Null Space and Nullity of a Matrix.

Let A be a $m \times n$ matrix

$$\text{Null Space}(A) = \{ v \in \mathbb{R}^n \,/\, Av = 0 \}.$$

It is evident that Null Sp(A) is a Vector Subsp. of $\mathbb{R}^n$

Def: $\text{Nullity}(A) = \text{Dim}(\text{Null Space}(A))$

Thm (**Rank-Nullity Thm**): For a $m \times n$ matrix A

$$\text{Rank } A + \text{Nullity } A = n = \# \text{ of Cols of } A.$$

Proof will be taken up later (if time permits).

Structure of Space of Solutions of $Ax = b$.

Let us Consider a System $Ax = b$.

Assume that the System has a sol. x_0

(Which means $\text{Rank } A = \text{Rank } [A : b]$).

Thm: The Set of all Solutions of $Ax=b$ (assuming a sol. x_0 exists)

is given by $\{ x_0 + n \,/\, n \in \text{Nullsp.}(A) \}$ (*)

proof: S = Set of all Solutions of $Ax = b$

and T = Set displayed in (*)

$A(x_0+n) = Ax_0 + An = Ax_0 = b$

$\therefore\ x_0 + n \in T \Rightarrow (x_0+n) \in S \quad \therefore\ T \subseteq S.$

Now Suppose $x \in S$ then $Ax = b$

but $Ax_0 = b$

$\therefore\ A(x - x_0) = 0$

$\therefore\ x - x_0 \in \text{Nullsp.} A$ Say $x - x_0 = n$

$\therefore\ x = x_0 + n \in T$

$\therefore\ S \subseteq T \qquad \therefore\ S = T.$

We see that for $Ax=b$, if Nullity $A > 0$ the Sys. has infinitely many Sol. or No. Sol.

If Nullity $A = 0$: System has a unique sol or no solutions.

Simple illustration: $A = [2, -1]$

Eq^n: $2x - y = 4$

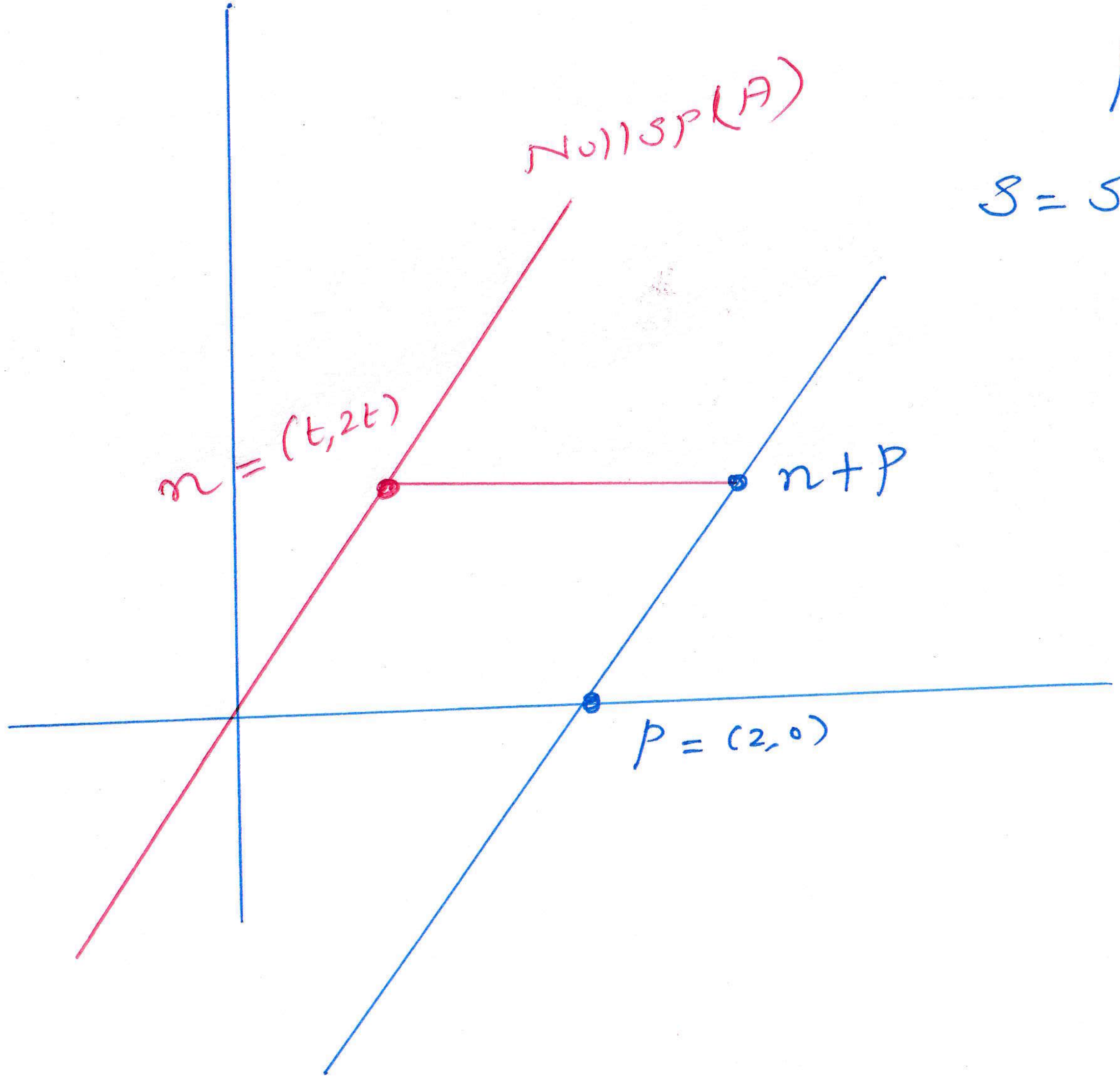
Null Space $N(A) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \Big/ \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \right\}$

