Expanding Square Complexes from Graph Products

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Abstract

Over the last two decades, several definitions of high-dimensional expanders have been proposed and studied, and the constructions that have come from this effort have led to many important results in computer science and mathematics. One recent example of this is the construction of c^3 locally testable codes by *Dinur et al*, which use an expanding left-right Cayley complex as a key component of their construction. In this paper we study square complexes, a generalization of these left-right Cayley complexes. We introduce the *corner product* of two graphs, and show that if the two graphs have good spectral expansion then the resulting square complex exhibits good 1-skeleteon expansion and constant parallel expansion. This product can be instantiated with graphs of any size.

1 Introduction

Expander graphs are mathematical objects that have found many applications in several areas of mathematics and computer science (see [HLW06] for a comprehensive reference many of these). Expansion can be defined in several ways that are related, but not equivalent. In this work we will focus on the usual spectral definition: given a graph G, its spectral expansion is given by $\max\{|\lambda_2(A_G)|, |\lambda_n(A_G)|\}$, where $\lambda_i(A_G)$ is the ith largest eigenvalue of the adjacency matrix of G. A good spectral expander is a graph that has small spectral expansion and is simultaneously sparse. To enforce sparseness, it is common to work with d-regular graphs, where d is some constant, and we will do the same simplifying assumption here.

New questions have motivated generalizations of these notions of expansion to higher-dimensional objects (see [Lub18] for a short survey on some of these). One example of such higher-dimensional objects are simplicial complexes: hypergraphs whose hyperedges satisfy downward-closure (i.e. all subsets of hyperedges are also hyperedges.) Just like in the case of graphs, there are many related definitions of high-dimensional expansion in simplicial complexes. We are mainly interested on two notions of high-dimensional expansion that are natural generalizations of the spectral one we described above: a global one, where expansion is defined in terms of the spectral expansion of the underlying graph; a local one, which requires that the underlying graph of the "neighborhood" of every non-maximal hyperedge is a spectral expander (these "neighborhoods" are formally known as links).

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More recently, a new type of high-dimensional object with its own definition of expansion has garnered interest. The breakthrough works of Dinur et al. [DEL+22] and Panteleev and Kalachev [PK22] solved two major open problems in the field of error-correcting codes: constructing locally testable codes with constant rate, distance and locality, and asymptotically optimal quantum LDPC codes (see also [LZ22] for a unified and simplified view of both results.) At the core of both constructions is an object known as a left-right Cayley complex, which in particular is a special case of a square complex. A follow up work of Dinur-Lin-Vidick [DLV24] describes a construction of a family of "almost-good" quantum locally testable codes, making use of cubical complexes, which are higher dimensional generalizations of square complexes.

A square complex is a generalization of a graph with cells of up to 2 dimensions: 0-dimensional cells are vertices, 1-dimensional cells are edges, and 2-dimensional cells are 4-cycles or *squares*, hence the name (we will formalize this in the next section.) Cubical complexes are generalizations of these to even higher dimensions, where an i-dimensional cell corresponds to a i-dimensional hypercube. In order to define the notion of expansion we will study, we first need to define the concept of parallel adjacency of edges. Two edges are parallel if they don't share any endpoints and there is a square that contains both of them. Now, the parallel adjacent edges to a given edge are given by the set of edges parallel to the given one. Naturally, this parallel adjacency defines a new graph, where vertices are the edges of the complex and the edges are given by parallel adjacency. We can now define two types of expansion of a square complex: the global expansion, which is given by the spectral expansion of the underlying graph (ignoring the squares) and the parallel expansion, which is given by the spectral expansion of the graph defined by the parallel adjacencies. We remark that this particular definition of expansion is the one used in [DEL+22], but we will mention other related ones also used in the literature.

1.1 Our contribution

Our main contribution is an explicit construction of square complexes with non trivial global expansion and constant parallel expansion. Specifically, we prove:

Theorem 1.1 (Informal, see Theorem 3.1 for full result)**.** *For any $d > 0$, there exists an explicit infinite family of square complexes with vertex regularity ($D^{|}$) and edge regularity ($D^{\square}$) both $\Theta(d^3)$, global expansion $\Theta((D^{|})^{3/4})$ and a constant parallel expansion of $\frac{1}{2}$.*

We obtain this by defining a product between two graphs called the *corner product*, and showing that if the two graphs are good expanders then the square complex resulting from their product has both good global and parallel expansion. With the appropriate choice of parameters we obtain the result above. See Theorem 3.1 for the full result.

