CS310M: Automata Theory (Minor)

Topic 3: Nondeterministic Finite Finite State Automata

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Course URL: https://cse.iitb.ac.in/∼pandya58/CS310M/automata.html

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Today's Topics:

- Nondeterminism in automata
- Syntax, Semantics of NFA, Examples
- Extended Transition Function
- Determinization: Subset Construction

Source: Kozen, Lectures 5 and 6.

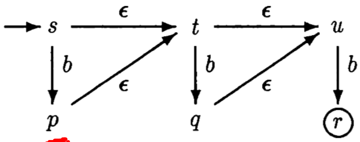
Automata with silent transitions

$C_\epsilon(\{p,r\}) = \{p,t,u,r\}$

$\cdot\ X \subseteq C_\epsilon(X)$

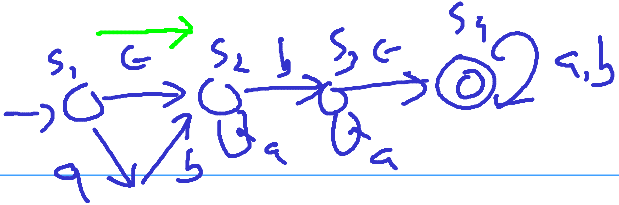
$\cdot\ X \subseteq Y$

$C_\epsilon(X) \subseteq C_\epsilon(Y)$



An ϵ-NFA is like an NFA $M = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$.

- Let $p \stackrel{\epsilon^*}{\Longrightarrow} q$ denote a finite seqeunce of epsilon moves from p to q. E.g. $s \stackrel{\epsilon^*}{\Longrightarrow} u$.
- Epsilon-closure: For $A \subseteq Q$, let $C_\epsilon(A) \;=\; \{q \mid \exists p \in A.\;\; p \stackrel{\epsilon^*}{\Longrightarrow} q\}$. It denotes all states reachable by epsilon paths from states in A.
  E.g. $C_\epsilon(\{p, r\}) = \{p, t, u, r\}$.



$$C_{\varepsilon}(\{s_1\}) = \{s_1, s_2\}$$

$$C_{\varepsilon}(\{s_3\}) = \{s_3, s_4\}$$

$$C_{\varepsilon}(\{s_1, s_3\}) = \{s_1, s_2, s_3, s_4\}$$

Equivalence

NFA → DFA

$Q \rightarrow \{X \subseteq Q\} = 2^Q$

Theorem

For every ϵ-NFA $N = (Q, \Sigma \cup \{\epsilon\}, \Delta, S, F)$ we can construct a DFA $M = (Q_M, \Sigma, \Delta_M, s_M, F_M)$ such that $L(M) = L(N)$.

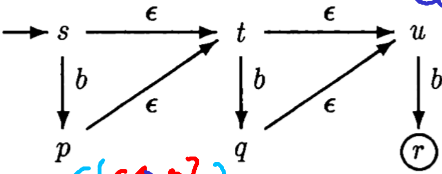
Modified Subset Construction of M

- $Q_M = \{X \subseteq Q \mid \underbrace{C_\epsilon(X) = X}\}$ epsilon closed subsets.
- $s_M = C_\epsilon(S)$.
- $F_M = \{X \in Q_M \mid X \cap F \neq \emptyset\}$
- $\Delta_M(X, a) = C_\epsilon(\hat{\Delta}(X, a))$.

$X \xrightarrow{a} Y \xrightarrow{\epsilon^*} Z$

Equivalent NFA construction Example

ϵ-NFA N



$Q = \{s,t,u,p,q,r\}$

2^6

$S = \{s\}$

$C_\epsilon(\{s\}) = \{s,t,u\}$

$C_\epsilon(\{p,r\})$

DFA M

	b
$\rightarrow \{s,t,u\}$	$\{p,t,u,q,r\}$
$\{p,t,u,q,r\}$ F	$\{q,u,r\}$
$\{q,u,r\}$ F	$\{r\}$
$\{r\}$ F	ϕ
ϕ	ϕ

