Introduction

"Hahn-Banach is why mathematics without the axiom of choice is really silly."

In this course we see many of the very fundamental results which allow us to do mathematics.

In Functional Analysis I, we prove critically the projection theorem, the Hahn-Banach theorem, and the Riesz representation theorem. Recall these as:

Theorem 1 (Hahn-Banach) *For $(X, \|\cdot\|)$ a normed space, $Y \subseteq X$ a subspace, there is an isometric extension map $E : Y^* \to X^*$ such that for $f \in Y^*$, $E(f)\big|_Y = f$.*

Theorem 2 (Riesz representation theorem) *For X a Hilbert space, there is an isometric isomorphism (or anti-isomorphism, if over $\mathbb{C}$) $\pi : X^* \to X$ such that $\ell \equiv \langle \cdot, \pi(\ell)\rangle$.*

Compact operators

Definition 1 (Compactness) *Let X, Y be normed spaces, $T \in \mathcal{B}(X,Y)$. Then T is compact if $\overline{T(\overline{B_X}(0,1))}$ is compact ($T(\overline{B_X}(0,1))$ is precompact).*

Critically, we can extend this definition from $\overline{B_X}(0,1)$ to any bounded $Y \subseteq X$, as given below:

Lemma 3 *Let $T \in \mathcal{B}(X,Y)$, $S \in \mathcal{B}(Y,Z)$ for X, Y, Z normed spaces. Then*
i. T is compact iff every bounded sequence (x_n) in X has a subsequence (x_{n_k}) such that (Tx_{n_k}) converges.
ii. If $\dim(T(X)) < \infty$ then T is compact.
iii. If S or T is compact then $S \circ T$ is compact.
iv. If Y is Banach, (T_n) is a sequence of compact operators with $T_n \to T$, then T is compact.

Proofs:
i. Follows immediately from equivalence of compactness and sequential compactness on metric spaces.
ii. $\overline{T(\overline{B}_X(0,1))}$ is bounded by the boundedness of $\overline{B}_X(0,1)$, and as for finite spaces a subset is closed and bounded iff it is compact, the result follows.
iii. If T is compact, by the continuity of S preserving compact sets, $\overline{S(\overline{T(\overline{B})})} = S(\overline{T(\overline{B})})$ is compact, and $T(\overline{B})$ is closed so $S \circ T$ is compact. If S is compact on the other hand, we just argue that, by (i), as $T(\overline{B})$ is bounded thus $\overline{S(T(\overline{B}))}$ is compact.
iv. Follows from a diagonal argument used in problem sheet 1.

Theorem 4 (Arzela-Ascoli) *Take $\Omega \subseteq \mathbb{R}^n$ compact, $\mathcal{F} \subseteq C(\Omega)$. $\overline{\mathcal{F}}$ is compact if and only if $\mathcal{F}$ is*
i. uniformly bounded, so $\sup_{f \in \mathcal{F}} \|f\| < \infty$, and
ii. equicontinuous, so for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $f \in \mathcal{F}$, $x, y \in \Omega$, if $\|x - y\| < \delta$, $|f(x) - f(y)| < \varepsilon$.

While the proof is non-examinable, it can be shown using the separability of $\mathbb{R}^n$ and compactness of closed bounded sets in $\mathbb{R}^n$.

This theorem is critical for demonstrating that the integral operator (and various associated operators) is compact. Once we know this, we get a variety of helpful results for these objects.

Lemma 5 *For X a Banach space, $T \in \mathcal{B}(X)$ a compact operator,*
1. $\dim(\ker(\mathrm{Id} - T)) < \infty$.
2. $(\mathrm{Id} - T)(X)$ is closed.
3. If X is a Hilbert space and T is adjoint then
$$\ker(\mathrm{Id} - T)^\perp = (\mathrm{Id} - T)(X)$$
The above is essentially a generalisation of the rank-nullity theorem to infinite dimensions. To prove it, one must use the lemma below.