In the next section we contextualize this result and also further motivate it's importance.

2 Background and Preliminaries

2.1 Spectral Expander Graphs

Let $G = (V, E)$ be a d-regular undirected graph with n vertices. We use the notation A_G to denote its *adjacency matrix* and we write its (real) eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. It is easy to see that $\lambda_1 = d$ and $\lambda_i < d$ for all $i \neq 1$ if G is connected. We say that G is a λ-*spectral expander* if

$\lambda(G) := \max\{|\lambda_2|, |\lambda_n|\} \leqslant \lambda$. This notion of expansion is known as *two-sided expansion*, since it controls both the second largest and smallest eigenvalues, but it is also common to work with a weaker notion of *one-sided expansion*, defined solely in terms of λ_2.

We will often work with a normalized version of the above, so we will write $\widetilde{A_G} = \frac{1}{d}A_G$ to denote the *normalized adjacency matrix* of G, whose eigenvalues are the same as the ones of A_G but scaled by $\frac{1}{d}$. This normalized adjacency matrix is exactly the Markov operator for the random walk on G. Thus, for G a $\lambda-$spectral expander, λ characterizes the rate of convergence of a random walk in G to the uniform distribution (its stationary distribution, assuming G is connected and not bipartite). We will occasionally use the language of random walk operators when referring to the normalized adjacency matrix of graphs, whenever it is convenient to think about $\widetilde{A_G}$ as a Markov operator.

A "good expander" is a graph for which $\lambda(G)$ is bounded away from d. The well-known Alon-Boppana [Alo86, Nil91, Fri93] bound asserts that $2\sqrt{d-1}$ is essentially a lower bound to expansion, and so an optimal spectral expander, also known as a *Ramanujan graph*, is a graph G for which $\lambda(G) \leqslant 2\sqrt{d-1}$. Perhaps rather surprisingly, Friedman's Theorem [Fri08, Bor20] shows that (almost) Ramanujan graphs are very common: for any $d \leqslant 3$, if G is a uniformly random d-regular graph, then $\lambda(G) \leqslant 2\sqrt{d-1} + o(1)$ with high probability. Many explicit constructions are also known (see [MOP22] for a comprehensive list of recent constructions). Some of these are algebraic, such as the results of Lubotzky-Phillips-Sarnak [LPS88], Margulis [Mar88] and Morgenstern [Mor94], who showed that when $d-1$ is a prime power there exist explicit constructions of d-regular Ramanujan graphs. Some are combinatorial, such as the derandomization of [Bor20] by Mohanty-O'Donnell-Paredes [MOP22] and Alon [Alo21].

As mentioned previously, there are many applications of these spectral expanders, but we will highlight one to the theory of error-correcting codes, which is a relevant motivation to our main result. Recall that a *binary linear code* $\mathcal{C}$ of block-length n is a subspace of $\mathbb{F}_2^n$. The *rate* of the code is given by k/n, where k is the dimension of the linear subspace, and the distance is given by the $\min_{c\in\mathcal{C}} |c|$, where $|c|$ is the Hamming weight of c. See [GRS12] for a comprehensive exposition on the theory of error-correcting codes. Given a n_0-regular graph $G = (V, E)$ and a linear code $\mathcal{C}_0$ of block length n_0, the Tanner code $T(G, \mathcal{C})$ is a linear code on $\mathbb{F}_2^E$, where each vertex enforces the condition that the bits corresponding to the edges in its neighborhood belong to $\mathcal{C}_0$. A well-known result of Sipser and Spielman [SS96] building on work of Tanner [Tan81] showed that if G is a good expander and $\mathcal{C}_0$ has sufficiently large distance, then $T(G, \mathcal{C}_0)$ has asymptotically good rate and distance.

2.2 High-dimensional expanders

A *simplicial complex* X on a vertex set V is a collection of subsets of V, known as faces, satisfying downward-closure, i.e. if $\tau \in X$ and $\sigma \subseteq \tau$ then $\sigma \in X$. We denote by $X(k)$ the set of *k-dimensional faces*, meaning the set of faces with cardinality $k+1$. In particular, this implies that $X(0)$ is the vertex set V. The *dimension* of a simplicial complex X is the maximum dimension of a face.

We are interested in constructions of infinite families of simplicial complexes that are both sparse and expanding. We enforce sparseness by requiring that all simplicial complexes have constant bounded degree, meaning that there is some constant $\overline{d}$ such that each vertex in the complex is in at most $\overline{d}$. This is a relaxation of the regularity condition imposed in the definition of good graph

expanders.

