A1.

Que 1 (5+5 marks). Red-blue s-t connectivity: In this problem, we are given an undirected graph G, with each edge colored either red or blue. We are also given a source vertex s and a destination vertex t. The goal is to find an alternating red-blue path between s and t. That is, a path that starts on s with a red edge, alternates between red and blue edges, and ends at t with a blue edge.

We try to reduce this problem to the matching problem as follows. Naturally, first we can delete any blue edges incident on s and any red edges incident on t. We will construct another graph H based on the given graph.

- For every vertex v in G other than s and t, create two vertices in H, v_r and v_b.
- Create two more vertices in H, s_r and t_b.
- For any edge (u, v) in G: if it is red then create an edge (u_r, v_r) in H and if it is blue then create an edge (u_b, v_b) in H.
- Create an edge (u_r, u_b) for every vertex u other than s and t.

Prove or disprove using a counter-example the following: graph G has an alternating red-blue path between s and t **if and only if** the new graph has a perfect matching.

($\Rightarrow$) Let G have an alternating red-blue path. Let the path be $(s, v_1, \ldots, v_n, t)$, consider matching

(s, v_{1r}), (v_{1b}, v_{2r}), (v_{2b}, v_{3r}), (v_{3b}, v_{4r}), ... (v_{nr}, t)

{if first and last edges are red}

($\Leftarrow$) Let H have a p.m.
To show G has red-blue alternating path

ALG.

```
u = s
P = [s]
while (u != t) {
    if π(u) = v_r for some v :
        u ← v_b
        P.append(v)
    else
        u ← v_r
        P.append(v)
}
return P
```

claim : P is a red-blue alternating path

Proof : Only u_r, v_r are matched,
if $\pi(u) = v_r$ then $u \leftarrow v_b$
hence, next-edge is of different color.

claim : P terminates in t

Proof : If it terminates at $v \neq t$, algorithm finds another vertex, i.e. it can't terminate at $v \neq t$.

Claim : This ALG gives path P

Proof : Let ALG have repeated vertices in P.
Then $\pi(u)$ has already occured in P at some point in ALG. {first instance}
But, $\pi(u) \in P \Rightarrow \pi(u)$ was added in P previously
$\Rightarrow \pi(u) = \pi(u')$ for $u \neq u'$
since $\pi(u)$ is first repeated instance.
This is a contradiction since π is a matching

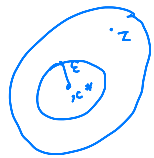
A2.

Que 2 (10 marks). Suppose S is a convex set and we are maximizing a linear function $w^T x$ over it. If a point $x^* \in S$ locally maximizes the function, then prove that it maximizes the function over all S. Locally maximizes means the following: there exists an $\epsilon > 0$ such that for all points $y \in S$ within distance ϵ from x^*, we have $w^T x^* \geq w^T y$. You will need to prove such an inequality for all points y in S.

S is convex. $x, y \in S \Rightarrow \lambda x + (1-\lambda) y \in S$, $\lambda \in [0,1]$

$x^* \in S$ is local maximum.

$\exists \varepsilon > 0$ s.t. $d(x^*, y) < \varepsilon \Rightarrow w^T x^* \geq w^T y$



Let $z \in S$, s.t. $w^T z > w^T x^*$

$$y = \lambda x^* + (1-\lambda) z$$

$$d(x^*, y) = d(x^*, \lambda x^* + (1-\lambda) z)$$

$$= (1-\lambda)\, d(x^* z)$$

Take $\lambda = \dfrac{1-\varepsilon}{2\, d(x^* z)}$. Then, $d(x^*, y) < \varepsilon$

But, $w^T y = \lambda w^T x^* + (1-\lambda) w^T z > w^T x^*$, but $d(x^*, y) < \varepsilon \Rightarrow w^T x^* \geq w^T y$

Hence, we get a contradiction to the premise $w^T z > w^T x^*$ ∎

A3.

Que 3 (10 marks). Use Fourier Motzkin procedure to compute the linear inequalities in variables x_1, x_2, x_3, which describe the cone $\{\lambda_1(1,2,3) + \lambda_2(2,3,1) + \lambda_3(3,1,2) : \lambda_1, \lambda_2, \lambda_3 \geq 0\} \subset \mathbb{R}^3$. Don't just write the final answer. You need to show the steps of Fourier Motzkin procedure.

$$x_1 = \lambda_1 + 2\lambda_2 + 3\lambda_3$$
$$x_2 = 2\lambda_1 + 3\lambda_2 + \lambda_3$$
$$x_3 = 3\lambda_1 + \lambda_2 + 2\lambda_3$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

$\rightarrow$

$$\lambda_1 = x_1 - 2\lambda_2 - 3\lambda_3$$
$$\lambda_1 = \frac{1}{2}(x_2 - 3\lambda_2 - \lambda_3)$$
$$\lambda_1 = \frac{1}{3}(x_3 - \lambda_2 - 2\lambda_3)$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

$\downarrow$

$$x_1 - 2\lambda_2 - 3\lambda_3 = \frac{1}{2}(x_2 - 3\lambda_2 - \lambda_3)$$
$$x_1 - 2\lambda_2 - 3\lambda_3 = \frac{1}{3}(x_3 - \lambda_2 - 2\lambda_3)$$
$$x_1 - 2\lambda_2 - 3\lambda_3 \geq 0$$
$$\lambda_2, \lambda_3 \geq 0$$

