MA 106 Spring 2022-2023
Quiz TSC

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Axioms for determinant functions

1. Suppose that the columns of $A \in \mathbb{R}^{n\times n}$ are $A_1, A_2, ..., A_n$
2. Define $d : \mathbb{R}^{n\times n} \to \mathbb{R}$ by $d(A) = d(A_1, A_2, ..., A_n)$
3. The function d is called **multilinear** function if for each $k = 1, 2, \cdots, n$; scalars α, β and column vectors $A_1, ..., A_{k-1}, A_{k+1}, ..., A_n, B, C \in \mathbb{R}^{n\times 1}$,

$$d(A_1, ..., A_{k-1}, \alpha B + \beta C, A_{k+1}, ..., A_n) = \alpha d(A_1, ..., A_{k-1}, B, A_{k+1}, ..., A_n)$$

4. d is called an alternating function if for some $i \neq j$ and $A_i = A_j$, then

$$d(A_1, A_2, ..., A_n) = 0$$

5. If $d(I) = d(e_1, e_2, \cdots, e_n) = 1$ then d is called **normalized** function

Definition (Determinant)

A normalized, alternating, and multilinear function d on $n \times n$ matrices is called a determinant function of order n.

Lemma

Suppose that $d(A_1, A_2, \cdots, A_n)$ is a multilinear alternating function on columns of $n \times n$ matrices. Then,

1. If some $A_k = 0$ then $d(A_1, A_2, ..., A_n) = 0$.
2. $d(A_1, \cdots, A_k, A_{k+1}, \cdots, A_n) = -d(A_1, \cdots, A_{k+1}, A_k, \cdots, An)$
3. $d(A_1, \cdots, A_i, \cdots, A_j, \cdots, A_n) = -d(A_1, \cdots, Aj, \cdots, A_i, \cdots, A_n)$

Exercise

The numbers 20604, 53227, 25755, 20927, 78421 are all divisible by 17. Show that the determinant of the matrix

$$\begin{bmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{bmatrix}$$

is also divisible by 17. (Don't need to expand it out!)

Uniqueness of the determinant function

Vanishing Lemma (for multilinear function)

Suppose f is a multilinear alternating function on $n \times n$ matrices and $f(e_1, e_2, \cdots, e_n) = 0$ Then $f = 0$.

Theorem

Let f be an alternating multilinear function on $\mathbb{R}^{n \times n}$ and d a determinant function on $\mathbb{R}^{n \times n}$.

$$f(A_1, A_2, \cdots, A_n) = d(A_1, A_2, \cdots, A_n) f(e_1, e_2, \cdots, e_n)$$

In particular, if f is also a determinant function then,

$$f(A_1, A_2, \cdots, A_n) = d(A_1, A_2, \cdots, A_n)$$

Lemma (Determinant)

Let $A = (a_{ij})$ be an $n \times n$ matrix, then the determinant is given by the function,

$$det(A) = \sum \epsilon_{i_1, i_2, \cdots, i_n} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}$$

This expression of the determinant actually turns up in the proof of uniqueness of determinant. But, how do you calculate $\epsilon_{i_1, i_2, \cdots, i_n}$? We do this using the expression

$$\epsilon_{i_1, i_2, \cdots, i_n} = \frac{(i_1 - i_2)(i_1 - i_3) \cdots (i_{n-1} - i_n)}{(1-2)(1-3) \cdots ((n-1) - n)}$$

This is called the parity of number of inversions present in the permutation.

Theorem

Let $A = (a_{ij})$ be an $n \times n$ matrix. Then the function,

$$f(A) = a_{11} detA_{11} - a_{12} detA_{12} + \cdots + (-1)^{n+1} a_{1n} detA_{1n}$$

is the determinant function on $n \times n$ matrices.

Theorem

1. Let U be an upper triangular or a lower triangular matrix. Then $det(U)$ is the product of diagonal entries of U.
2. If $E = [e_1, \cdots, e_i + me_j, \cdots, e_n]$, for some $i \neq j$. Then $det(E) = 1$.
3. If $F = [e_1, e_2, \cdots, e_j, \cdots, e_i, \cdots, e_n]$, for some $i \neq j$. Then $det(F) = -1$.
4. If $G = [e_1, e_2, \cdots, me_i, \cdots, e_n]$ then $det(G) = m$.

Theorem

Let A, B be two $n \times n$ matrices. Then,

$$det(AB) = det(A)det(B)$$

Proposition (Determinant and invertibility)

1. If A is invertible then $det(A) \neq 0$ and $det(A^{-1}) = \frac{1}{det(A)}$
2. If $det(A) \neq 0$ then A is invertible.
3. If $AB = I$ then A is invertible and $B = A^{-1}$.

Theorem

For any $n \times n$ matrix A,

$$det(A) = det(A^t)$$

Questions

1. (2014 Quiz) Let L be a $n \times n$ lower triangular matrix with diagonal entries $l_1, l_2, \cdots, l_n$
(i) When is L invertible?
(ii) If L is invertible, must L^{-1} be lower triangular? Why?
2. (2019-20 Quiz) For given $r, s, t \in \mathbb{R}$ and distinct $a, b, c \in R$, find all possible polynomials p(x) of degrees at most 2 which satisfy the conditions $p(a) = r, p(b) = s, p(c) = t$

Theorem

Let $A = (a_{ij})$ be an $n \times n$ matrix and let $1 \leq k \leq n$. Then,

$$det(A) = \sum_{i=1}^{n} (-1)^{k+i} a_{ik} det(A_{ik})$$

Turns out, you can also compute the determinant of a matrix using the Gauss-Jordan method.

