Insights into the Core of the Assignment Game via Complementarity

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Abstract

The classic paper of Shapley and Shubik [SS71] characterized the core of the assignment game using ideas from matching theory and LP-duality theory and their highly non-trivial interplay. The pristine structural properties of this game make it a paradigmatic setting for studying the solution concept of core, with a view to using insights developed for providing guidance on profit-sharing in real-life situations.

We provide additional insights by viewing imputations in the core of this game through the lens of complementarity. Our exploration yields new insights: we obtain a relationship between the competitiveness of individuals and teams of agents and the amount of profit they accrue. Additionally, we shed light on the phenomenon of degeneracy, i.e., when the optimal assignment is not unique.

The core is a quintessential solution concept in cooperative game theory. It contains all ways of distributing the total worth of a game among agents in such a way that no sub-coalition has incentive to secede from the grand coalition.

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1 Introduction

The *core* is a quintessential solution concept in cooperative game theory. It captures all possible ways of distributing the total worth of a game among individual agents in such a way that the grand coalition remains intact, i.e., a sub-coalition will not be able to generate more profits by itself and therefore has no incentive to secede from the grand coalition. The core provides profound insights into the negotiating power of individuals and sub-coalitions. In particular, the profit allocated to an agent is consistent with their negotiating power, i.e., their worth, and therefore the core is viewed as a "fair" profit-sharing mechanism, e.g., see Section 4.2. For an extensive coverage of this notion, see the book by Moulin [Mou14].

The classic paper of Shapley and Shubik [SS71] characterized profit-sharing methods that lie in the core of an assignment game by using ideas from matching theory and LP-duality theory and their highly non-trivial interplay. The assignment game can also be viewed as a matching market in which utilities of agents are stated in monetary terms and side payments are allowed, i.e., it is a *transferable utility (TU) market*. The main theorem of [SS71] shows that the set of imputations in the core of this game is precisely the set of optimal solutions to the dual of the LP-relaxation of the optimal assignment problem. As a corollary, the core of this game is always non-empty[1]. These facts, and additional insights provided in [SS71], make the assignment game a paradigmatic setting. Its insights provide valuable guidance on profit-sharing in real-life situations, which of course will not have such a clean structure.

Our paper goes further in this direction. The worth of an assignment game is determined by an optimal solution to the primal LP, and a core imputation distributes it using an optimal solution to the dual LP. This fact naturally raises the question of viewing core imputations through the lens of complementarity. Our exploration yields new insights: we obtain a relationship between the competitiveness of individuals and teams of agents and the amount of profit they accrue, where by *competitiveness* we mean whether an individual or a team is matched in every/some/no optimal assignment.

A second aspect of this paper is to shed light on the phenomenon of *degeneracy* in an assignment game; a game is degenerate if its optimal assignment is not unique. Shapley and Shubik had mentioned this phenomenon; however, they claimed that "in the most common case" the optimal assignment will be unique. Furthermore, their suggestion for dealing with degeneracy was to perturb the edge weights of the underlying graph to make the optimal assignment unique, and they only addressed this case. However, perturbing the weights destroys crucial information contained in the original instance. We use complementarity to study this phenomenon as well; Section 3 discusses past approaches.

The following setting, taken from [EK01] and [BKP12], vividly captures the issues underlying profit-sharing in an assignment game. Suppose a coed tennis club has sets U and V of women and men players, respectively, who can participate in an upcoming mixed doubles tournament. Assume $|U| = m$ and $|V| = n$, where m, n are arbitrary. Let $G = (U, V, E)$ be a bipartite graph whose vertices are the women and men players and an edge (i, j) represents the fact that agents $i \in U$ and $j \in V$ are eligible to participate as a mixed doubles team in the tournament. Let w

[1] In general, non-emptiness of core is rare, see Section 3 for a discussion and alternative solution concepts proposed in the literature.

be an edge-weight function for G, where $w_{ij} > 0$ represents the expected earnings if i and j do participate as a team in the tournament. The total worth of the game is the weight of a maximum weight matching in G.

