Sharp Inner Product Correlations for Hypercube Bijections

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Abstract

We resolve a conjecture of Rob Morris concerning bijections on the hypercube. Specifically, we show that for any bijection $f : \{-1,1\}^n \to \{-1,1\}^n$,

$$\mathbf{Pr}_{x,y\in\{-1,1\}^n}\left[\langle x,y\rangle \geqslant 0 \text{ and } \langle f(x), f(y)\rangle \geqslant 0\right] \quad\geqslant\; \tfrac{1}{4} - O(1/\sqrt{n}),$$

implying the same lower bound for the joint event under any two bijections. Our proof proceeds by applying the spectral decomposition of the Hamming association scheme, which allows us to reformulate the problem as a linear program over the Birkhoff polytope. This makes it possible to isolate the contribution of the nontrivial spectrum, which we show is asymptotically negligible, leaving the dominant contribution arising from the principal eigenvalue.

1 Introduction

Recently, Balister et al ([BBC+24]) proved that the r-colour Ramsey number $R_r(k)$, defined as the minimum $n \in \mathbb{N}$ such that every r-colouring of the edges of the complete graph K_n contains a monochromatic copy of K_k, satisfies $R_r(k) \leqslant e^{-\delta k^r} r^k$ for each fixed $r > 2$, where $\delta = \delta(r) > 0$ and k is sufficiently large.

Key to their proof is the following geometric lemma:

Lemma 1.1 (Lemma 3.1 in [BBC+24]). *Let U and U' be i.i.d. random variables taking values in a finite set X, and let $\sigma_1, \ldots, \sigma_r : X \to \mathbb{R}^n$ be arbitrary functions. There exists $\lambda \geqslant -1$ and $i \in [r]$ such that*

$$\mathbf{Pr}\left[\langle \sigma_i(U), \sigma_i(U')\rangle \geqslant \lambda \text{ and } \langle \sigma_j(U), \sigma_j(U')\rangle \geqslant -1 \text{ for all } j \neq i\right] \geqslant \beta e^{-C\sqrt{\lambda+1}}.$$

Applying their result to the hypercube immediately yields that for all collections of bijections $f_1, f_2, \cdots, f_r : \{-1,1\}^n \to \{-1,1\}^n$, $\mathbf{Pr}_{x,y\in\{-1,1\}^n}\left[\wedge_{i=1}^r \langle f_i(x), f_i(y)\rangle \geqslant 0\right] \geqslant C$ for some constant $C > 0$. Determining the exact constant C in the hypercube bijection case is an interesting open problem. Note that when each f_i is taken to be a uniformly random bijection, we have that

$$\mathbf{Pr}_{x,y\in\{-1,1\}^n}\left[\bigwedge_{i=1}^r \langle f_i(x), f_i(y)\rangle \geqslant 0\right] = \left(\frac{1}{2}\right)^r + o(1)$$

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A natural question is that of analyzing the worst case bijections, that is, can we lower bound $\min_{f_1,\dots,f_r} \mathbf{Pr}_{x,y\in\{-1,1\}^n}\left[\bigwedge_{i=1}^{r}\langle f_i(x), f_i(y)\rangle \geqslant 0\right]$? At PCMI 2025, Rob Morris conjectured that for any hypercube bijection f, $\mathbf{Pr}_{x,y\in\{-1,1\}^n}\left[\langle x,y\rangle \geqslant 0 \text{ and } \langle f(x), f(y)\rangle \geqslant 0\right] \geqslant \frac{1}{4} - o(1)$ We resolve this conjecture in the affirmative. Our main result is the following theorem.

Theorem 1.2. *Let $f : \{-1,1\}^n \to \{-1,1\}^n$ be any bijection on the hypercube. Then,*

$$\mathbb{P}_{x,y\in\{-1,1\}^n}\left[\langle x,y\rangle \geqslant 0 \;\; \textit{and} \;\; \langle f(x), f(y)\rangle \geqslant 0\right] \geqslant \frac{1}{4} - O(\frac{1}{\sqrt{n}}),$$

where the probability is over uniformly random independent $x, y \in \{-1,1\}^n$.

We believe that this result is independently interesting in boolean function analysis and could have further applications. In the next subsection, we provide an overview of our techniques. Section 2 contains some preliminaries while Section 3 is the proof of Theorem 1.2.