Lemma 6 *Let X be Banach, $S \in \mathcal{B}(X)$ such that*
$$\inf_{x \in X \setminus \{0\}} \frac{\|Sx\|}{\|x\|} > 0$$
Then S is injective and $S(X)$ is closed.

Fourier Series

Recall that for complex-valued $f \in L^1([-\pi,\pi])$, we can write
$$\mathcal{F}(f)(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \langle f, e^{inu}\rangle e^{inx},$$
or, with $e_n(x) := e^{inx}/\sqrt{2\pi}$,
$$\mathcal{F}(f)(x) = \sum_{n=-\infty}^{\infty} \langle f, e_n\rangle e_n.$$
We want to demonstrate that $\mathcal{F}$ is well-defined on L^2 with respect to the L^2 norm, and that $\mathcal{F}(f) = f$ for $f \in L^2$. Additionally, we want to show that pointwise convergence is guaranteed with particular conditions on continuous functions.

Recall, for the purpose of this section, that we have $C(-\pi,\pi) \subseteq L^2(-\pi,\pi) \subseteq L^1(-\pi,\pi)$.

Lemma 7 (Termwise differentiation in L^1) *With $f \in L^1(-\pi,\pi)$ s.t. $\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = 0$, define*
$$F(x) = \int_0^x f(t)\,\mathrm{d}t.$$
Then $F \in C^0([-\pi,\pi])$ satisfies $F(\pi) = F(-\pi)$, and
$$\langle f, e_n\rangle = in\langle F, e_n\rangle.$$

To see this, use integration by parts.

Theorem 8 (Completeness of the trigonometric system in L^2) *For $f \in L^2(-\pi,\pi)$,*
$$\sum_{k=-n}^{n} \langle f_n, e_n\rangle e_n \to f$$
in $L^2(-\pi,\pi)$ as $n \to \infty$.

In other words, the closed span of $\{e_n : n \in \mathbb{Z}\}$ is $L^2(-\pi,\pi)$. In order to prove this, we need to demonstrate that $f = 0$ iff $\langle f, e_n\rangle = 0$ for all $n \in \mathbb{Z}$. This requires some creativity to demonstrate that no part of f can 'escape' the inner product and thus be orthogonal to all but non-zero.

It's worth pointing out that due to Parseval's identity, we have that
$$\int_{-\pi}^{\pi} |f|^2 = \sum_{n=-\infty}^{\infty} |\langle f_n, e_n\rangle|.$$
This allows us to claim the below result for $L^2(-\pi,\pi)$:

Lemma 9 (Riemann-Lebesgue) *For $f \in L^1(-\pi,\pi)$, $\langle f, e_n\rangle \to 0$ as $|n| \to \infty$.*

Theorem 10 *There exists a function $f \in C_{\mathrm{per}}[-\pi,\pi]$ whose Fourier series diverges at $x = 0$.*

This is a consequence of the uniform boundedness theorem proven later in the course. Assuming that for any $f \in C(-\pi,\pi)$, $\mathcal{F}(f)(0)$ is well-defined under pointwise convergence, we have
$$A_n(f) = \int_{-\pi}^{\pi} f(u) \sum_{k=-n}^{n} e^{iku}\,\mathrm{d}u$$
a sequence in $C_{\mathrm{per}}[-\pi,\pi]^*$ such that $A_n \to A$ pointwise. Yet we can see that $\|A_n\|_* \to \infty$ as $n \to \infty$, giving us a contradiction.

Consequently we see that continuity is not sufficient to give us pointwise convergence, and in fact pointwise convergence is a more difficult thing to guarantee.

