Sequences

Definition 1 (Convergence) *Where (a_n) is a real sequence, it is said to converge to L as $n \to \infty$ if for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$.*

The only inequality in the above definition that is necessarily strict is that $\varepsilon > 0$. Apart from that, $n \geq N$ vs $n > N$ is often switched between, although out of ease the inequality $|a_n - L| < \varepsilon$ is usually kept strict.

Elementary Results

Theorem 1 (Sandwiching) *Let (a_n) and (b_n) be real sequences with $0 \leq a_n \leq b_n$ for all $n \geq 1$. If $b_n \to 0$ as $n \to \infty$, then $a_n \to 0$ as $n \to \infty$.*

The proof of this is a definition chase.

A potentially useful result for dealing with oscillating sequences is that $\sin, \cos : \mathbb{N} \to [-1, 1]$ are injective. This follows trivially using their periodic properties.

Lemma 1 *(i) For $c \in \mathbb{R}$, $|c| < 1$, $c^n \to 0$ as $n \to \infty$.*
(ii) Let $a_n = \frac{n}{2^n}$ for $n \geq 1$. Then $a_n \to 0$ as $n \to \infty$.

(i) is proved using Bernoulli's inequality, by rewriting $|c|$. By the binomial theorem $2^n \geq \binom{n}{2}$, from which the proof of *(ii)* follows.

Theorem 2 (Uniqueness of limits) *If (a_n) is convergent, then it has a unique limit.*

Proof: given distinct limits L_1 and L_2, observe results at $\varepsilon \leq \frac{|L_1 - L_2|}{2}$.

Lemma 2 *If (a_n) is convergent, then so is $(|a_n|)$. Moreover, if $a_n \to L$ as $n \to \infty$, then $|a_n| \to |L|$.*

Lemma 3 (Preservation of weak inequalities) *If (a_n) and (b_n) are real sequences with limits L and M respectively, and $a_n \leq b_n$ for all n, then $L \leq M$.*

Prove by contradiction, with $\varepsilon = \frac{L-M}{2}$.

Theorem 3 (Sandwiching v2) *Let (a_n), (b_n) and (c_n) be real sequences with $a_n \leq b_n \leq c_n$ for all $n \geq 1$. If $a_n \to L$ and $c_n \to L$ as $n \to \infty$, then $b_n \to L$ as $n \to \infty$.*

Series Tests

Alternating Series

The series $\sum(-1)^k u_k$ converges if
- $u_k \to 0$ as $k \to \infty$.
- $u_k \geq 0$.
- u_k is decreasing.

Prove that s_{2n} is monotonic increasing and bounded by grouping with $u_1 - \sum(u_k - u_{k+1})$, then show that $s_{2n+1} = s_{2n} + u_{2n+1} \to s + 0$ if $s_{2n} \to s$.

Ratio

For a positive sequence a_k, if $\frac{a_{k+1}}{a_k} \to L$:
- If $L > 1$, then $\sum a_k$ diverges.
- If $0 \leq L < 1$, then $\sum a_k$ converges.

For this proof, use $\alpha = \frac{1+L}{2}$, and the respective intervals within which this lies. Set $\varepsilon = L - \alpha$, then from the definition of the limit find a relation between a_k and $\alpha^{k-N} a_N$ for some N. This leads to conclusions based on the comparison test with $\sum a_k$.

The conclusion for $L = \infty$ is only slightly more involved, using $\alpha = 2$.

Fields

Any field $\mathbb{F}$ is a set with closed commutative and associative operations $+$ and $\cdot$, an additive identity 0, multiplicative identity 1, additive invertibility and multiplicative invertibility (except for denominator 0). Additionally $\cdot$ must distribute over $+$, and to rule out the possibility that $\mathbb{F} = \{0\}$, $0 \neq 1$.

To define $\mathbb{R}$, an ordering on the set is defined by identifying a partition $\{\mathbb{P}, \{0\}, -\mathbb{P}\}$ such that for any $a, b \in \mathbb{R}$ if $a, b \in \mathbb{P}$ then $a + b \in \mathbb{P}$ and $a \cdot b \in \mathbb{P}$. Note that if any other singleton had been selected this would not be a partition.

Theorem 4 (Bernoulli's Inequality) *Let x be a real number with $x > -1$. If $n \in \mathbb{Z}^+$, then $(1 + x)^n \geq 1 + nx$.*

The proof of this follows immediately from induction on n.

Theorem 5 (Triangle Inequality) *For any $a, b \in \mathbb{R}$,*

$$||a| - |b|| \leq |a + b| \leq |a| + |b|.$$

To prove, use $|a| \leq b \Leftrightarrow -b \leq a \leq b$.

Completeness

Definition 2 (Supremum and Infimum) *For a set $S \subseteq \mathbb{R}$,*

$$\sup S = \min \{\alpha \in \mathbb{R} \mid \alpha \geq s \text{ for all } s \in S\}$$
$$\inf S = \max \{\alpha \in \mathbb{R} \mid \alpha \leq s \text{ for all } s \in S\}$$

if defined.

Axiom 1 (Completeness) *For any non-empty subset $S \subset \mathbb{R}$, if S is bounded above then S has a supremum.*

To get a bit of intuition for what this axiom means: every irrational number can be understood as equivalent to a sequence of rational numbers, converging to some point. For any increasing bounded sequence we get a non-empty subset of $\mathbb{R}$ with its supremum the limit of the sequence, so by completeness the above follows. Helpfully, although completeness technically says that the limit of any real sequence is real, we can still get every real number just from rational sequences.

Theorem 6 (Approximation) *Let $S \subseteq \mathbb{R}$ be non-empty and bounded above. For any $\varepsilon > 0$, there exists s_ε such that $\sup S - \varepsilon < s_\varepsilon \leq \sup S$.*

Prove this by contradiction (this is where the $<$ comes from). By this theorem, one may prove the existence of roots through showing that both $(\sup S)^2 > 2$ and $(\sup S)^2 < 2$ give a contradiction. The idea here is to show that assuming $\sup S$ is on either side of 2, there is a value contradicting $\sup S$ closer to 2.

Theorem 7 (Archimedean Property) *$\mathbb{N}$ is not bounded above.*

If $\mathbb{N}$ is bounded above, then (as a non-empty subset of $\mathbb{R}$), $\sup \mathbb{N}$ exists. By the approximation property there is an element of n immediately below this, to which we may add 1 to find a number greater than $\sup \mathbb{N}$ in $\mathbb{N}$.

In general, the majority of statements regarding completeness are best proven by contradiction.

Countability

A set A is countable where there exists an injection from A to $\mathbb{N}$.

The following are countably infinite:
- $\mathbb{N}$ (trivially).
- $\mathbb{N} \times \mathbb{N}$ (using $f((m, n)) = 2^{m-1}(2n - 1)$).
- $\mathbb{N}^n$ for all $n \in \mathbb{N}$ (by induction).
- $A \times B$ where A and B are countable (using $h((a, b)) = 2^{f(a)} 3^{g(b)}$).
- $\mathbb{Q}$ (as $\mathbb{Q}^{>0}$ is countable and bijects $\mathbb{N} \times \mathbb{N}$).
- $A \cup B$ where A and B are countable.