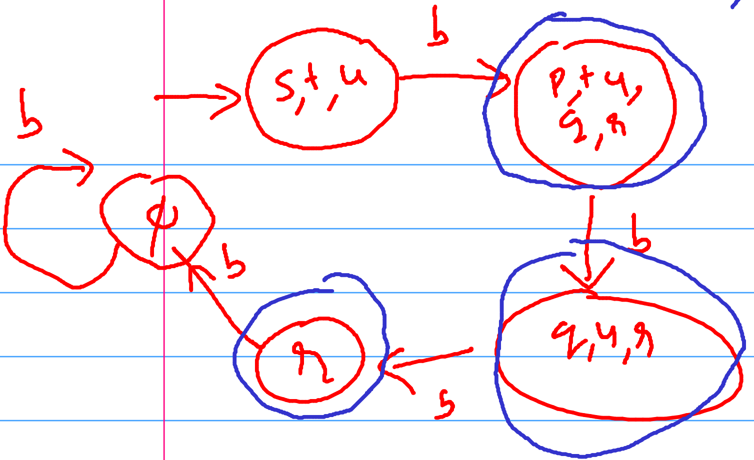
$\Delta_M(\{s,t,u\}, b) =$

$C_\epsilon(\hat{\Delta}(\{s,t,u\}, b))$

$C_\epsilon(\{p,q,r\}) =$

$\{p,t,u,q,r\}$

b, bb, bbb



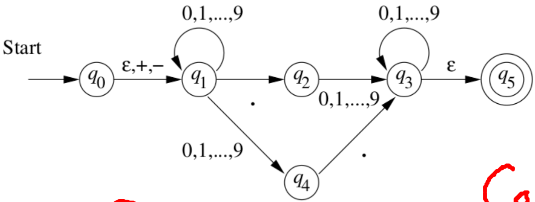
Deterministic FA — DFA

Nondeterminstic FA — NFA

Nondet. FA with silent tran. FA — ϵ-NFA

Example: Decimal numbers

$C_\epsilon(\{q_0\}) = \{q_0, q_1\}$

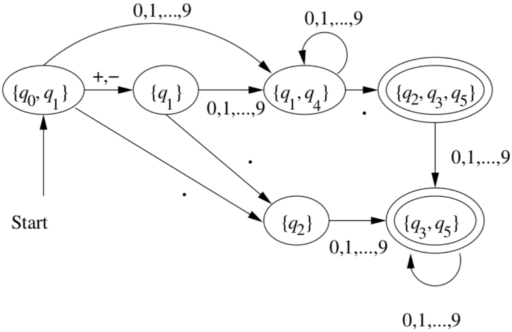


DFA

Complete the table

	$\pm$	D	.
$\rightarrow \{q_0, q_1\}$	$\{q_1\}$	$\{q_1, q_4\}$	$\{q_2\}$
$\{q_2\}$	ϕ	$\{q_3 q_5\}$	ϕ

Constructed DFA for Decimal NFA



Size of the DFA constructed using Subset Method

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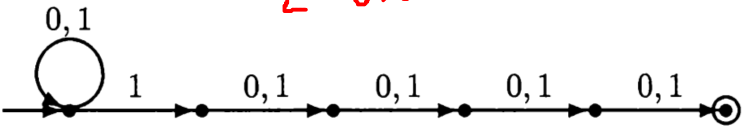
- Given ϵ-NFA with size n, the size of DFA using subset construction is at most 2^n.
  Is there a better construction giving smaller size DFA?
- Answer No. Consider the following automaton.

$\Sigma = \{0,1\}$



Thinking Question: can we have DFA of size less than 2^5?

$L = \{x \mid \exists u,v.\ x = u \cdot 1 \cdot v \text{ and } v \in \Sigma^4\}$

$= \Sigma^* \cdot \{1\} \cdot \Sigma^4$

Claim: There is no DFA M with $|M| < 2^5$ and $L(M) = L$.

Proof: Assume to contrary. M .... exist

Language Operations

Let $A, A_1, A_2 \subseteq \Sigma^*$. Define

- Union $A_1 \cup A_2$
- Intersection $A_1 \cap A_2$
- Complementation $\sim A \;=\; \Sigma^* - A$.
- Catenation $A_1 \cdot A2$.
- Kleene Closure A^*
- Reverse $rev(A)$

Closure of Regular languages under operations

Theorem If A_1, A_2 are regular then $A_1 \cap A_2$ is regular.

We showed that regular languages are closed under $\cap, \;\; \sim, \;\; \cup$.

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• Pay attention to size of automaton !

$|M_3| = |M_1| \times |M_2|$

Closure of Regular languages under operations

Theorem If A_1, A_2 are regular then $A_1 \cap A_2$ is regular.
Proof Method: Given DFA M_1, M_2 s.t. $A_1 = L(M_1)$ and $A_2 = L(M_2)$ we construct DFA M_3 s.t. $L(M_3) = A_1 \cap A_2$.

We showed that regular languages are closed under $\cap$, $\sim$, $\cup$.

Normalized ϵ-NFA

An ϵ-NFA is normalized if

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Given any ϵ-NFA $N_1 = (Q, \Sigma, \Delta, S, F)$ we can construct a normalized ϵ-NFA $N_2 = (Q \cup \{s, f\}, \Sigma, \Delta_2, s, f)$ s.t. $L(N_1) = L(N_2)$.

Closure under Union

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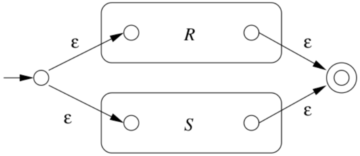

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- $Q_3 \; = \; Q_1 \cup Q_2 \cup \{s_3, f_3\}$ fresh states s_3, f_3.

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- $\Delta_3(s_3, \epsilon) = \{s_1, s_2\}$, and $\forall a \in \Sigma.\ \ \Delta_3(s_3, a) = \emptyset$.

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  $\forall s \in Q_2.\ \ \Delta_3(s, a) = \Delta_2(s, a)$.
- $\forall s \in Q_1 - \{f_1\}.\ \ \Delta_3(s, \epsilon) = \Delta_1(s, \epsilon)$, and
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- $\Delta_3(f_1, \epsilon) \; = \; \Delta_1(f_1, \epsilon) \cup \{f_3\}$ and
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