Definition 2 (Hölder continuity) *With $\alpha \in (0,1]$, $f : \mathbb{R} \to \mathbb{R}$ is α-Hölder continuous if for $x \in \mathbb{R}$ there is $M > 0$, $\delta > 0$ such that for $y \in \mathbb{R}$ with $|x - y| < \delta$,*
$$|f(x) - f(y)| < M|x - y|^\alpha.$$

Using this definition, we write for any compact interval $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$,
$$[f]_\alpha = \sup_{\substack{x,y \in I\\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha},$$
and then define correspondingly
$$C^{0,\alpha}(I) = \{f \in C^0(I) : [f]_\alpha < \infty\}.$$
In particular, this is a Banach space when equipped with $\|\cdot\|_{0,\alpha} = \|\cdot\|_{\sup} + [\cdot]_\alpha$.

Theorem 11 *For $f \in L^1_{\mathrm{loc}}(-\pi,\pi)$, 2π-periodic and α-Hölder continuous for some $\alpha \in (0,1]$ at $x_0 \in [-\pi,\pi]$, $\mathcal{F}(f)(x_0) = f(x_0)$.*

This is a very slightly stronger statement than if made for $f \in L^1(-\pi,\pi)$, which just results from the fact that our proof only relies on integrating f within compact subsets of $(-\pi,\pi)$. Note in particular that this is a statement about pointwise convergence, rather than L^1 or L^2 convergence.

Big theorems

Baire category theorem

Definition 3 *A subset S of a metric space M*
i. is nowhere dense in M if $\mathrm{int}(\overline{S}) = \varnothing$.
ii. has Baire category 1 if there is a sequence of nowhere dense sets (A_n) such that $S = \bigcup_{n\in\mathbb{N}} A_n$.
iii. has Baire category 2 if it does not have category 1.
iv. residual if S^c has category 1.

More intuitively, with these definitions we're trying to get a notion of when things are tiny (category 1), not tiny (category 2), or when sets are "essentially everything" (residual).

Lemma 12 *For A a subset of a metric space, A is closed and nowhere dense iff A^c is open and dense.*

Theorem 13 (Baire category theorem) *Let (M,d) be a complete non-trivial metric space. Then $\mathrm{Cat}(M) = 2$, and every residual set is dense.*

Note firstly that if every residual set is dense, then immediately $\mathrm{Cat}(M) = 2$, as if $\mathrm{Cat}(M) = 1$ then $\varnothing$ would be residual and thus dense, which is a contradiction.

Thus we want to claim for our proof that for any countable collection (U_n) of open dense sets, that $\bigcap_{n\in\mathbb{N}} U_n$ is dense. To do this, take $x \in M$, $\varepsilon > 0$. We want a sequence (y_n) such that $y_n \in B(x,\varepsilon) \cap \bigcap_{k=1}^n U_k$, and to do this we take y_{n+1} such that $d(y_n, y_{n+1}) < \varepsilon_n/2$. Critically, to ensure that the limit of (y_n) is in the intersection of (U_n), we take for each $n \in \mathbb{N}$, $F_n = B(y_n, \varepsilon_n)$.

Fundamentally, we want to construct for each $x \in M$, $\varepsilon > 0$, a decreasing sequence of closed sets (F_n) with $F_n \subseteq U_n$ and $\mathrm{dist}(x, F_n) < \varepsilon$.

From this, we then take any residual set $S = \bigcap_{n\in\mathbb{N}} A_n^c$ where $\varnothing = \mathrm{int}(\overline{A_n}) = \overline{(\overline{A_n}^c)}^c = \overline{\mathrm{int}(A_n^c)}^c$, so $\mathrm{int}(A_n^c)$ is dense in M and thus by our claim S is dense (being larger than the intersection of $\mathrm{int}(A_n^c)$).