1.1 Outline of Proof

Our method is to prove this theorem via linear-algebraic techniques. As, such, we fix an enumeration of the hypercube: let $x_1, \dots, x_N \in \{-1,1\}^n$, where $N = 2^n$, represent all possible hypercube vectors. We define $M \in \mathbb{R}^{N\times N}$ by

$$M_{i,j} = \mathbf{1}\left[\langle x_i, x_j\rangle \geqslant 0\right],$$

so that M encodes whether the inner product between two hypercube vectors is nonnegative. At a high-level, our proof can be broken down into three main steps

1. Observing that $\langle x_i, x_j\rangle \geqslant 0$ is exactly equivalent to the event $\Delta(x_i, x_j) \leqslant n/2$, where $\Delta(\cdot)$ denotes the hamming distance, the matrix M can be expressed as a sum of matrices corresponding to the Hamming Association Scheme. This implies that M can be diagonalized in the Fourier Basis, and its eigenvalues correspond to sums of Krowchuk polynomials. With some computation, we get that

$$\begin{aligned}\mathbf{Pr}\left[\langle x,y\rangle \geqslant 0 \text{ and } \langle f(x), f(y)\rangle \geqslant 0\right] &= \min_{P\in\mathcal{P}} \frac{1}{N^2}\operatorname{Tr}(P^\top M P M) \\ &\geqslant \frac{1}{4} + \min_{P\in\mathcal{P}} \frac{1}{N^2} \sum_{\substack{S\neq\emptyset \\ T\neq\emptyset}} \lambda_S\lambda_T \left(U^\top P U\right)^2_{S,T}.\end{aligned}$$

2. The rest of the proof is towards showing the remainder term is lower bounded by $-o(1)$. To do this, we observe that the matrix B, where $B_{ij} = (U^\top P U)^2_{ij}$ is a doubly stochastic matrix. We then relax the remainder term to the following LP:

$$\min_{B\in\mathcal{B}} \frac{1}{N^2} \sum_{\substack{S\neq\emptyset \\ T\neq\emptyset}} \lambda_S\lambda_T B_{S,T}$$

   where $\mathcal{B}$ denotes the Birkhoff polytope (set of doubly stochastic matrices). Because the vertices of $\mathcal{B}$ are given by the permutation matrices, we have that the remainder term is lower bounded by

$$\min_{B\in\mathcal{B}} \frac{1}{N^2} \sum_{\substack{S\neq\emptyset \\ T\neq\emptyset}} \lambda_S\lambda_T B_{S,T} = \min_{\pi\in\mathrm{Sym}(2^{[n]}\setminus\{\emptyset\})} \frac{1}{N^2} \sum_{\substack{S\subseteq[n] \\ S\neq\emptyset}} \lambda_S\lambda_{\pi(S)} = \frac{1}{N^2}\sum_{i=1}^{N-1} \lambda_i\lambda_{N-i}$$

where the last equality follows from the rearrangement inequality (as we order the eigenvalues corresponding to $\lambda_S, S \in [n]\setminus\emptyset$ as $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{N-1}$).

3. We finish off the proof by showing that $r_n := \frac{1}{N^2}\sum_{i=1}^{N-1}\lambda_i\lambda_{N-i} \geqslant -o(1)$. By means of generating function and coefficient extraction, we find a closed formula for λ_S involving the Beta function. Applying known properties of the Beta function, we are able to sort the eigenvalues and count their multiplicity. Using Stirling's approximation, we conclude with an asymptotically tight lower bound of r_n, namely $-O(\frac{1}{\sqrt{n}})$.

2 Preliminaries

2.1 Hamming Association Scheme

Let $\Omega = \{0,1\}^n$ denote the hypercube. The Hamming association scheme on Ω is generated by the symmetric matrices $A_0, A_1, \ldots, A_n \in \mathbb{R}^{2^n\times 2^n}$, where each A_k is defined entrywise by

$$(A_k)_{x,y} = \mathbf{1}[d(x,y) = k],$$

with $d(x,y)$ denoting the Hamming distance. The matrices A_k commute and are simultaneously diagonalizable; together they span the Bose–Mesner algebra of the scheme. The Hamming Association Scheme has been well-studied and has numerous applications in coding theory, for a more detailed treatment (see e.g. Chapter 10 of [GM16]).

A common orthonormal eigenbasis for all A_k is given by the Walsh–Hadamard vectors $\chi_S \in \mathbb{R}^{2^n}$, indexed by subsets $S \subseteq [n]$. Note that these are defined by evaluations of the Fourier Characters:

$$\chi_S(x) := (-1)^{\sum_{i\in S} x_i}, \quad x \in \Omega,$$

and we treat χ_S as a vector in $\mathbb{R}^{2^n}$ whose coordinates are indexed by $x \in \Omega$. These vectors form an orthonormal basis under the normalized inner product

$$\langle f, g\rangle := \frac{1}{2^n}\sum_{x\in\Omega} f(x)g(x).$$

Each matrix A_k acts diagonally in this basis. Specifically, the Walsh–Hadamard vector χ_S is an eigenvector of A_k with eigenvalue

$$A_k\chi_S = K_k(|S|)\chi_S,$$

where $|S|$ is the cardinality of the subset S, and $K_k(i)$ is the degree-k Krawtchouk polynomial evaluated at i, given by

$$K_k(i) = \sum_{j=0}^{k} (-1)^j \binom{i}{j}\binom{n-i}{k-j}.$$

Letting $U \in \mathbb{R}^{2^n \times 2^n}$ denote the orthogonal matrix whose columns are the normalized Walsh–Hadamard vectors $\chi_S/\sqrt{2^n}$, we obtain the spectral decomposition

$$A_k = U\Lambda_k U^\top,$$

where $\Lambda_k \in \mathbb{R}^{2^n \times 2^n}$ is diagonal with entries $(\Lambda_k)_{S,S} = K_k(|S|)$.

