A1. Let $G=(V,E)$ be obtained by adding bi-directional edges and $d_u$ self loops.

Take $\tilde{p}_{v,t}(u) = \frac{\#\text{random walks ending at } u}{N}$

define $\tilde{p}(e) = \tilde{p}_{v,t}(u)/d_u$

Now, sort $\tilde{p}(e)$, in the "natural" way (i.e. club edges outgoing from same vertex together)

as, $\tilde{p}(e_1) \geq \tilde{p}(e_2) \geq \dots \geq \tilde{p}(e_{2m})$

Define $\tilde{G}_p(x) = \sum_{i=1}^{x} p(e_i)$ and define $\tilde{g}_t$ as linear interpolation of $\tilde{G}_p$ on $[0,2m]$

<u>Claim</u>: $\tilde{g}_t$ is concave

<u>Pf</u>: Let the set of points where slope changes be $H=\{h_1, \dots, h_k\}$

then for $h_i \leq x \leq y \leq h_{i+1}$, both points lie on same line segment, so concavity is trivially satisfied.

If not, let $x < h_i < y$, with slope before $h_i = m_i$, after $m_{i+1}$

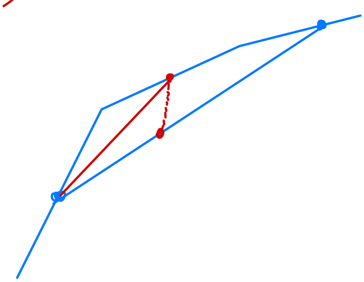
<u>Claim</u>: Slope of piecewise linear segments of $\tilde{g}_t$ is decreasing

<u>Pf</u>: for $h_i \in H$,

$$m_i = \tilde{G}_p(h_i) - \tilde{G}_p(h_i - 1) = p(e_{h_i})$$
$$m_{i+1} = \tilde{G}_p(h_i+1) - \tilde{G}_p(h_i) = p(e_{h_i+1})$$

Since $p(e_{h_i}) \geq p(e_{h_i+1})$

$$\Rightarrow m_i \geq m_{i+1}$$



Now, for some $\sum l_i < (y-x)$

$$\tilde{g}_t(x+\sum l_i) = \tilde{g}_t(x) + \sum l_i m_i$$

$$\Rightarrow \frac{\tilde{g}_t(x+\sum l_i) - \tilde{g}_t(x)}{\sum l_i} = \frac{\sum l_i m_i}{\sum l_i}$$

and $\frac{\tilde{g}_t(y) - \tilde{g}_t(x)}{(y-x)} = \frac{\sum_{i\in S} l_i m_i + \sum_{j\in S'} l_i' m_i'}{\sum_{i\in S} l_i + \sum_{j\in S'} l_i'}$ where $\sum l_i' = (y-x) - \sum l_i$

where $\dfrac{\sum l_i' m_i'}{\sum l_i'} \le \max_j m_j \le \min_i m_i \le \dfrac{\sum l_i m_i}{\sum l_i}$

Hence, it follows that

$$\frac{\lambda \tilde{g}_t(x) + (1-\lambda)\tilde{g}_t(y) - \tilde{g}_t(x)}{(\lambda x + (1-\lambda) y - x)} = \frac{\tilde{g}_t(y) - \tilde{g}_t(x)}{y - x} \le \frac{\tilde{g}_t(\lambda x + (1-\lambda) y) - \tilde{g}_t(x)}{(\lambda x + (1-\lambda) y - x)}$$

$$\Rightarrow \lambda \tilde{g}_t(x) + (1-\lambda)\tilde{g}_t(y) \le \tilde{g}_t(\lambda x + (1-\lambda) y) \quad \blacksquare$$

A2. $\alpha, \delta > 0$ are small, $w = c \cdot \frac{1}{\alpha} \cdot \log n$ $\left(c = \frac{1}{\delta^2}\right)$