Principle of uniform boundedness

Theorem 14 *Let X be a Banach space, Y a normed space, and $\mathcal{F} \subseteq \mathcal{B}(X,Y)$ is pointwise bounded, so for $x \in X$,*
$$\sup_{T\in\mathcal{F}} \|Tx\| < \infty$$
Then $\mathcal{F}$ is uniformly bounded, so in fact
$$\sup_{T\in\mathcal{F}} \|T\| < \infty.$$

To prove this, consider the collection of sets (A_n) with
$$A_n = \left\{x \in X : \sup_{T\in\mathcal{F}} \|Tx\|_Y \le n\right\}.$$
As T and $\|\cdot\|_Y$ are continuous, thus A_n is closed as an intersection of closed sets. If $\mathcal{F}$ is pointwise bounded, then $X = \bigcup_{n=1}^\infty A_n$ so by the Baire category theorem there is an $n \in \mathbb{N}$ such that A_n is not nowhere dense, so there is an open ball $B(x,r) \subseteq A_n$. Thus we have a bound on $\|Tx\|$ for $x \in B(x,r)$, which we can use to get a bound on $\|Tx\|$ for $x \in B(0,r)$, and thus a bound on $\|T\|$.

Theorem 15 *With X, Y Banach spaces, (T_n) a sequence in $\mathcal{B}(X,Y)$, TFAE:*
i. There is $T \in \mathcal{B}(X,Y)$ such that $T_n \to T$ pointwise as $n \to \infty$.
ii. For $x \in X$, $(T_n x)$ is convergent.
iii. There is $M \in [0,\infty)$ and a dense $Z \subseteq X$ such that $\|T_n\| \le M$ and $(T_n z)$ is convergent for $z \in Z$.

Theorem 16 *Let X be a Hilbert space, $\mathcal{F} \subseteq \mathcal{B}(X)$ such that for $x, y \in X$,*
$$\sup_{T\in\mathcal{F}} |\langle Tx, y\rangle| < \infty.$$
Then $\sup_{T\in\mathcal{F}} \|T\| < \infty$.

Open mapping theorem

Definition 4 *For X, Y topological spaces, $f : X \to Y$ is open if $f(U)$ is open for every open $U \subseteq X$.*

Lemma 17 *Let X, Y be normed spaces, $T : X \to Y$ linear. TFAE:*
i. T is open.
ii. There is $\delta > 0$ such that $B_Y(0,\delta) \subseteq T(B_X(0,1))$.
iii. There is $\varepsilon > 0$, $y \in Y$, $r > 0$ such that $B_Y(y,\varepsilon) \subseteq T(B_X(0,r))$.

Theorem 18 (Open mapping theorem) *Let X, Y be Banach spaces and $T \in \mathcal{B}(X,Y)$ be surjective. Then T is an open map.*

To prove this, we first show that there is $\varepsilon > 0$ such that $B_Y(0,2\varepsilon) \subseteq \overline{T(B_X(0,1))}$ using the Baire category theorem. We then show that for this ε, $B_Y(0,\varepsilon) \subseteq T(B_X(0,1))$. This gives our result.

Theorem 19 (Inverse mapping theorem) *Let X, Y be Banach spaces and $T \in \mathcal{B}(X,Y)$ a bijection. Then $T^{-1} \in \mathcal{B}(Y,X)$.*

Theorem 20 *Let X, Y be Hilbert spaces, $T \in \mathcal{B}(X,Y)$. Then $T(X)$ is closed iff $T^*(Y)$ is closed.*

Closed graph theorem

Theorem 21 (Closed graph theorem) *Let X, Y be Banach spaces, $T : X \to Y$ linear. Then $T \in \mathcal{B}(X,Y)$ iff its graph $\Gamma(T) = \{(x, Tx) : x \in X\}$ is closed in $X \times Y$.*

One direction of this proof (that $T \in \mathcal{B}(X,Y)$ implies $\Gamma(T)$ is closed) is clear from the continuity of bounded linear operators. The other direction involves the construction of T in terms of projections on $\Gamma(T)$, which are continuous by IMT and thus give the continuity of T.