The first definitions of expansion in simplicial complex were topological, and based on the seminal work of Lubotzky-Samuels-Vishne [LSV05] that defined and constructed *Ramanujan complexes*. In this work we will focus on a spectral notion of expansion, but before we can define it we need to introduce some more definitions.

The 1-*skeleton* of a simplicial complex X is the graph defined by $(X(0), X(1))$, which means it corresponds to the underlying graph of X. Given a d-dimensional simplicial complex X, the *link* of a k-face τ for $k \leqslant d-1$ is a $(d-k-1)$-dimensional simplicial complex defined as $X_\tau := \{\sigma \setminus \tau : \sigma \in X, \tau \subseteq \sigma\}$. Links encapsulate the "local information" of a face. For example, if X is a 1-dimensional (i.e. a graph) then the link of a vertex is its neighborhood.

Definition 2.1. The *global expansion* of a simplicial complex X is given by $\widetilde{\lambda_2}(1\text{-skeleton}(X))$. The *$k$-local expansion* of a dimension d simplicial complex X is given by $\min_{\tau \in X(k)} \widetilde{\lambda_2}(1\text{-skeleton}(X_\tau))$. The *local expansion* of X is given by the minimum k-local expansion for $0 \leqslant k \leqslant d-1$.

Note that in simplicial complexes spectral expansion is usually defined in terms of normalized one-sided expansion, as is evident in the above definition. This exact definition was introduced in [DK17] motivated by the study of PCPs and agreement tests, where they also showed that picking appropriate parameters of Ramanujan complexes of [LSV05], one can construct constant bounded degree simplicial complexes with global and local expansion bounded by λ, for any $\lambda > 0$[1]. Later, Kaufman and Oppenheim [KO23] introduced a new simpler construction that also obtains constant bounded degree and arbitrary constant expansions, based on what they called coset complexes.

The previous paragraph shows that much like graph expanders, the construction of expanding simplicial complexes has a rich literature. Both of the mentioned constructions are heavily algebraic and rely on quite involved mathematical tools. Contrary to the case of graph expanders, more elementary constructions relying mostly on combinatorics or randomization seem to be harder to find. Notably, the analog of Friedman's theorem isn't true, that is, random simplicial complexes are not good high dimensional expanders. A few constructions obtaining suboptimal parameters (when compared to the algebraic ones mentioned above) have been proposed, of which we will highlight two that follow a product based approach that our main result is inspired by. Liu-Mohanty-Yang [LMY20] and Golowich [Gol21] defined a variant of a tensor product between a graph and a small simplicial complex and by picking the appropriate parameters construct bounded degree simplicial complexes with global expansion $1/2$ and k-local expansion $\Theta(1/k)$.

2.2.1 Square Complexes

The discussion of high-dimensional expanders so far has focused on simplicial complexes, but there are other interesting high-dimensional objects, in particular square complexes, which is the focus of this work.

Definition 2.2. A *square complex* Z is a 2-dimensional object defined as 3 collections of subsets of a vertex set V:

- The vertices, denoted by $Z(0)$, are exactly the vertex set V.

[1] There is a trade-off between the maximum degree of these simplicial complexes and λ, but they can both be simultanously constant.

- The edges, denoted by $Z(1)$, are sets of two vertices.
- The squares, denoted by $Z(2)$, are ordered sets of four vertices, such that if $(v_1, v_2, v_3, v_4) \in Z(2)$, then $\{v_i, v_{i+1}\} \in Z(1)$ (where we use a slightly abuse of notation so that $v_{4+1} = v_1$).

This definition is very similar to the definition of a 2-dimensional simplicial complex. In a 2-dimension simplicial complex, 2-faces are triangles or 3-cycles, whereas in this definition they are squares or 4-cycles. Also, the property enforced in the definition of squares is analogous to downward closure. There are a few additional interesting things to remark about this definition.

Remark 2.3. Under this notation, we have that $(v_1, v_2, v_3, v_4) = (v_4, v_1, v_2, v_3) = (v_3, v_4, v_1, v_2) = (v_2, v_3, v_4, v_1)$, but $(v_1, v_2, v_3, v_4) \neq (v_2, v_1, v_3, v_4)$.

$Z(2)$ is a set of 4-cycles of Z, but not that not all 4-cycles (in the graph sense) are necessarily in $Z(2)$[2].

Remark 2.4. Square complexes are a special case of a high-dimensional object known as *cubical complex*. We can naturally extend the above definition to contain sets $Z(3)$, $Z(4)$, etc as sets containing "cubes", or 4-dimensional hypercubes, under a similar downward closure property. In this sense, square complexes are 2-dimensional examples of cubical complexes.

