The Riemann Integral

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We consider step functions ϕ_- and ϕ_+ satisfying $\phi_- \le f \le \phi_+$. Any ϕ_- is a minorant for f if $f \ge \phi_-$ pointwise, and any ϕ_+ a majorant if vice-versa. We define the integral $I(\phi)$ of any step function, and give the following definition of integrability:

Definition 1 (Integrability) *f is integrable if*

$$\sup_{\phi_-} I(\phi_-) = \inf_{\phi_+} I(\phi_+)$$

Specifically, we label a function $\phi : [a, b] \to \mathbb{R}$ a step function if there is a finite increasing sequence $(x_n) \in [a, b]$ beginning at a and ending at b such that ϕ is constant on each open interval (x_i, x_{i+1}).

Lemma 1 *A function $\phi : [a, b] \to \mathbb{R}$ is a step function if and only if it is a finite linear combination of indicator functions of intervals (open and closed).*

With $\phi(x) = c_i$ on the interval (x_{i-1}, x_i)

$$I(\phi) = \sum_{i=1}^{n} c_i(x_i - x_{i-1})$$

Lemma 2 *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\varepsilon > 0$, there exists a majorant ϕ_+ and a minorant ϕ_- such that $I(\phi_+) - I(\phi_-) < \varepsilon$.*

This is shown via the approximation property.

Theorem 3 *Any continuous function $f : [a, b] \to \mathbb{R}$ is integrable.*

This follows from the uniform continuity of f on $[a, b]$, ensuring that as we make the mesh of $\mathcal{P}$ smaller, $I(\phi^+) - I(\phi^-)$ decreases.

Mean Value Theorem

Theorem 4 *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is some $c \in [a, b]$ such that $\int_a^b f = (b - a)f(c)$. When $a \neq b$, we then have*

$$\frac{1}{b-a}\int_a^b f = f(c)$$

As f is continuous, it attains its bounds, and so we have that the mean value of f is between its minimum and maximum, we know $f[a, b]$ is an interval by the IVT.

Theorem 5 (A second MVT) *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, and that $w : [a, b] \to \mathbb{R}$ is a nonnegative integrable function. Then there is some $c \in [a, b]$ such that*

$$\int_a^b fw = f(c)\int_a^b w$$

Roughly the same proof as above follows here, with the caveat to check if $\int_a^b w = 0$.

Monotone functions

Theorem 6 *Any monotone function $f : [a, b] \to \mathbb{R}$ is integrable.*

Assume increasing, then divide $\mathcal{P}$ into n parts. On (x_i, x_{i+1}) define $\phi^+(x) = f(x_{i+1})$, $\phi^-(x) = f(x_i)$. We then show that with the parts equal and size $\frac{1}{n}$, we get a telescoping in $I(\phi^+) - I(\phi^-)$ that allows us to get the difference as small as desired.

Differentiation

Theorem 7 (First fundamental theorem) *Suppose that f is integrable on (a, b). Define $F : [a, b] \to \mathbb{R}$ by*

$$F(x) := \int_a^x f.$$

Then F is continuous. Moreover, if f is continuous at $c \in (a, b)$ then F is differentiable at c and $F'(c) = f(c)$.

This follows fairly immediately from writing $|F(c + h) - F(c) - hf(c)|$ with $h < \delta$ for which $|f(c + h) - f(c)| < \varepsilon$.

Theorem 8 (Second fundamental theorem) *Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Suppose furthermore that its derivative F' is integrable on (a, b). Then*

$$\int_a^b F' = F(b) - F(a)$$

Note that we may have a derivative which is non-integrable, for example $x \sin\frac{1}{x}$ with unbounded derivative $\sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$ on $(-1, 1)$.

By the mean value theorem we can select ξ_i such that $F'(\xi_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1})$, giving us a telescoping sum that returns the above result.

Technically this result can also be used for F differentiable on all but finitely many elements of (a, b).

Riemann sums

If $\mathcal{P}$ is a partition of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is a function then by a Riemann sum adapted to $\mathcal{P}$ we mean an expression of the form

$$\Sigma(f; \mathcal{P}; \boldsymbol{\xi}) = \sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1})$$

where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ for $\xi_i \in [a, b]$.

Theorem 9 *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Fix a sequence of partitions $\mathcal{P}^{(n)}$. For each n, let $\Sigma(f; \mathcal{P}^{(n)}; \boldsymbol{\xi}^{(n)})$ be a Riemann sum adapted to $\mathcal{P}^{(n)}$. Suppose that there is some constant c such that as $n \to \infty$, $\Sigma(f; \mathcal{P}^{(n)}; \boldsymbol{\xi}^{(n)}) \to c$ irrespective of how $\boldsymbol{\xi}^{(n)}$ is chosen. Then f is integrable and $c = \int_a^b f$.*

The proof of his follows immediately from selecting ξ^i to get $f(\xi^i)$ ε-close to its supremum on $[x_i, x_{i+1}]$, then using this to define majorants and minorants.

