MA 109 Autumn 2022
Endsem TSC

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Sets

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Some notation.

- $\mathbb{N}$: The set of natural numbers.
- $\mathbb{Z}$: The set of real numbers.
- If a set S contains some element a, we write $a \in S$.
- To refer to all the elements in the set S, we use $\forall s \in S$.
- 'There exists s in S': $\exists s \in S$.
- $\mathbb{Q}$: The set of rational numbers (numbers of the form p/q for $p, q \in \mathbb{Z}$).
- $\mathbb{R}$: The set of real numbers.

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The cardinality of a finite set S, denoted as $|S|$, is defined as

1. $|S| = 0$ if $S = \emptyset$.
2. $|S| = n$ if a bijection $f : S \to \{1, 2, \dots n\}$ exists.

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Can we talk about cardinality of infinite sets?

Intervals

A set $X \subseteq \mathbb{R}$ is an interval iff for all $a, b \in X$ and $c \in \mathbb{R}$, $a \leq c \leq b$ implies $c \in X$.

Open Interval

If the endpoints are not included, the interval is called open and denoted (a,b).

Closed Interval

An interval which contains it's endpoints is called a closed interval and denoted by [a, b].

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Similarly, the Greatest Lower Bound (GLB) is defined. More commonly, we refer to LUB as the supremum, and the GLB as the infimum.

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then f_n converges, and the limit is $f_0 = a_0 = b_0$.

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We use monotonic and eventually monotonic synonymously.

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Does the converse of the MCT hold? No. Take $a_n := (-1)^n/n$.

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  2. [Tutorial Q3] $d_n := (-1)^n(1/2 - 1/n)$. Does this converge?

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- **Showing Convergence**: [2019 Midsem] Consider the sequence $a_n := 2^n/n!$. Does this sequence converge? If so, what is the limit?
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- [2019 Quiz 1] Let a_n be a sequence of real numbers such that $a_1 \in (3, 4)$ and $a_{n+1} = \sqrt{12 + a_n} \forall n \in \mathbb{N}$. Determine if $(a_n)_n$ is convergent.

The Idea

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We defined and spent time understanding convergence of sequences. Now, we will apply those properties to talk about functions over $\mathbb{R}$.

Continuity

We give *two* definitions.

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Further, we have that, both the definitions are **equivalent**! Can you give a proof?

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Let $x_0 \in \mathbb{R}$, and let $A \subset \mathbb{R}$ such that $N_r(x_0) \subset A$ for some $r > 0$.

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Again, both definitions are **equivalent**! Note that f need not be defined at x_0 to talk about the limit at x_0.

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Proposition

Let $x_0 \in \mathbb{R}$, and let $A \subset \mathbb{R}$ such that $N_r(x_0) \subset A$ for some $r > 0$. The function $f : A \to \mathbb{R}$ is continuous at x_0 iff $\lim_{x \to x_0} f(x)$ exists and is equal to $f(x_0)$. That is,

$$\boxed{\text{Continuity at } x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0)}$$

- **Continuity is a local property**:

Properties of Continuous functions

- **Continuity is a local property**:
  It is determined by the values of the function in an ϵ-neighbourhood of p (for every $\epsilon > 0$). Hence, we assume $f : U \to \mathbb{R}$ where U is an open interval (of the type (a, b),(a, ∞),(∞, b) and $\mathbb{R}$).
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Define the Dirichlet Function, denoted $1_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$, as,

$$1_{\mathbb{Q}} := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

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$$f(x) = \begin{cases} \cos x & \text{if } x \text{ is rational,} \\ \sin x & \text{if } x \text{ is irrational.} \end{cases}$$

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  Show that f is discontinuous at each rational in $(0, 1)$ and it is continuous at each irrational in $(0, 1)$.

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  Show that f is discontinuous at each rational in $(0, 1)$ and it is continuous at each irrational in $(0, 1)$.

Thus, the $\epsilon - N$ (sequential) way is usually good for disproving continuity and the $\epsilon - \delta$ way is usually good for showing continuity.

