Continuous-time models for a single species

In order to model population dynamics, we establish a law governing the conservation of population:

$$\begin{aligned}\text{rate of increase of population } =& \text{birth rate} - \text{death rate}\\ & \text{rate of immigration} - \text{rate of emigration}.\end{aligned}$$

We generally assume in this:
- The system is closed, so there is no immigration, and no emigration.
- There is no spatial dependence. This is the *well-mixed assumption*.
- Time is continuous.

With the population density function $N : [0, \infty) \to [0, \infty)$, we can write the population conservation law as

$$\frac{\mathrm{d}N}{\mathrm{d}t} = Ng(N)$$

where g is the net intrinsic per growth rate for population N. We can take this as relatively reasonable in the abstract, because in practice we would hope that our model is invariant to time-shifts.

A basic example of this is the Malthus model, which takes $g(N) = b - d$, with b, d constant birth and death rates respectively. This gives us $N(t) = N_0 e^{(b-d)t}$. This doesn't really make sense in the long-term, although we might view it as accurate for small populations (or even as locally accurate).

The Verhulst model adapts this slightly, suggesting that g ought to be a decreasing function in N:

$$g(N) = r\left(1 - \frac{N}{K}\right).$$

We label K the 'carrying capacity'. This gives the equation

$$\begin{aligned}\int \frac{N'}{N(1 - N/K)}\,\mathrm{d}t &= \int \frac{N'}{N}\,\mathrm{d}t + \int \frac{N'}{K - N}\,\mathrm{d}t\\ &= \log \frac{N}{K - N} + \log N_0\\ &= rt,\end{aligned}$$

from which one can derive the result.

A key point of consideration when looking at a model is the steady states, and their stability. A steady state, in general, is a point where the dynamics do not change in time. Thus when we have differentiable functions, these are just the stationary points. A *stable* steady state is one where beginning at the steady state means the system remains close to the steady state across time. More rigorously, a steady state N_s is stable if and only if, for all $t > 0$, $\varepsilon > 0$ there is a δ such that $|N_0 - N_s| < \delta$ means $|N_{N_0}(t) - N(s)| < \varepsilon$.