At first glance, the closure of $\Gamma(T)$ seems like a far weaker statement than $T \in \mathcal{B}(X,Y)$. We only need to show that for $(x_n, y_n) \in \Gamma(T)$, x_n and y_n both converge in their own spaces (and don't need to check whether $\lim y_n \in T(X)$). Thus we apply the closed graph theorem almost exclusively in order to show that linear maps are bounded linear maps.

Corollary 1 *Let X be a Hilbert space and $T : X \to X$ linear. If $\langle Tx, y\rangle = \langle x, Ty\rangle$ for $x, y \in X$, then $T \in \mathcal{B}(X)$ and is self-adjoint.*

Lemma 22 *If h is measurable such that for all $f \in L^1(I)$ we have $f \cdot h \in L^1(I)$, then $h \in L^\infty(I)$.*

We can prove this by any of the above theorems. By PUB, we take $h_n(x)\chi_{|h(x)| \le n}$.

Prove with all of the above theorems.

Weak Convergence

A distinct point of annoyance in applied maths is the inability to guarantee that bounded sequences in an infinite-dimensional space have a convergent subsequence – in particular, that bounded functions may not have a convergent subsequence.

A key idea, therefore, is to weaken the notion of convergence to one that retains key properties of convergence (uniquenesss of limits, boundedness, and other good interactions with strong convergence), while gaining that for reflexive Banach spaces, any bounded sequence has a convergent subsequence.

Definition 5 (Weak convergence) *A sequence (x_n) in a normed space X is said to converge weakly to $x \in X$, $x_n \rightharpoonup x$, if for all $\ell \in X^*$,*
$$\lim_{n\to\infty} \ell(x_n) = \ell(x).$$

Without too much difficulty, we can see that the weak limit is unique if it exists, and its norm bounded above by $\liminf \|x_n\|$. The former is a corollary of Hahn-Banach extending any non-zero $\ell \in \langle x - y\rangle^*$ to X^*. The latter utilises the identification of X and X^{**} by $\iota(x)(\ell) = \ell(x)$ so that if $x_n \rightharpoonup x$, thus $\iota(x_n) \to \iota(x)$ pointwise. Consequently the sequence is pointwise bounded, and thus uniformly bounded so $\sup \|\iota(x_n)\| = \sup \|x_n\| < \infty$. We then find $\ell \in X^*$ s.t. $\ell(x) = \|x\|$ and $\|\ell\| = 1$, so $|\ell(x_n)| \le \|x_n\|$ and the result follows.

In fact, this is a special case of a wider statement about sets closed under weak convergence.

Theorem 23 (Mazur) *Let K be a closed convex subset of a normed space X, (x_n) a sequence in K which converges weakly to $x \in X$. Then $x \in K$.*

Otherwise stated: if K is closed and convex with respect to standard convergence, then K is sequentially weakly closed (note this is a non-metrizable topology).

Theorem 24 (Hyperplane separation) *Let X be a normed space, $A, B \subseteq X$ disjoint and convex such that $\mathrm{int}(B) \neq \varnothing$. Then A and B can be separated by a hyperplane.*

Note we define a hyperplane by $\ell^{-1}(c + i\mathbb{R})$ for some $c \in \mathbb{R}$, so its sides are $\ell^{-1}((-\infty, c] + i\mathbb{R})$ and $\ell^{-1}([c,\infty) + i\mathbb{R})$.

Theorem 25 (Geometric version of Mazur) *Every closed convex subset K of a normed space can be written as an intersection of half-spaces.*

Theorem 26 *For X a reflexive Banach space, the closed unit ball is weakly sequentially compact.*

This is a major result, as it gives us immediately that bounded sequences in a reflexive Banach space have weakly convergent subsequences.