We note that it is common in the literature (especially the pure mathematics one) to define square complexes algebraically in the language of chain complexes. Since we won't make use of the extra algebraic properties that come with this definition, we opted to use the simpler combinatorial one, which is also implicitly in [DEL+22].

We define the *1-skeleton* of a square complex Z to be the graph $(Z(0), Z(1))$. To characterize the relationships between layers of Z, we define the *vertex-association* and *edge-association* as follows:

Definition 2.5 (Vertex- and Edge-Association)**.** Given a square complex Z and a vertex v, we define the *vertex-association* L_v of v to be the set of squares of which v is a member. Mathematically, $L_v = \{\tau : v \in \tau, \tau \in Z(2)\}$. Similarly, (with some abuse of notation), for an edge e, we define its *edge-association* to be $L_e = \{\tau : e \in \tau, \tau \in Z(2)\}$.

Note that these quantities are analogous to links in simplicial complexes, in the sense that they represent the "local neighborhoods" of vertices and edges.

We can furthermore extend the notion of regularity to square complexes. That is, we say a square complex is *d-vertex regular* if its 1-skeleton forms a d-regular graph, and *k-edge regular* if $\forall e \in Z(1), |L_e| = k$. We denote the vertex-regularity of a square complex Z as $d_Z^{|}$ and the edge regularity as $d_Z^{\square}$.

To define our notion of expansion in square complexes, we first need to define parallelism.

Definition 2.6. Given a square complex Z, we say that an edge e is parallel to e' if both e and e' are members of a common square, but share no vertices. Thus, each square $s = (v_1, v_2, v_3, v_4)$ has exactly two pairs of parallel edges: $(v_1, v_2) \parallel (v_3, v_4)$ and $(v_2, v_3) \parallel (v_1, v_4)$. Additionally, given an edge e, we define its *parallel adjacency* $E^{\parallel}(e)$ to be the set of all edges it is parallel to. That is, $E^{\parallel}(e) = \{e' : e' \parallel e, \{e, e'\} \subset s \in Z(2)\}$.

[2] In the world of simplicial complexes the notion of *pure simplicial complex* plays a similar role in making this distinction.

The above definition induces a notion of parallel-skeleton, which is a graph G whose vertices are the edges $Z(1)$ and edges are given by the sets $E^{\parallel}$.

Definition 2.7. The global expansion of a square complex Z is given by the $\widetilde{\lambda}(\text{1-skeleton}(Z))$. The parallel expansion of a square complex Z is given by the $\widetilde{\lambda}(\text{parallel-skeleton}(Z))$.

We will use the notation of Markov operators to represent the above expansions. Let $M_Z^{\downarrow}$ be the Markov operator of the random walk on the 1-skeleton, i.e. $\lambda(M_Z^{\downarrow})$ is the global expansion of Z. The *parallel random walk* on a square complex is given as the random walk defined on edges, such that given an edge we pick one parallel edge uniformly at random to go to. This is a natural generalization of the random walk on G, as the two vertices $(v_1, v_2) \in E(G)$ are parallel to each-other with respect to edges. So, let $M_Z^{\square}$ to be the Markov operator of the parallel random walk, i.e. $\lambda(M_Z^{\square})$ is the parallel expansion of Z.

The definition of expansion we use is also implicitly used in [DEL+22], and it is equivalent to other definitions seen in the literature. For example, in [LZ22] parallel adjacency is replaced by a form of "corner adjacency", defined between vertices, where two vertices are "corner adjacent" if they are on opposite corners of a square. We prefer to use the definition of parallel adjacency since it gives us an adjacency between edges that is very similar to the definition of a "up" operator from edges to squares that given an edge it picks a random square containing that edge. This "up" operator is not very nice to work with since it is not symmetric, so the parallel operator is a better choice.

Non spectral notions of expansion have also been studied. The work of Jordan and Livné [JL00] defined and constructed *Ramanujan cubical complexes*, in a way analogous to the work of [LSV05]. Additionally, in [DLV24] certain topological expansion definitions of cubical complexes are studied, motivated by the construction of quantum locally testable codes.

2.2.2 Left-right Cayley complexes and local codes

The main motivation for studying square complexes comes from their application to the construction of locally testable codes. A code is *q-locally testable* if it has a property-tester that reads q bits at random from a given word, and rejects words with probability proportional to their distance from the code.

