Comment on A new chaotic attractor and its robust function projective synchronization by Kim et al.

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Abstract

In the paper entitled "A new chaotic attractor and its robust function projective synchronization" in [Nonlinear Dyn., 73 (2013) 1883-1893], D. Kim, P. Chang and S. Kim proposed a new three-dimensional chaotic system: $\dot{x} = a(y - x)$, $\dot{y} = -xz + cy$, $\dot{z} = y^2 - bz$. Combining theoretical analysis, numerical simulation and circuit realisation, they studied the complex dynamical properties of that system, i.e., chaotic attractor, stability, bifurcation, robust function projective synchronization, circuit design and implementation, etc. In particular, the authors formulated a conclusion on the stability and Hopf bifurcation of $P_{\pm} = (\pm\sqrt{bc}, \pm\sqrt{bc}, c)$. However, by theory analyzing, we contend that both the conclusion itself and the derivation of its proof are wrong. To this end, we firstly derive the right result by Routh-Hurwitz criterion and Projection Method. Secondly, that system undergoes Bautin bifurcation (generalized or degenerate Hopf bifurcation) at $P_{\pm}$ when parameters a, b, c satisfy the golden proportion $a = \frac{1+\sqrt{5}}{2}c$, $b = \frac{-1+\sqrt{5}}{2}c$. Finally, a hidden Lorenz-like attractor coexisting with one saddle in the origin and two stable equilibria is coined based on bifurcation diagrams.

2020 Mathematics Subject Classification: 34C45; 34C37; 65P20; 65P30; 65P40.

Keywords: New Lorenz-like system; Hopf bifurcation; hidden attractor; Projection Method.

1 Introduction

In 2013, Kim et al. [1] reported a new three-dimensional quadratic autonomous chaotic system given by

$$\begin{cases} \dot{x} &= a(y - x), \\ \dot{y} &= cy - xz, \\ \dot{z} &= -bz + y^2, \end{cases} \tag{1.1}$$

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where $a, c, b \in \mathbb{R}$. In contrast to the generalized Lorenz system [2], system (1.1) is not topologically equivalent. It displays a chaotic attractor when $(a, b, c) = (35, 3, 25)$ and $(x_0, y_0, z_0) = (1, 2, 1)$, whose Lyapunov exponents are $(\lambda_{LE_1}, \lambda_{LE_2}, \lambda_{LE_3}) = (2.7545, 0.0165, -15.7712)$ and Lyapunov dimension is $D_{KY} = 2.1757$. Refer to [1], Kim et al. uncovered some rich dynamics of it from viewpoint of local and global behaviors, including stability, bifurcation and function projective synchronization, etc.

Except for generating limit cycles, Hopf bifurcation is an important route to chaos for the case of either self-excited or hidden chaotic attractors [3,4]. From the application point of view, Hopf bifurcation theory involves the nonlinear oscillation circuits, railway vehicles, food chain systems, among others, as shown in the references [5–10].

However, as for what one points out next, the result on stability and Hopf bifurcation is erroneous. Detailed discussions follow.

The rest of this comment's structure is the following: Section 2 presents the main results. Finally, some conclusions are drawn.

2 The main results

Referring to [1, Eq. (8), p. 1885], the right Jacobian matrix of system (1.1) at $P_\pm = (\pm\sqrt{bc}, \pm\sqrt{bc}, c)$ is

$$\begin{bmatrix} -a & a & 0 \\ -c & c & \mp\sqrt{bc} \\ 0 & \pm 2\sqrt{bc} & -b \end{bmatrix},$$

rather than

$$\begin{bmatrix} -a & a & 0 \\ -c & c & \mp\sqrt{bc} \\ \pm 2\sqrt{bc} & 0 & -b \end{bmatrix},$$

which suggests the breakdown of the story of the stability and Hopf bifurcation of $P_\pm$.

In fact, the characteristic equation of $P_\pm$ is

$$\lambda^3 + (a + b - c)\lambda^2 + b(a + c)\lambda + 2abc = 0. \tag{2.1}$$

Notice that the parameters a, b and c belong to the set

$$W = \{(a, b, c) \in \mathbb{R}^3 | a \neq 0, bc > 0\}.$$

For convenience of discussion in the sequel, one divides the set W into W_1 and W_2 with $W_1 = \{(a, b, c) \in W : a + b - c > 0, b(a + c) > 0, 2abc > 0\}$ and $W_2 = W \setminus W_1$.

Again divide the set W_1 into the subsets W_1^1, W_1^2 and W_1^3 as follows.

$$\begin{aligned} W_1^1 &= \{(a, b, c) \in W_1 : \Delta < 0\}, \\ W_1^2 &= \{(a, b, c) \in W_1 : \Delta = 0, b = b_*\}, \\ W_1^3 &= \{(a, b, c) \in W_1 : \Delta > 0\}, \end{aligned}$$

where $\Delta = b[(a + c)(a + b - c) - 2ac]$ and $b_* = c - a + \frac{2ac}{a+c}$. The following assertion holds concerning the stability and Hopf bifurcation of $P_\pm$.