We prove this only for Hilbert spaces. To do this, we show first that there is a subsequence for any (x_n), (x_{n_j}) such that $\langle x_{n_j}, x_m\rangle$ converges for every m (using a diagonal argument with Bolzano-Weierstrass). **Complete proof later.**

Theorem 27 (Eberlein) *The closed unit ball in a Banach space is weakly sequentially compact iff the space is reflexive.*

Theorem 28 (Closed point in a closed convex subset) *Let K be a non-empty closed convex subset of a reflexive Banach space X. Then, for every $x \in X$, there is a point $y \in K$ such that*
$$\mathrm{dist}(x, K) = \|x - y\|.$$

The proof of this is an instance of a more general method, whereby for a function $E : \Omega \to \mathbb{R}$ we demonstrate that a minimising sequence $E(u_n) \to \inf E(u)$ is bounded and thus that a weakly convergent subsequence exists by sequential compactness, to a point in Ω such that $E(u) \le \liminf E(u_n)$ (**why is this necessary?**). Thus $E(u) = \inf E$.

Weak convergence in Hilbert spaces

Lemma 29 *For X a Hilbert space, $x_n \to x$ iff $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$.*

If $x_n \to x$, then immediately we have that (x_n) converges weakly and $\|x_n\| \to \|x\|$. The other direction can be seen from the fact that if $x_n \rightharpoonup x$, $\langle x_n, y\rangle \to \langle x, y\rangle$, and combined with $\langle x_n, x_n\rangle \to \langle x, x\rangle$, $\|x_n - x\| \to 0$.

Due to being able to identify X with X^* by Riesz, we can characterise weak convergence of (x_n) in terms of the convergence of $\langle x_n, y\rangle$ for $y \in X$, and consequently for Hilbert spaces by behaviour on a basis.

A corollary of this is that any orthonormal sequence converges weakly to zero (but not strongly to zero).

Spectral theory

Fundamentally, spectral theory is concerned with answering the question: for $T \in \mathcal{B}(X)$, $\lambda \in \mathbb{F}$, is $T - \lambda\,\mathrm{Id}$ invertible?

Definition 6 *Let X be a complex Banach space, $T \in \mathcal{B}(X)$.*
i. The resolvent set $\rho(T)$ is
$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda\,\mathrm{Id} \text{ is invertible}\}.$$
ii. The resolvent operator is, for $\lambda \in \rho(T)$,
$$R_\lambda(T) := (T - \lambda\,\mathrm{Id})^{-1}.$$
iii. The spectrum set $\sigma(T)$ is $\rho(T)^c$,
$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda\,\mathrm{Id} \text{ is not invertible}\}.$$
iv. The point spectrum $\sigma_p(T)$ is
$$\sigma_p(T) := \{\lambda \in \mathbb{C} : \mathrm{Ker}(T - \lambda\,\mathrm{Id}) \neq \{0\}\}.$$
As normal, these are called the eigenvalues of T, and the non-trivial elements of $\mathrm{Ker}(T - \lambda\,\mathrm{Id})$ are eigenvectors of T (or eigenfunctions if X is a function space).
v. The residual spectrum $\sigma_r(T)$ of T is
$$\sigma_r(T) := \{\lambda \in \mathbb{C} : \mathrm{Ker}(T - \lambda\,\mathrm{Id}) = \{0\},\ \overline{\mathrm{Im}(T - \lambda\,\mathrm{Id})} \neq X\}.$$
vi. The continuous spectrum $\sigma_c(T)$ of T is
$$\sigma_r(T) := \{\lambda \in \mathbb{C} : \mathrm{Ker}(T - \lambda\,\mathrm{Id}) = \{0\},\ \mathrm{Im}(T - \lambda\,\mathrm{Id}) \neq \overline{\mathrm{Im}(T - \lambda\,\mathrm{Id})} = X\}.$$
vii. The approximate point spectrum $\sigma_{ap}(T)$ of T is
$$\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : \exists (x_n) \in \partial B_X(0,1) \text{ s.t. } \|Tx_n - \lambda x_n\| \to 0\}.$$
viii. The spectral radius $r_\sigma(T)$ is
$$r_\sigma(T) := \sup_{\lambda\in\sigma(T)} |\lambda|$$

Immediately, $\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T)$ is a disjoint union. We also get that $\sigma_p(T) \subseteq \sigma_{ap}(T)$, $\sigma_c(T) \subseteq \sigma_{ap}(T)$, and $\sigma(T) \subseteq \overline{B}(0, \|T\|)$.