$X_1, \dots, X_n$ s.t. $a_i \le X_i \le b_i$ almost surely, $S_n = X_1 + \dots + X_n$ then } hoeffding

$$\Pr(|S_n - \mathbb{E}[S_n]| \ge t) \le 2e^{\frac{-2t^2}{\sum_{i=1}^n (b_i - a_i)^2}}$$

where $X_i$ is a bernoulli variable representing if random walk terminates at $u$.
$0 \le X_i \le 1$

$$\Pr(|\tilde{p}_{v,t}(u) - p_{v,t}(u)| \ge \delta_u)$$
$$= \Pr(|S_n - \mathbb{E}(S_n)| \ge w\delta_u) \le 2e^{-\frac{2w^2\delta_u^2}{w}}$$

Putting $\delta_u = \delta(p_{v,t}(u) + \alpha)$, $w = \frac{c}{\alpha}\log n$,

$$= 2\exp(-2w\delta_u^2) = 2\exp(-2w\delta^2(p_{v,t}(u) + \alpha)^2)$$
$$= 2\exp\left(-2\frac{\cancel{c}}{\alpha}\log n \cancel{\delta^2}(p_{v,t}(u) + \alpha)^2\right)$$
$$= 2n^{-\frac{2}{\alpha}(p_{v,t}(u)+\alpha)^2}$$

But $\alpha$ is sufficiently small, such that

$$\le n^{-10} \quad \blacksquare$$

A3. Since, showed in A2 that $\Pr(|\tilde{p}_{v,t}(u) - p_{v,t}(u)| \geq \delta_u) \leq n^{-10} \leq n^{-9}$

$$\Rightarrow \Pr(|\tilde{p}_{v,t} - p_{v,t}| \leq \delta_u)$$

$$= \Pr\left( (1-\delta) p_{v,t}(u) - \delta\alpha \leq \tilde{p}_{v,t}(u) \leq (1+\delta) p_{v,t}(u) + \delta\alpha \right)$$

$$= 1 - \Pr(|\tilde{p}_{v,t}(u) - p_{v,t}(u)| \geq \delta_u)$$

$$\geq 1 - n^{-9}$$ ■

A4. Now, given

$$(1-\delta) p_{v,t}(u) - \delta\alpha \leq \tilde{p}_{v,t}(u) \leq (1+\delta) p_{v,t}(u) + \delta\alpha$$

$$\Rightarrow \sum_{u \in S} (1-\delta) p_{v,t}(u) - \delta\alpha|S| \leq \sum_{u \in S} \tilde{p}_{v,t}(u) \leq \sum_{u \in S} (1+\delta) p_{v,t}(u) + \delta\alpha|S|$$

Since $|S| \leq \sum_{u \in S} d_u$ (since $G$ is connected, $d_u \geq 1$)

and putting $\sum_{u \in S} p_{v,t}(u) = p_{v,t}(S)$, $\sum_{u \in S} \tilde{p}_{v,t}(u) = \tilde{p}_{v,t}(S)$

$$\Rightarrow (1-\delta) p_{v,t}(S) - \delta\alpha \, vol(S) \leq \tilde{p}_{v,t}(S) \leq (1+\delta) p_{v,t}(S) + \delta\alpha \, vol(S) \quad (*)$$

A5. Now, it is sufficient to show the inequality

$$(1-\delta) g_t(x) - \delta\alpha x \leq \tilde{g}_t(x) \leq (1+\delta) g_t(x) + \delta\alpha x$$

for points $x$ s.t. $x$ represents end of a set of edges outgoing from $u$.

for such $x$, $g_t(x) = \sum_u \rho(e) 2d_u = \sum_u \frac{p_u 2d_u}{2d_u} = \sum_u p_u$ for $u \in S$.

$$= p_{v,t}(S)$$

Similarly, $\tilde{g}_t(x) = \tilde{p}_{v,t}(S)$.

and $vol(S) = \sum_{u \in S} d_u = x$

Putting these in $(*)$ {result of A4}, we get the desired result

$$H = \left\{ u \in V \;\middle|\; \frac{P_{v,t-1}(u)}{2d_u} \geq \delta\alpha \right\}$$

B1. $|H| = h$, then for $u \in H$, $e \in U$, $p(e) \geq \delta\alpha$.