Proposition 2.1. *(1) $P_\pm$ are unstable when $(a,b,c) \in W_2 \cup W_1^1$ whereas $P_\pm$ are asymptotically stable when $(a,b,c) \in W_1^3$.*

(2) For $(a,b,c) \in W_1^2$, system (1.1) undergoes Hopf bifurcation at $P_\pm$. Further, the first Lyapunov constant (or coefficient) of system (1.1) at $P_\pm$ is given by

$$l_1(a,c,b_*) = -\frac{a(a+c)^2(a^2-ac-c^2)(a^2-2ac-c^2)(3a^3+a^2c-3ac^2-2c^3)}{c(a^4-5a^2c^2-4ac^3-c^4)(a^4-8a^2c^2-4ac^3-c^4)}. \tag{2.2}$$

Further,

(i) If $l_1(a,c,b_*) < 0$, then the Hopf bifurcation at $P_\pm$ is subcritical, and the stable periodic orbit exists for $b < b_*$.

(ii) If $l_1(a,c,b_*) = 0$, one has to compute the second Lyapunov constant (or coefficient) to determine the stability of the bifurcated periodic orbit.

(iii) If $l_1(a,c,b_*) > 0$, then the Hopf bifurcation at $P_\pm$ is supercritical, and the unstable periodic orbit exists for $b > b_*$.

Proof. (1) The stability of $P_\pm$ easily follows from the Routh-Hurwitz criterion.

(2) Due to the symmetry of $P_\pm$, one only needs to verify the following two conditions for the occurrence of Hopf bifurcation of P_+.

(2.1) Transversality

It follows from the relation between roots and coefficients of a polynomial equation with degree 3 that Eq. (2.1) has one negative real root $\lambda_1 = -\frac{2ac}{a+c}$ and a pair of conjugate purely imaginary roots $\lambda_{2,3} = \pm\omega i$ with $\omega = \sqrt{2ac+c^2-a^2}$. Then, calculating the derivative on both sides of Eq. (1.1) with respect to the parameter b and substituting $\lambda = \omega i$ into the derivative yield

$$\left.\frac{dRe(\lambda_2)}{db}\right|_{b=b_*} = -\frac{w^2}{2[w^2+(b_*+a-c)^2]} < 0.$$

Hence, the transversal condition holds.

(2.2) Nondegeneracy

First of all, the linear transformation

$$T : (x,y,z) \to (x+\sqrt{b_*c}, y+\sqrt{b_*c}, z+c)$$

converts system (1.1) into the resulting equivalent system

$$\begin{cases} \dot{x} &= a(y-x), \\ \dot{y} &= -cx+cy-\sqrt{b_*c}z-xz, \\ \dot{z} &= 2\sqrt{b_*c}y-b_*z+y^2, \end{cases} \tag{2.3}$$

and its equilibria P_+ are thus transformed into the origin P_0 of the system (2.3).

Therefore, the Jacobian matrix of the system (2.3) at P_0 is given by

$$A = \begin{pmatrix} -a & a & 0 \\ -c & c & -\sqrt{b_*c} \\ 0 & 2\sqrt{b_*c} & -b_* \end{pmatrix}$$

and the eigenvalues corresponding to matrix A are

$$\lambda_1 = -\frac{2ac}{a+c}, \quad \lambda_{2,3} = \pm\omega i.$$

By some tedious calculations, one finds that the vectors

$$p = \frac{1}{J}\begin{pmatrix} c \\ -a+\omega i \\ \frac{\sqrt{b_* c}[ab+\omega^2+(a-b_*)\omega i]}{b_*^2+\omega^2} \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} a \\ a+\omega i \\ \frac{2\sqrt{b_* c}[ab_*+\omega^2+(b_*-a)\omega i]}{b_*^2+\omega^2} \end{pmatrix}$$

satisfy $Aq = i\omega q, \quad A^T p = -i\omega p, \quad \langle p, q\rangle = \sum_{i=1}^{3} \bar{p}_i q_i = 1$, where

$$J = ac - (a-\omega i)^2 + \frac{2b_* c[ab_* + \omega^2 + (a-b_*)\omega i]^2}{b_*^2+\omega^2}.$$

Meanwhile, one arrives at the following multi-linear symmetric functions

$$B(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} 0 \\ -(x_1 y_3 + x_3 y_1) \\ 2x_2 y_2 \end{pmatrix}$$

and

$$C(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Performing further computations, one obtains

$$h_{11} = \begin{pmatrix} -\frac{\sqrt{a+c}(-a^3+2a^2c+3ac^2+c^3)}{\sqrt{c^3}\sqrt{-a^2+2ac+c^2}} \\ -\frac{\sqrt{a+c}(-a^3+2a^2c+3ac^2+c^3)}{\sqrt{c^3}\sqrt{-a^2+2ac+c^2}} \\ -\frac{2a(a+c)}{c} \end{pmatrix},$$

h_{20} and others, whose expressions are too long to present here.