2.2 LP Geometry and the Birkhoff Polytope

A linear program (LP) optimizes a linear objective over a polyhedron defined by linear constraints. The standard form of a linear program is:

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax = b, \\ & x \geqslant 0, \end{aligned}$$

where $c \in \mathbb{R}^d$, $A \in \mathbb{R}^{m \times d}$, and $b \in \mathbb{R}^m$. The feasible region $\mathcal{P} = \{x \in \mathbb{R}^d : Ax = b,\ x \geqslant 0\}$ is a convex polytope when bounded.

Fact 2.1 (Extreme Point Principle)**.** *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a polytope and $c \in \mathbb{R}^d$. Then the minimum of the linear program $\min_{x \in \mathcal{P}} c^\top x$ is attained at an extreme point (vertex) of $\mathcal{P}$.*

In our setting, the relevant polytope is the Birkhoff polytope, which is defined as the set of all doubly stochastic matrices. More formally, we denote this polytope by $\mathcal{D}_n \subseteq \mathbb{R}^{n \times n}$ where

$$\mathcal{D}_n := \left\{ X \in \mathbb{R}^{n \times n} \;\middle|\; X_{ij} \geqslant 0,\ \sum_{j=1}^n X_{ij} = 1,\ \sum_{i=1}^n X_{ij} = 1 \right\}.$$

Fact 2.2 (Birkhoff–von Neumann Theorem)**.** *The vertices of $\mathcal{D}_n$ are exactly the set of $n \times n$ permutation matrices.*

Note that the result implies that linear optimization over $\mathcal{D}_n$ reduces to optimization over permutations, which is critical to our proof.

3 Proof of Theorem 1.2

We begin by translating

$$\mathbf{Pr}\left[\langle x, y\rangle \geqslant 0 \text{ and } \langle f(x), f(y)\rangle \geqslant 0\right], \tag{1}$$

into the language of linear algebra, which will allow us to establish a universal lower bound on this probability for all bijections f. We fix an enumeration of the hypercube: let $x_1, \ldots, x_N \in \{-1, 1\}^n$, where $N = 2^n$, represent all possible hypercube vectors. Define the matrix $M \in \mathbb{R}^{N \times N}$ by

$$M_{i,j} = \mathbf{1}\left[\langle x_i, x_j\rangle \geqslant 0\right],$$

so that M encodes whether the inner product between two hypercube vectors is nonnegative. The bijection f induces a permutation matrix $P \in \mathbb{R}^{N \times N}$ such that $P_{ij} = \mathbf{1}[f(x_i) = x_j]$.

Then the condition $\langle x_i, x_j \rangle \geqslant 0$ and $\langle f(x), f(y) \rangle \geqslant 0$ holds if and only if $M_{i,j} = M_{f(i),f(j)} = 1$ for a random pair i, j. This yields that:

$$\mathbf{Pr}\left[\langle x, y\rangle \geqslant 0 \text{ and } \langle f(x), f(y)\rangle \geqslant 0\right] = \frac{1}{N^2}\sum_{i,j=1}^{N} M_{i,j}\cdot M_{f(i),f(j)} = \frac{1}{N^2}\operatorname{Tr}(P^\top M P M).$$

Thus, lower bounding (1) over all bijections is equivalent to lower bounding

$$\min_{P\in\mathcal{P}} \frac{1}{N^2}\operatorname{Tr}(P^\top M P M), \tag{2}$$

where $\mathcal{P}$ denotes the set of $N \times N$ permutation matrices.

To analyze (2), we first fully characterize the spectrum of the matrix M via its relation to the Hamming Association Scheme. This is done in the following Lemma:

Lemma 3.1. *Let $M \in \mathbb{R}^{2^n\times 2^n}$ be the matrix with entries given by $M_{x,y} = \mathbf{1}\left[\langle x, y\rangle \geqslant 0\right]$, for $x, y \in \{-1, 1\}^n$. Then M is diagonalized by the Walsh–Hadamard basis, with eigenvalues*

$$\lambda_S = \sum_{d=0}^{\lfloor n/2\rfloor} K_d(|S|),$$

where K_d is the degree-d Krawtchouk polynomial. Each λ_S has multiplicity $\binom{n}{|S|}$.