$i_h$ = number of edges out of $H$. Now, since $\sum_e p(e) = 1$,

we can't have too many edges with large weight

$$1 \geq \sum_{e \in out(H)} p(e) \geq (i_h)\,\delta\alpha$$

$$\Rightarrow \boxed{i_h \leq \frac{1}{\delta\alpha}}$$

Notation : $\hat{x} = \min(x, 2m - x)$

B2. <u>Case 1</u> : $k \geq b$

Now, $i_k - 2\phi\hat{i_k} \geq i_k - 2\phi i_k$

To show $i_k - 2\phi\hat{i_k} \geq i_h$, it is enough to show $i_k - 2\phi i_k \geq \frac{1}{\delta\alpha}$

(since $i_h \leq \frac{1}{\delta\alpha}$). i.e. $i_k \geq \frac{1}{\delta\alpha(1-2\phi)}$

Now, $k \geq b \Rightarrow i_k \geq i_b$, so, it is enough to show $i_b \geq \frac{1}{\delta\alpha(1-2\phi)}$

<u>Claim</u> : $i_b \geq \frac{1}{\delta\alpha(1-2\phi)}$

<u>Proof</u> : $i_b \geq b$ holds.

If claim is false, $\frac{1}{\delta\alpha(1-2\phi)} > i_b \geq b \Rightarrow b \leq \frac{1}{\delta\alpha(1-2\phi)}$

$$\Rightarrow \frac{t}{2(1-2\phi)\alpha} \leq \frac{1}{\delta\alpha(1-2\phi)}$$

$$\Rightarrow t\delta \leq 2 \quad \text{---} (*)$$

Now, equating $w$ in questions 1, 2. I take $w = \frac{1}{16}\frac{t^2 \log n}{\alpha}$ instead (for reasons that will become apparent in the next step)

$$\Rightarrow w = \frac{1}{16}\frac{t^2 \log n}{\alpha} = \frac{1}{\delta^2}\frac{\log n}{\alpha}$$

$$\Rightarrow t\delta = 4 > 2.$$

Hence, claim holds by contradiction ■

Now, since claim holds, $i_k - 2\phi\hat{i_k} \geq i_h$. Hence for all subsequent edges, $p(e) < \delta\cdot\alpha$

Hence, slope of LS curve $\leq \delta\cdot\alpha \;\; \forall x \geq i_h$.

Hence, slope of secant joining two points $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sum l_i m_i}{\sum l_i} \leq \max m_j \leq \delta\alpha$.

where $l_i, m_i$ are length of $x$-interval, slope for a section of LS curve with const-slope

{Now, using sir's hint :)}

Hence, $g_{\ell-1}(i_k) \le g_{\ell-1}(i_k - 2\phi\hat{i}_k) + 2\phi\hat{i}_k\alpha\delta$

$g_{\ell-1}(i_k) \le g_{\ell-1}(i_k + 2\phi i_k)$

(+)

$\Rightarrow g_\ell(i_k) \le \frac{1}{2}\left(g_{\ell-1}(i_k - 2\phi\hat{i}_k) + g_{\ell-1}(i_k + 2\phi\hat{i}_k)\right) + \phi\hat{i}_k\alpha\delta$ — (1)

And, $P_{v,\ell}(\tilde{S}_{\ell,k}) = \sum_{u \in \tilde{S}_{\ell,k}} P_{v,\ell}(u) = \sum_{e \in out(\tilde{S}_{\ell,k})} \rho(e)$ where $|out(\tilde{S}_{\ell,k})| = i_k$

But $g(i_k)$ is the largest sum $\sum_{e \in S} \rho(e)$ where $|S| = i_k$.

Hence, $P_{v,\ell}(\tilde{S}_{\ell,k}) \le g_\ell(i_k)$ — (2)

Using (1),(2)

$\Rightarrow P_{v,\ell}(\tilde{S}_{\ell,k}) \le \frac{1}{2}\left(g_{\ell-1}(i_k - 2\phi\hat{i}_k) + g_{\ell-1}(i_k + 2\phi\hat{i}_k)\right) + \phi\hat{i}_k\alpha\delta$ $\blacksquare$

<u>Case 2</u> : $k \le b$

Using LS formula on $\tilde{g}$ at $i_k$, since $i_k$ is a hinge point, $\phi_{\tilde{S}_{\ell,k}} \ge \phi$

$\tilde{g}_\ell(i_k) \le \frac{1}{2}\left(\tilde{g}_{\ell-1}(i_k - 2\phi\hat{i}_k) + \tilde{g}_{\ell-1}(i_k + 2\phi\hat{i}_k)\right)$