At last, substituting the above calculated results into the expression in [12, Definition, Eq. (3.20), p. 99], one obtains the first Lyapunov constant (or coefficient) given by (2.2). □

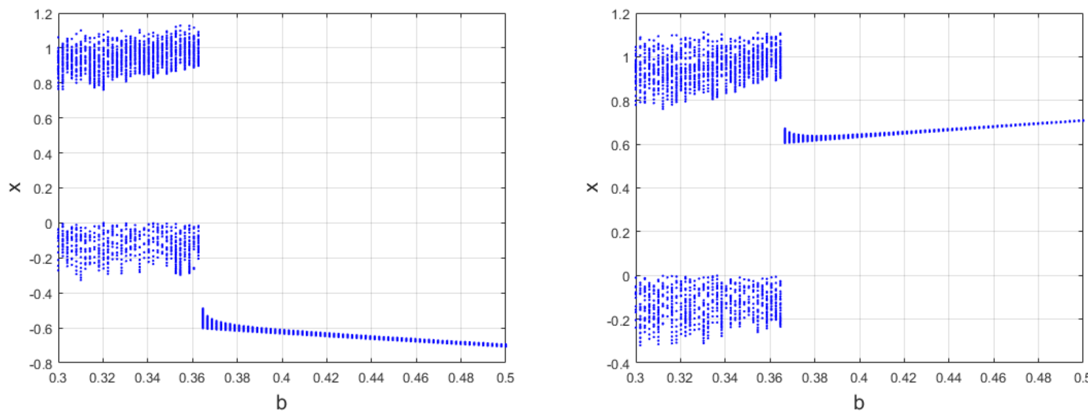
Applying Projection Method in [11–15] in the following proposition, one identifies the sign of the second Lyapunov coefficient on the straight line where the first coefficient vanishes.

Proposition 2.2. *For the parameter values on the straight line* $\{a, c, b\} = \{\frac{1+\sqrt{5}}{2}c, c, \frac{-1+\sqrt{5}}{2}c\}$ *determined in Proposition 2.1, the three-parameter family of differential Eq. (1.1) has a transversal Hopf point of codimension 2 at* $P_\pm$*. The second Lyapunov coefficient or constant:*

$$l_2(c) = -15.0249c^3 - 22.9894c^2 - 108.7893c - 1.4656. \tag{2.4}$$

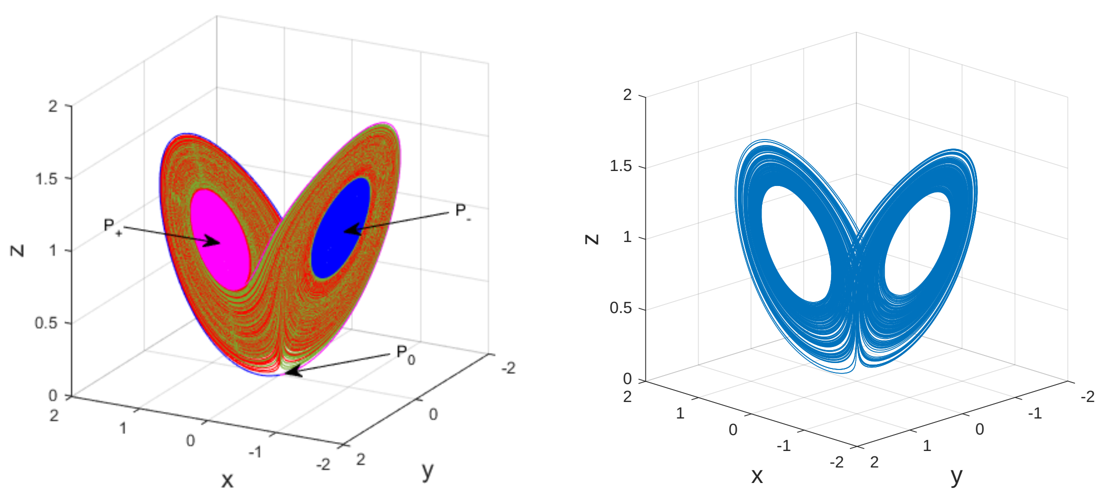
Since $l_2 < 0$ *for* $c > 0$*,* $P_\pm$ *are stable.*

Set $(a,b)=(2,1)$. It follows from Proposition 2.1 that $P_{\pm}$ are asymptotically stable when $b>\frac{1}{3}$. Choosing the initial point $(x_0^1,y_0^1,z_0^1)=(10,1,160)\times 10^{-9}$ and $(x_0^2,y_0^2,z_0^2)=(0.3833,0.4033,0.63)$ and using MATLAB's procedure ode45 with relative tolerance 10^{-8} and absolute tolerance 10^{-8}, Fig. 2.1 demonstrates bifurcation diagrams versus b. Based on Fig. 2.1, a hidden Lorenz-like attractor is coined, as depicted in Fig. 2.2-2.3.