Proof. For $x, y \in \{-1, 1\}^n$, the inner product satisfies $\langle x, y\rangle = n - 2d_H(x, y)$. Thus,

$$M_{x,y} = \mathbf{1}[\langle x, y\rangle \geqslant 0] = \mathbf{1}[d_H(x, y) \leqslant n/2],$$

so we may write

$$M = \sum_{d=0}^{\lfloor n/2\rfloor} A_d = U\left(\sum_{d=0}^{\lfloor n/2\rfloor}\Lambda_d\right)U^\top = U\Lambda U^\top$$

where A_d corresponds to the distance d matrix of the Hamming Association Sceheme. The result follows from the fact that Λ is diagonal with $\Lambda_{S,S} = \sum_{d=0}^{\lfloor n/2\rfloor} K_d(|S|)$. □

Applying Lemma 3.1 to (2), we have:

$$\begin{aligned}
\frac{1}{N^2}\operatorname{Tr}(P^\top M P M) &= \frac{1}{N^2}\operatorname{Tr}(P^\top U\Lambda U^\top P U\Lambda U^\top) \\
&= \frac{1}{N^2}\operatorname{Tr}\left((U^\top P^\top U)\Lambda(U^\top P U)\Lambda\right) \\
&= \frac{1}{N^2}\sum_{S,T\subseteq[n]} \lambda_S\lambda_T\left(U^\top P U\right)^2_{S,T}.
\end{aligned} \tag{3}$$

Furthermore, a direct implication of Lemma 3.1 of is the following

$$\lambda_\varnothing = \sum_{k=0}^{\lfloor n/2\rfloor} K_k(0) = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k} \geqslant 2^{n-1}$$

Note that the bulk of the spectrum of M lies in its largest eigenvalue. As such, to understand the above sum (3), we require an understanding of the coefficients that correspond to the terms $\lambda_S\lambda_T$ where one or both of S and T equal the empty set. This is given by the following lemma.

Lemma 3.2. *Let $U \in \mathbb{R}^{N\times N}$, with $N = 2^n$, be the orthogonal Walsh–Hadamard matrix whose columns are $\chi_S/\sqrt{N}$ for $S \subseteq [n]$, and let $P \in \mathbb{R}^{N\times N}$ be any permutation matrix. If $R := U^\top PU$, then:*

$$R_{\varnothing,\varnothing} = 1,$$
$$R_{\varnothing,T} = R_{T,\varnothing} = 0 \quad \text{for all } T \neq \varnothing.$$

Proof. Since $U_{x,S} = \frac{1}{\sqrt{N}}(-1)^{\langle S,x\rangle}$, we have:

$$R_{\varnothing,T} = \sum_{a,b} U_{a,\varnothing}P_{a,b}U_{b,T} = \sum_{a,b} \frac{1}{\sqrt{N}}P_{a,b} \cdot \frac{1}{\sqrt{N}}(-1)^{\langle T,x_b\rangle} = \frac{1}{N}\sum_b (-1)^{\langle T,x_b\rangle},$$

where we used that $\sum_a P_{a,b} = 1$ since P is a permutation matrix.

If $T \neq \varnothing$, then the character $x \mapsto (-1)^{\langle T,x\rangle}$ is nontrivial and satisfies:

$$\sum_{x\in\{0,1\}^n} (-1)^{\langle T,x\rangle} = 0.$$

Thus, $R_{\varnothing,T} = 0$. The same argument shows $R_{T,\varnothing} = 0$ by symmetry.

Finally, for $R_{\varnothing,\varnothing}$, we compute:

$$R_{\varnothing,\varnothing} = \sum_{a,b} U_{a,\varnothing}P_{a,b}U_{b,\varnothing} = \sum_{a,b} \frac{1}{\sqrt{N}}P_{a,b} \cdot \frac{1}{\sqrt{N}} = \frac{1}{N}\sum_{a,b} P_{a,b} = 1,$$

since P is a permutation matrix and hence has exactly one 1 in each row and column. ☐

Applying Lemma 3.2, we have that:

$$\begin{aligned}
(3) &= \frac{1}{N^2}\lambda_\varnothing^2 \left(U^\top PU\right)^2_{\varnothing,\varnothing} + \frac{1}{N^2}\sum_{S=\varnothing,\, T\neq\varnothing} \lambda_S\lambda_T \left(U^\top PU\right)^2_{S,T} + \frac{1}{N^2}\sum_{S\neq\varnothing,\, T=\varnothing} \lambda_S\lambda_T \left(U^\top PU\right)^2_{S,T} \\
&\quad + \frac{1}{N^2}\sum_{S\neq\varnothing,\, T\neq\varnothing} \lambda_S\lambda_T \left(U^\top PU\right)^2_{S,T} \\
&= \frac{1}{N^2}\lambda_\varnothing^2 + \frac{1}{N^2}\sum_{S\neq\varnothing, T\neq\varnothing} \lambda_S\lambda_T \left(U^\top PU\right)^2_{S,T} \\
&\geqslant \frac{1}{4} + \frac{1}{N^2}\sum_{S\neq\varnothing, T\neq\varnothing} \lambda_S\lambda_T \left(U^\top PU\right)^2_{S,T}.
\end{aligned}$$