In order to construct locally testable codes, Dinur et al. [DEL+22] employed a generalization of expander codes to square complexes given as follows. Given a square complex Z and a linear code $\mathcal{C}_0$ of block length $d_Z^{\square}$, define a code on $\mathbb{F}_2^{Z(2)}$ such that every edge $e \in Z(1)$ enforces the condition that the the bits corresponding to L_e form a word belonging to $\mathcal{C}_0$. This definition implies that the bits corresponding to L_v should form a word belonging to the tensor code $\mathcal{C} \otimes \mathcal{C}$. The local tester that one can define for this code is given by picking a vertex at random and then accepting if the bits corresponding to L_v form a word belonging to the tensor code $\mathcal{C} \otimes \mathcal{C}$.

Dinur et al. [DEL+22] use a special case of square complexes known as left-right Cayley complexes, which are defined given a group G and two symmetric sets of generators A and B, which must follow the *Total No-Conjugacy* property that states that $\forall a \in A, b \in B, g \in G, ag \neq gb$. When the Cayley graphs $\text{Cay}(G, A)$ and $\text{Cay}(G, B)$ are of degree Δ and are Ramanujan (which can be obtained using [LPS88, Mar88, Mor94]), then the square complex they build has $d_Z^{\downarrow} = 2\Delta$ and nearly optimal unnormalized global expansion of $4\sqrt{\Delta}$. However, its parallel expansion is

trivial, since its parallel-skeleton is disconnected. Instead, the parallel-skeleton has two connected components of degree Δ^2, both of which have nearly optimal spectral expansion of 4Δ.

This left-right Cayley complex has an additional property that the edges of the complex can be labeled in such a way that allows for a consistent local view of squares among edges. Formally, this means that each edge e has a labeling map $\sigma_e \colon Z(2) \to [d_Z^{\square}]$ of the L_e edges in its edge-association, such that if $e_1, e_2 \in \tau$, then $\sigma_{e_1}(\tau) = \sigma_{e_2}(\tau)$. This local consistency lets one use a code $\mathcal{C}_0$ such that $\mathcal{C} \otimes \mathcal{C}$ is robust testable code, and [DEL+22] show that this is enough to obtain locally testable codes of constant rate, distance and testability (the parameter q in the above definition). We remark that the square complexes we build in this work do not have this labeling map. In fact, it seems like obtaining a square complex using combinatorial method that has such a map is a very challenging problem, since it requires a lot of symmetry.

3 Expanding Square Complexes

3.1 Generalized Corner Product

Consider two graphs $G = (V_G, E_G)$ and a graph $H = (V_H, E_H)$. We define the *General Corner Product* between these two graphs to take in these graphs as input and output a square complex with:

- $Z(0) = V_G \times V_H$
- $Z(2) = \{((v_1, u_1), (v_2, u_2), (v_3, u_3), (v_2, u_4)) : (v_1, v_2) \in E_G, (v_2, v_3) \in E_G, (u_1, u_3) \in E_H, (u_2, u_4) \in E_H\}$

and $Z(1)$ is defined to be all the edges implied by the existence of the 4-cycles in $Z(2)$. Our result revolves around determining the properties of the square complex resulting from this operation. Indeed, our main result is the below theorem.

Theorem 3.1. *Suppose that $G = (V_G, E_G)$ is a d_G- regular λ_G- expanding graph with $|V_G| = n_G$ and that $H = (V_H, E_H)$ is d_H- regular, λ_H- expanding graph with $|V_H| = n_H$. The square complex Z resulting from the General Corner Product of G and H has the following properties.*

- $d_Z^{|} = d_G n_H$, *in other words, the 1-skeleton of Z is a $d_G n_H$ regular graph.*
- *The (normalized) expansion of the random walk on the 1-skeleton of Z is $\lambda(M_Z^{|}) \leqslant \lambda_G$*
- *The edge regularity $d_Z^{\square}$ is $2(d_G - 1)d_H^2$.*
- *The (normalized) expansion of the parallel random walk on Z is $\lambda(M_Z^{|}) \leqslant \max(\frac{d_G + d_G\lambda_G - 2}{2d_G - 2}, \lambda(\tilde{M}_H))$*

We prove this Theorem in the next section. An immediate consequence demonstrating its utility is the following.