(a) (b)

Fig. 2.1: When $(a,c)=(2,1)$, bifurcation diagrams versus b with initial conditions (a) $(x_0^1,y_0^1,z_0^1)=(10,1,160)\times 10^{-9}$, (b) $(x_0^2,y_0^2,z_0^2)=(0.3833,0.4033,0.63)$.



(a) (b)

Fig. 2.2: When $(a,c,b)=(2,1,0.364)$, (a) initial points $(x_0^{1,3},y_0^{1,3},z_0^{1,3})=(\pm 10,\pm 1,\pm 160)\times 10^{-9}$ and $(x_0^{2,4},y_0^{2,4},z_0^2)=(\pm 0.3833,\pm 0.4033,0.63)$, time interval: $T=[0,5000]$, (b) $(x_0^2,y_0^2,z_0^2)=(0.3833,0.4033,0.63)$, $T=[0,6\times 10^6]$, we locate a hidden Lorenz-like attractor with Lyapunov exponents $(\lambda_{LE_1},\lambda_{LE_2},\lambda_{LE_3})=(0.090144,0.000001,-1.454144)$ shown in Fig. 2.3.

3 Conclusion

In the commented paper [1], Kim et al. investigated complex dynamics of a 3D Lorenz-like system, i.e., chaotic attractor, stability, bifurcation, robust function projective synchronization, circuit design and implementation, etc. Unfortunately, in the present work, we point out that the result on the stability

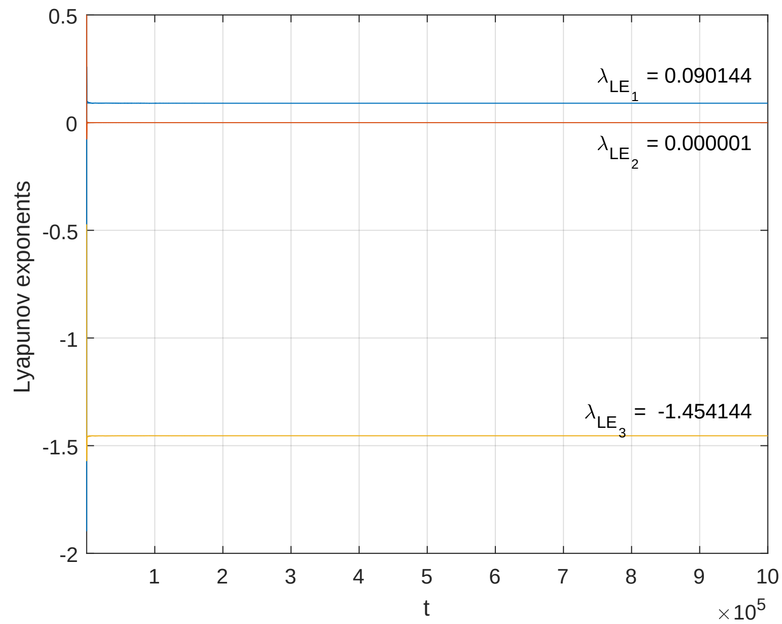


Fig. 2.3: Lyapunov exponents of the hidden Lorenz-like attractor for $(a, c, b) = (2, 1, 0.364)$ and $(x_0^2, y_0^2, z_0^2) = (0.3833, 0.4033, 0.63)$.

and Hopf bifurcation is completely wrong. Moreover, the right one is derived by Routh-Hurwitz criterion and Projection Method. To one's surprise, for the golden proportion $a = \frac{1+\sqrt{5}}{2}c$ and $b = \frac{-1+\sqrt{5}}{2}c$, that system undergoes Bautin bifurcation (or generalized or degenerate Hopf bifurcation) at $P_{\pm}$. In addition, a hidden Lorenz-like attractor is numerically found based on bifurcation diagrams versus b.

Acknowledgements

This work is supported in part National Natural Science Foundation of China under Grant 12001489, in part Zhejiang public welfare Technology Application Research Project of China under Grant L-GN21F020003, in part Natural Science Foundation of Taizhou University under Grant T20210906033, in part NSF of Zhejiang Province under Grant LQ18A010001.

Conflicts of Interest

The authors declare no conflict of interest. The founding sponsors had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript and in the decision to publish the results.

Data availability

There is no data because the results obtained in this paper can be reproduced based on the information given in this paper.

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