Poisson Processes

Definition 1 (Markov property) *The sequence of random variables (Y_n) is a discrete time Markov chain with state space $\mathbb{S}$ if it satisfies the Markov property*

$$\mathbb{P}(Y_n = y_n \,|\, Y_0 = y_0, \ldots, Y_{n-1} = y_{n-1}) = \mathbb{P}(Y_n = y_n \,|\, Y_{n-1} = y_{n-1}).$$

As shown in Part A, provided the Markov chain is time-homogeneous, then the distribution of (Y_n) is entirely specified by an initial distribution $\mu_s = \mathbb{P}(Y_0 = s)$, and a transition matrix $P = (P_{st})_{s,t\in\mathbb{S}}$.

Definition 2 *Let (Z_n) be a sequence of i.i.d. $\mathrm{Exp}(\lambda)$ distributed random variables for $\lambda > 0$. With*

$$T_n = \sum_{k=1}^{n} Z_k,$$

we define (X_t) for $t \geq 0$ by

$$X_t = \sup\{n \geq 1 : T_n \leq t\}.$$

(X_t) is called a Poisson process of rate λ, written $(X_t) \sim \mathrm{PP}(\lambda)$.

Theorem 1 *Take $(X_t) \sim \mathrm{PP}(\lambda)$. For $t \geq 0$, $(X_r)_{r\leq t}$ and $(X_r)_{r\geq t}$ are conditionally independent on X_t, and $(X_{s+t})_{s\geq 0} \sim \mathrm{PP}(\lambda)$ started from X_t.* ***rewrite this***.

Continuous-time Markov chains

Definition 3 *A Q-matrix or generator is a matrix $Q = (q_{ij})$ for $i, j \in \mathbb{S}$ a countable state space, such that*

i. for $i \in \mathbb{S}$, $q_{ii} \in (\infty, 0]$,
ii. for $i, j \in \mathbb{S}$, $i \neq j$, $q_{ij} \in [0, \infty)$, and
iii. for $i \in \mathbb{S}$, $\sum_{j\in\mathbb{S}} q_{ij} = 0$.

We often write $q_i := -q_{ii}$.

We construct from this a stochastic matrix $\Pi = (\pi_{ij})$ such that for $q_i \neq 0$, $j \in \mathbb{S}$,

$$\pi_{ij} := (1 - \delta_{ij})\frac{q_{ij}}{q_i}$$

and for $q_i = 0$, $j \in \mathbb{S}$,

$$\pi_{ij} = \delta_{ij}.$$

Definition 4 (Jump chain) *A minimal right-continuous process (X_t) is a continuous time Markov chain with initial distribution ν and Q-matrix Q, $(X_t) \sim \mathrm{Markov}(\nu, Q)$, if*

i. (Y_n) is a discrete Markov chain with initial distribution ν, transition matrix Π;
ii. conditional on $Y_0 = i_0, Y_1 = i_1, \ldots, Y_{n-1} = i_{n-1}$, the holding times $Z_1, Z_2, \ldots, Z_{n-1}$ are independent exponential random variables distributed with parameters $q_{i_0}, q_{i_1}, \ldots, q_{i_{n-1}}$; and
iii. with $T_n := \sum_{k=1}^{n} Z_k$,

$$X_t = \begin{cases} Y_n & \text{if } T_n \leq t < T_{n+1} \\ \infty & \text{otherwise.} \end{cases}$$