Ans 1 $G = (V, E)$, $n = \#\text{vertices}$, $m = \#\text{edges}$

D = degree matrix

$M = D^{-1}A$, A = adjacency matrix

(M is not symmetric since $m_{ij} = \dfrac{a_{ij}}{\deg(i)}$, $m_{ji} = \dfrac{a_{ij}}{\deg(j)}$)

$\overline{M} = D^{-1/2} A D^{-1/2}$ is symmetric

(1) **Claim**: eigenvalues of M are equal with multiplicities to those of $\overline{M}$.

Proof: Let eigenbasis of $\overline{M} = v_1, v_2, \ldots, v_n$. Then,

let $w_i = D^{-1/2} v_i$

$$\Rightarrow M w_i = D^{-1} A D^{-1/2} v_i = D^{-1/2}(\overline{M} v_i) = D^{-1/2} . \lambda_i v_i = \lambda_i w_i$$

Now, to show that w_i are linearly independent.

$$W = [w_1 \cdots w_n] = D^{-1/2}[v_1 \cdots v_n]$$

$$\det W = (\det D) \det V \neq 0$$

$\blacksquare$

(2) **Claim**: Eigenvalues of $\overline{M}$ lie between -1 and 1.

Proof: $I + \overline{M}$ and $I - \overline{M}$ being p.s.d will finish the proof.

Case 1: $I - \overline{M} = I - D^{-1/2} A D^{-1/2} = D^{-1/2}(D - A) D^{-1/2}$

$$v^T D^{-1/2}(D-A) D^{-1/2} v = w^T (D-A) w$$

where $D - A = \sum_{e \in E} L_e$, $L_e = \begin{pmatrix} 1 & \cdots & -1 \\ \vdots & & \vdots \\ -1 & \cdots & 1 \end{pmatrix}$

$$\Rightarrow w^T (D-A) w = \sum_{e=(u,v)\in E} w^T L_e w = \sum_{(u,v)\in E} (w_u - w_v)^2 \geq 0.$$ Hence, $I - \overline{M}$ is p.s.d.

Similarly, $I + \overline{M}$ is p.s.d. as $v^T(I + \overline{M})v = \sum_{(u,v)\in E} (w_u + w_v)^2 \geq 0$.

Ans 2

(1) **Claim**: v is an eigenvector of M then $D^{1/2} v$ is an eigenvector of $\overline{M}$

Proof: $\overline{M} D^{1/2} v = D^{-1/2} A v = D^{1/2}(Mv) = \lambda D^{1/2} v$

(2) **Claim**: $\forall i \in V(G)$, $\lim_{t\to\infty} \| \overline{M}^t 1_i - x_i \|_2 = 0$ where $x_i(j) = \sqrt{d_i d_j}/2m$

Proof: Let $1_i = \sum_{i=1}^{n} \alpha_i v_i$ where v_i is orthonormal eigenbasis of $\overline{M}$.

$\overline{M}^t 1_i = \sum_i \alpha_i \overline{M}^t v_i = \sum_i \alpha_i \lambda_i^t v_i$. Hence, as $t \to \infty$ only $|\lambda_i| = 1$ survives.

$\mathbb{1}$ is eigenvector of M then, $\dfrac{D^{1/2}\mathbb{1}}{\|D^{1/2}\mathbb{1}\|}$ is eigenvector with $\lambda = 1$ for $\overline{M}$

As given in question, $\lambda = 1$ is a unique eigenvalue ($\max(\lambda_2, |\lambda_n|) \leq 1-\varepsilon$)

Hence, $\lim_{t \to \infty} \overline{M}^t 1_i = \lim_{t \to \infty} \alpha_1 v_1 = \frac{\langle 1_i, D^{1/2} 1 \rangle D^{1/2} 1}{\| D^{1/2} 1 \|_2^2} = \frac{\sqrt{d_i} \cdot \{\sqrt{d_j}\}_{j=1}^n}{\left(\sum_{i=1}^n d_i\right)}$

Hence, $x_i(j) = \sqrt{d_i d_j}/2m$.

