Sequences

Definition 1 (Convergence) *Where (a_n) is a real sequence, it is said to converge to L as $n \to \infty$ if for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$.*

Elementary Results

Theorem 1 (Sandwiching) *Let (a_n) and (b_n) be real sequences with $0 \leq a_n \leq b_n$ for all $n \geq 1$. If $b_n \to 0$ as $n \to \infty$, then $a_n \to 0$ as $n \to \infty$.*

The proof of this is a definition chase.

A potentially useful result for dealing with oscillating sequences is that $\sin, \cos : \mathbb{N} \to [-1, 1]$ are injective. This follows trivially using their periodic properties.

Lemma 1 *(i) For $c \in \mathbb{R}$, $|c| < 1$, $c^n \to 0$ as $n \to \infty$.*
(ii) Let $a_n = \frac{n}{2^n}$ for $n \geq 1$. Then $a_n \to 0$ as $n \to \infty$.

(i) is proved using Bernoulli's inequality, by rewriting $|c|$. By the binomial theorem $2^n \geq \binom{n}{2}$, from which the proof of *(ii)* follows.

Theorem 2 (Uniqueness of limits) *If (a_n) is convergent, then it has a unique limit.*

Proof: given distinct limits L_1 and L_2, observe results at $\varepsilon \leq \frac{|L_1 - L_2|}{2}$.

Lemma 2 *If (a_n) is convergent, then so is $(|a_n|)$. Moreover, if $a_n \to L$ as $n \to \infty$, then $|a_n| \to |L|$.*

Lemma 3 (Preservation of weak inequalities) *If (a_n) and (b_n) are real sequences with limits L and M respectively, and $a_n \leq b_n$ for all n, then $L \leq M$.*

Prove by contradiction, with $\varepsilon = \frac{L-M}{2}$.

Theorem 3 (Sandwiching v2) *Let (a_n), (b_n) and (c_n) be real sequences with $a_n \leq b_n \leq c_n$ for all $n \geq 1$. If $a_n \to L$ and $c_n \to L$ as $n \to \infty$, then $b_n \to L$ as $n \to \infty$.*

Series Tests

Alternating Series

The series $\sum (-1)^k u_k$ converges if
- $u_k \to 0$ as $k \to \infty$.
- $u_k \geq 0$.
- u_k is decreasing.

Prove that s_{2n} is monotonic increasing and bounded by grouping with $u_1 - \sum (u_k - u_{k+1})$, then show that $s_{2n+1} = s_{2n} + u_{2n+1} \to s + 0$ if $s_{2n} \to s$.

Ratio

For a positive sequence a_k, if $\frac{a_{k+1}}{a_k} \to L$:
- If $L > 1$, then $\sum a_k$ diverges.
- If $0 \leq L < 1$, then $\sum a_k$ converges.

For this proof, use $\alpha = \frac{1+L}{2}$, and the respective intervals within which this lies. Set $\varepsilon = L - \alpha$, then from the definition of the limit find a relation between a_k and $\alpha^{k-N} a_N$ for some N. This leads to conclusions based on the comparison test with $\sum a_k$.

The conclusion for $L = \infty$ is only slightly more involved, using $\alpha = 2$.

Fields

Any field $\mathbb{F}$ is a set with closed commutative and associative operations $+$ and $\cdot$, an additive identity 0, multiplicative identity 1, additive invertibility and multiplicative invertibility (except for denominator 0). Additionally $\cdot$ must distribute over $+$, and to rule out the possibility that $\mathbb{F} = \{0\}$, $0 \neq 1$.

To define $\mathbb{R}$, an ordering on the set is defined by identifying a partition $\{\mathbb{P}, \{0\}, -\mathbb{P}\}$ such that for any $a, b \in \mathbb{R}$ if $a, b \in \mathbb{P}$ then $a + b \in \mathbb{P}$ and $a \cdot b \in \mathbb{P}$. Note that if any other singleton had been selected this would not be a partition.

Question: What if the multiplicative inverse were used instead of additive to form the partition?

Theorem 4 (Bernoulli's Inequality) *Let x be a real number with $x > -1$. If $n \in \mathbb{Z}^+$, then $(1 + x)^n \geq 1 + nx$.*

The proof of this follows immediately from induction on n.

Theorem 5 (Triangle Inequality) *For any $a, b \in \mathbb{R}$,*

$$||a| - |b|| \leq |a + b| \leq |a| + |b|.$$

To prove, use $|a| \leq b \Leftrightarrow -b \leq a \leq b$.

Completeness

Definition 2 (Supremum and Infimum) *For a set $S \subseteq \mathbb{R}$,*

$$\sup S = \min \{\alpha \in \mathbb{R} \,|\, \alpha \geq s \text{ for all } s \in S\}$$
$$\inf S = \max \{\alpha \in \mathbb{R} \,|\, \alpha \leq s \text{ for all } s \in S\}$$

if defined.

Axiom 1 (Completeness) *For any non-empty subset $S \subset \mathbb{R}$, if S is bounded above then S has a supremum.*

Theorem 6 (Approximation) *Let $S \subseteq \mathbb{R}$ be non-empty and bounded above. For any $\varepsilon > 0$, there exists s_ε such that $\sup S - \varepsilon < s_\varepsilon \leq \sup S$.*

Prove this by contradiction (this is where the $<$ comes from). By this theorem, one may prove the existence of roots through showing that both $(\sup S)^2 > 2$ and $(\sup S)^2 < 2$ give a contradiction. The idea here is to show that assuming $\sup S$ is on either side of 2, there is a value contradicting $\sup S$ closer to 2.

Theorem 7 (Archimedean Property) *$\mathbb{N}$ is not bounded above.*

If $\mathbb{N}$ is bounded above, then (as a non-empty subset of $\mathbb{R}$), $\sup \mathbb{N}$ exists. By the approximation property there is an element of n immediately below this, to which we may add 1 to find a number greater than $\sup \mathbb{N}$ in $\mathbb{N}$.

In general, the majority of statements regarding completeness are best proven by contradiction.

Countability

A set A is countable where there exists an injection from A to $\mathbb{N}$.

The following are countably infinite:
- $\mathbb{N}$ (trivially).
- $\mathbb{N} \times \mathbb{N}$ (using $f((m, n)) = 2^{m-1}(2n - 1)$).
- $\mathbb{N}^n$ for all $n \in \mathbb{N}$ (by induction).
- $A \times B$ where A and B are countable (using $h((a, b)) = 2^{f(a)} 3^{g(b)}$).
- $\mathbb{Q}$ (as $\mathbb{Q}^{>0}$ is countable and bijects $\mathbb{N} \times \mathbb{N}$).
- $A \uplus B$ where A and B are countable.