Corollary 3.2. *There exists a square complex with (un-normalized) $\lambda_Z^{|} \leqslant \mathcal{O}((d_Z^{|})^{0.751})$ and $\lambda_Z^{\square} \leqslant \frac{1}{2} + \epsilon$. The vertex and edge-regularity of this construction will both be $\mathcal{O}(d_G^3)$.*

Proof of Corollary 3.2 (assuming Theorem 3.1): Set G and H to both be Ramanujan expanders with $n_H = d_G^{1+\epsilon}$ and $d_H = d_G$. Take the generalized corner product of G and H. By Theorem 3.1, this results in a square complex with 1-skeleton-expansion $\frac{2}{\sqrt{d_G - 1}}$ and parallel expansion $\max(\frac{d_G + d_G\lambda_G - 2}{2d_G - 2}, \lambda(\tilde{M}_H))$. This complex has $d_Z^{\downarrow} = d_G n_H$ and $d_Z^{\square}$ is $2(d_G - 1)d_H^2$, so the un-normalized expansion of $M_Z^{\downarrow}$ is $\frac{2d_G n_H}{\sqrt{d_G}}$. Plugging in n_H and d_H and analyzing the asymptotics yields

$$\lambda(M_Z^{\downarrow}) = \mathcal{O}\left(d_G^{3/2+\epsilon}\right) \text{ and } d_Z^{\downarrow} = \mathcal{O}\left(d_G^{2+\epsilon}\right) \implies \lambda(M_Z^{\downarrow}) = \mathcal{O}\left(d_Z^{\downarrow\,\frac{3/2+\epsilon}{2+\epsilon}}\right)$$

as desired □

3.2 Relevant Operator Decompositions

Given an operator M, we denote $\tilde{M}$ to be the operator normalized. Thus, for a graph $G = (V, E)$, we define the random walk operator to be $\tilde{M}_G$ (note that M_G is simply the adjacency matrix of G). We equip every operator with the following norm ([RV05]) pertaining to its spectrum

Definition 3.3 ([RV05])**.** For an $N \times N$ matrix C, define the matrix norm $\|C\|$ as $\|C\| = \max_{v \in \mathbb{R}^n} \frac{\|Cv\|}{\|v\|}$. The matrix norm satisfies:

- $\|AB\| \leqslant \|A\| \cdot \|B\|$ for every pair of matrices A, B.
- $\|A \otimes B\| \leqslant \|A\| \cdot \|B\|$.
- If A is the transition matrix of a graph, then $\|A\| = 1$.

Proposition 3.4. *Let A be a $N \times N$ symmetric, stochastic matrix with second largest eigenvalue λ. Then $A = \frac{(1-\lambda)}{N} J_N + \lambda C$ where $\|C\| \leqslant 1$.*

Proof. Let $C = \frac{A - (1-\lambda)\frac{J}{N}}{\lambda}$. Given that $A\mathbb{1}_N = \frac{J}{N}\mathbb{1}_N = \mathbb{1}_N$, it follows that $C\mathbb{1}_N = \mathbb{1}_N$. For any v orthogonal to $\mathbb{1}_N$, both $\frac{J}{N}v$ and Av are orthogonal to $\mathbb{1}_N$, which means that Cv is also orthogonal to $\mathbb{1}_N$. Thus, all that remains is to show that $\|Cv\| \leqslant \|v\|$ for all v orthogonal to $\mathbb{1}_N$. However, this is clearly true, as $\frac{J}{N}v = 0$ and $\|Av\| \leqslant \lambda\|v\|$, implies $\|Cv\| \leqslant \|v\|$, yielding our desired result. □

Observe that on an undirected graph G, both the normalized random walk operator and the normalized random walk operator of its line graph $l(G)$ are symmetric, stochastic operators. Thus we immediately attain the following decompositions.

Corollary 3.5 (Decomposition of Random walk operator)**.** *Let M_G be the random walk operator (transition matrix) of a $n-$ vertex, $d-$ regular, $\lambda-$ expanding graph and denote $\tilde{J}_n$ to be the normalized all ones matrix (so $\tilde{J}_n = \frac{1}{n}J_n$). Then $\tilde{M}_G = (1-\lambda)\tilde{J}_n + \lambda C$ where $\|C\| \leqslant 1$.*

Corollary 3.6 (Decomposition of Random walk operator on line graph)**.** *Let $\tilde{M_{l(G)}}$ be the random walk operator for the line graph of G. Denote $\tilde{J}_n$ to be the normalized all ones matrix (so $\tilde{J_{dn}} = \frac{1}{dn}J_{dn}$). If the second largest eigenvalue of $\tilde{M_{l(G)}}$ is λ, then $\tilde{M_{l(G)}} = (1-\lambda)\tilde{J_{dn}} + \lambda C$ where $\|C\| \leqslant 1$.*

3.3 Spectra of Line Graphs

The proof of Theorem 3.1 depends on the use of line graphs. As such, we provide all necessary definitions and tools pertaining to these objects.

