Banach and Hilbert spaces

Definition 1 (Banach space) *A normed space* $(X, ||\cdot||)$ *is a Banach space if it is complete (every Cauchy sequence in* X *converges).*

Note that this means it is sufficient for completeness if a subspace is closed.

As a point to consider initially, we analyse certain vector spaces of bounded functions $f : \Omega \to \mathbb{F}$, such as:

- $\mathcal{F}^b(\Omega) := \{f : \Omega \to \mathbb{F} \text{ bounded}\}$,
- $C_b(\Omega) := \{f : \Omega \to \mathbb{F} \text{ continuous and bounded}\}$,

under the supremum norm, or the $L^p(\Omega)$ spaces under the L^p norm. Note that in order to make this a norm, the vector space is not precisely one of functions, but a quotient space $L^p(\Omega) := \mathcal{L}^p(\Omega)/\mathcal{N}$ where

$$\mathcal{L}^p := \left\{ f : \Omega \to \mathbb{F} \,\middle|\, \int_\Omega |f|^p \,\mathrm{d}x < \infty \right\}$$
$$\mathcal{N} := \left\{ f : \Omega \to \mathbb{F} \,\middle|\, \int_\Omega f \,\mathrm{d}x = 0 \right\}$$

Definition 2 (Hilbert space) *An inner product space is complete with the induced norm, it is a Hilbert space.*