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Definition (Open Neigbourhoods)

For any $x \in \mathbb{R}$, and for any $\epsilon \in \mathbb{R}_+$ define the open neighbourhood, denoted $N_x(\epsilon)$ as

$$N_x(\epsilon) := \{y \in \mathbb{R} \mid |x - y| < \epsilon\}$$

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$$N_x(\epsilon) := \{y \in \mathbb{R} \mid |x - y| < \epsilon\}$$

Definition (Open Sets)

Any $U \subset \mathbb{R}$ is called an *open* subset of $\mathbb{R}$ if for all $u \in U$ there exists a $\epsilon > 0$ such that $N_u(\epsilon) \subset U$.

A subset which is said to be closed if its complement is open. When we talk about differentiability, we do so using open sets.

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Lemma

A function $f : U \to \mathbb{R}$ is **differentiable** at point $c \in U$ if and only if there exists a function $f_1 : U \to \mathbb{R}$ that is **continuous** at c and satisfies $f(x) = f(c) + f_1(x)(x - c)$ for all $x \in U$.

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$$L(f, \mathcal{P}) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \; ; \; U(f, \mathcal{P}) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) \text{ (Draw!)}$$

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Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then, f is said to be Riemann Integrable on $[a, b]$ if $U(f) = L(f)$. Further, we say that the quantity $U(f)$ $(= L(f))$ is the *Riemann Integral* of f on $[a, b]$, and denote it as

$$U(f) = L(f) = \int_a^b f(x)dx$$

The Riemann Condition

Theorem (The RC)

Given a bounded real function f on $[a, b]$,

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Given a bounded real function f on $[a, b]$, it holds that f is RI as per the previous definition **if and only if** the following holds,

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Theorem (The RC)

Given a bounded real function f on $[a, b]$, it holds that f is RI as per the previous definition **if and only if** the following holds,

$$\forall \epsilon > 0, \text{ there exists a partition } \mathcal{P} \text{ of } [a, b]$$

$$\text{such that } U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$$

Two Funny Functions

- The Dirichlet Function, denoted $1_{\mathbb{Q}} : [0,1] \to \mathbb{R}$, defined as,

$$1_{\mathbb{Q}} := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

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- The Thomae Function, denoted $\mathrm{T} : [0,1] \to \mathbb{R}$, defined as,

$$\mathrm{T}(x) := \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q}, x \neq 0, x = p/q \text{ with } p, q \text{ coprime} \\ 0 & \text{if } x = 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Properties

Proposition (Domain Additivity)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and on $[c, b]$. In this case,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

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Proposition (Order Relations)

Let $f, g : [a, b] \to \mathbb{R}$ be integrable. (i) If $f \leq g$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. (ii) The function $|f|$ is integrable and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f|(x)dx$.

Properties

Proposition (Algebraic and order relations)

Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions. Then

- $f + g$ is integrable and $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$,
- rf is integrable for any $r \in \mathbb{R}$ and $\int_a^b (rf)(x)dx = r \int_a^b f(x)dx$,
- fg is integrable,
- if there is $\delta > 0$ such that $|f(x)| \geq \delta$ and all $x \in [a, b]$, then $1/f$ is integrable,
- if $f(x) \geq 0$ for all $x \in [a, b]$, then for any $k \in \mathbb{N}$, the function $f^{1/k}$ is integrable.

Integrable functions

Lemma

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- Integrability $\implies$ continuity? No!
- Integrability $\implies$ boundedness? **Yes**.
- Integrability $\implies$ monotonicity? No!

Questions

- [2019 Midsem] Let $f : [0, 2] \to \mathbb{R}$ be defined as follows:
$f(x) = \begin{cases} x, x \in [0, 1] \\ 1, x \in [1, 2] \end{cases}$ Show that f is Riemann integrable from first principles and evaluate the integral $\int_0^2 f(x)dx$.

Riemann Sums

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Definition (Tagged Partition)

Let $\mathcal{P} := \{x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$. Let $t_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$ be arbitrary, and denote $t := \{t_1 < t_2 < \cdots < t_n\}$. We call the tuple $(\mathcal{P}, t)$ a tagged partition.

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Definition (Riemann Sum)

Given $f : [a, b] \to \mathbb{R}$, and a tagged partition $(\mathcal{P}, t)$ of $[a, b]$, define

$$R(f, \mathcal{P}, t) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

to be the associated Riemann Sum.

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to be the associated Riemann Sum.

Note that, $L(f, \mathcal{P}) \leq R(f, \mathcal{P}, t) \leq U(f, \mathcal{P})$.