So, we have that over all hypercube bijections $f : \{-1,+1\}^n \to \{-1,+1\}^n$

$$\begin{aligned}
\mathbf{Pr}\left[\langle x,y\rangle \geqslant 0 \text{ and } \langle f(x), f(y)\rangle \geqslant 0\right] &\geqslant \min_{P\in\mathcal{P}} \frac{1}{N^2}\operatorname{Tr}(P^\top MPM) \\
&\geqslant \frac{1}{4} + \min_{P\in\mathcal{P}} \frac{1}{N^2}\sum_{\substack{S\neq\varnothing \\ T\neq\varnothing}} \lambda_S\lambda_T \left(U^\top PU\right)^2_{S,T}.
\end{aligned}$$

Lemma 3.3. *Let M be as defined above, with eigendecomposition $M = U\Lambda U^\top$, where $\Lambda = \operatorname{diag}(\lambda_S)$, and $\mathcal{P}$ the set of $N \times N$ permutation matrices. Then*

$$\min_{P\in\mathcal{P}} \frac{1}{N^2}\sum_{\substack{S\neq\varnothing \\ T\neq\varnothing}} \lambda_S\lambda_T \left(U^\top PU\right)^2_{S,T} \geqslant -o(1).$$

Proof. We begin by rewriting our optimization problem as follows

$$
\begin{aligned}
\min_{P\in\mathcal{P}} \ \frac{1}{N^2}\sum_{\substack{S\neq\varnothing\\ T\neq\varnothing}} \lambda_S\lambda_T \left(U^\top PU\right)^2_{S,T} &= \min_{P\in\mathcal{P}} \ \frac{1}{N^2}\sum_{\substack{S\neq\varnothing\\ T\neq\varnothing}} \lambda_S\lambda_T \left[(U^\top PU)\circ(U^\top PU)\right]_{S,T} \\
&= \begin{cases} \min\limits_{B} & \dfrac{1}{N^2}\sum\limits_{\substack{S\neq\varnothing\\ T\neq\varnothing}} \lambda_S\lambda_T B_{S,T} \\ \text{s.t.} & B\in\mathcal{B}' := \left\{(U^\top PU)\circ(U^\top PU) : P\in\mathcal{P}\right\} \end{cases}
\end{aligned} \tag{4}
$$

Our strategy for lower bounding this quantity is to consider a sufficient LP relaxation of (4). We do this by observing that $\mathcal{B}'\subseteq\mathcal{D}_N$, where $\mathcal{D}_N$ denotes the Birkhoff polytope. This follows from the fact that since both U and P are orthogonal, the matrix $U^\top PU$ is also orthogonal. In particular, for any matrix $B\in\mathcal{B}'$, we have that for each row i,

$$\sum_{j=1}^{N} B_{ij} = \sum_{j=1}^{N}(U^\top PU)^2_{ij} = \left\|(U^\top PU)_{i\cdot}\right\|_2^2 = 1,$$

and similarly, for each column j,

$$\sum_{i=1}^{N} B_{ij} = \sum_{i=1}^{N}(U^\top PU)^2_{ij} = \left\|(U^\top PU)_{\cdot j}\right\|_2^2 = 1.$$

Each entry $B_{ij}\geqslant 0$ since it is a square. Therefore, B is nonnegative with all row and column sums equal to 1, implying $B\in\mathcal{D}_N$. Furthermore, from Lemma 3.2, we know that every matrix $B\in\mathcal{B}'$ has $B_{\varnothing,\varnothing}=1$ and $B_{\varnothing,T}=B_{T,\varnothing}=0$. Thus,

$$\mathcal{B}'\subseteq\left\{\begin{bmatrix}1 & \mathbf{0}^\top\\ \mathbf{0} & D\end{bmatrix}\ \middle|\ D\in\mathcal{D}_{N-1}\right\} = \mathcal{D}_N^{(0)}$$

Defining $\mathcal{S} := 2^{[n]}\setminus\varnothing$, we now have the following LP relaxation-based lower bound for (4)

$$
\begin{aligned}
(4) \geqslant \min_{B\in\mathcal{D}_N^{(0)}} \frac{1}{N^2}\sum_{\substack{S\subseteq[n]\\ S\neq\varnothing}}\sum_{\substack{T\subseteq[n]\\ T\neq\varnothing}} \lambda_S\lambda_T B_{S,T} \\
= \min_{D\in\mathcal{D}_{\mathcal{S}}} \frac{1}{N^2}\sum_{S\in\mathcal{S}}\sum_{T\in\mathcal{S}} \lambda_S\lambda_T D_{S,T} \\
= \min_{\pi\in\mathrm{Sym}(\mathcal{S})} \frac{1}{N^2}\sum_{S\in\mathcal{S}} \lambda_S\lambda_{\pi(S)}
\end{aligned}
$$

where the last equality follows from the fact that LPs achieve their objective at their vertices, and that the vertices of the Birkhoff polytope are the permutation matrices (Fact 2.1 and Fact 2.2). Let $\lambda_1\geqslant\lambda_2,\cdots\geqslant\lambda_{N-1}$ be the ordering of the eigenvalues $\{\lambda_S\}_{S\in\mathcal{S}}$. By the rearrangement inequality, we have

$$\min_{\pi\in\mathrm{Sym}(\mathcal{S})} \frac{1}{N^2}\sum_{S\in\mathcal{S}} \lambda_S\lambda_{\pi(S)} = \frac{1}{N^2}\sum_{i=1}^{N-1}\lambda_i\lambda_{N-i}$$

Hence, Lemma 3.3 is reduced to Lemma 3.4, which we will prove in the next section.