Assume that the club picks such a matching for the tournament. The question is how to distribute the total profit among the agents — strong players, weak players and unmatched players — so that no subset of players feel they will be better off seceding and forming their own tennis club.

2 Definitions and Preliminary Facts

The *assignment game*, $G = (U, V, E)$, $w : E \rightarrow \mathcal{R}_+$, has been defined in the Introduction. We start by giving definitions needed to state the Shapley-Shubik Theorem.

Definition 1. The set of all players, $U \cup V$, is called the *grand coalition*. A subset of the players, $(S_u \cup S_v)$, with $S_u \subseteq U$ and $S_v \subseteq V$, is called a *coalition* or a *sub-coalition*.

Definition 2. The *worth* of a coalition $(S_u \cup S_v)$ is defined to be the maximum profit that can be generated by teams within $(S_u \cup S_v)$ and is denoted by $p(S_u \cup S_v)$. Formally, $p(S_u \cup S_v)$ is the weight of a maximum weight matching in the graph G restricted to vertices in $(S_u \cup S_v)$ only. $p(U \cup V)$ is called the *worth of the game*. The *characteristic function* of the game is defined to be $p : 2^{U \cup V} \rightarrow \mathcal{R}_+$.

Definition 3. An *imputation*[2] gives a way of dividing the worth of the game, $p(U \cup V)$, among the agents. It consists of two functions $u : U \rightarrow \mathcal{R}_+$ and $v : V \rightarrow \mathcal{R}_+$ such that $\sum_{i \in U} u(i) + \sum_{j \in V} v(j) = p(U \cup V)$.

Definition 4. An imputation (u, v) is said to be in the *core of the assignment game* if for any coalition $(S_u \cup S_v)$, the total worth allocated to agents in the coalition is at least as large as the worth that they can generate by themselves, i.e., $\sum_{i \in S_u} u(i) + \sum_{j \in S_v} v(j) \geq p(S)$.

We next describe the characterization of the core of the assignment game given by Shapley and Shubik [SS71][3].

As stated in Definition 2, the worth of the game, $G = (U, V, E)$, $w : E \rightarrow \mathcal{R}_+$, is the weight of a maximum weight matching in G. Linear program (1) gives the LP-relaxation of the problem of finding such a matching. In this program, variable x_{ij} indicates the extent to which edge (i, j) is picked in the solution. Matching theory tells us that this LP always has an integral optimal solution [LP86]; the latter is a maximum weight matching in G.

[2]Some authors prefer to call this a pre-imputation, while using the term imputation when individual rationality is also satisfied.

[3]Shapley and Shubik had described this game in the context of the housing market in which agents are of two types, buyers and sellers. They had shown that each imputation in the core of this game gives rise to unique prices for all the houses. In this paper we will present the assignment game in a variant of the tennis setting given in the Introduction; this will obviate the need to define "prices", hence leading to simplicity.

$$
\begin{aligned}
\max \quad & \sum_{(i,j)\in E} w_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{(i,j)\in E} x_{ij} \leq 1 \quad \forall i \in U, \\
& \sum_{(i,j)\in E} x_{ij} \leq 1 \quad \forall j \in V, \\
& \qquad\quad x_{ij} \geq 0 \quad \forall (i,j) \in E
\end{aligned} \tag{1}
$$

Taking u_i and v_j to be the dual variables for the first and second constraints of (1), we obtain the dual LP:

$$
\begin{aligned}
\min \quad & \sum_{i\in U} u_i + \sum_{j\in V} v_j \\
\text{s.t.} \quad & u_i + v_j \geq w_{ij} \quad \forall (i,j) \in E, \\
& u_i \geq 0 \qquad \forall i \in U, \\
& v_j \geq 0 \qquad \forall j \in V
\end{aligned} \tag{2}
$$

Theorem 1. *(Shapley and Shubik [SS71]) The imputation (u, v) is in the core of the assignment game if and only if it is an optimal solution to the dual LP, (2).*

3 Related Works

The core is a key solution concept in cooperative game theory for several reasons. First, imputations in the core give a way of distributing the worth of the game among players so the grand coalition remains intact, i.e., no sub-coalition can do better by itself and hence has any incentive to secede. Second, this way of distributing the worth can be considered to be *fair*, since each agent gets as much as her worth to the grand coalition and all sub-coalitions she is in.