Now, $(1-\delta)g_t(x) - \delta\alpha x \le \tilde{g}_t(x) \le (1+\delta)g_t(x) + \delta\alpha x.$ w.h.p

$\Rightarrow (1-\delta)\, g_\ell(i_k) - \delta\alpha i_k \le \frac{1}{2}\Big(\big((1+\delta)g_{\ell-1}(i_k - 2\phi\hat{i}_k) + \delta\alpha(i_k - \cancel{2\phi}\hat{i}_k)$

$+ (1+\delta)g_{\ell-1}(i_k + 2\phi\hat{i}_k) + \delta\alpha(i_k + \cancel{2\phi}\hat{i}_k)\Big)$

w.h.p.

Since $\delta$ is small,

$\Rightarrow g_\ell(i_k) \le \frac{1}{2}\left(g_{\ell-1}(i_k - 2\phi\hat{i}_k) + g_{\ell-1}(i_k + 2\phi\hat{i}_k)\right) + 2\delta\alpha i_k$ w.h.p.

Since $\phi\delta\alpha\hat{i}_k \approx O(2\delta\alpha\hat{i}_k)$, we have

$\Rightarrow g_\ell(i_k) \le \frac{1}{2}\left(g_{\ell-1}(i_k - 2\phi\hat{i}_k) + g_{\ell-1}(i_k + 2\phi\hat{i}_k)\right) + \delta\alpha\phi\hat{i}_k$ w.h.p $\blacksquare$

B3.

1. We know for all $0 \le l \le t$,

$$P_{v,l}(\tilde{S}_{l,k}) \le \frac{1}{2}\left(g_{l-1}(i_k - 2\phi \hat{i}_k) + g_{l-1}(i_k + 2\phi \hat{i}_k)\right) + \phi \hat{i}_k \alpha \delta \quad -(*)$$

and, from (1.5), we have

$$(1-\delta)g_t(x) - \delta\alpha x \le \tilde{g}_t(x) \le (1+\delta)g_t(x) + \delta\alpha x.$$

i.e. $g_t(x) \le \dfrac{\tilde{g}_t(x) + \delta\alpha x}{1-\delta}$ for every $t$ w.h.p $-①$

and from $(1-\delta)\cdot \boldsymbol{p}_{v,t}(u) - \delta\alpha \le \tilde{\boldsymbol{p}}_{v,t}(u) \le (1+\delta)\cdot \boldsymbol{p}_{v,t}(u) + \delta\alpha.$

$$P_{v,t}(u) \ge \frac{\tilde{P}_{v,t}(u) - \delta\alpha}{1+\delta} \quad \text{w.h.p.} \quad -②$$

Putting ①, ② in (*) and using $\sum_{u \in \tilde{S}_{l,k}} \tilde{p}_{v,t}(u) = \tilde{g}_t(i_k)$

$$\Rightarrow \frac{\tilde{g}_l(i_k) - i_k \delta\alpha}{1+\delta} \le \frac{1}{2}\left(\frac{\tilde{g}_{l-1}(i_k - 2\phi\hat{i}_k) + \tilde{g}_{l-1}(i_k + 2\phi\hat{i}_k) + 2\delta\alpha i_k}{1-\delta}\right) + \alpha\delta\phi\hat{i}_k$$

$$\Rightarrow \tilde{g}_l(i_k) \le \frac{(1+\delta)}{2(1-\delta)}\left(\tilde{g}_{l-1}(i_k - 2\phi\hat{i}_k) + \tilde{g}_{l-1}(i_k + 2\phi\hat{i}_k)\right) + \frac{2\delta\alpha i_k(1+\delta)}{2(1-\delta)} + \alpha\delta\phi\hat{i}_k(1+\delta) + \delta\alpha i_k$$

Now, using the fact $\delta$ is small, $\phi < 1 \Rightarrow \delta\frac{1+\delta}{1-\delta} + \delta(1+\delta) + \delta \le 4$

Hence, $\tilde{g}_l(i_k) \le \dfrac{(1+\delta)}{2(1-\delta)}\left(\tilde{g}_{l-1}(i_k - 2\phi\hat{i}_k) + \tilde{g}_{l-1}(i_k + 2\phi\hat{i}_k)\right) + 4\alpha\delta i_k$ ∎