Theorem 10 *Let $\mathcal{P}^{(n)}$ be a sequence of partitions with $\operatorname{mesh}\mathcal{P}^{(n)} \to 0$ as $n \to \infty$. If f is integrable, then $\lim_{n\to\infty} \Sigma(f; \mathcal{P}^{(n)}; \boldsymbol{\xi}^{(n)}) = \int_a^b f$ irrespective of $\boldsymbol{\xi}^{(n)}$.*

To prove this, we take a small enough δ, then compare the majorant / minorant used by the Riemann sum with the optimal majorant / minorant on some arbitrary partition. Show that we can only get so many overlapping intervals, and on the "good" intervals we can have the former step functions be good approximations, meaning the overall approximation is good.

The former theorem serves to provide a criteria for integrability - that if there is a partition sequence for which the Riemann sum converges independently of $\boldsymbol{\xi}^{(n)}$, f is integrable - while the latter theorem provides that integrability implies this to be true for any $\mathcal{P}^{(n)}$ with $\operatorname{mesh}\mathcal{P}^{(n)} \to 0$. A candidate method of dealing with this is to use the step-function definition of integrability to then apply the latter theorem. Thus we get the following theorem:

Theorem 11 *Let $f : [a, b] \to \mathbb{R}$ be a function. Let $(\mathcal{P}^{(n)})$ be a sequence of partitions with $\operatorname{mesh}\mathcal{P}^{(n)} \to 0$. Then f is integrable if and only if $\lim_{n\to\infty} \Sigma(f, \mathcal{P}^{(n)}, \boldsymbol{\xi}^{(n)})$ is constant in the choice of $\boldsymbol{\xi}^{(n)}$. If so, it is equal to $\int_a^b f$.*

Basic results

Proposition 1 (Integration by parts) *Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions, differentiable on (a, b). Suppose that f', g' are integrable on (a, b). Then*

$$\int_a^b fg' + \int_a^b f'g = f(b)g(b) - f(a)g(a)$$

The above is clear from the second fundamental theorem.

Proposition 2 (Substitution rule) *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and that $\varphi : [c, d] \to [a, b]$ is continuous on $[c, d]$, has $\varphi(c) = a$ and $\varphi(d) = b$, and maps (c, d) to (a, b). Suppose moreover that φ is differentiable on (c, d) and that its derivative φ' is integrable on this interval. Then*

$$\int_a^b f(x)\,\mathrm{d}x = \int_c^d f(\varphi(t))\varphi'(t)\,\mathrm{d}t.$$

We have that $f \circ \varphi$ is continuous and hence integrable on $[c, d]$, so $(f \circ \varphi)\varphi'$ is integrable on $[c, d]$. By the fundamental theorem $\int_c^d (f \circ \varphi)\varphi' = (F \circ \varphi)(d) - (F \circ \varphi)(c) = \int_a^b f$.

Limits and the integral

Theorem 12 *Suppose that $f_n : [a, b] \to \mathbb{R}$ are integrable, and that $f_n \to f$ uniformly on $[a, b]$. Then f is also integrable, and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n.$$

to prove this, first show integrability by comparing $I(\phi_n^+)$ and $I(\phi_n^-)$ when selecting an n for which $I(\phi_n^+) - I(\phi^+) \le \varepsilon$, then take the difference of integrals and show it converges to zero.

As corollary, the weierstrass M-test then allows us to take the integral of a sum where each term of the sum is bounded with convergent bound.

We may use the fundamental theorem of calculus to prove further results about differentiation.

Theorem 13 *Suppose that $f_n : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$ is a sequence of functions where f_n is continuously differentiable on (a, b), $f_n \to f$ on $[a, b]$, and that $f_n' \to g$ uniformly where g is bounded on (a, b). Then f is differentiable and $f' = g$. In particular, $\lim_{n\to\infty} f_n' = (\lim_{n\to\infty} f_n)'$.*

We must have g continuous by uniform convergence of continuous functions. As g is bounded then we must also have that g is integrable. We have with $F(x) = \int_a^x g(t)\mathrm{d}t$, $F' = g$, $\int_a^x f_n'(t)\mathrm{d}t = f_n(x) - f_n(a)$. By the previous theorem we then get $F(x) = f(x) - f(a)$.