An imputation in the core has to ensure that *each* of the exponentially many sub-coalitions is "happy" — clearly, that is a lot of constraints. As a result, the core is non-empty only for a handful of games, those with very good structural properties. Besides the assignment game, which is described in Section 2, this holds for the stable matching solution concept, given by Gale and Shapley [GS62], which lies in the core of their game. The only coalitions that matter in this game are ones formed by one agent from each side of the bipartition; a stable matching ensures that no such coalition has the incentive to secede.

Next, let us consider a generalization of the assignment game to the *general graph matching game*. It consists of an undirected graph $G = (V, E)$ and an edge-weight function w. The vertices $i \in V$ are the agents and an edge (i, j) represents the fact that agents i and j are eligible for an activity or a trade. Continuing with the analogy of tennis, given for the assignment game in the Introduction, for this generalization, we will assume that any two agents can form a doubles team. If $(i, j) \in E$, w_{ij} represents the profit generated if i and j play in the tournament. The worth of a coalition $S \subseteq V$, denoted $p(S)$, is given by the weight of a maximum matching in G restricted to S. This game may have an empty core.

To deal with the question of emptiness of core, the following two notions have been given in the past. The first is that of *least core*, defined by Mascher et al. [MPS79]. If the core is empty, there will necessarily be sets $S \subseteq V$ such that $v(S) < p(S)$ for any imputation v. The least core maximizes the minimum of $v(S) - p(S)$ over all sets $S \subseteq V$, subject to $v(\emptyset) = 0$ and $v(V) = p(V)$. This involves solving an LP with exponentially many constraints, though, if a separation oracle can be implemented in polynomial time, then the ellipsoid algorithm will accomplish this in polynomial time [GLS88]; see below for a resolution for the case of the matching game.

A more well known notion is that of *nucleolus* which is contained in the least core. After maximizing the minimum of $v(S) - p(S)$ over all sets $S \subseteq V$, it does the same for all remaining sets and so on. A formal definition is given below.

Definition 5. For an imputation $v : V \rightarrow \mathcal{R}_+$, let $\theta(v)$ be the vector obtained by sorting the $2^{|V|} - 2$ values $v(S) - p(S)$ for each $\emptyset \subset S \subset V$ in non-decreasing order. Then the unique imputation, v, that lexicographically maximizes $\theta(v)$ is called the *nucleolus* and is denoted $\nu(G)$.

The nucleolus was defined in 1969 by Schmeidler [Sch69], though its history can be traced back to the Babylonian Talmud [AM85]. It has several modern-day applications, e.g., [BST05]. In 1998, [FKFH98] stated the problem of computing the nucleolus of the matching game in polynomial time. For the assignment game with unit weight edges, this was done in [SR94]; however, since the assignment game has a non-empty core, this result was of little value. For the general graph matching game with unit weight edges, this was done by Kern and Paulusma [KP03]. Finally, the general problem was resolved by Konemann et al. [KPT20]. However, their algorithm makes extensive use of the ellipsoid algorithm and is therefore neither efficient nor does it give deep insights into the underlying combinatorial structure. They leave the open problem of finding a combinatorial polynomial time algorithm. We note that the difference $v(S) - p(S)$ appearing in the least core and nucleolus has not been upper-bounded for any standard family of games, including the general graph matching game.