Lemma 3.4. *Let $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_{N-1}$ be the multiset $\{\lambda_S : \emptyset \neq S \subseteq 2^{[n]}\}$ sorted in descending order, here $N = 2^n$.*

$$r_n := \frac{1}{N^2} \sum_{k=1}^{N} \lambda_i \lambda_{N-i}$$

Then $r_n \geqslant -o(1)$ as $n \to \infty$

□

4 Bounding the remainder term

In this section, we prove Lemma 3.4, thus finishing the proof of Theorem 1.2. For simplicity, we assume $n = 4m$ for some $m \in \mathbb{N}$. The proof works for all n with minor adjustment. Our proof utilizes the following well-known facts, which we state without proof.

Fact 4.1. $\sum_{j=0}^{D} (-1)^j \binom{n}{j} = (-1)^D \binom{n-1}{D}$

Fact 4.2 (Extraction of coefficients)**.** $[x^k]P(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} P(e^{i\theta}) d\theta$

Fact 4.3. $\mathrm{B}(x, y) = 2 \int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1}\, d\theta$

Fact 4.4. *$\Gamma(x)$ is log-convex. When $x + y$ is fixed, $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is unimodal and minimizes when $x = y$*

Fact 4.5 (Stirling's Approximation)**.** $\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z \left(1 + O\left(\frac{1}{z}\right)\right)$

Corollary 4.6.

$$B(x, y) = \frac{\sqrt{2\pi}\, x^{x-1/2} y^{y-1/2}}{(x+y)^{x+y-1/2}} \frac{(1 + O(\frac{1}{x}))(1 + O(\frac{1}{y}))}{1 + O(\frac{1}{x+y})}$$

In particular, there exist constant $C > 0$ such that $\frac{1}{C} \frac{\sqrt{2\pi}\, x^{x-1/2} y^{y-1/2}}{(x+y)^{x+y-1/2}} \leqslant |B(x, y)| \leqslant C \frac{\sqrt{2\pi}\, x^{x-1/2} y^{y-1/2}}{(x+y)^{x+y-1/2}}$ for all $x, y \geqslant \frac{1}{2}$

4.1 Explicit formulae for eigenvalues

Proposition 4.7. $\lambda_S = [x^{2m}](1 - x)^{|S|-1}(1 + x)^{4m-|S|}$

Proof.

$$\begin{aligned}
\lambda_S &= \sum_{k=0}^{2m} \sum_{j=0}^{k} (-1)^j \binom{|S|}{j} \binom{n - |S|}{k - j} \\
&= \sum_{0 \leqslant j \leqslant k \leqslant 2m} (-1)^j \binom{|S|}{j} \binom{n - |S|}{k - j} \\
&= \sum_{0 \leqslant j + k \leqslant 2m} (-1)^j \binom{|S|}{j} \binom{n - |S|}{k} \\
&= \sum_{k=0}^{2m} \binom{n - |S|}{k} \sum_{j=0}^{2m-k} (-1)^j \binom{|S|}{j}
\end{aligned}$$

$$= \sum_{k=0}^{2m} \binom{n-|S|}{k} (-1)^{2m-k} \binom{|S|-1}{2m-k} \ (\textit{Fact 4.1})$$
$$= \sum_{k=0}^{2m} (-1)^k \binom{|S|-1}{k} \binom{n-|S|}{2m-k}$$
$$= [x^{2m}](1-x)^{|S|-1}(1+x)^{n-|S|}$$

□

Proposition 4.8.

$$\lambda_S = \begin{cases} \frac{2^{4m-1}}{\pi}(-1)^{\frac{|S|}{2}} B(\frac{|S|+1}{2}, \frac{4m+1-|S|}{2}), & |S| \textit{ even} \\ \frac{2^{4m-1}}{\pi}(-1)^{\frac{|S|-1}{2}} B(\frac{|S|}{2}, \frac{4m+2-|S|}{2}), & |S| \textit{ odd} \end{cases}$$