Every graph G, has a line graph $l(G)$ where $V(l(G)) = E(G)$ and $E(l(G)) = \{(u,v) : u \in E(G), v \in E(G), |u \cap v| = 1\}$. That is, u and v are adjacent in $l(G)$ if and only if they are adjacent edges in G. The spectra of a graph G and its line graph $l(G)$ are similar, as characterized by the following lemma from [Spi19]. For completeness, we include a self-contained proof of it.

Lemma 3.7 ([Spi19])**.** *Let G be a d-regular graph with n vertices, and let H be its line graph. Then the spectrum of the Laplacian of H is the same as the spectrum of the Laplacian of G, except that it has $\frac{dn}{2} - n$ extra eigenvalues of $2d$.*

Proof. Define U to be the signed edge-vertex adjacency matrix of a graph G. That is, U is a $E(G) \times V(G)$ matrix where each entry takes on value:

$$U((a,b),c) = \begin{cases} 1 & \text{if } a = c \\ -1 & \text{if } b = c \\ 0 & \text{otherwise} \end{cases}.$$

Note that the Laplacian L_G can be expressed as $U_G^\intercal U_G$. Denote the graph $|U|$ to be the $E(G) \times V(G)$ matrix where each element $|U|_{ij} = |U_{ij}|$. The matrix $|U|^\intercal |U|$ is equal to the Laplacian, except that its off-diagonal entries are 1 instead of -1. So, we have that $|U|^T|U| = D_G + M_G = dI + M_G$

Now, consider $|U||U|^T$. This is a matrix with $\frac{nd}{2}$ rows and $\frac{nd}{2}$ columns, indexed by edges of G. Observe that the entry at the intersection of row (u,v) and column (w,z) takes on value 2 if they are the same edge, 1 if they share a vertex, and 0 otherwise. That is $|U||U|^T = 2I_{\frac{nd}{2}} + M_H$. Thus, $|U||U|^T$ and $|U|^T|U|$ have the same eigenvalues (besides for the $\frac{nd}{2} - n$ extra eigenvalues of for $|U|^T|U|$).

Suppose that λ_i is an eigenvalue of $L_G = dI - M_G$. Then, $2d - \lambda_i$ is an eigenvalue of $D_G + M_G = |U|^\intercal |U|$ and thus an eigenvalue of $|U|^\intercal |U| = 2I_{\frac{nd}{2}} + M_H$. It follows that $2d - \lambda_i - 2$ is an eigenvalue of M_H. Because H is $(2d-1)$-regular, we attain that λ_i is an eigenvalue of $D_H - M_H = L_H$.

By identical reasoning, the extra $\frac{dn}{2} - n$ zero eigenvalues of $2I_{\frac{nd}{2}} + M_H$ map to $2d$ in L_H, yielding the desired result. □

Corollary 3.8. *Let G be a d-regular graph with n vertices, and denote the un-normalized and normalized adjacency matrices of G to be M_G and $\tilde{M}_G$ respectively. Let H be the line graph of G and similarly, denote the un-normalized and normalized adjacency matrices of H to be M_H and $\tilde{M}_H$. If λ is the second largest eigenvalue of $\tilde{M}_G$, then the second largest eigenvalue of $\tilde{M}_H$ is $\frac{d+d\lambda-2}{2d-2}$.*

Proof. H is $2(d-1)$ regular, so its (un-normalized) Laplacian can be expressed as $2(d-1)I - M_H$. Now, denote γ to be an eigenvalue of M_H. It must follow that $d - \gamma$ is an eigenvalue of L_G and (by Lemma 3.7) an eigenvalue of L_H. Therefore, $2(d-1) - (d-\gamma) = d + \gamma - 2$ is an eigenvalue of M_H. So, we have that α being an eigenvalue of $\tilde{M}_G$ implies that $\frac{d+d\alpha-2}{2d-2}$ is an eigenvalue of $\tilde{M}_H$.