A different notion was recently proposed in [Vaz22], namely *approximate core*. That paper gives an imputation in the 2/3-approximate core for the general graph matching game, i.e., the total profit allocated to a sub-coalition is at least 2/3 factor of the profit which it can generate by seceding. Moreover, this imputation can be computed in polynomial time and the bound is best possible, since it is the integrality gap of the natural underlying LP. This method used methodology developed in field of approximation algorithms, e.g., see [Vaz01], which uses multiplicative approximation as a norm, and yields polynomial time algorithms.

Whereas the previous two notions involve additive approximation, [Vaz22] uses multiplicative approximation. As argued in [Vaz22], this is more fair: the worst set for least core, say S, may have small $p(S)$. If so, the members of this set are unfairly treated, since they may end up getting almost no part of the worth of the game. On the other hand, under multiplicative approximation, the profit of *every* coalition is guaranteed to be at least 2/3 fraction of its worth. It therefore appears to be a simpler, more direct and more effective way of arriving at a "fair" division of the worth of a game in the face of an empty core.

Over the years, researchers have approached the phenomenon of degeneracy in the assignment game from directions that are different from ours. Nunez and Rafels [NR08], studied relationships between degeneracy and the dimension of the core. They defined an agent to be *active* if her profit is not constant across the various imputations in the core, and non-active otherwise. Clearly, this notion has much to do with the dimension of the core, e.g., it is easy to see that if all agents are non-active, the core must be zero-dimensional. They prove that if all agents are active, then the core is full dimensional if and only if the game is non-degenerate. Furthermore, if there are exactly two optimal matchings, then the core can have any dimension between 1 and $m-1$, where m is the smaller of $|U|$ and $|V|$; clearly, m is an upper bound on the dimension.

In another work, Chambers and Echenique [CE15] study the following question: Given the entire set of optimal matchings of a game on $m = |U|$, $n = |V|$ agents, is there an $m \times n$ surplus matrix which has this set of optimal matchings. They give necessary and sufficient conditions for the existence of such a matrix.

4 The Core via the Lens of Complementarity

In Theorem 2 we provide new insights into core imputations of an assignment game by studying them through the lens of complementarity. We present a relationship between the competitiveness of individuals and teams of agents and the amount of profit they accrue in imputations that lie in the core, where by *competitiveness* we mean whether an individual or a team is matched in every/some/no optimal assignment. Additionally, it sheds light on the phenomenon of degeneracy in assignment games, i.e., when the maximum weight matching is not unique.

Definition 6. By a *mixed doubles team* we mean an edge in G; a generic one will be denoted as $e = (u, v)$. We will say that e is:

1. *essential* if e is matched in every maximum weight matching in G.
2. *viable* if there is a maximum weight matching M such that $e \in M$, and another, M' such that $e \notin M'$.
3. *subpar* if for every maximum weight matching M in G, $e \notin M$.

Definition 7. Let y be an imputation in the core of the game. We will say that e is *fairly paid in* y if $y_u + y_v = w_e$ and it is *overpaid* if $y_u + y_v > w_e$[4]. Finally, we will say that e is *always paid fairly* if it is fairly paid in every imputation in the core.

Definition 8. A generic player in $U \cup V$ will be denoted by q. We will say that q is:

1. *essential* if q is matched in every maximum weight matching in G.
2. *viable* if there is a maximum weight matching M such that q is matched in M and another, M' such that q is not matched in M'.
3. *subpar* if for every maximum weight matching M in G, q is not matched in M.

[4]Observe that by the first constraint of the dual LP (2), these are the only possibilities.

Definition 9. Let y be an imputation in the core. We will say that q *gets paid in* y if $y_q > 0$ and *does not get paid* otherwise. Furthermore, q is *paid sometimes* if there is at least one imputation in the core under which q gets paid, and it is *never paid* if it is not paid under every imputation.