Proof. By Proposition 4.7 and Fact 4.2

$$\begin{aligned}
\lambda_S &= \frac{1}{2\pi}\int_0^{2\pi} e^{-2mi\theta}(1-e^{i\theta})^{|S|-1}(1+e^{i\theta})^{4m-|S|}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{-2mi\theta}(2\sin\frac{\theta}{2}e^{i(\frac{\theta}{2}-\frac{\pi}{2})})^{|S|-1}(2\cos\frac{\theta}{2}e^{i\frac{\theta}{2}})^{4m-|S|}d\theta \\
&= \frac{2^{4m-1}}{2\pi}i^{1-|S|}\int_0^{2\pi} e^{-i\frac{\theta}{2}}(\sin\frac{\theta}{2})^{|S|-1}(\cos\frac{\theta}{2})^{4m-|S|}d\theta \\
&= \frac{2^{4m-2}}{\pi}i^{1-|S|}(\int_0^{2\pi}(\sin\frac{\theta}{2})^{|S|-1}(\cos\frac{\theta}{2})^{4m+1-|S|}d\theta - i\int_0^{2\pi}(\sin\frac{\theta}{2})^{|S|}(\cos\frac{\theta}{2})^{4m-|S|}d\theta) \\
&= \frac{2^{4m-1}}{\pi}i^{1-|S|}(\int_0^{\pi}(\sin\theta)^{|S|-1}(\cos\theta)^{4m+1-|S|}d\theta - i\int_0^{\pi}(\sin\theta)^{|S|}(\cos\theta)^{4m-|S|}d\theta)
\end{aligned}$$

Note that when $|S|$ is odd, $\int_0^\pi(\sin\theta)^{|S|}(\cos\theta)^{4m-|S|}d\theta = 0$, when $|S|$ is even, $\int_0^\pi(\sin\theta)^{|S|-1}(\cos\theta)^{4m+1-|S|}d\theta = 0$, the result follows from Fact 4.3, parity casework and symmetry. □

Remark 4.9. When $n = 4m+1$ or $4m+3$, there's a uniform expression.

For notational convenience, we define $\lambda(|S|) := \lambda_S$, as it depends only on $|S|$.

Proposition 4.10. *The positive eigenvalues are:*

$$\mu_1 := \lambda(1) = \lambda(4m)$$

$$\mu_{4k+1} := \lambda(4k) = \lambda(4k+1) = \lambda(4m-4k+1) = \lambda(4m-4k)$$

The negative eigenvalues are:

$$-\mu_{4k+3} := \lambda(4k+2) = \lambda(4k+3) = \lambda(4m-4k-1) = \lambda(4m-4k-2)$$

where

$$\mu_{2k+1} = \frac{2^{4m-1}}{\pi}B(\frac{2k+1}{2}, \frac{4m+1-k}{2})$$
$$\mu_1 > \mu_3 > \mu_5 > ... > \mu_{2m+1}$$

are the distinct absolute values of eigenvalues, sorted in descending order.

Proof. Consequence of Proposition 4.8 and Fact 4.4 □

Proposition 4.11.

$$n_{2k+1} := |\{S \subseteq 2^{[n]} : \lambda_S = (-1)^k \mu_{2k+1}\}|$$

Then

$$n_1 < n_3 < \ldots < n_{2m+1} \text{ and } n_{2k+1} < 4\binom{4m}{2k+1}$$

4.2 Proof of Lemma 3.4

By Proposition 4.10 and Proposition 4.11, we have

$$-r_n \leqslant \frac{2}{N^2}\sum_{k=0}^{m} n_{2k+1}\mu_{2k+1}\mu_{2k+3}$$

By Proposition 4.8, Proposition 4.10, Proposition 4.11 and Corollary 4.6,

$$\begin{aligned}
\frac{2}{N^2} n_{2k+1}\mu_{2k+1}\mu_{2k+3} &= \frac{1}{2\pi^2} n_{2k+1} B\Big(\frac{2k+1}{2}, \frac{4m+1-2k}{2}\Big) B\Big(\frac{2k+3}{2}, \frac{4m-1-2k}{2}\Big) \\
&\leqslant \frac{2}{\pi^2}\binom{4m}{2k+1} B\Big(\frac{2k+1}{2}, \frac{4m+1-2k}{2}\Big) B\Big(\frac{2k+3}{2}, \frac{4m-1-2k}{2}\Big) \\
&= \frac{2}{\pi^2}\frac{1}{4m+1}\frac{B(\frac{2k+1}{2}, \frac{4m+1-2k}{2}) B(\frac{2k+3}{2}, \frac{4m-1-2k}{2})}{B(2k+2, 4m-2k)} \\
&\leqslant \frac{2C^3}{\pi^2}\frac{1}{4m+1}\frac{\frac{(\frac{2k+1}{2})^k(\frac{4m+1-2k}{2})^{2m-k}}{(2m+1)^{2m+\frac{1}{2}}}\frac{(\frac{2k+3}{2})^{k+1}(\frac{4m-1-2k}{2})^{2m-k-1}}{(2m+1)^{2m+\frac{1}{2}}}}{\frac{(2k+2)^{2k+\frac{3}{2}}(4m-2k)^{4m-2k-\frac{1}{2}}}{(4m+2)^{4m+\frac{3}{2}}}} \\
&= \frac{4C^3}{\pi^2}\frac{\sqrt{4m+2}}{4m+1}\frac{1}{\sqrt{(2k+2)(4m-2k)}}\frac{(2k+1)^k(2k+3)^{k+1}}{(2k+2)^{2k+1}} \\
&\quad \frac{(4m+1-2k)^{2m-k}(4m-1-2k)^{2m-k-1}}{(4m-2k)^{4m-2k-1}} \\
&\leqslant \frac{C_1}{\sqrt{4m}}\frac{1}{2m+1}\frac{1}{\sqrt{\frac{2k+2}{4m+2}(1-\frac{2k+2}{4m+2})}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
-r_n &\leqslant \frac{2}{N^2}\sum_{k=0}^{m} n_{2k+1}\mu_{2k+1}\mu_{2k+3} \\
&\leqslant \frac{C_1}{\sqrt{4m}}\sum_{k=0}^{m}\frac{1}{2m+1}\frac{1}{\sqrt{\frac{2k+2}{4m+2}(1-\frac{2k+2}{4m+2})}}
\end{aligned}$$