Note that (by Lemma 3.7), we have that all eigenvalues of L_H are either eigenvalues of L_G or have value $2d$. All the eigenvalues of $2d$ of L_H correspond to eigenvalues of $-2/(2d-1)$ of $\tilde{M}_H$. It follows that the second largest eigenvalue of $\tilde{M}_H$ is $\frac{d+d\lambda-2}{2d-2}$. □

3.4 Proof of Theorem 3.1

Proof (for vertex-regularity and 1-skeleton expansion): Note that the definition of $Z(2)$ implies that $Z(1) = \{((v_1, u_1), (v_2, u_2)) : (v_1, v_2) \in E_G\}$. It thus follows that every vertex has exactly $d_G n_H$ neighbors. These observations enable us to quickly deduce that the 1-skeleton of Z is λ_G-expanding. This is because a random walk starting from a vertex (v, u) is equivalent to first sampling a neighbor of $v \in G$ and then a random vertex in H. Because this sampling process is expressed by the operator $\tilde{M_G} \otimes \tilde{J_{n_H}}$, we attain an upper bound on its second largest eigenvalue as follows:

$$\|\tilde{M_G} \otimes \tilde{J}_{n_H} - \tilde{J}_{d_G n_H}\| = \left\|\tilde{M_G} \otimes \frac{1}{n_H} J_{n_H} - \frac{1}{d_G n_H}\left(J_{d_G} \otimes J_{n_H}\right)\right\|$$

$$= \left\|\left(\tilde{M_G} - \frac{1}{d_G} J_{d_G}\right) \otimes \frac{1}{n_H} J_{n_H}\right\| \leqslant \left\|\tilde{M_G} - \frac{1}{d_G} J_{d_G}\right\| \cdot \left\|\frac{1}{n_H} J_{n_H}\right\| \leqslant \lambda_G$$

□

Proof (for edge-regularity): We claim that Z is $2(d_G - 1)d_H^2$ edge-regular. To see this, we apply the following constructive counting argument. We begin with some edge $((v_1, u_1), (v_2, u_2))$, so every square containing this edge must be of the form: $((v_1, u_1), (v_2, u_2), (\cdot, \cdot), (\cdot, \cdot))$. Note that every edge has one clone vertex, WLOG, let it be (v_2, u_2), this means that our squares can take the form $((v_1, u_1), (v_2, u_2), (\cdot, \cdot), (v_2, \cdot))$. Note that the first missing coordinate can take on any of the $d_G - 1$ neighbors of v_2 besides v_1, and the second missing coordinate can be any neighbor of u_1, so it has d_H choices and the third missing coordinate is required to be a neighbor of u_2, so it has d_H choices. Putting this together (and multiplying by 2 due to choice of clone vertex) gives us an edge-regularity of $2(d_G - 1)d_H^2$. □

Proof (for parallel-expansion): Our strategy is to simulate the parallel random walk via a combinatorial process $\mathcal{P}$, express the process in terms of operators, and then bound the spectrum. The following process $\mathcal{P}$represents the parallel random walk.

1. We begin with some edge $e = (u, v) \in Z(1)$. Note that $v_1 = (v_{1G}, v_{1H})$ and $v_2 = (v_{2G}, v_{2H})$. Take a step on the line graph of G (using $l(G)$), call the output $e'_G = (v'_{1G}, v'_{2G})$
2. Take a random walk step on the graph H using the vertex v_{1H}, call the output vertex v'_{1H}. Similarly, take a random walk step on the graph H using the vertex v_{2H}, call the output vertex v'_{2H}.
3. Output the edge $((v'_{1G}, v'_{1H}), (v'_{2G}, v'_{2H}))$.

We can see that $\mathcal{P}$ exactly replicates the constructive argument used in the proof for edge-regularity, by noting that we start with the edge $((v_1, u_1), (v_2, u_2))$ and then filling out the blanks uniformly in the set of possible squares: $((v_1, u_1), (v_2, u_2), (\cdot, \cdot), (v_2, \cdot))$.

Note that the operator $\tilde{M}_{l(G)} \otimes (\tilde{M}_H \otimes \tilde{M}_H)$ encapsulates $\mathcal{P}$. The operator $\tilde{M}_{l(G)}$ enacts step 1, step 2 is done by $\tilde{M}_H \otimes \tilde{M}_H$. Finally, the coordinates are joined together by the Kronecker product in step 3.

The eigenvalues of the matrix $\tilde{M}_{l(G)} \otimes (\tilde{M}_H \otimes \tilde{M}_H)$ are given by the products of the eigenvalues of $\tilde{M}_{l(G)}, \tilde{M}_H$, and$\tilde{M}_H$. It follows that the second largest eigenvalue is $\max(\lambda(\tilde{M}_{l(G)}), \lambda(\tilde{M}_H) = \max(\frac{d_G + d_G\lambda_G - 2}{2d_G - 2}, \lambda(\tilde{M}_H)$

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