2. Now, for any $x \in [i_k, i_{k+1})$

Let $x = \alpha i_k + (1-\alpha) i_{k+1}$

Now, $\tilde{g}_l(i_k) \le \dfrac{1+\delta}{2(1-\delta)}\left(\tilde{g}_{l-1}(i_k - 2\phi\hat{i}_k) + \tilde{g}_{l-1}(i_k + 2\phi\hat{i}_k)\right) + 4\alpha\delta i_k \quad -①$

$\tilde{g}_l(i_{k+1}) \le \dfrac{1+\delta}{2(1-\delta)}\left(\tilde{g}_{l-1}(i_{k+1} - 2\phi\hat{i}_{k+1}) + \tilde{g}_{l-1}(i_{k+1} + 2\phi\hat{i}_{k+1})\right) + 4\alpha\delta i_{k+1} \quad -②$

$\alpha \times ① + (1-\alpha) \times ②$ and $\tilde{g}(\lambda x + (1-\lambda)y) \ge \lambda\tilde{g}(x) + (1-\lambda)\tilde{g}(y)$.

$$\Rightarrow \tilde{g}_l(x) \le \frac{1+\delta}{2(1-\delta)}\left(\tilde{g}_{l-1}(x - 2\phi(\alpha\hat{i}_k + (1-\alpha)\hat{i}_{k+1})) + \tilde{g}_{l-1}(x + 2\phi(\alpha\hat{i}_k + (1-\alpha)\hat{i}_{k+1}))\right) + 4\alpha\delta x$$

Now, $\hat{x} = \min(x, 2m - x)$, $i_{k+1} \le m$ (or) $i_k \ge m$ then $\alpha\hat{i}_k + (1-\alpha)\hat{i}_{k+1} = \hat{x}$.

$$\Rightarrow \tilde{g}_l(x) \le \frac{1+\delta}{2(1-\delta)}\left(\tilde{g}_{l-1}(x - 2\phi\hat{x}) + \tilde{g}_{l-1}(x + 2\phi\hat{x})\right)$$

Let $i_k < m < i_{k+1}$

Construction: Make a new graph $\tilde{G}$ with a dummy vertex of $k+1$ such that $\tilde{i}_{k+1} = m$. Now, $k+1$ and $\widetilde{k+1}$ vertices collectively have same # edges as $k+1$ which are split now b/w $k+1, \widetilde{k+1}$. Now for $e \in N(k+1)$ or $N(\widetilde{k+1})$, $p(e)$ does not change as prob of $k+1 \cup \widetilde{k+1}$ remains same as a "unit" and the ratio of their probabilities is same as ratio of degrees. Hence, LS curve does not change and $x = m$ is a hinge point making this case impossible. ∎

Bonus

(a)
$$\varphi = -\log\left(\frac{1}{2}\left(\sqrt{1-2\phi} + \sqrt{1+2\phi}\right)\right)$$

Putting $l = 0$,

Claim: $\tilde{g}_0(x) \le \left[\sqrt{\hat{x}} + \frac{x}{2m}\right] + \frac{4x}{\sqrt{m}}$

It is enough to show this for hinge points.

let $x = i_k$.

Then $\tilde{g}_0(i_k) \le \left[\sqrt{\hat{i}_k} + \frac{i_k}{2m}\right] + \frac{4 i_k}{\sqrt{m}}$

where $\tilde{g}_0(i_k) \le 1$ and $\hat{i}_k \in \mathbb{N} \Rightarrow \sqrt{\hat{i}_k} \ge 1$.

So, base case holds for hinge points.

