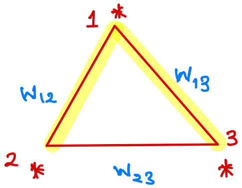
Assignment - 2

Ans 1

Consider the graph H given as



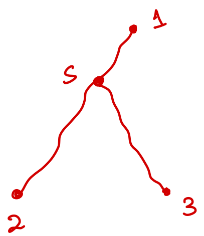
Without loss of generality, consider

$$w_{12} \le w_{13} \le w_{23}$$

Then weight of Steiner tree obtained,

$$ALG \le w_{12} + w_{13} .$$

Now, consider the optimal Steiner tree and let 3 vertex be connected to path joining 1, 2 at s.



Let path length from s to $1 = x$

s to $2 = y$

s to $3 = z$

Then $OPT = x + y + z$ — $(*)$

But minimum path length arguement shows,

$$x + y \ge w_{12} \quad \text{—(1)}$$

$$y + z \ge w_{23} \quad \text{—(2)}$$

$$x + z \ge w_{13} \quad \text{—(3)}$$

(1) + (2) + (3)

$$\Rightarrow (x+y+z)\, 2 \ge w_{12} + w_{23} + w_{13} \quad \text{—}(**)$$

Now $ALG \le w_{12} + w_{13}$

and $w_{12} \le w_{13} \le w_{23}$

$$\Rightarrow ALG \le w_{12} + w_{23}$$

$$ALG \le w_{13} + w_{23}$$

$$\Rightarrow 3\,ALG \le 2(w_{12} + w_{13} + w_{23})$$

$$\Rightarrow ALG \le \frac{2}{3}(w_{13} + w_{12} + w_{23})$$

Using $(*)$ and $(**)$, we get

$$ALG \le \frac{2}{3}(w_{12} + w_{13} + w_{23}) \le \frac{4}{3}(x+y+z) \le \frac{4}{3} OPT$$

$$\Rightarrow OPT \le ALG \le 4/3\, OPT .$$

■

Ans 2 Def^n: $U \subseteq V$ is a separating cut if for some i, $s_i \in U, t_i \notin U$

LP relaxation

$$\min \Sigma_e w_e x_e$$
$$x_e \geq 0$$
$$\sum_{\underbrace{e \in \delta(U)}_{\text{set of cut-edges}}} x_e \geq 1, \quad U \text{ is sep. cut}$$

(similar to spanning tree limiting $\cup s$)

Dual LP

$$\max \Sigma_U y_U$$
$$y_U \geq 0$$
$$\sum_{\substack{U: \\ e \in \delta(U) \\ \text{and } U \text{ is sep. cut}}} y_U \leq w_e$$

Primal dual Algo.

① $F \leftarrow \emptyset$

$y \leftarrow 0$

do {

- Let $C_1, C_2, \ldots, C_n$ be the connected components in the tight edge subgraph, which separate some terminal pair, then increase $y_{C_1}, y_{C_2}, \ldots, y_{C_n}$ simultaneously
- Include tight edges one-by-one not creating cycles in F

} till every (s_i, t_i) pair is connected

Pruning: Remove any edge from F that can be removed

Now, analysis of this algorithm for 2 pairs.

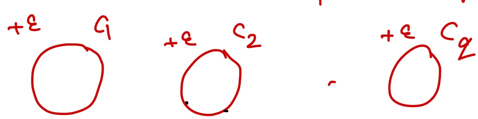
Proof of $\frac{3}{2}$-approx

Claim: $w(F) \leq \frac{3}{2} \Sigma_U y_U$

Proof: We show, in each step, $\frac{\Delta w(F)}{\Delta D} \leq \frac{3}{2}$.

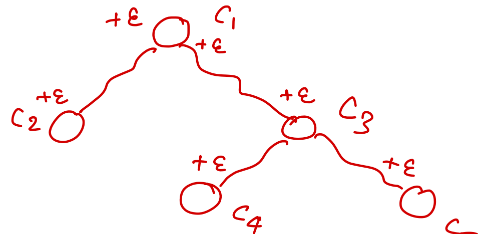
ΔD = increment in dual objective
$= q \cdot \varepsilon$

Let number of connected components be q



$$w(F) = \sum_{e \in F} \overbrace{w_e}^{\text{tight.}}$$
$$= \sum_{e \in F} \sum_{\substack{U: \\ e \in \delta(U)}} y_U$$

Now, for the final forest, these components each have exactly 1 path b/w any pair.