Theorem

Let $A = (a_{ij})$ be an $n \times n$ matrix and let $1 \leq k \leq n$. Then,

$$det(A) = \sum_{i=1}^{n} (-1)^{k+i} a_{ik} det(A_{ik})$$

Turns out, you can also compute the determinant of a matrix using the Gauss-Jordan method. **how?**

Matrix inverse and the cofactor matrix

Definition

Let $A = (a_{ij})$ be an $n \times n$ matrix. The cofactor of a_{ij}, denoted by $cof(a_{ij})$ is defined as

$$cof(a_{ij}) = (-1)^{i+j} det(A_{ij})$$

The cofactor matrix of A is defined as the matrix $cofA = (cofa_{ij})$:

Theorem

For any $n \times n$ matrix A,

$$A(cofA)^t = (detA)I = (cofA)^t A$$

Cramer's Rule for solving linear equations

Cramer's Rule

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$$

be a system of n linear equations in n unknowns, $x_1, x_2, \cdots, x_n$. Suppose the coefficient matrix $A = (a_{ij})$ is invertible. Let C_j be the matrix obtained from A by replacing the j^{th} column of A by $b = (b_1, b_2, \cdots, b_n)^t$. Then for $j = 1, 2, \cdots, n$, $x_j = \frac{det(C_j)}{det(A)}$

Exercise

Show that a necessary condition for the quadratic equations $x^2 + ax + b = 0$ and $x^2 + px + q = 0$ to have a common root is that the following matrix is singular:

$$\begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 1 & p & q & 0 \\ 0 & 1 & p & q \end{bmatrix}$$

Multiplication of Block matrices

Consider a $2n \times 2n$ matrix. This can be thought of as consisting of 4 $n \times n$ matrices given as A, B, C, D

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Now, how to multiply two $2n \times 2n$ block matrices? As it turns out, the following result holds

Lemma

Two product of two block matrices is given as,

$$MN = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} AP + BR & AQ + BS \\ CP + DR & CQ + DS \end{bmatrix}$$

Questions

1. Prove that $det(I - AB) = det(I - BA)$ (hint : use block matrices)
2. Can you further extend this to show $det(xI - AB) = det(xI - BA)$?
3. (2014 Quiz) Let A and B be $n \times n$ matrices with determinants α and β respectively. Let O_n and I_n denote the $n \times n$ matrix of all zeros and the $n \times n$ identity matrix respectively. Using the definition of the determinant function given in the class, find the determinant of the $2n \times 2n$ matrix

$$C := \begin{bmatrix} A & B \\ O_n & I_n \end{bmatrix}$$

Wronskian

Note : This will be covered as a part of MA108 and hence a difficult problem given the concept covered till now, that being said, you are free to take a swing at this problem :)

Let $f_1, f_2, \cdots, f_n$ be functions over some interval (a, b). Their Wronskian is another function on (a, b) defined by a determinant involving the given functions and their derivatives upto the order $n - 1$.

$$M = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1^{(1)} & f_2^{(2)} & \cdots & f_n^{(2)} \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix}$$

the wronskian is defined as $det(M)$. Prove that if $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ holds over the interval (a, b) for some constants $c_1, c_2, \cdots, c_n$ and $W(f_1, f_2, \cdots, f_n)(x_0) \neq 0$ at some x_0, then $c_1 = c_2 = \cdots = c_n = 0$. In other words, non-vanishing of $W(f_1, f2, \cdots, f_n)$ at a single point establishes linear independence of $f_1, f_2, \cdots, f_n$ on (a, b).
Caution: The converse is false. $W \neq 0 \not\Leftarrow f_1, f_2, \cdots, f_n$ linearly dependent on (a, b). Though one can prove existence of a sub-interval of (a, b) where linear dependence holds.

Exercise

Consider the equation $x^2 + y^2 - z^2 + 7xy - 3yz + 6xz = 3$. Write it in the form $\mathbf{x}A\mathbf{x}^\mathbf{T}$, where $x = [x\,y\,z]$, and A is a real symmetric matrix. Is such a matrix unique? What if we drop the symmetry requirement?
(Unrelated Note: This representation is called the quadratic form representation, and will be analysed in the latter part of the course)

Exercise

Find two mutually perpendicular vectors on the plane $x + y + z = 0$. Consider the intersection of the plane with the sphere $x^2 + y^2 + z^2 = 1$. What is the shape formed by the points of intersection? Can you *parametrize* all such points using the two vectors above?

Exercise

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Consider $p = \begin{bmatrix} x \\ y \end{bmatrix}$. What can you say about the set $\{p, Ap, A^2p, A^3p \cdots\}$? What is it's cardinality?

Exercise

Let $\mathbf{u}$ be some unit column vector in $\mathbb{R}^3$. Is the matrix $I - uu^T$ invertible?

Exercise

Consider the equation $\hat{a} \times \vec{x} = \hat{b}$, where $\hat{a}$ and $\hat{b}$ are given unit vectors in $\mathbb{R}^3$. Is the equation linear? What are it's solutions?

Exercise

Consider the following augmented matrix:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & 1 & -2 & -1 & 1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 & -3 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \end{array}\right]$$

1. Is the system of equations solvable?
2. Find all vectors in the Null Space of A.
3. Find all vectors in the Column Space of A.
4. Which are the Free variables of A?
5. Find a square submatrix of A with non-zero determinant. What is the largest size you can do this for?