= Line through origin with slope 2

$p = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ a special sol.

S = Set of all solutions

$= \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 2t \end{bmatrix} \Big/ t \in \mathbb{R} \right\}$



Example: Given 3 planes in $\mathbb{R}^3$.

Possible geometric Configurations:

3 planes parallel

2 parallel third meets trans.

3 planes meet at a unique pt (Concurrence)

3 planes pass through a line

4. 3 planes form a prism

Any others?

Write the eqn 3 planes in matrix form

$$Ax = b$$

Rank of $[A : b] = 3$

Rank $A = 2$

Can you decide from this the possible geometric Configuration of 3 planes?

Determinantal Rank

We shall use only the basic ideas of determinants such as switching two rows / cols simply changes sign. In short effect of Elem. Row op. on determinants.

Also $\det A = \det A^T$ (this is non trivial but you could try proving it now with whatever you have learnt so far)

<u>Lemma:</u> If A is an $n \times n$ matrix and a $(k \times k)$ submatrix has non zero det then Rank $\geq k$.

Conversely if Rank $A \geq k$ then some $(k \times k)$ submatrix has non zero determinant.

<u>proof:</u> First part follows from a simple observation: Suppose $c_1, \ldots, c_k$ are linearly <u>indep</u> columns and we extend these columns to $\begin{bmatrix} c_1 \\ D_1 \end{bmatrix}, \ldots \begin{bmatrix} c_k \\ D_k \end{bmatrix}$

then $\begin{bmatrix} c_1 \\ D_1 \end{bmatrix}, \ldots, \begin{bmatrix} c_k \\ D_k \end{bmatrix}$ are <u>also</u> <u>lin. indep</u>.

(Convince yourself)

Now assume A_0 is a $k \times k$ Submatrix of A
det $A_0 \neq 0$.

Then Columns of A_0 lin. Indep. (we have proved this)

if $c_{i_1}, \ldots, c_{i_k}$ are the Columns of A_0 then these are parts of taller Columns $\tilde{c}_{i_1}, \ldots, \tilde{c}_{i_k}$ from A

By the simple observation made above, $\tilde{c}_{i_1}, \ldots, \tilde{c}_{i_k}$ lin. indep

$\therefore$ Col Rank $A \geq k$.

Converse: Suppose Rank $A \geq k$

Select k lin indep Columns of A and form $m \times k$ Sub matrix B.