Final tree chosen by algorithm. with connected comp. contracted.

Hence, $\Delta w(F)$ = increase in weight of edges included in the final tree
$= 2(q-1)\varepsilon$

But, since we have only two pairs (s_1, t_1), (s_2, t_2) hence, number of connected components in tight edge subgraph ≤ 4.

$$\Rightarrow \frac{\Delta w(F)}{\Delta D} = \frac{(q-1)2\varepsilon}{q\varepsilon} \leq \frac{3}{4} \cdot \frac{2\varepsilon}{\varepsilon} = \frac{3}{2}$$

Ans 3

Linear Program can be written as,

$$\max \sum_{i=1}^{n} (1-y_i) P_i + \sum_{(i,j)\in E} z_{i,j} Q_{i,j}$$

$$z_{i,j} \leq y_i + y_j \quad \forall (i,j) \in E$$

$$0 \leq z_{i,j} \leq 1$$

$$0 \leq y_i \leq 1 \quad \forall i$$

Let the optimal solution be given as (y^*, z^*)

Rounding Scheme

Pick vertex i w.p. $(1-\lambda) + \lambda y_i^*$

Now, since algorithm is randomized, we consider expectation value of objective

$$\mathbb{E}\Big[\sum_{i\notin S} P_i + \sum_{\substack{(i,j)\in E \\ i\in S \text{ or } j\in S}} Q_{i,j}\Big] = \sum_i \Pr(i\notin S)\, P_i + \sum_{(i,j)\in E} \Pr(i\in S \text{ or } j\in S)\, Q_{i,j}$$

where $\Pr(i\notin S) = 1-(1-\lambda) - \lambda y_i^* = \lambda(1-y_i^*)$

$\Pr(i\in S \text{ or } j\in S) = 1 - \Pr(i\notin S)\Pr(j\notin S)$. (Since $i\in S$ are independent events)

$= 1 - \lambda^2(1-y_i^*)(1-y_j^*)$

$$= \sum_i \lambda(1-y_i^*)\, P_i + \sum_{(i,j)\in E} \left(1-\lambda^2(1-y_i^*)(1-y_j^*)\right) Q_{i,j}$$

Claim: $1-\lambda^2(1-y_i^*)(1-y_j^*) \geq \left(1-\frac{\lambda^2}{4}\right) z_{i,j}^*$

Case 1: $y_i^* + y_j^* \geq 1$

$$\Rightarrow \frac{1}{2} \geq \frac{1-y_i^* + 1-y_j^*}{2} \geq \sqrt{(1-y_i^*)(1-y_j^*)}$$

$$\Rightarrow (1-y_i^*)(1-y_j^*) \leq 1/4 \Rightarrow 1-\lambda^2(1-y_i^*)(1-y_j^*) \geq 1-\frac{\lambda^2}{4} \geq \left(1-\frac{\lambda^2}{4}\right) z_{ij}$$

Case 2: $y_i^* + y_j^* < 1$

Then, $z_{i,j}^* = y_i^* + y_j^*$ since increasing $z_{i,j}$ increases objective function.

Now, $1 - \lambda^2(1-y_i^*)(1-y_j^*) - \left(1 - \frac{\lambda^2}{4}\right)(y_i^* + y_j^*)$

$$\geq 1 - \lambda^2\left(\sqrt{(1-y_i^*)(1-y_j^*)}\right)^2 - \left(1 - \frac{\lambda^2}{4}\right)(y_i^* + y_j^*)$$

$$\geq 1 - \lambda^2\left(\frac{1-y_i^* + 1 - y_j^*}{2}\right)^2 - \left(1 - \frac{\lambda^2}{4}\right)(y_i^* + y_j^*)$$

$$= 1 - \lambda^2\left(1 - \frac{y_i^* + y_j^*}{2}\right)^2 - \left(1 - \frac{\lambda^2}{4}\right)(y_i^* + y_j^*)$$

Putting $y_i^* + y_j^* = x$

$$= 1 - \lambda^2\left(1 - \frac{x}{2}\right)^2 - \left(1 - \frac{\lambda^2}{4}\right)x$$

$$= -\lambda^2\left(\frac{x^2}{4} - x + 1\right) - x\left(1 - \frac{\lambda^2}{4}\right) + 1$$

$$= \frac{-\lambda^2 x^2}{4} + x\left(\frac{5}{4}\lambda^2 - 1\right) + (1 - \lambda^2) = f(x)$$

