambda_2, \dots, \lambda_n\}$

$$\det(L+J) = n \prod_{i=2}^{n} (\lambda_i)_L$$

We would show $\det(L+J) = n^2 \det(L^{(1)})$

$$\underbrace{\begin{pmatrix} | & & | \\ l_1 & \cdots & l_n \\ | & & | \end{pmatrix}}_{L} \qquad \underbrace{\begin{pmatrix} n^2 & | & & | \\ 0 & & & \\ \vdots & l_2 & \cdots & l_n \\ 0 & | & & | \end{pmatrix}}_{L+J \text{ after transform}}$$

Starting with $L+J$

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \xrightarrow{r_1 \to r_1 + \cdots + r_n} \begin{pmatrix} n & n & \cdots & n \\ l_1+1 & l_2+1 & \cdots & l_n+1 \end{pmatrix}$$

$\downarrow c_1 \to c_1 + \cdots + c_n$

$$n^2 \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & | & & | \\ 1 & l_2 & \cdots & l_n \\ 1 & | & & | \end{pmatrix} \xleftarrow{} \begin{pmatrix} n^2 & n & \cdots & n \\ n & & & \\ \vdots & & & \\ n & & & \end{pmatrix}$$

factor n from row, col and subtract c_1 from rest of cols

Directed Matrix Tree Theorem

Def [In laplacian of directed graph]

$G(V,E)$ directed with

$$L_{in}(i,j) = \begin{cases} \text{indegree}(i) & \text{if } i=j \\ -\#\text{ of dir edges } i \to j & \text{o/w} \end{cases}$$

row sums of $L_{in} \neq 0$
col sums of $L_{in} = 0$

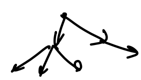
$$L_{out}(i,j) = \begin{cases} \text{outdeg}(i) & \text{if } i=j \\ -\#\text{ of dir edges } i \to j & \text{o/w} \end{cases}$$

row sums of $L_{out} = 0$
col " ——— " $\neq 0$

$In_v(G)$ = # in-trees rooted at v
$out_v(G)$ = # out-trees rooted at v

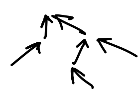
Spanning tree

e.g.



out-tree

e.g.



in-tree

Thm [Directed MTT]

Let v be vertex of G
Pick any $1 \le i \le n$

① $In_v(G) = (-1)^{v+i} \det L_{out}^{(v,i)}$

② $Out_v(G) = (-1)^{i+v} \det L_{in}^{(i,v)}$

Note: For undirected, #in-trees = $n \times$ ST. (choose root, orientation)

Pf : By induction on # directed edges whose starting point is NOT vertex v.

Base case : Sps $\delta_{out}(u)=0$ when $u \neq v$, then $\mathrm{in}_v(G)=0$, verify $\det L_{out}^{(v,i)} = 0$

Induction: Pick $e = i \to j$ where $i \neq v$ remove e from i^{th} vertex

$$L_{in}(G) = \underset{\substack{\uparrow \\ \text{remove } e}}{L(G_1)} + \underset{\substack{\uparrow \\ \text{keep } e}}{L(G_2)}$$

only affected row

$$i \to \det\begin{pmatrix} a_1 & a_2 & \overset{a_i\, a_j}{\cdots} & a_n \end{pmatrix} = i \to \det\begin{pmatrix} a_1 & a_2 \cdots a_i - 1 & \cdots & a_j + 1 & \cdots & a_n \end{pmatrix} + \det\begin{pmatrix} \cdot\, 0 & \overset{i}{1} & 0 & \overset{j}{-1} & 0 \cdots \end{pmatrix}$$

$L_{out}(G)$ $\qquad L(G_1) \qquad L(G_2)$

By induction on $G_1 = G \setminus e$, $G_2 = G/e$, we get result (vertex has out deg ≤ 1)

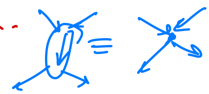
Point : if only one outgoing edge from $i \to j$ then we get $L_{out}(G) = L(G_2)$

So, we argue this case separately.

If $|\delta_{out}(i)| = 1$.

Then, contract edge $i \to j$ and apply induction., $i \neq v$, so $|\delta_{out}(i)| = 1$.

bijection for trees b/w contracted graph, original graph.