A Definition

Further for a partion $\mathcal{P} := \{x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$, define $||\mathcal{P}|| := \max\{x_i - x_{i-1}\}$ to be the *norm* of that partition.

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whenever $\|\mathcal{P}\| < \delta$

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A weaker definition

Definition (RS Def II)

A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if there exists some $R \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ **and a** partition $\mathcal{P}$ such that for every tagged refinement $(\mathcal{P}', t')$ of $\mathcal{P}$ with $\|\mathcal{P}'\| \leq \delta$ we have,

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The nice thing about the above definition is that one only has to check that $|R\left(f, P', t'\right) - R|$ is small for for refinements of a fixed partition, and not for all partitions.

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- $||\mathcal{P}^*|| \leq ||\mathcal{P}||$
- $L(f, \mathcal{P}) \leq L(f, \mathcal{P}^*) \leq R(f, \mathcal{P}^*, t^*) \leq U(f, \mathcal{P}^*) \leq U(f, \mathcal{P})$

A Result

Theorem (Tying things up)

The three definitions, viz. The RI, RS Def I, and the RS Def II are **equivalent**.

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The RI definition, along with the RC help us in proving things about integrals. The RS Def (II) helps us in computing integrals (rigorously).

- [2019 Quiz 1] Determine if

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cos^2\left(\frac{i\pi}{n}\right)$$

exists. If so, find the limit.

The Fundamental Theorem of Calculus

The FTC connects together the notions of integrability and differentiability. That is,

$$\text{Integrability} \xleftrightarrow{\textbf{FTC}} \text{Differentiability}$$

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Also, a helpful definition.

Definition (Antiderivative)

Given a function $f : D \to \mathbb{R}$, we say that f has a antiderivative on D if there exists a differentiable function $F : D \to \mathbb{R}$ such that,

$$F'(x) = f(x) \ \forall x \ \in \ D$$

FTC I

The FTC I connects integration to differentiation.

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- If f is continuous at $c \in [a, b]$, then F is differentiable at c. Further, $F'(c) = f(c)$.

That is, if your f is continuous, the proposed F is an antiderivative. Thus a continuous function on an interval always possesses an antiderivative.

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Another way to look at this, is that,

$$F(x) = F(a) + \int_a^x f(x)dx$$

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Another way to look at this, is that,

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for *any* antiderivative F of f. This also relates to the fact that two antiderivatives differ by a constant.