Here, $f(0) = 1 - \lambda^2 \geq 0$

$$f(1) = -\frac{\lambda^2}{4} + \frac{5}{4}\lambda^2 - 1 + 1 - \lambda^2 = 0$$

Hence, $f(x) \geq 0 \;\; \forall x \in [0,1]$

Thus, $1 - \lambda^2(1-y_i^*)(1-y_j^*) - \left(1 - \frac{\lambda^2}{4}\right)(y_i^* + y_j^*) \geq 0$. $\blacksquare$

Hence, $\sum_i \lambda(1-y_i^*) P_i + \sum_{(i,j)\in E} \left(1 - \lambda^2(1-y_i^*)(1-y_j^*)\right) Q_{i,j}$

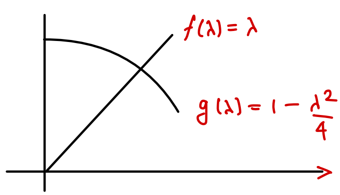
$$\geq \min\left\{\lambda, 1 - \frac{\lambda^2}{4}\right\}\left(\sum_i (1-y_i^*) P_i + \sum_{(i,j)\in E} z_{ij}^* Q_{i,j}\right) = \min\left\{\lambda, 1-\frac{\lambda^2}{4}\right\} \text{LP-OPT}$$

$$\geq \min\left\{\lambda, 1-\frac{\lambda^2}{4}\right\} \text{OPT} \quad \blacksquare$$

Now, $\alpha = \min\left\{\lambda, 1 - \frac{\lambda^2}{4}\right\}$

λ is an increasing f^n, $1 - \frac{\lambda^2}{4}$ is decreasing

Hence best approx. factor is λ s.t.



$$\lambda = 1 - \frac{\lambda^2}{4}$$

$$\Rightarrow 4\lambda = 4 - \lambda^2$$

$$\Rightarrow \lambda^2 + 4\lambda - 4 = 0$$

$$\Rightarrow \lambda = \frac{-4 + \sqrt{32}}{2} = -2 + 2\sqrt{2} = 2(\sqrt{2} - 1)$$

i.e. $\boxed{\alpha = 2(\sqrt{2}-1)}$ is best.

Ans 4 Quadratic program is given as

$$\max \sum_{i,j} d_{i,j}\left(\frac{1-z_i z_j}{2}\right) + \sum_{i,j} s_{i,j}\left(\frac{1+z_i z_j}{2}\right)$$

$$z_i \in \{-1,+1\} \quad \forall\ 1 \le i \le n$$

then SDP relaxation is given as

$$\max \sum_{i,j} d_{i,j}\left(\frac{1-\langle v_i, v_j\rangle}{2}\right) + \sum_{i,j} s_{i,j}\left(\frac{1+\langle v_i, v_j\rangle}{2}\right)$$

$$\langle v_i, v_i\rangle = 1, \quad v_i \in \mathbb{R}^n \quad \forall\ 1 \le i \le n$$

Now, randomized hyperplane rounding gives

$$\sum_{i,j} d_{i,j}\ \Pr(i,j \text{ lie on opp sides}) + \sum_{i,j} s_{i,j}\ \Pr(i,j \text{ lie on same side})$$

$$= \sum_{i,j} d_{i,j}\ \frac{\theta_{i,j}}{\pi} + \sum_{i,j} s_{i,j}\left(1-\frac{\theta_{i,j}}{\pi}\right)$$

$$\min_{\theta} \frac{2\theta}{(1-\cos\theta)\pi} = \alpha$$ gives $\alpha = 0.878$ (from proof of MAX-CUT)

Putting $\theta = \pi - \theta'$

$$\min_{\theta'} \frac{2\left(1-\frac{\theta}{\pi}\right)}{1+\cos\theta} = \alpha$$

$$\ge \sum_{i,j} d_{i,j}\ \alpha\cdot\left(\frac{1-\cos\theta}{2}\right) + \sum_{i,j} s_{i,j}\cdot\alpha\left(\frac{1+\cos\theta}{2}\right)$$

$$= \alpha\left(\sum_{i,j} d_{i,j}\left(\frac{1-\langle v_i,v_j\rangle}{2}\right) + \sum_{i,j} s_{i,j}\left(\frac{1+\langle v_i,v_j\rangle}{2}\right)\right.$$

Hence, $ALG \ge \alpha\ \text{OPT-SDP} \ge \alpha\cdot\text{OPT-QP} \ge \alpha\cdot\text{OPT}$.

i.e. $ALG \ge \alpha\cdot OPT$ ∎