Measurable sets and functions

Definition 1 (σ-algebras) *Let Ω be a set and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a collection of subsets of Ω:*
1. *$\mathcal{A}$ is an* algebra *if $\varnothing \in \mathcal{A}$ and for $A, B \in \mathcal{A}$, $\mathcal{A}^c = \Omega \setminus A \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.*
2. *$\mathcal{A}$ is a* σ-algebra *if $\varnothing \in \mathcal{A}$, for $A \in \mathcal{A}$, $A^c \in \mathcal{A}$, and for (A_n) a sequence of sets in $\mathcal{A}$, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.*

A collection of sets is an algebra subject to being closed under finite applications of the basic operators. The σ-algebra concept extends this slightly to infinite ones. Consider where this distinction is relevant?

Note that if we have $\{\mathcal{F}_i : i \in I\}$ are σ-algebras, then

$$\mathcal{F} = \bigcap_{i \in I} \mathcal{F}_i$$

is a σ-algebra. This allows us to consider the notion of a smallest σ-algebra containing a set (the σ-algebra 'generated' by a set). We write the σ-algebra generated by a collection of collections of sets $\mathfrak{A}$ as $\sigma(\mathfrak{A})$.

Definition 2 (Borel σ-algebra) *Let $(E, \mathcal{T})$ be a topological space. The σ-algebra generated by the open sets in E is called the* Borel σ-algebra *on E and is denoted $\mathcal{B}(E) = \sigma(\mathcal{T})$.*

Definition 3 *Suppose $(\Omega_i, \mathcal{F}_i)_{i \in I}$ are measurable spaces. With $\Omega = \prod_{i \in I} \Omega_i$, $\mathcal{F}$ the σ-algebra generated by $A = \prod_{i \in I} A_i$ where $A_i \in \mathcal{F}_i$ for all $i \in I$ and for all but finitely many $i \in I$, $A_i = \Omega_i$: $(\Omega, \mathcal{F})$ is the product space.*

This space is measurable, and $\mathcal{F}$ is a σ-algebra.

Definition 4 (π and λ-systems) *A collection of sets $\mathcal{A}$ is called a π-system if it is closed under intersections.*

A collection of sets $\mathcal{M}$ is called a λ-system if $\Omega \in \mathcal{M}$, if $A, B \in \mathcal{M}$, $A \subseteq B$, then $B \setminus A \in \mathcal{M}$, and if $(A_n) \subseteq \mathcal{M}$ with $A_n \subseteq A_{n+1}$ increasing then $\bigcup_{n \geq 1} A_n \in \mathcal{M}$.

A collection of sets is a σ-algebra if and only if it is both a π-system and a λ-system.

Lemma 1 (π-λ systems lemma) *Let $\mathcal{A}$ be a π-system and $\mathcal{M}$ a λ-system. Then if $\mathcal{A} \subseteq \mathcal{M}$ then $\sigma(\mathcal{A}) \subseteq \mathcal{M}$.*

We can use this with a convenient π-system to show that our λ-system contains more than is immediately obvious.

Let $\lambda(\mathcal{A})$ be the smallest λ-system containing $\mathcal{A}$. This is a subset of $\mathcal{M}$ and $\sigma(\mathcal{A})$, so we just need to show that $\lambda(\mathcal{A})$ is a σ-algebra (for which we just have to show that it is a π-system).

Definition 5 (Random variables) *With measurable spaces $(\Omega, \mathcal{F})$, $(E, \mathcal{E})$, a function $f : \Omega \to E$ is said to be an E-valued random variable (or a measurable function) if for all $A \in \mathcal{E}$, $f^{-1}(A) \in \mathcal{F}$.*

We get immediately that random variables can be composed as one would expect. We can also use random variables to define new σ-algebras. Note that $(\Omega, \{f^{-1}(A) : A \in \mathcal{E}\})$ is a σ-algebra.

Definition 6 *With $\{f_i : i \in I\}$ a family of functions $\Omega \to E$, $\sigma(f_i : i \in I)$ is the smallest σ-algebra on Ω for which all f_i are measurable.*

This is initially a slightly intimidating definition, but the intuition is just that we need our $\sigma(f_i : i \in I) = \sigma(f_i^{-1}(A) : A \in \mathcal{E}, i \in I)$.

Theorem 2 (Monotone Class Theorem) *Let $\mathcal{H}$ be a class of bounded functions from $\Omega \to \mathbb{R}$ such that*
- *$\mathcal{H}$ is a vector space over $\mathbb{R}$,*
- *The constant function $1 \in \mathcal{H}$,*
- *If $(f_n) \subseteq \mathcal{H}$, $f_n \to f$ monotonically increasing, then $f \in \mathcal{H}$,*

then if $\mathcal{C} \subseteq \mathcal{H}$, and $\mathcal{C}$ is closed under multiplication, then all $\sigma(\mathcal{C})$-bounded functions are in $\mathcal{H}$.