Application: Area

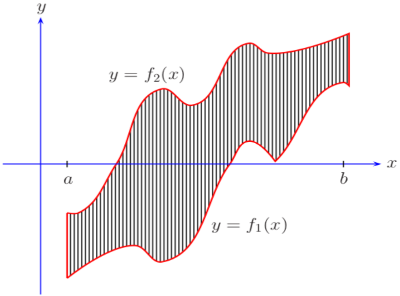

Figure: Area: Type 1

Application: Area

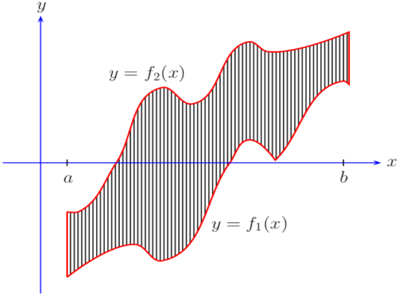

Figure: Area: Type 1

We compute the area as,

$$A = \int_a^b (f_2(x) - f_1(x))dx$$

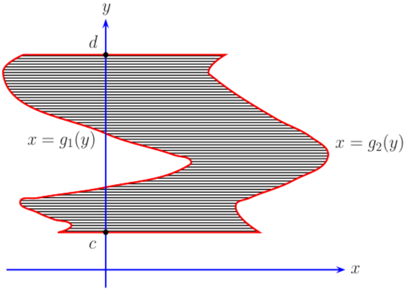

Figure: Area: Type 2

Application: Area

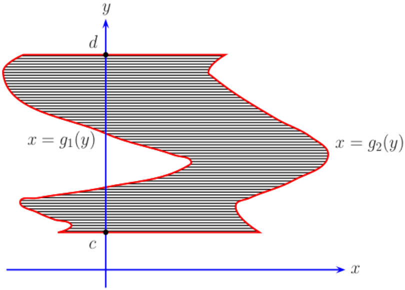

Figure: Area: Type 2

We compute the area as,

$$A = \int_c^d (g_2(y) - g_1(y))dy$$

Application: Area

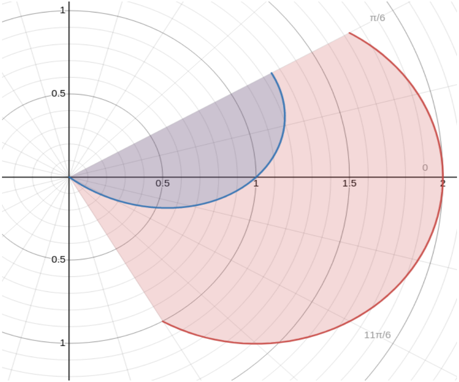

Figure: Area: Type 3, $\rho_1(\theta) = 2\cos\theta$, $\rho_2(\theta) = \cos^2\theta + \sin\theta$

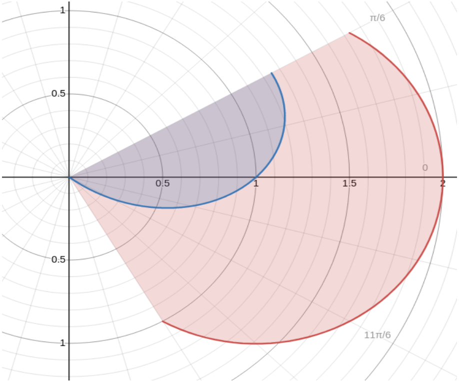

Figure: Area: Type 3, $\rho_1(\theta) = 2\cos\theta$, $\rho_2(\theta) = \cos^2\theta + \sin\theta$

We compute the area as,

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (\rho_2(\theta)^2 - \rho_1(\theta)^2)d\theta$$

Application: Arc Length

Definition (Curve)

A *curve* is a tuple $C = (x(t), y(t))$ wherein $x, y : [a, b] \to \mathbb{R}$ are continuous functions.

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Steps:

- Sum of lengths, $\sum_{i=1}^{n} \sqrt{\left(x\left(t_i\right) - x\left(t_{i-1}\right)\right)^2 + \left(y\left(t_i\right) - y\left(t_{i-1}\right)\right)^2}$

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- Apply MVT, $\sum_{i=1}^{n} \sqrt{\left(x'\left(s_i\right)\right)^2 + \left(y'\left(u_i\right)\right)^2}\left(t_i - t_{i-1}\right)$;

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- Apply MVT, $\sum_{i=1}^{n} \sqrt{\left(x'\left(s_i\right)\right)^2 + \left(y'\left(u_i\right)\right)^2}\left(t_i - t_{i-1}\right)$; recognize the Riemann Sum!

Application: Arc Length

Definition (Curve)

A *curve* is a tuple $C = (x(t), y(t))$ wherein $x, y : [a, b] \to \mathbb{R}$ are continuous functions.

That is, $C : \mathbb{R} \to \mathbb{R}^2$. We will assume that C is smooth. The image of a curve is called a *path*.

Steps:

- Sum of lengths, $\sum_{i=1}^{n} \sqrt{\left(x\left(t_i\right) - x\left(t_{i-1}\right)\right)^2 + \left(y\left(t_i\right) - y\left(t_{i-1}\right)\right)^2}$
- Apply MVT, $\sum_{i=1}^{n} \sqrt{\left(x'\left(s_i\right)\right)^2 + \left(y'\left(u_i\right)\right)^2}\left(t_i - t_{i-1}\right)$; recognize the Riemann Sum!
- Conclude,

$$\boxed{\text{Arc Length}(C) := \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}dt}$$

Application: Surface Area

- Identify the area of a frustum, $A_F = \pi\lambda_2(d_1 + d_2)$

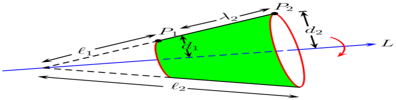

Figure: A frustum

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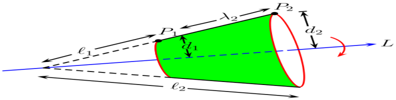
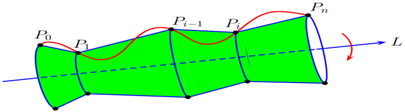


Figure: A frustum

- Use this to form the Riemann Sum, $\pi \sum(\rho(t_{i-1}) + \rho_{t_i})\lambda_i$