Now, $x \in [i_k, i_{k+1})$
$$\Rightarrow \lambda \tilde{g}_0(i_k) + (1-\lambda)\tilde{g}_0(i_{k+1}) \le 1 \le \sqrt{\hat{x}} + \frac{x}{2m} + \frac{4x}{\sqrt{m}}$$

Assuming the hypothesis for all $x \in [0, 2m]$ and $\tilde{g}_l$ that is,
$$\tilde{g}_l(x) \le e^{10\delta l}\left[\sqrt{\hat{x}}\, e^{-\psi l} + \frac{x}{2m}\right] + \frac{4}{\sqrt{m}} e^{4\delta l} x$$

Using 2.3.2
$$\tilde{g}_{l+1}(x) \le \exp(O(\delta))\, \frac{1}{2}\left(\tilde{g}_l(x - 2\phi\hat{x}) + \tilde{g}_l(x + 2\phi\hat{x})\right) + 4\delta\alpha x$$
$$\le \exp(O(\delta))\; e^{10\delta l}\left[\frac{x}{2m} + e^{-\psi l}\, \frac{\sqrt{(x - 2\phi\hat{x})} + \sqrt{(x + 2\phi\hat{x})}}{2}\right] + \frac{4}{\sqrt{m}} e^{4\delta l} \cdot x$$

Lemma: $f(x) = \sqrt{\hat{x}}$ is concave.

$$\lambda f(x) + (1-\lambda) f(y) \le \sqrt{\lambda\hat{x} + (1-\lambda)\hat{y}} \le \sqrt{(\widehat{\lambda x + (1-\lambda) y})}$$ ∎

Using this,
$$\tilde{g}_{l+1}(x) \le e^{O(\delta)}\, e^{10\delta l}\left[\frac{x}{2m} + e^{-\psi l}\sqrt{\hat{x}}\right] + \frac{4}{\sqrt{m}} e^{4\delta l} \cdot x$$

Now, $$e^{o(\delta)} e^{10\delta(l+1)}\left[\frac{x}{2m} + e^{-\psi(l+1)}\sqrt{\hat{x}}\right] + \frac{4}{\sqrt{m}} e^{4\delta(l+1)} x$$

$$- e^{o(\delta)} e^{10\delta l}\left[\frac{x}{2m} + e^{-\psi l}\sqrt{\hat{x}}\right] - \frac{4}{\sqrt{m}} e^{4\delta l} \cdot x$$

$$= \frac{x}{2m}\left[e^{o(\delta)} e^{10\delta l}(e^{\delta l}-1)\right] \quad (a)$$

$$+ e^{-\psi l} e^{o(\delta)} e^{10\delta l}\sqrt{\hat{x}}\left[e^{10\delta-\psi} - 1\right] \quad (b)$$

$$+ \frac{4}{\sqrt{m}} x e^{4\delta l}\left(e^{4\delta}-1\right) \quad (c)$$

a, c are +ve with increasing $f^n$ in $l$ and (b) is negative with decreasing mag. in $l$ since $e^{(10\delta-\psi)l}$ decreases with $l$.

So, showing that difference $\geq 0$ for $l=1$ suffices to show the claim.

Putting $l=1$

$$\frac{x}{2m}\left[e^{o(\delta)} e^{10\delta}(e^{\delta}-1)\right] + e^{-\psi} e^{o(\delta)} e^{10\delta} \overset{(b)}{\sqrt{\hat{x}}}\left[e^{10\delta-\psi} - 1\right]$$
$$+ \frac{4}{\sqrt{m}} x e^{4\delta}(e^{4\delta}-1)$$

Let $u = e^{-\psi+10\delta}$

Then (b) $\simeq \sqrt{\hat{x}}\,(u^2-u)$

and since $e^{-\psi} = \frac{1}{2}\left(\sqrt{1+2\not{\delta}} + \sqrt{1-2\not{\delta}}\right) = e^{-\delta/2} \in \left(e^{-1/16}, 1\right)$

$\Rightarrow u$ is close to $1 \Rightarrow u^2-u$ is close to $0$.

Hence, (b) subtracted (since it's negative) is small and, claim will hold.

(b). Putting $\delta = 1/t$. $(\delta t = 1)$

$$\tilde{g}_t(x) \le e^{10\cancel{\delta t}}\left[\sqrt{\hat{x}}\, e^{-\psi \cdot t} + \frac{x}{2m}\right] + \frac{4}{\sqrt{m}}\, e^{4\cancel{\delta t}} \cdot x$$

Putting $x = 1$

$$\tilde{g}_t(1) \le e^{10}\left[e^{-\psi t} + \frac{1}{2m}\right] + \frac{4}{\sqrt{m}}\, e^{4}$$

So, $\tilde{g}_t(1) \le O\left(\frac{1}{\sqrt{m}}\right)$ $\blacksquare$