Itô calculus

Theorem 1 *Let (X_t) be a super-martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, $D \subseteq [0, \infty)$ countable:*

- *The process of right limits $Z_t = \lim_{\substack{s \in D \\ s \downarrow t}} X_s$ of (X_t) in D is well-defined a.s., is $\mathcal{F}_t^+$-measurable, and has left limits a.s..*
- $\mathbb{E}\big[Z_t \,|\, \mathcal{F}_t\big] \leq X_t$ *a.s..*
- *(Z_t) is a super-martingale a.s. on $(\Omega, \mathcal{F}, (\mathcal{F}_t^+), \mathbb{P})$.*

Note that (Z_t) can still differ somewhat from (X_t) if (X_t) is not a martingale (and in particular, if $t \mapsto \mathbb{E}\big[X_t\big]$ is not right-continuous), as we possibly don't have that (Z_t) is a modification of (X_t).

Definition 1 (Usual conditions) *A filtration $(\mathcal{F}_t)$ satisfies the usual conditions if $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $(\mathcal{F}_t)$ is right-continuous, and $\mathbb{P}^{-1}(\{0\}) \subseteq \mathcal{F}_t$ for all $t \geq 0$.*

Definition 2 (Local martingales) *(M_t) is a local martingale if it is right-continuous (to ensure we can talk about M_T for T a stopping time), and there is a sequence (T_n) of stopping times such that $T_n \to \infty$ as $n \to \infty$ a.s., and $(M_t^{T_n})$ is a martingale for all $n \geq 1$.*

Right-continuity is necessary here to ensure that $X_T \chi_{T<\infty}$ is $\mathcal{F}_T$-measurable, because otherwise we may have stopping times which provide information that is strictly known in the future, and consequently the value of the stopped random variable is conditional on the future. For an example, take $T = \inf\{t : X_t = 1\}$, and if X_t **Finish**

A continuous local martingale stopped at hitting times is a martingale, since **fill**.

A continuous local martingale stopped at deterministic times is a martingale, since **fill**.

Theorem 2 (Itô's formula)

The proof of this formula is derived almost entirely via the Stone-Weierstrass theorem stating that polynomials on compact sets $K \subseteq \mathbb{R}^n$ are dense in $C(K)$. Consequently, it should be emphasised that, contrary to how the proof is often stated, we do not require that $f \in C^2(\mathbb{R})$, but merely $C^2(K)$ where K is of sufficient size to be well-defined in the context of the formula.

Theorem 3 *Let X be a continuous local martingale starting at 0, $T > 0$:*

- *for $p > 1$*

$$(4p)^{-p}\mathbb{E}\,[\langle]\, X\rangle_T^p \leq \mathbb{E}\,[|]\sup_{t \leq T} X_t|^{2p} \leq (2ep^2)^p \mathbb{E}\,[\langle]\, X\rangle_T^p$$

- *for $p = 1$*

$$\mathbb{E}\,[\langle]\, X\rangle_T \leq \mathbb{E}\,[|]\sup_{t \leq T} X_t|^2 \leq 4\mathbb{E}\,[\langle]\, X\rangle_T$$

- *For $0 < p < 1$*

$$p^{2p}\mathbb{E}\,[\langle]\, X\rangle_T^p \leq \mathbb{E}\,[|]\sup_{t \leq T} X_t|^{2p} \leq \left(\frac{16}{p}\right)^p \mathbb{E}\,[\langle]\, X\rangle_T^p$$

Stochastic Differential Equations

Lemma 4 *Let $X_t = M_t + A_t$ with M a continuous local martingale, A an adapted continuous process with finite total variation, $X_0 = 0$. Then the equation*

$$\mathrm{d}Z_t = Z_t \mathrm{d}X_t, \quad Z_0 = 1$$

is solved by

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}\langle M\rangle_t\right).$$

Theorem 5 (Novikov) *Let (M_t) be a continuous local martingale with $M_0 = 0$. If*

$$\mathbb{E}\left[\,]\,[\exp\left(\frac{1}{2}\langle M\rangle_T\right)\right]$$

then $\mathcal{E}(M)$ is a martingale up to time T.

Theorem 6 (Girsanov's theorem) *Let (M_t) be a continuous local martingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ up to time T. Then*

$$X_t = M_t - \int_0^t \frac{\mathrm{d}\langle M, Z\rangle_s}{Z_s}$$

is a continuous local martingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ up to time T.

Note we require that $\mathbb{Q} \ll \mathbb{P}$ in order for this quantity to be defined.

Equivalently, $M_t - \langle M, L\rangle_t$ is a continuous local martingale with respect to $\mathbb{Q}$, where (L_t) satisfies

$$\mathrm{d}L_t = \frac{\mathrm{d}Z_t}{Z_t}$$

where $Z_t = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t}$.

In practise we get Z_t a supermartingale, and so we need Z_t to be positive to get equivalence.

The point of this theorem is to say that when we change our measure for one not so different (strictly absolutely continuous with respect to), we maintain the class of semimartingales. Every semimartingale with respect to $\mathbb{P}$ is also a semimartingale with respect to $\mathbb{Q}$, with an added drift.

Theorem 7 (Martingale representation theorem) *Let (M_t) be a square-integrable martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Then there is a square-integrable process (F_t) such that*

$$M_t = \mathbb{E}\big[M_0\big] + \int_0^t F_s \,\mathrm{d}B_s$$

a.s. for any $t \geq 0$. In particular, any martingale with respect to the Brownian filtration $(\mathcal{F}_t)$ has a continuous version.

