CS 774: Homework hw1

Akash Kumar

August 26, 2023

Grading Rules

- This homework has 4 problems each of which are worth 25 points. There is a fifth problem which is worth 0 points, it is just that you do not know which one it is. If you are feeling adventurous, you can play Russian Roulette with the assignment and you may skip a problem you think is the *dud*. Just make sure you don't skip one of the ones I plan to grade :-). I will reveal which problem is worth zero points immediately after the submission deadline has passed.
- You are *encouraged* to discuss. Feel free to consult online sources. Just make sure that when you sit down (or is it lie down?) to write solutions, you are not copy/pasting the solution. In general, the policy of the class is **be reasonable**. I trust that you all have an instinct which tells you what constitutes a reasonable behavior.
- **Due date: Sep 11, 2023, Monday in class.** You can also email your submissions if you LaTex it (preferred, but not mandated).

1 Problem 1: Irregular Graphs

In this problem, we will see how to think about spectral properties of irregular graphs. Recall that for an undirected graph $G = (V, E)$ with n vertices and m edges with degree matrix $\boldsymbol{D}$ the random walk matrix is $\boldsymbol{M} = \boldsymbol{D}^{-1}\boldsymbol{A}$. Note that $\boldsymbol{M}$ is not a symmetric matrix. In this exercise, you will show the following.

1. All the eigenvalues of $\boldsymbol{M}$ are still real. To show this, you might find it convenient to use the following symmetric matrix $\overline{\boldsymbol{M}} = \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}$. Show that all eigenvalues of $\boldsymbol{M}$ are equal, with multiplicities, to the eigenvalues of $\overline{\boldsymbol{M}}$.
2. Show that all the eigenvalues of $\overline{\boldsymbol{M}}$ are between -1 and 1. By the previous exercise, note that this means all eigenvalues of $\boldsymbol{M}$ lie between -1 and 1 as well.

2 Problem 2: Stationary distribution of random walks on Irregular Graphs

Let the eigenvalues of $\boldsymbol{M}$ (or $\overline{\boldsymbol{M}}$) be $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -1$. Suppose $\max(\lambda_2, |\lambda_n|) \leq 1 - \varepsilon$ for some $\varepsilon > 0$. In this exercise, you will show that the stationary distribution of random walks on G is given by the distribution $\boldsymbol{\pi}$ where for $u \in V(G)$, you have $\boldsymbol{\pi}(u) = d_u/2m$. To this end, show the following

1. If $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{M}$, then $\boldsymbol{D}^{1/2}\boldsymbol{v}$ is an eigenvector of $\overline{\boldsymbol{M}}$.

2. Fix vertex $i \in V(G)$. Show the following statement about limits of doing power iterations on $\overline{\boldsymbol{M}}$:

$$\lim_{t\to\infty} \|\overline{\boldsymbol{M}}^t \mathbf{1}_i - \boldsymbol{x}_i\|_2 = 0 \text{ where } \boldsymbol{x}_i(j) = \sqrt{d_i \cdot d_j}/2m.$$

You might want to use the result from item (1) above. Note that the power iterations on $\overline{\boldsymbol{M}}$ converge to a vector that depends on the start vector. It might be helpful to write the vector $\mathbf{1}_i$ in the eigenbasis of $\overline{\boldsymbol{M}}$.

3. Put the two items above together to conclude that the stationary distribution for random walks on $\boldsymbol{M}$ is the vector $\boldsymbol{\pi}$ defined above.

3 Problem 3: A connection to bipartiteness

In this problem, let us go back to regular graphs. So, you are given a d-regular graph $G = (V, E)$ with normalized Laplacian being $\boldsymbol{L} = \boldsymbol{I} - \boldsymbol{A}/d$. Let the eigenvalues of $\boldsymbol{L}$ be denoted as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$. Show that G is bipartite *iff* $\lambda_n = 2$. Please note there are two directions in this problem.

4 Problem 4: More on spectrum of bipartite graphs

Again, let $G = (V, E)$ denote a d-regular bipartite graph. Let $\boldsymbol{L} = \boldsymbol{I} - \boldsymbol{A}/d$ denote the normalized Laplacian of G with eigenvalues being $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n = 2$. Show that the spectra is in fact symmetric around 0. That is, $\lambda_i = \lambda_{n-i+1}$.

5 Problem 5: More Cheeger like phenomena

Let $G = (V, E)$ be a graph which has two prominent clusters C_1 and C_2 with $|C_1| = |C_2| = n/2$ where $|V| = n$. Suppose $G[C_1]$ and $G[C_2]$ both have conductance at least ϕ. Also, suppose $|E(C_1, C_2)| \leq \varepsilon nd$ where $\varepsilon \ll \phi^2$. Show that $\lambda_3 \geq \phi^2/2$.