$\leftarrow$

$$2x_1 - x_2 - \lambda_2 - 5\lambda_3 = 0$$
$$3x_1 - x_3 - 5\lambda_2 - 7\lambda_3 = 0$$
$$x_1 - 2\lambda_2 - 3\lambda_3 \geq 0$$
$$\lambda_2, \lambda_3 \geq 0$$

$\downarrow$

$$\lambda_2 = 2x_1 - x_2 - 5\lambda_3$$
$$\lambda_2 = \frac{1}{5}(3x_1 - x_3 - 7\lambda_3)$$
$$\lambda_2 \leq \frac{1}{2}(x_1 - 3\lambda_3)$$
$$\lambda_2, \lambda_3 \geq 0$$

$\rightarrow$

$$2x_1 - x_2 - 5\lambda_3 = \frac{1}{5}(3x_1 - x_3 - 7\lambda_3)$$
$$2x_1 - x_2 - 5\lambda_3 \leq \frac{1}{2}(x_1 - 3\lambda_3)$$
$$2x_1 - x_2 - 5\lambda_3 \geq 0$$
$$\lambda_3 \geq 0$$

$\downarrow$

$$7x_1 - 5x_2 + x_3 = 18\lambda_3$$
$$3x_1 - 2x_2 \leq 7\lambda_3$$
$$2x_1 - x_2 \geq 5\lambda_3$$
$$\lambda_3 \geq 0$$

$\leftarrow$

$$3x_1 - 2x_2 \leq \frac{7}{18}(7x_1 - 5x_2 + x_3)$$
$$\frac{5}{18}(7x_1 - 5x_2 + x_3) \leq 2x_1 - x_2$$
$$7x_1 - 5x_2 + x_3 \geq 0$$

A4.

Que 4 (5+5 marks). We proved the following Farkas' lemma in the class. For any given $k \times n$ matrix A and $b \in \mathbb{R}^k$, the system

$$Ax = b, x \geq 0$$

has no feasible solution if and only if the system

$$A^T y \geq 0, b^T y = -1$$

has a feasible solution. Use this lemma (or any other way) to prove that for any given numbers b_1, b_2, b_3, b_4, the system

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &\leq b_1 \\ -x_1 + x_2 + 2x_3 &\leq b_2 \\ x_1 - x_2 &= b_3 \\ x_2 - 2x_3 &= b_4 \\ x_1, x_2 &\geq 0 \\ x_3 &\in \mathbb{R} \end{aligned}$$

has no feasible solution **if and only if** there exists $y_1 \geq 0, y_2 \geq 0, y_3, y_4 \in \mathbb{R}$ such that

$$\begin{aligned} 2y_1 - y_2 + y_3 &\geq 0 \\ -3y_1 + y_2 - y_3 + y_4 &\geq 0 \\ y_1 + 2y_2 - 2y_4 &= 0 \\ b_1y_1 + b_2y_2 + b_3y_3 + b_4y_4 &= -1. \end{aligned}$$

You need to show both the directions.

A4. Taking cases,

(I) $x_3 \geq 0$

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &\leq b_1 \\ -x_1 + x_2 + 2x_3 &\leq b_2 \\ x_1 - x_2 &= b_3 \\ x_2 - 2x_3 &= b_4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Equivalently,

$$\begin{aligned} 2x_1 - 3x_2 + x_3 + w &= b_1 \\ -x_1 + x_2 + 2x_3 + v &= b_2 \\ x_1 - x_2 &= b_3 \\ x_2 - 2x_3 &= b_4 \\ x_1, x_2, x_3, w, v &\geq 0 \end{aligned}$$

i.e. $Ax = b,\ x \geq 0$ where,

$$A = \begin{pmatrix} 2 & -3 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ w \\ v \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

is infeasible iff $A^T y \geq 0,\ b^T y = -1$ is feasible (Farkas lemma)

i.e.

$$\begin{aligned} 2y_1 - y_2 + y_3 &\geq 0 \\ -3y_1 + y_2 - y_3 + y_4 &\geq 0 \\ y_1 + 2y_2 - 2y_4 &\geq 0 \\ b_1 y_1 + b_2 y_2 + b_3 y_3 + b_4 y_4 &= -1 \\ y_1, y_2 \geq 0,\ y_3, y_4 &\in \mathbb{R} \end{aligned}$$

(II) $x_3 \leq 0$, $Ax = b,\ x \geq 0$ where,

$$A = \begin{pmatrix} 2 & -3 & -1 & 1 & 0 \\ -1 & 1 & -2 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & +2 & 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \\ w \\ v \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

is not feasible iff $A^T y \geq 0,\ b^T y = -1$. is feasible i.e. (by farkas lemma)

$$
\begin{aligned}
2y_1 - y_2 + y_3 &\geq 0 \\
-3y_1 + y_2 - y_3 &\geq 0 \\
y_1 + 2y_2 - 2y_4 &\leq 0 \\
b_1 y_1 + b_2 y_2 + b_3 y_3 + b_4 y_4 &= -1 \\
y_1, y_2 \geq 0,\ y_3, y_4 &\in \mathbb{R}
\end{aligned}
$$

Now, for the primals in (I) and (II) to be infeasible, the intersection of corresponding dual conditions hold, i.e.

$$
\begin{aligned}
2y_1 - y_2 + y_3 &\geq 0 \\
-3y_1 + y_2 - y_3 &\geq 0 \\
y_1 + 2y_2 - 2y_4 &= 0 \\
b_1 y_1 + b_2 y_2 + b_3 y_3 + b_4 y_4 &= -1 \\
y_1, y_2 \geq 0,\ y_3, y_4 &\in \mathbb{R}
\end{aligned}
$$