Theorem 2. *The following hold:*

1. *For every team $e \in E$:*

$$e \text{ is always paid fairly} \iff e \text{ is viable or essential}$$

2. *For every player $q \in (U \cup V)$:*

$$q \text{ is paid sometimes} \iff q \text{ is essential}$$

Proof. The proofs follow by applying complementary slackness conditions and strict complementarity to the primal LP (1) and dual LP (2); see [Sch86] for formal statements of these facts. We will use Theorem 1 stating that the set of imputations in the core of the game is precisely the set of optimal solutions to the dual LP.

1). Let x and y be optimal solutions to LP (1) and LP (2), respectively. By the Complementary Slackness Theorem, for each $e = (u, v) \in E: \quad x_e \cdot (y_u + y_v - w_e) = 0$.

Suppose e is viable or essential. Then there is an optimal solution to the primal, say x, under which it is matched, i.e., $x_e > 0$. Let y be an arbitrary optimal dual solution. Then, by the Complementary Slackness Theorem, $y_u + y_v = w_e$. Varying y over all optimal dual solutions, we get that e is always paid fairly. This proves the forward direction.

not covered in 602, explain. (holds if primal, dual feasible ∃ x,y)

For the reverse direction, we will use strict complementarity. It implies that corresponding to each team e, there is a pair of optimal primal and dual solutions x, y such that either $x_e = 0$ or $y_u + y_v = w_e$ but not both.

For team e, assume that the right hand side of the first statement holds and that x, y is a pair of optimal solutions for which strict complementarity holds for e. Since $y_u + y_v = w_e$ it must be the case that $x_e > 0$. Now, since the polytope defined by the constraints of the primal LP (1) has integral optimal vertices, there is a maximum weight matching under which e is matched. Therefore e is viable or essential and the left hand side of the first statement holds.

2). The proof is along the same lines and will be stated more succinctly. Again, let x and y be optimal solutions to LP (1) and LP (2), respectively. By the Complementary Slackness Theorem, for each $q \in (U \cup V): \; y_q \cdot (x(\delta(q)) - 1) = 0$.

Suppose q is paid sometimes. Then, there is an imputation in the core, say y, such that $y_q > 0$. Therefore, for every primal optimal solution x, $x(\delta(q)) = 1$ and in every maximum weight matching in G, q is matched. Hence q is essential, proving the reverse direction.

Strict complementarity implies that corresponding to each player q, there is a pair of optimal primal and dual solutions x, y such that either $y_q = 0$ or $x(\delta(q)) = 1$ but not both. Since we have already established that the second condition must be holding for x, we get that $y_q > 0$ and hence q is paid sometimes. $\square$

4.1 Consequences of Theorem 2

In this section, we will derive several useful consequences of Theorem 2.

1). Negating both sides of the first statement proved in Theorem 2 we get the following double-implication. For every team $e \in E$:

$$e \text{ is subpar} \iff e \text{ is sometimes overpaid}$$

Clearly, this statement is equivalent to the first statement of Theorem 2 and hence contains no new information. However, it may provide a new viewpoint. These two equivalent statements yield the following assertion, which at first sight seems incongruous with what we desire from the notion of the core and the just manner in which it allocates profits:

Whereas viable and essential teams are always paid fairly, subpar teams are sometimes overpaid.

How can the core favor subpar teams over viable and essential teams? Here is an explanation: Even though u and v are strong players, the team (u, v) may be subpar because u and v don't play well together. On the one hand, u and v are allocated high profits, since they generate large earnings while playing with other players. On the other hand, w_{uv} is small. Thus, this subpar team does not get overpaid while playing together, but by teaming up with others.

2). The second statement of Theorem 2 is equivalent to the following. For every player $q \in (U \cup V)$:

$$q \text{ is never paid} \iff q \text{ is not essential}$$

Thus core imputations pay only essential players. Since we have assumed that the weight of each edge is positive, so is the worth of the game, and all of it goes to essential players. This gives the next conclusion; in contrast, the set of essential teams may be empty.

