Probability Space

Any probability space is a triple $\langle \Omega, \mathcal{F}, \mathbb{P} \rangle$ where
- Ω is the sample space – it exists as a set from which $\mathcal{F}$ may be formed.
- $\mathcal{F}$ is the set of events – an event being a subset of Ω, satisfying
  F$_1$ (universal set): $\Omega \in \mathcal{F}$.
  F$_2$ (closed under complementation): $A \in \mathcal{F}$ implies that $\Omega \setminus A \in \mathcal{F}$.
  F$_3$ (closed under countable unions): If $A_k \in \mathcal{F}$ for every $k \in \mathbb{N}$, then $\bigcup\limits_{k \in \mathbb{N}} A_k \in \mathcal{F}$.

Equivalently, $\mathcal{F}$ is a σ-algebra with respect to Ω. This means $\mathcal{F}$ is measurable, so a non-negative real value may be assigned to each element of $\mathcal{F}$.

- $\mathbb{P}$ is a measure on $\mathcal{F}$ satisfying
  P$_1$ (non-negative): For all $A \in \mathcal{F}$, $\mathbb{P}(A) \geq 0$.
  P$_2$ (1 on whole space): $\mathbb{P}(\Omega) = 1$.
  P$_3$ (σ-additivity): If $A_k \in \mathcal{F}$ for every $k \in \mathbb{N}$, and $A_i \cap A_j = \varnothing$ for every $i \neq j$, then $\mathbb{P}\left(\bigcup\limits_{k \in \mathbb{N}} A_k\right) = \sum\limits_{k \in \mathbb{N}} \mathbb{P}(A_k)$.

The first immediate consequence of **P**$_3$ is the formula for $\mathbb{P}(A \setminus B)$: $\mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) = \mathbb{P}(A)$. Consequently, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. The inclusion-exclusion formula is then derived with $S_n = \bigcup_{k=1}^{n} A_k$ by induction.

Random Variables

A discrete random variable is a function from samples to values $X : \Omega \to \mathbb{R}$, such that for every value in ImX there is an event to which it corresponds (Equivalently, there is a well-defined $f : \mathbb{R} \to \mathcal{F}$ where $f(x) = \{\omega \in \Omega \,|\, X(\omega) = x\}$, $f(\mathbb{R})$ is a partition of $\mathcal{F}$). Additionally, ImX is countable.

In the case where the random variable is continuous, then the notion of $X : \Omega \to \mathbb{R}$ as a function becomes more general. In this case we redefine $f : \mathbb{R} \to \mathcal{F}$ to $f(x) = \{\omega \in \Omega \,|\, X(\omega) \leq x\}$. Note here that Im$X$ need not be countable.

Given that this occurs in conjunction with **F**$_3$ and **P**$_3$, if $\mathcal{F}$ was merely a set of uncountable singletons, then one would not be able to apply σ-additivity. Instead, we ensure that $\mathcal{F}$ is formed of uncountable sets.

Difference Equations

A nth order difference equation has the form

$$\sum_{j=0}^{k} a_j u_{n+j} = f(n)$$

with $a_0 \neq 0$, $a_k \neq 0$, a_k independent of n for all k.

To solve equations of this form one initially solves the homogeneous equation using the $u_j = A\lambda^j$ ansatz, then finds a particular solution with respect to f. The ansatz gives k solutions, and the auxiliary equation gives an additional solution.

The set of solutions to any homogeneous kth order difference equation is k-dimensional.