Since

$$\lim_{m\to+\infty}\sum_{k=0}^{m}\frac{1}{2m+1}\frac{1}{\sqrt{\frac{2k+2}{4m+2}(1-\frac{2k+2}{4m+2})}} = \int_0^{\frac{1}{2}}\frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{2}$$

We conclude that

$$-r_n \leqslant \frac{C_2}{\sqrt{4m}} = \frac{C_2}{\sqrt{n}} = o(1)$$

as desired.

5 Conclusion

In this work, we resolved the conjecture of Rob Morris, showing that for any bijection $f : \{-1,1\}^n \to \{-1,1\}^n$, the probability

$$\mathbf{Pr}_{x,y\in\{-1,1\}^n}\left[\langle x,y\rangle \geqslant 0 \text{ and } \langle f(x),f(y)\rangle \geqslant 0\right] \geqslant \frac{1}{4} - O(\frac{1}{\sqrt{n}}).$$

Our proof proceeds by analyzing the spectrum of the Hamming association scheme, and reducing the problem to optimization over the Birkhoff polytope. Some interesting future directions and open questions are as follows:

Problem 5.1. Extend the result to the setting of $r > 2$ bijections. That is, given any collection of r bijections $f_1, f_2, \cdots, f_r$, what can we say about

$$\mathbf{Pr}_{x,y\in\{-1,1\}^n}\left[\bigwedge_{i=1}^{r} \langle f_i(x), f_i(y)\rangle \geqslant 0\right]?$$

Our methodology completely resolves the $r = 2$ case, but for $r > 2$, it is not clear how to extend it. In particular, it is not too difficult to generalize our method to the r-variable case and attain the lower bound

$$\mathbf{Pr}_{x,y\in\{-1,1\}^n}\left[\bigwedge_{i=1}^{r} \langle f_i(x), f_i(y)\rangle \geqslant 0\right] \geqslant \left(\frac{1}{2}\right)^r + \frac{1}{N^r}\min_{X\in\mathcal{D}_N^{(r)}} \langle X, \lambda^{\otimes r}\rangle$$

where $\lambda \in \mathbb{R}^N$ has its i^{th} entry corresponding to the i^{th} largest eigenvalue of M and $\mathcal{D}_N^{(r)}$ is the set of tensors given by

$$\mathcal{D}_N^{(r)} = \left\{ T \in \mathbb{R}_{\geqslant 0}^{N^r} : \sum_{j=1}^{N} T_{i_1,\ldots,i_{k-1},j,i_{k+1},\ldots,i_r} = 1 \text{ for all } k \in [r], \ (i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_r) \in [N]^{r-1} \right\},$$

which is the natural extension of doubly stochastic matrices to higher-order tensors, requiring that every axis-parallel line sums to 1. The main difficulty lies in lower bounding the linear program

$$\frac{1}{N^r}\min_{X\in\mathcal{D}_N^{(r)}} \langle X, \lambda^{\otimes r}\rangle,$$

since for $r > 3$ no analytic solution is available. Numerical evidence suggests that $-o(1)$ remains a valid lower bound, but a rigorous proof is still missing. Moreover, while the tensor structure of the LP ensures—via a standard cycling argument—the existence of an integral solution, the absence of a suitable "high-dimensional rearrangement inequality" prevents us from characterizing the objective explicitly.

Another related research direction (proposed by Rob Morris) is the following:

Problem 5.2. Can we characterize subsets $A, B \subseteq \{0,1\}^n$ for which analogous lower bounds continue to hold? In particular, the lemma in [BBC+24] implies that if $f : A \to B$ is a bijection and

$$\mathbf{Pr}\big[\langle x, y\rangle \geqslant 0 \text{ and } \langle f(x), f(y)\rangle \geqslant 0\big] = o(1),$$

then one of A or B must be "clustered." Can this condition be weakened to only requiring the probability to be less than $1/4 - c$ for some constant $c > 0$?

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References

[BBC+24] Paul Balister, Béla Bollobás, Marcelo Campos, Simon Griffiths, Eoin Hurley, Robert Morris, Julian Sahasrabudhe, and Marius Tiba. Upper bounds for multicolour ramsey numbers, 2024. 1, 12

[GM16] Chris Godsil and Karen Meagher. In *Erdős–Ko–Rado Theorems: Algebraic Approaches*. Cambridge University Press, Cambridge, 2016. 3