$\{ L_{out}\, v = L_{out}\, v_{contracted} \}$

Sampling Uniformly Random Spanning Trees

Fix $e = (i,j) \in E(G)$, G undirected.

$$\Pr(e \in T) = 1 - \frac{K(G-e)}{K(G)} = p_e$$

$T \sim$ all spanning trees

$e_1 \cdots\cdots e_m$

p_1

depending on edge taken or not update $\Pr(e_2 \in T)$

Prove this gives uniform [HW]

Induction: e_1 chosen or not $\to$ uniform on rest of trees e_1 taken & another uniform e_1 left

$P(T \mid e_1 = 1)$, $P(T \mid e_1 = 0) \leftarrow$ uniform

and $P(T) = P(e_1 = 1)\, P(T \mid e_1 = 1) + P(e_1 = 0)\, P(T \mid e_1 = 0)$

$=$ uniform!

	Decision Tree	MCMC
Uniform Sampling	✓	✗
Fast Sampling	✗	✓

runtime $O(n^2)$

$T_1 \cdots T_n$

Go from one ST to another ST by
. swapping edge,
perform random walk,
~ almost uniform sample.

Generating Random Spanning Trees

26/08/23

Aldos asked: How do you prove thms about uniform spanning trees [USTs] ?

Spielman & Srivastava: tear down graph to sparsify $n^2 \to 10n$ and size of cuts is approximately preserved (mult-approx)

desirability of edge $= \Pr_{T \sim UST}(e \in T)$

Aaron Schild: UST in almost linear time.

Answer: First sample UST

Think: r/walk $\equiv$ random oriented trail.

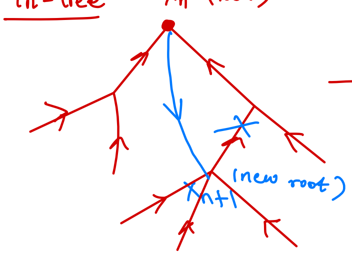
$G = (V, E)$ connected, undirected

$G_M = (V, E_m)$, each edge both ways

let (X_n) denote the r/walk.

$X_{n,n+1}$ = oriented edge X traverses b/w step n and $n+1$

in-tree



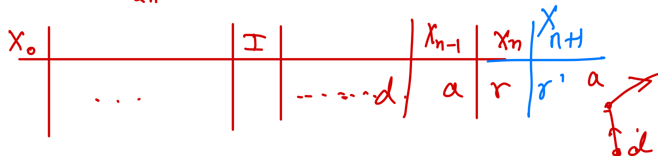
new in-tree rooted at X_{n+1}

Forward chain: $(X_n)_{n \in \mathbb{N}}$

"last-time rule"

T_n: rooted at X_n (in-tree)

For $v \in V$ $\quad \ell_n(v)$: last time I exit v before time n.



all vertices show up (only consider latest out-edge)

This construction is spanning tree $\to$ $n-1$ edges, connected.

for $X_n \Rightarrow X_{n+1}$ use forward rule!

r' is root, delete outgoing edge from r', include (r, r')

Questions: Is the state space connected? (single recurrent class)

Claim: Underlying graph of forward chain is strongly connected. (show later)

Markov chain Matrix Tree Theorem

Thm: Stationary distribution of forward chain is $\underset{(\text{of } G)}{\pi} \otimes \underset{\substack{(\text{of } G) \\ \text{undir.}}}{\text{UST}}$

(space of fwd chain = # intrees = # ST $\times$ n)

Remark: $\pi \times$ UST can be thought of as dist. supported on in-trees of G_M

Proof: We saw any dist. that satisfies detailed balance in a stationary dist.
lets propose D^{in} supported on in-trees as a potential candidate that satisfies detailed balance.

$D^{in} = \pi \times \text{UST}$

Call a pair of in-trees S, T of G_M compatible if it is possible to go from $S \to T$ in one-step.

Take compatible pair (S, T). let's see

$$Q(S,T) = \frac{1}{\deg(\rho(S))} \qquad \rho(S) = \text{root of } S.$$

For any in-tree T

consider
$$\sum_{S \sim T} D^{in}(S)\, Q(S,T) = \sum_{S\sim T} \frac{\pi(\rho(S))}{\#\text{in-trees}} \cdot \frac{1}{\deg(\rho(S))}$$

$$= \sum_{S\sim T} \frac{\deg(\cancel{\rho(S)})}{2m \ \#\text{in-trees}} \cdot \frac{1}{\deg \cancel{\rho(S)}}$$

$$= \frac{\deg(\rho(T))}{2m \ \#\text{in-trees}} = \frac{\rho(T)}{\#\text{in-trees}} = D^{in}(T)$$

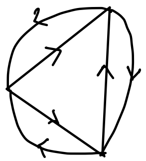
Backward-chain

$(X_n)_{n\in\mathbb{N}}$. T_n = Tree rooted at X_n.

(v) = first edge used to enter v after time n $\leftarrow$ to get in-tree.
(reverse)

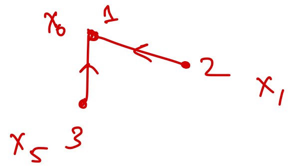
Algorithm : Once all vertices are covered, return backward chain.