Lemma 30 (Neumann series) *If X is Banach, $S \in B_{\mathcal{B}(X)}(0,1)$, then*
$$(\mathrm{Id} - S)^{-1} = \sum_{n=0}^\infty S^n$$
and the RHS is a well-defined bounded linear operator in $\mathcal{B}(X)$.

Theorem 31 *Let $(X, \|\cdot\|)$ be a complex Banach space. For $T \in \mathcal{B}(X)$, $\rho(T)$ is open and $R_\lambda(T)$ is analytic in $\lambda \in \rho(T)$, and $\varnothing \neq \sigma(T) \subseteq \overline{B}(0, \|T\|)$.*

Lemma 32 *Let $(X, \|\cdot\|)$ be a normed space, $S, T \in \mathcal{B}(X)$. If $ST = TS$ and ST is invertible, then S and T are invertible.*

Theorem 33 *Let $(X, \|\cdot\|)$ be a complex Banach space, $T \in \mathcal{B}(X)$, $p : \mathbb{C} \to \mathbb{C}$ a complex polynomial. Then $\sigma(p(T)) = p(\sigma(T))$.*

Corollary 2 *Let $(X, \|\cdot\|)$ be a Banach space, $T \in \mathcal{B}(X)$. For $k \in \mathbb{N}$, $\lambda \in \sigma(T)$, then $\lambda^k \in \sigma(T^k)$, and*
$$|\lambda| \le \inf_{j\in\mathbb{N}} \|T^j\|^{1/j}.$$

Theorem 34 (Gelfand's formula) *Let $(X, \|\cdot\|)$ be a complex Banach space. For $T \in \mathcal{B}(X)$,*
$$r(T) = \lim_{j\to\infty} \|T^j\|^{1/j} = \inf_{j\in\mathbb{N}} \|T^j\|^{1/j}.$$

Corollary 3 *Let $(X, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space, $T \in \mathcal{B}(X)$ self-adjoint, then*
$$r_\sigma(T) = \|T\|.$$
Theorem 35 *Let $(X, \|\cdot\|)$ be a Banach space, $T \in \mathcal{B}(X)$, $T' \in \mathcal{B}(X^*)$. Then*
$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T').$$

Theorem 36 *Let $(X, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space, $T \in \mathcal{B}(X)$. Then*
$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_p'(T^*)$$
where for $A \subseteq \mathbb{C}$, $A' = \{\overline{x} : x \in A\}$.

Theorem 37 *Let $(X, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space, $T \in \mathcal{B}(X)$ self-adjoint. Then*
- *With $a = \inf_{\|x\|=1}\langle x, Tx\rangle$, $b = \sup_{\|x\|=1}\langle x, Tx\rangle$,*
$$\{a, b\} \subseteq \sigma(T) = \sigma_{ap}(T) \subseteq [a, b]$$
.
- *$\sigma(T) = \sigma_p(T) \cup \sigma_c(T)$.*
- *Eigenvectors corresponding to different eigenvalues of T are orthogonal.*

Let $X = \ell^2(\mathbb{C})$, $(a_j) \in \ell^\infty(\mathbb{R})$, with $T((x_j)) = (a_j x_j)$ ...

Lemma 38 *Let $(X, \|\cdot\|)$ be Banach, $T \in \mathcal{B}(X)$ compact, $\mathrm{Id} - T$ injective. Then $\mathrm{Id} - T$ is also surjective.*