

Probability Space

Any probability space is a triple $\langle \Omega, \mathcal{F}, \mathbb{P} \rangle$ where

- Ω is the sample space – it exists as a set from which \mathcal{F} may be formed.
- \mathcal{F} is the set of events – an event being a subset of Ω , satisfying
 - \mathbf{F}_1 (universal set): $\Omega \in \mathcal{F}$.
 - \mathbf{F}_2 (closed under complementation): $A \in \mathcal{F}$ implies that $\Omega \setminus A \in \mathcal{F}$.
 - \mathbf{F}_3 (closed under countable unions): If $A_k \in \mathcal{F}$ for every $k \in \mathbb{N}$, then $\bigcup_{k \in \mathbb{N}} A_k \in \mathcal{F}$.

Equivalently, \mathcal{F} is a σ -algebra with respect to Ω . This means \mathcal{F} is measurable, so a non-negative real value may be assigned to each element of \mathcal{F} .

- \mathbb{P} is a measure on \mathcal{F} satisfying
 - \mathbf{P}_1 (non-negative): For all $A \in \mathcal{F}$, $\mathbb{P}(A) \geq 0$.
 - \mathbf{P}_2 (1 on whole space): $\mathbb{P}(\Omega) = 1$.
 - \mathbf{P}_3 (σ -additivity): If $A_k \in \mathcal{F}$ for every $k \in \mathbb{N}$, and $A_i \cap A_j = \emptyset$ for every $i \neq j$, then $\mathbb{P}\left(\bigcup_{k \in \mathbb{N}} A_k\right) = \sum_{k \in \mathbb{N}} \mathbb{P}(A_k)$.

The first immediate consequence of \mathbf{P}_3 is the formula for $\mathbb{P}(A \setminus B)$: $\mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) = \mathbb{P}(A)$. Consequently, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. The inclusion-exclusion formula is then derived with $S_n = \bigcup_{k=1}^n A_k$ by induction.

Random Variables

A discrete random variable is a function from samples to values $X : \Omega \rightarrow \mathbb{R}$, such that for every value in $\text{Im}X$ there is an event to which it corresponds (Equivalently, there is a well-defined $f : \mathbb{R} \rightarrow \mathcal{F}$ where $f(x) = \{\omega \in \Omega \mid X(\omega) = x\}$, $f(\mathbb{R})$ is a partition of \mathcal{F}). Additionally, $\text{Im}X$ is countable.

In the case where the random variable is continuous, then the notion of $X : \Omega \rightarrow \mathbb{R}$ as a function becomes more general. In this case we redefine $f : \mathbb{R} \rightarrow \mathcal{F}$ to $f(x) = \{\omega \in \Omega \mid X(\omega) \leq x\}$. Note here that $\text{Im}X$ need not be countable.

Given that this occurs in conjunction with \mathbf{F}_3 and \mathbf{P}_3 , if \mathcal{F} was merely a set of uncountable singletons, then one would not be able to apply σ -additivity. Instead, we ensure that \mathcal{F} is formed of uncountable sets.

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