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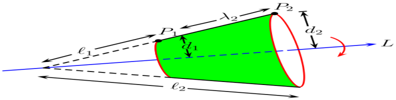

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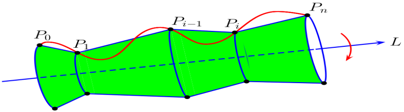

Figure: A general area

Thus,

$$\boxed{\text{Area}(S) := 2\pi\int_{\alpha}^{\beta}\rho(t)\sqrt{x'(t)^2 + y'(t)^2}dt,}$$

Application: Volume

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We will deal with solids of revolution. Two methods, funny names,

1. Washer,
2. Shell.

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Slices take **perpendicular to rotation axis**.

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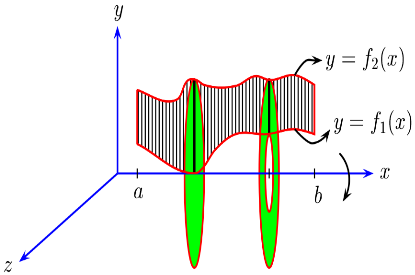

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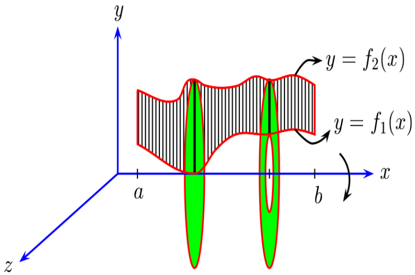

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$$\boxed{Vol(D) = \int_a^b \pi(f_2(x)^2 - f_1(x)^2)dx}$$

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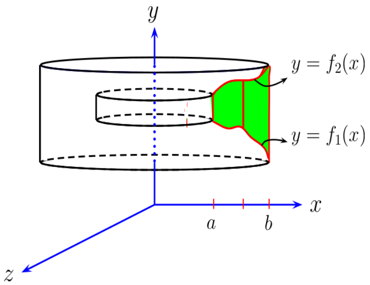

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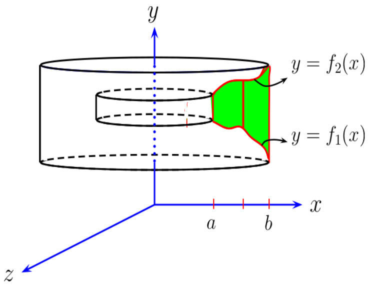

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$$\boxed{Vol(D) = 2\pi \int_a^b x(f_2(x) - f_1(x))dx}$$

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  5. But note that, we will only deal with *interior points* here.

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Gradient

Gradient

Definition (Gradient)

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, and let $(x_0, y_0) \in D$ be an interior point of D. Suppose both partial derivatives of f exist at (x_0, y_0), then we define the gradient of f at (x_0, y_0)

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Definition (Directional Derivatives)

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, and let $(x_0, y_0) \in D$ be an interior point of D. For some $u = (u_1, u_2)^T \in \mathbb{R}^2$, we say that f has a directional derivative along u at (x_0, y_0) if

$$\lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

exists, and denote it by $D_u f(x_0, y_0)$.

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1D Definition:

$$\lim_{h \to 0} \frac{|f(x_0 + h) - f(x_0) - f'(x_0)h|}{|h|} = 0$$

What about 2D?

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A function $f : D \to \mathbb{R}$ is said to be differentiable at an interior point (x_0, y_0) if, $\partial_x f$ and $\partial_y f$ exist at that point and,

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Differentiability at a point *implies* the following,

- Existence of partial derivatives at that point.
- Existence of ALL directional derivatives at that point.
- The fact that $D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u$ for every unit vector u.

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Definition (Differentiability)

A function $f : D \to \mathbb{R}$ is said to be differentiable at an interior point (x_0, y_0) if, $\partial_x f$ and $\partial_y f$ exist at that point and,

$$\lim_{(h,k)\to 0} \frac{|f(x_0+h, y_0+k) - f(x_0, y_0) - \partial_x(x_0, y_0)h - \partial_y(x_0, y_0)k|}{||(h,k)||} = 0$$

Differentiability at a point *implies* the following,

- Existence of partial derivatives at that point.
- Existence of ALL directional derivatives at that point.
- The fact that $D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u$ for every unit vector u.
- Continuity of f at that point.