So Rank $A \geq k =$ Rank B
$=$ Row Rank B

So k rows of B must be Lin. Indep. Take these k-rows and create a $(k \times k)$ Submatrix C

det $C \neq 0$ and C is the desired $(k \times k)$ Submatrix

Thm: (Determinantal Rank)

Let A be a $m \times n$ matrix

We say A has determinental rank k if

(i) $\exists$ some $(k \times k)$ submatrix of A with non zero det.

(ii) All $(k+1) \times (k+1)$ matrices have zero determinant

Row Rank A = Col. Rank A = det. Rank A.

Proof: Assume Col Rank A = Rank A = k

We have seen that $\exists$ a $(k \times k)$ submatrix with non zero determinant.

Now if $\exists$ a $(k+1) \times (k+1)$ submatrix with non zero det. then the lemma says Rank $A \geq k+1$ Contradiction

So all $(k+1) \times (k+1)$ submatrices have zero det.

In books written in the ~~early~~ first part of 20th cent. the determinantal characterization of Rank was more ~~[illegible]~~ common.

Consider $A = \begin{bmatrix} 1 & -1 & 2 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

without computing matrix products can you find out Rank (AA^T) and Rank (A^TA) ?

<u>Sol</u>: Rows of A obviously lin. indep.

Rank $A = 2$, Nullity $A = 2$

$\therefore$ $\exists$ two lin. indep vectors v_1, v_2 s.t

$Av_1 = 0, \quad Av_2 = 0$

$\therefore$ $A^TAv_1 = 0, \quad A^TAv_2 = 0$ $\therefore$ Rank $(A^TA) \le 2$

$\therefore$ Nullity $A^TA \ge 2$

Is it evident that Rank $A^TA = 2$?

Can you simply think of the (2×2) subdet in

$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ ? & \end{bmatrix} \begin{bmatrix} 1 & -1 & ? \\ 1 & 1 & \end{bmatrix} = ?$ $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Complete the problem.

Observations:

(1) Performing a row operation on A does not change the null space of A.

That is if $\tilde{A}$ is obtained from A through Elem. Row Op.

$\text{Null sp } A = \text{Null sp } \tilde{A} \Rightarrow$ ~~me~~ $\text{Nullity } A = \text{Nullity } \tilde{A}$

(2) Performing Elem. Row op. on A does not change the Row rank and hence does not change the Column rank either (!)

$\therefore \ \text{Rank } A = \text{Rank } \tilde{A} = k$ say

Now take $\tilde{A} = REF(A)$

$\text{Rank } \tilde{A} = \#$ of pivots $= k$

(3) Let $x_{i_1}, \ldots, x_{i_\ell}$ be the free variables. Thus $k + \ell = n$

Assign $x_{i_1} = 1, \ x_{i_2} = \cdots = x_{i_\ell} = 0$ and solve

$Ax = 0 \longrightarrow$ Solution is n_1

Assign $x_{i_1} = 0, \ x_{i_2} = 1, \ x_{i_3} = \cdots = x_{i_\ell} = 0$. Solve $Ax = 0$ Sol. $= n_2$

Proceeding thus we get l vectors $n_1, \ldots, n_l$ in Null Sp of A

It is clear these are lin. Indep (how?)

Next, $v \in$ Nullsp A $\therefore$ $Av = 0$ $\therefore$ $\tilde{A}v = 0$

From this you need to show v is a lin. comb. of $n_1, \ldots, n_l$

This needs a little thought but you will figure it out. (Optional Exercise)

Hint: The system $\tilde{A}v = 0$ is equivalent to

$$C \begin{bmatrix} x_{j_1} \\ \vdots \\ x_{j_k} \end{bmatrix} = x_{i_1} w_1 + \cdots + x_{i_l} w_l \quad (*)$$

$x_{j_1}, \ldots, x_{j_k}$ pivotal variables
$x_{i_1}, \ldots, x_{i_l}$ free variables.

For each choice of $x_{i_1}, \ldots, x_{i_l}$, $(*)$ has a unique sol. C is $(k \times k)$ invertible

You have already solved $(*)$ in special cases

e.g $x_{i_1} = 1$, $x_{i_2} = \cdots = x_{i_l} = 0$ etc;

$\therefore$ Nullity $A = l$ $\quad$ $n = k + l$ translates to Rank A + Nullity $A = n$