Claim : Exact sample from uniform distribution.

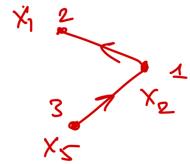


x_0	x_1	x_2	x_3	x_4	x_5	x_6
1	2	1	2	1	3	

$T_0 \equiv$ (depends till cover time)



$T_1 \equiv$



$E(T_0) = \{e_0(v) : v \neq x_0\}$

For RV X, let μ_X denote distribution.

Thm [Aldous/Broder]

Let $X_0 \sim \pi$ then $\mu_{(X_0, \{e_0(v), v \neq X_0\})} = \pi \times \text{UST}$.

$e_0(v)$ = first entry

$l_0(v)$ = last exit

Pf : (obs 1)

$$\mu_{(X_0, \{e \cdots\})} = \mu_{(X_0, \{l_0(v), v \neq X_0\})}$$

$(X_0, X_1, X_2, \ldots)$

$(X_0, X_{-1}, X_{-2}, \ldots)$

$\}$ independent events, identical backward.

LHS $\equiv$ dist. μ_1 can be explicitly described using a f^n f

$$f(X_0, [\text{Randomness in } X_0, X_1, \ldots])$$

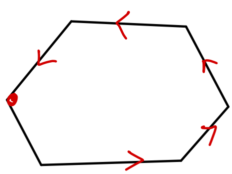
RHS $\equiv$ can be described by f^n f

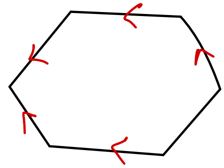
$$f(X_0, [\text{Randomness in } X_0, X_{-1}, X_{-2}, \ldots])$$

[obs 2] As $n \to \infty$, fwd chain $\to \pi \times \text{UST}$

In fwd chain if $X_0 \sim \pi$

$$(X_0, \{l_0(v), v \neq X_0\}) \sim \pi \times \text{UST}$$





equal probability ?

Random Target Lemma

$x \in V$ $\quad \tau^+(x) = \min_{t \geq 1} \{t : X_t = x\}$

(general : ergodic)

$x, y \in V$ $\quad \mathbb{E}_{a \sim V}[\tau_x^+(a)] = \mathbb{E}_{b \sim n}[\tau_y^+(b)]$

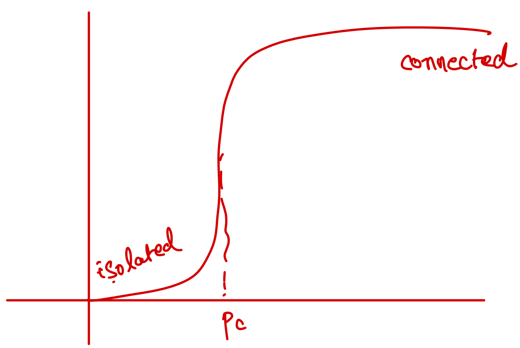
{lazy random walk on graphs}

Lecture

02/11/23

Graph Sparsification

$$G \longrightarrow H$$
$$(V, E) \qquad (V, E_H, w_H)$$
$$|E| = m \geq n^{1.1}$$



$G(n, p)$

Karger : number of min-cuts $\leq n^2$.

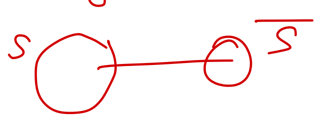
[Thm] (Benczúr - Karger '96)

G : min-cut$(G) \geq k$, $p : c \log n / k$ Then $H = G(n, p)$ satisfies

$\forall S \subseteq [n]$ $\quad w_H(S, \bar{S}) \in (1 \pm \varepsilon)|E_G(S, \bar{S})|$ w.p. $\geq 1 - \frac{1}{n}$

Pf : missing min-cut $= (1-p)^k$ $\quad w \Rightarrow 1/p$

$$(1-p)^k \approx 1 - p^k$$
$$\leq 1 - e^{-pk}$$
$$\leq 1 - e^{-c}$$



conc. $\longrightarrow$ union bound

$\Pr(\bigcup$ missing all cut$) \leq \#$cut $(1-p)^k$

at most n^2 $\leftarrow$ maximised by a cycle.

cuts b/w $(\alpha-1)k, \alpha k$ is $\le n^{\alpha}$.

Now, do a naive union bound. (is delicate)

- **Bucketing** $\leftarrow$ Bourgain style bucketing

Bad event : $\Pr\left(|w_n(S,\bar{S}) - E_G(S,\bar{S})| > \varepsilon . E(S,\bar{S}) \right) \le 2e^{-\varepsilon^2 \frac{E_G(S,\bar{S})}{3} p}$

(chernoff)

Karger proved

cuts with cost $\in [(\alpha-1)k, \alpha k] \le n^{O(\alpha)}$

So, union bound gives $n^{-1000\alpha} \times n^{10\alpha}$. so good to go.