Corollary 1. *In the assignment game, the set of essential players is non-empty.*

To do: Make comments on bipartite polytope integrality gap.

3). Clearly the worth of the game is generated by teams that do play. Assume that (u, v) is such a team in an optimal assignment. Since $x_{uv} > 0$, by complementary slackness we get that $y_u + y_v = w_{uv}$, where y is a core imputation. Thus core imputations distribute the worth generated by a team among its players only. In contrast, the $\frac{2}{3}$-approximate core imputation for the general graph matching game given in [Vaz22] distributes the worth generated by teams which play to non-playing agents as well, thereby making a more thorough use of the TU aspect.

4). Next we use Theorem 2 to get insights into degeneracy. Clearly, if an assignment game is non-degenerate, then every team and every player is either always matched or always unmatched in the set of maximum weight matchings in G, i.e., there are no viable teams or players.

Corollary 2. *Imputations in the core of an assignment game treat viable and essential teams in the same way. Additionally, they treat viable and subpar players in the same way.*

4.2 Insights Provided by the Core into the Negotiating Power of Agents

Consider an assignment game whose bipartite graph has two edges, $(u, v_1), (u, v_2)$ on the three agents u, v_1, v_2. Clearly, one of v_1 and v_2 will be left out in any matching. First assume that the weight of both edges is 1. If so, the unique imputation in the core gives zero to v_1 and v_2, and 1 to u. Next assume that the weights of the two edges are 1 and $1 + \epsilon$ respectively, for a small $\epsilon > 0$. If so, the unique imputation in the core gives $0, \epsilon$ and 1 to v_1, v_2 and u, respectively.

How fair are these imputations? As stated in the Introduction, imputations in the core have a lot to do with the negotiating power of individuals and sub-coalitions. Let us argue that when the imputations given above are viewed from this angle, they are fair in that the profit allocated to an agent is consistent with their negotiating power, i.e., their worth. In the first case, whereas u has alternatives, v_1 and v_2 don't. As a result, u will squeeze out all profits from whoever she plays with, by threatening to partner with the other player. Therefore v_1 and v_2 have to be content with no rewards! In the second case, u can always threaten to match up with v_2. Therefore v_1 has to be content with a profit of ϵ only.

In an arbitrary assignment game $G = (U, V, E), w$, by Theorem 2,

$$q \text{ is never paid} \iff q \text{ is not essential}$$

Thus core imputations reward only those agents who always play. This raises the following question: Can't a non-essential player, say q, team up with another player, say p, and secede, by promising p almost all of the resulting profit? The answer is "No", because the dual (2) has the constraint $y_q + y_p \geq w_{qp}$. Therefore, if $y_q = 0$, $y_p \geq w_{qp}$, i.e., p will not gain by seceding together with q.

Next, consider an assignment game whose bipartite graph has four edges, $(u_1, v_1), (u_1, v_2)$, $(u_2, v_2), (u_2, v_3)$ on the five agents u_1, u_2, v_1, v_2, v_3. Let the wights of these four edges be $1, 1.1, 1.1$ and 1, respectively. The worth of this game is clearly 2.1.

At first sight, v_2 looks like the dominant player, since he has two choices of partners, namely u_1 and u_2, and because teams involving him have the biggest earnings, namely 1.1 as opposed to 1. Yet, the unique core imputation in the core awards $1, 1, 0, 0.1, 0$ to agents u_1, u_2, v_1, v_2, v_3, respectively.

The question arises: "Why is v_2 allocated only 0.1? Can't he negotiate a higher profit, given his favorable circumstance?" The answer is "No". The reason is that u_1 and u_2 are in an even stronger position than v_2, since both of them have a ready partner available, namely v_1 and v_3, respectively, with whom each can earn 1. Therefore, the core imputation awards 1 to each of them, giving the leftover profit of 0.1 to v_2. Hence the core imputation has indeed allocated profits according to the negotiating power of each agent.

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