Definition 3 *An adapted continuous $\mathbb{R}^N$-valued stochastic process (X_t) on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a weak solution of*

$$\mathrm{d}X_t^j = \sum_{i=1}^n f_i^j(t, X_t)\,\mathrm{d}B_t^i + f_0^j(t, X_t)\,\mathrm{d}t \quad j \in \{1, \ldots, N\}$$

if there is a Brownian motion (B_t) in $\mathbb{R}^N$ adapted to $(\mathcal{F}_t)$ such that

$$X_t^j - X_0^j = \sum_{i=1}^n \int_0^t f_i^j(s, X_s)\,\mathrm{d}B_t^i + \int_0^t f_0^j(s, X_s)\,\mathrm{d}s.$$

It is a strong solution if we have the same, but $(\mathcal{F}_t)$ is the natural filtration of the Brownian motion.

Martingale problem

In some cases, we want to convert the problem of finding a weak solution of an SDE where for $i \in [d]$,

$$\mathrm{d}X_t^i = \sum_{k=1}^d \sigma_k^i(X_t)\mathrm{d}B_t^k + b^i(X_t)\mathrm{d}t,$$

into a problem of determining if a functional of X is a local martingale.

To do this, note that for any $f \in C^2(\mathbb{R}^d)$,

$$\begin{aligned}
\mathrm{d}f(X_t) &= \sum_{i=1}^d \frac{\partial f}{\partial x^i}(X_t)\mathrm{d}X_t^i + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \frac{\partial^2 f}{\partial x^i \partial x^j}\mathrm{d}\langle X^i, X^j\rangle_t \\
&= \sum_{i=1}^d \frac{\partial f}{\partial x^i}(X_t)\left(\sum_{k=1}^d \sigma_k^i(X_t)\mathrm{d}B_t^k + b^i(X_t)\mathrm{d}t\right) + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \frac{\partial^2 f}{\partial x^i \partial x^j}(X_t)\left(\sum_{k=1}^d \sigma_k^i(X_t)\sigma_k^j(X_t)\mathrm{d}t\right) \\
&= \sum_{i=1}^d\sum_{k=1}^d \frac{\partial f}{\partial x^i}(X_t)\sigma_k^i(X_t)\mathrm{d}B_t^k + Lf(X_t)\mathrm{d}t
\end{aligned}$$

where

$$Lf(x) = \sum_{i=1}^d b^i(x)\frac{\partial f}{\partial x^i}(x) + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \frac{\partial^2 f}{\partial x^i \partial x^j}(x)\sum_{k=1}^d \sigma_k^i(x)\sigma_k^j(x)$$

and thus defining $M_t^{[f]}$ to solve, with $M_0^{[f]} = 0$,

$$\mathrm{d}M_t^{[f]} = \mathrm{d}f(X_t) - Lf(X_t)\mathrm{d}t$$

we get

$$\mathrm{d}M_t^{[f]} = \sum_{i=1}^d \frac{\partial f}{\partial x^i}(X_t)\sum_{k=1}^d \sigma_k^i(X_t)\mathrm{d}B_t^k$$

and thus M_t is a continuous local martingale.

We then say that a process (X_t) on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ solves the L-martingale problem if for $f \in C^2(\mathbb{R}^d)$,

$$M_t^{[f]} = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\mathrm{d}s$$

is a continuous local martingale. Indeed this occurs if and only if the same (X_t) is a solution to the corresponding SDE.

An advantage of this formulation is that we don't require a definition of Brownian motion, and thus this approach can be generalised to manifolds.

Local times

We now seek to find cases where we can generalise the functions f for which Itô's formula applies. Indeed it is possible to show that if f is convex and X is a continuous semimartingale, then $f(X)$ is again a semimartingale with a similar formula, known as the generalised Itô's formula.

To show this is essentially a task of prelims and part A analysis using a mollifier $\alpha \in C_0^\infty(\mathbb{R})$ with $\int_0^\infty \alpha(s)\,\mathrm{d}s = 1$. Writing $f_\varepsilon = f * \alpha_\varepsilon$ where $\alpha_\varepsilon(s) = \frac{1}{\varepsilon}\alpha(s/\varepsilon)$, one can then take limits in probability on Itô's formula to get an adapted continuous process for any continuous semimartingale X,

$$L_t^a = \frac{1}{2}\left(|X_t - a| - |X_0 - a| - \int_0^t \operatorname{sgn}(X_s - a)\,\mathrm{d}X_s\right).$$

This is a definition of the local time of X at a.

Fill in - mainly need to figure out how we show that $f(X_t)$ is a semimartingale.

Theorem 8 (Skorohod's equation) *If $y \in C[0, \infty)$ is a continuous path in $\mathbb{R}$ with $y(0) \geq 0$, $k(t) = \left(\inf_{0 \leq s \leq t} y(s)\right)^-$, then k is the unique continuous non-decreasing function on $[0, \infty)$:*

- *beginning at 0;*
- *such that $y + k \geq 0$;*
- *and which increases only on $(y + k)^{-1}(\{0\})$, so $\int_0^\infty \chi_{(y+k)^{-1}(0,\infty)}(t)\,\mathrm{d}k(t)$.*

This proof is just a matter of deterministic analysis.

The consequence for our purposes comes from noting the relation to Tanaka's formula. $2L_t^a$ in this case reflects exactly the definition of k in relation to

$$y(t) = \int_0^t \operatorname{sgn}(X_s - a)\ \mathrm{d}X_s + |X_0 - a|$$

Thus we get

$$L_t^a = \frac{1}{2}\left(|X_0 - a| + \inf_{0 \leq s \leq t}\int_0^s \operatorname{sgn}(X_r - a)\ \mathrm{d}X_r\right)^-$$