Alternate Condition

Proposition

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Then there is $r > 0$ such that $S := \{(x, y) \in \mathbb{R}^2 :\mid x - x_0 \mid< r \text{ and } \mid y - y_0 \mid< r\} \subset D$. Consider $f : S \to \mathbb{R}$, and Suppose one of the partial derivatives of f exists on S and is continuous at (x_0, y_0), while the other exists at (x_0, y_0). Then f is differentiable at (x_0, y_0).

Mixed Partials

The following theorem relates the mixed partial derivatives of a function :

Theorem

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Then there is $r > 0$ such that $S := \{(x, y) \in \mathbb{R}^2 :| x - x_0 |< r \text{ and } | y - y_0 |< r\} \subset D$. Consider $f : S \to \mathbb{R}$, and suppose f_x and f_y exist on S. If one of the mixed partials f_{xy} or f_{yx} exists on S, and it is continuous at (x_0, y_0), then the other mixed partial exists at (x_0, y_0), and $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

Questions

- [2019 Endsem] Show that $f(x, y) := xy + |xy|$ is differentiable at $(0, 0)$.
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$$\mathrm{f}(x) := \begin{cases} x^3 sin(\frac{1}{x^2}) + y^3 sin(\frac{1}{y^2}) & \text{if } x \neq 0, y \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

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Tangent Lines and Planes

- Let $F : D \subset \mathbb{R}^2 \to \mathbb{R}$ have partial derivatives at an interior point $P = (x_0, y_0)$ and let $\nabla F(x_0, y_0) \neq (0, 0)^T$. Consider the curve, $C := \{(x, y) \mid F(x, y) = 0\}$ and assume that P lies on C.

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- Consider a C lying on the surface defined by $F(x, y, z) = 0$. Then, for any $t_0 \in [a, b]$, we have,

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Extrema & Saddle Points

Recall the concept of **local extrema**.

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Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Suppose $f : D \to \mathbb{R}$ has a local extremum at (x_0, y_0). If u is a unit vector and $(\nabla_u f)(x_0, y_0)$ exists, then $(\nabla_u f)(x_0, y_0) = 0$.

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Definition (Saddle Points)

For a 'nice' function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, and some interior point $P \in D$, we call P to be a saddle point of f if $\nabla F(P) = (0, 0)^T$ but P is **not** a local extrema.

Questions

- [2019 Quiz 2] Find the absolute minimum and the absolute maximum values of the function $f(x, y) = (x^2 - 3x)\cos y$ over the region $x \in [1, 3], y \in [-\pi/4, \pi/4]$.
- [2019 Quiz 2] Does $f(x, y) = x^2 y$ have a local extrema at $(0, 1)$?
- [2017 Midsem] Show that the tangent plane to the surface $z = x^2 - y^2$ at $(3, 3, 0)$ intersects the surface in two perpendicular lines.

Hessian Test

Proposition (The Hessian Test)

Let $D \subset \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Suppose $f : D \to \mathbb{R}$ is such that the first-order and second-order partial derivatives of f exist and are continuous in a neighbourhood of (x_0, y_0), and $(\nabla f)(x_0, y_0) = (0, 0)$.

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The discrimimant test is inconclusive if $(\Delta f)(x_0, y_0) = 0$.

Constrained Extrema

Setting: Let $f, g : D \subset \mathbb{R}^2 \to \mathbb{R}$ be nice functions. We aim to find extrema of f on D constrained to the fact that $g = 0$.

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2. $\left(\nabla g\right)\left(x_0, y_0\right) \neq (0, 0)$

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2. $\left(\nabla g\right)\left(x_0, y_0\right) \neq (0, 0)$
3. f, when restricted to C, has a local extremum at (x_0, y_0).

Constrained Extrema

Setting: Let $f, g : D \subset \mathbb{R}^2 \to \mathbb{R}$ be nice functions. We aim to find extrema of f on D constrained to the fact that $g = 0$.

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The real number λ_0 is called a Lagrange multiplier.

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6. Note that this procedure only tells us that these points are the extrema – does not comment on whether they are minima or maxima.

- [2017 Endsem] Let $f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + F$ where $A > 0$ and $AC - B^2 > 0$.
  - Show that there is a unique critical point of $f(x, y)$, say (x_1, y_1).
  - Show that $f(x, y)$ has a relative minimum at (x_1, y_1).