Upgrade : $G = (V, E)$

Sample edge e w.p. p_e

You know $\forall S \subseteq V$

$\mathbb{E}\left[\begin{matrix} \text{\# of edges sampled} \\ \text{from } E_G(S,\bar{S}) \end{matrix} \right] \ge \frac{\Omega(\log n)}{\varepsilon^2}$

Then all cuts in graph are within $(1 \pm \varepsilon)$ of expectation w.p. $\ge 1 - \frac{1}{n}$

Def [K- strong component]

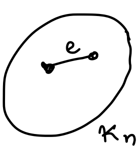
Maximal $\wedge$ induced subgraph that is k-edge connected.
(vertex)

Def [Strength of edge]

Maximum edge connectivity of an induced subgraph containing e

$K(e) = \max_{k'} \exists\, k'$ strong component that contains e

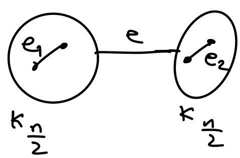
$= \max_{e \in E(G(U))} \{ K(G(U)) \}$



$K(e) = n-1$

note : $\uparrow$ degree $= n-1$

so $K(e) \ge n-1$



$K(e_1) = \frac{n}{2} - 1$

$K(e_2) = n/2 - 1$

$K(e) = 1$

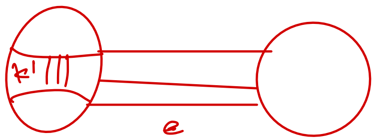
Lemma : 1. For each edge e, $K(e)$ is well-defined.

2. Take k_1, k_2 with $k_2 \geq k_1$

k_2 strong components are refinements of k_1

3. $\sum_{e \in E} \frac{1}{k_e} \leq n-1$

Pf : (3)



[Min-cut]

$|E(S,\bar{S})| = k$

k-edge connected.

observe : $K(e) = k$

$$\sum_{e \in E(S,\bar{S})} \frac{1}{k_e} \leq \frac{1}{k} \cdot k = 1.$$

Now, remove these edges.

Now, for subgraph of component 1, again has min-cut $\sum_{(S',\bar{S}')} \frac{1}{k_e} \leq 1$.

Hence, repeating this arguement $n-1$ times (# times cut graph)

$k^{(2)}(e) \geq k'$ — strength, connec.

$$\sum_{e \in (T,\bar{T})} \frac{1}{k^2(e)} \leq \frac{1}{k'} \cdot k' = 1$$

hence, this makes sense.

NTS : $K^{(1)}(e) \geq K^{(2)}(e)$, $e \in (T, \bar{T})$ ← Koi component 2 mein hai toh 1 mein toh hoga hi

Algo :

For $e \in E(G)$ — Think $q = O(\log n / \varepsilon^2)$

pick edge e w.p. $p_e = \min\left\{\frac{q}{K(e)}, 1\right\}$

Set $w_H(e) = 1/p_e$

$G \to H$ (subsampling)

$G_w \to G_{w'}$ (subsampling) ← auxiliary graph $\left(\frac{K_e}{q} = wt(e) \text{ in } G_w\right)$

let $0 < k_1 < k_2 < \cdots < k_{m'}$ denote diff values of edge strengths

define subgraphs $F_1, F_2, \ldots, F_{m'}$ (F_i is subgraph that contains all edges with strength atleast $\geq k_i$)

(F_1 = entire graph)

write $G_{W'} = \sum_{i=1}^{m'} \left(\frac{k_i - k_{i-1}}{q}\right) \underbrace{F_i}_{\text{adj matrices}}$

telescopic sum, check for each e.

k_i appears in all lower F_i : & $F_1 \cdots F_{i'}$.

Lemma : $G \rightarrow H$ (sampled graph)

① # of edges in $H \leq O\left(\frac{n \log n}{\varepsilon^2}\right)$

② $\forall\, S \subseteq V$, $|w_H(S, \bar{S})| \in (1 \pm \varepsilon)|E_G(S, \bar{S})|$

Proof :

① $\mathbb{E}[\#\text{edges } H] = \mathbb{E} \sum \mathbb{1}_{e \in H}$

$= \sum_e p(e \in H) \leq \sum q/k_e$

$= q\, n. = O\left(\frac{n \log n}{\varepsilon^2}\right)$.

② Take some edge $e \in E$.

Sps e was sampled from G_w

e appears in $F_1, \ldots, F_i$