$\blacksquare$

<u>Claim</u>: Stationary distribution for random walks on M is $\pi(u) = d_u/2m$

<u>Proof</u>: Interpretation: Stationary dist $\equiv (\pi M = M)$

Let $\pi(u) = \frac{d_u}{2m}$.

$$(\pi^t M)_i = \sum_{j=1}^n \pi_j M_{ji} = \sum_{j=1}^n \frac{\deg(j)}{2m} \cdot \frac{1}{\deg(j)} a_{ji} = \sum_{j=1}^n \frac{a_{ji}}{2m}$$

$$= \frac{\deg(i)}{2m}$$

$\blacksquare$

Interpretation: stationary distribution = converging distribution

Let $p = \sum_{i=1}^n \alpha_i 1_i^T$, where $\sum \alpha_i = 1$.

$$\text{Then, } \tilde{p} = \lim_{t \to \infty} p M^t = \lim_{t \to \infty} p D^{-1/2} \overline{M}^t D^{1/2}$$

$$= \lim_{t \to \infty} \sum_{i=1}^n \alpha_i 1_i^T D^{-1/2} \overline{M}^t D^{+1/2}$$

$$= \lim_{t \to \infty} \left(\sum_{i=1}^n \alpha_i D^{+1/2} \overline{M}^t D^{-1/2} 1_i \right)^T$$

$$= \left(\sum_{i=1}^n \alpha_i D^{+1/2} \left(\lim_{t \to \infty} \overline{M}^t \frac{1}{\sqrt{d_i}} 1_i \right)\right)^T$$

$$= \sum_{i=1}^n \alpha_i D^{+1/2} \frac{1}{\sqrt{d_i}} x_i \quad \text{where} \quad x_i(j) = \frac{\sqrt{d_i d_j}}{2m}$$

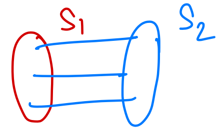
$$\text{Hence, } \tilde{p}(j) = \sum_{i=1}^n \frac{\alpha_i}{\sqrt{d_i}} \left(D^{+1/2} x_i\right)(j) = \sum_{i=1}^n \sqrt{\frac{d_j}{d_i}} x_i(j) \alpha_i$$

$$= \sum_{i=1}^n \frac{d_j}{2m} \alpha_i = \frac{\deg(j)}{2m}$$

$\blacksquare$

Ans3 $G=(V,E)$ is d-regular, $L = I - \frac{A}{d}$, eigenvalues of L are

$0=\lambda_1 \le \lambda_2 \le \dots \le \lambda_n \le 2$



Claim : G is bipartite iff $\lambda_n = 2$

($\Rightarrow$) Let G be bipartite.

Let $S_1=\{1,2,\dots,i\}$, $S_2=[n]\setminus S_1$. (WLOG)

We just need to show existence of -1 eigenvalue for $\frac{A}{d}$.

$$p(i) = \begin{cases} p\dfrac{d_i}{\sum_{i\in S_1} d_i}, & i\in S_1 \\[2ex] -p\dfrac{d_i}{\sum_{i\in S_2} d_i}, & i\in S_2\end{cases}$$

where $\sum_{i\in S_1} d_i = \sum_{i\in S_2} d_i = D$

Then, after one step i.e. $\tilde{p} = pM$ equals

$i\in S_1$

$$\tilde{p}_i = \sum_{j\in S_2 \cap N(i)} \frac{-pd_j}{\sum_{i\in S_2} d_i}\cdot\frac{1}{d_j} = \frac{-pd_i}{\sum_{i\in S_1} d_i} = -\frac{pd_i}{D}$$

Similarly for S_2, we have

$$\tilde{p}_i = \sum_{j\in S_1}\frac{pd_j}{\sum_{i\in S_1} d_i}\cdot\frac{1}{d_j}, \quad i\in S_2$$

$$= +\frac{pd_i}{D} \blacksquare$$

($\Leftarrow$) Now, given $\lambda_n=2$ i.e. $\frac{A}{d}$ has eigenvalue -1.

To show G is bipartite.

Let $v^T\frac{A}{d} = -v^T$, $\left(v^T\frac{A}{d}\right)_i = \sum_{j=1}^n v_j\frac{a_{ji}}{d} = -v_i$

$\Rightarrow \mathbb{E}_{j\in N(i)}[v_j] = -v_i$, Let $S_1=\{i\in[n] \mid v_i\ge 0\}$

$S_2=\{i\in[n]\mid v_i<0\}$

claim : if $\arg\max_i |v_i| = i^*$, $|v_{i^*}|=k$, then $|v_i|=k\ \forall i$

Let $\arg\max_i |v_i| = i^*$. Then, $\mathbb{E}_{j\in N(i^*)}[v_j] = -v_{i^*} = -k$.

Hence $\forall j\in N(i^*)$, $v_j=-v_{i^*}$ otherwise atleast one i has $|v_i|>k$.

Hence, $\forall j\in N(i^*)$, if $i^*\in S_1 \Leftrightarrow j\in S_2$ and $i\in S_2 \Leftrightarrow j\in S_1$.

Repeating this arguement, the absolute value of $v_i = k$ for all i in G since all nodes are in same connected component as i^*, and we get that

if $i\in S_1 \Rightarrow \forall j\in N(i), j\in S_2$, $v_j=-v_i$ (if not, $\left|\mathbb{E}_{j\in N(i)} v_j\right| < \mathbb{E}|v_j| = k$)

Hence, S_1, S_2 give partitions of the bipartite graph.

Ans 4

$G = (V, E)$ is a d-regular bipartite graph.

$L = I - \frac{A}{d}$, with eigs $0 = \lambda_1 \le \lambda_2 \le \dots \le \lambda_n = 2$

Claim: $\lambda_i + \lambda_{n-i+1} = 2$

Proof: It is enough to show that if v is an eigenvector with eigenvalue λ, $v' = f(v)$ with eigenvalue $-\lambda$ where f is a bijection.

$$f(v)_i = \begin{cases} v_i & , i \in S_1 \\ -v_i & , i \in S_2 \end{cases}$$ (Clearly f is a bijection).

Now, let $\frac{Av}{d} = \lambda v$

$$\left(\frac{A}{d} f(v)\right)_i = \sum_{j=1}^{n} \frac{a_{ij}}{d} \cdot v_j' = \sum_{j \in N(i)} \frac{v_j'}{d} = -\sum_{j \in N(i)} \frac{v_j}{d}$$

$$\left\{ \text{But } \left(\frac{A}{d} v\right)_i = \sum_{j \in N(i)} \frac{v_j}{d} = \lambda v_i \right\}$$

$$= -\lambda v_i$$ ∎

Ans 5

$G = (V, E)$, $|C_1| = |C_2| = n/2$, $|V| = n$, $(C_1 \cap C_2 = \phi)$

$G[C_1], G[C_2]$ have conductance $\ge \phi$

$|E(C_1, C_2)| \le \varepsilon n d$, $\varepsilon << \phi^2$

$0 = \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \le \lambda_n = 2$.

Claim: If $H \subseteq G$ then $L_H \preceq L_G$

Proof: Showing $L_G - L_H$ is p.s.d is enough.

$$x^T (L_G - L_H) x = \sum_{e \in E(G) \setminus E(H)} x^T L_e x = \sum_{(u,v) \in E(G) \setminus E(H)} (x_u - x_v)^2 \ge 0$$ ∎

Now, let's remove $E(C_1, C_2)$ and replace the cut-edges with self-loops.

Now, in the resultant graph's Laplacian, $\lambda_1 = \lambda_2 = 0$ with eigenspace span $\langle v_1, v_2 \rangle$ where, $(v_1)_i = \begin{cases} 1, & i \in C_1 \\ 0, & i \in C_2 \end{cases}$, $(v_2)_i = \begin{cases} 0, & i \in C_1 \\ 1, & i \in C_2 \end{cases}$

Then, v_3 corresponding to λ_3 is $\perp^{er}$ to v_1, v_2. Hence,

$$\sum_{i \in C_1} (v_3)_i = \sum_{j \in C_2} (v_3)_j = 0$$

Consider vectors $u \in \mathbb{R}^{|C_1|}$, $v \in \mathbb{R}^{|C_2|}$, s.t. $\sum u_i = \sum v_i = 0$.

Let L' denote laplacian of new graph with $E(C_1, C_2)$ removed.

Then, $\lambda_3 = \min \dfrac{[u^T \; v^T] \, L' \begin{bmatrix} u \\ v \end{bmatrix}}{u^T u + v^T v}$, where columns of L' are ordered appropriately

$$\lambda_3 = \min \frac{u^T L_{C_1} u + v^T L_{C_2} v}{u^T u + v^T v}, \quad \sum u_i = \sum v_i = 0.$$

Now, by cheeger's inequality on C_1, C_2 individually,

$$\lambda_2^{C_1} = \min \frac{u^T L_{C_1} u}{u^T u}, \quad \phi \leq \phi_{C_1} \leq \sqrt{2 \lambda_2^{C_1}}$$

$$\Rightarrow \lambda_2^{C_1} \geq \frac{\phi^2}{2}.$$

Similarly $\lambda_2^{C_2} \geq \dfrac{\phi^2}{2}$.

$$\lambda_3 = \min \frac{u^T L_{C_1} u + v^T L_{C_2} v}{u^T u + v^T v} \quad \text{where} \quad \begin{aligned} u^T L_{C_1} u &\geq \frac{\phi^2}{2} u^T u \\ v^T L_{C_1} v &\geq \frac{\phi^2}{2} v^T v \end{aligned}$$

$$\Rightarrow \lambda_3 \geq \phi^2/2$$

Now, since λ_3 calculated here is for H,

$\lambda_3 \leq (\lambda_3)^G$ from an earlier claim

$$\Rightarrow (\lambda_3)^G \geq \frac{\phi^2}{2}. \qquad \blacksquare$$