Question Number	Mark Award
1	
2	
3	
Total	

APPLIED MATHEMATICS

Introduction to Numerical Analysis APM2B02/APM2B10

Test 2 V1: 07/10/2021

Duration: 80 minutes + 20 minutes download/upload time **Marks:** 36
Assessor: Dr F. Chirove and Mr J Homann
Moderator: Prof E. Momoniat

Surname: ______________________ **Assigned Number:** _______________

Instructions:

1. Check that this question paper consists of 2 pages in total.
2. **Carefully read and follow the instructions of each question.**
3. Calculators are permitted.
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5. Answer the questions in order (from 1 to 3) and write out your solutions on sheets of paper. Cross out answers that are not to be marked. **Your assigned number, surname and student number must be written at the top of each page.**
6. All calculations must be shown.
7. Use a scanning app (CamScanner is a good option) to scan your solutions into a PDF. Your solutions must be one PDF. Pages must be oriented correctly, i.e. not upside down or on their sides. Do not upload JPEG files.
8. Check that your PDF is not too big (it should be around 1MB/page). SAVE YOUR PDF AS "assignednumber-surname" eg 123-Mango
9. To submit - Scroll down to "ASSIGNMENT SUBMISSION" and then "Attach files". Select your PDF and click "Submit".
10. FOR ANY ISSUES, EMAIL apm02a2@uj.ac.za

Question 1 (15 marks)

(a) Consider the task of approximating $\int_0^1 e^{-2x^2}dx$ using the Composite Trapezoidal rule. How large should n and h be chosen in order to ensure that the error is at most 0.001? (9)

Solution: (a) $f'(x) = -4e^{-2x^2}x$, $f''(x) = 4e^{-2x^2}(4x^2-1)$, $f'''(x) = -16e^{-2x^2}x(4x^2-3)$
$f'''(x) = 0$ *at* $x = 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}$ *and only* $x = 0, \frac{\sqrt{3}}{2} \in [0,1]$
$|f''(0)| = 4$, $|f''(\frac{\sqrt{3}}{2})| = 1.7850$, $|f''(1)| = 1.6240$ *and so*
$M = max\{4, 1.7850, 1.6240\} = 4$
$\frac{h^2(b-a)M}{12} \leq 0.001 \implies h^2 \leq 3 \times 0.001 = 0.003$
$\implies h \leq 0.0548 \implies n \geq 18.2574$
Choose $n = 19$ *and* $h = 0.0526$ 1/2 1/2

(b) The composite trapezoidal rule is applied with $h = 0.2$ to approximate the integral

$$I = \int_0^1 x(1-x^2)dx.$$

i. Complete the table below leaving your solutions correct to four decimal places. (3)

x_i	0	0.2	0.4	0.6	0.8	1
$f(x_i)$	A=0.0000	B=0.1920	C=0.3360	D=0.3840	E=0.2880	F=0.0000
	1/2	1/2	1/2	1/2	1/2	1/2

ii. Find the approximate value of I on $[0,1]$ using the composite trapezoidal rule. (3)

Solution:
$I \approx \frac{h}{2}[A+E+2(B+C+D)] = \frac{0.2}{2}(2(2(0.1920+0.3360+0.3840+0.2880)+0+0)) = 0.2400$

Question 2 (10 marks)

(a) Find an approximate value for the integral $\int_0^1 \cos^2 x\ dx$ using the Composite Simpson's rule with $N = 2$ correct to four decimal points. (6)

Solution:
$a = 0$ $b = 1$, $N = 2$ *and* $h = \frac{b-a}{2N} = 1/4$

x_i	0	0.25	0.5	0.75	1
$f(x_i)$	1.0000	0.9388	0.7702	0.5354	0.2919

3 | 3 2 1/2

$\int_0^1 \cos^2 x\ dx \approx \frac{0.25}{3}[1.0000 + 0.2919 + 2(0.7702) + 4(0.9388 + 0.5354)]$
$= 0.7274$ 1/2

(b) Use the solution to part (a) and an appropriate trigonometrical identity to deduce an approximate value for $\int_0^1 \sin^2 x\ dx$ correct to four decimal points. (4)

Solution: $\sin^2 x = 1 - \cos^2 x$ *and so* $\int_0^1 \sin^2 x \; dx = \int_0^1 (1 - \cos^2 x) \; dx = \int_0^1 \; dx - \int_0^1 \cos^2 x \; dx = 1 - 0.7274 =$ 0.2726

Question 3 (11 marks)

(a) Develop a first-order method for approximating $f''(x)$ which uses the data $f(x-h)$, $f(x)$ and $f(x+3h)$. (4)

Solution

We have

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(\xi_1), \quad \xi_1 \in [x-h, x]$$

and

$$f(x+3h) = f(x) + 3hf'(x) + \frac{9h^2}{2} f''(x) - \frac{9h^3}{2} f'''(\xi_2), \quad \xi_2 \in [x, x+3h]$$

Hence,

$$3f(x-h) + f(x+3h) = 4f(x) + 6h^2 f''(x) + \frac{h^3}{2}\left[9f'''(\xi_2) - f'''(\xi_1)\right].$$

Thus,

$$f''(x) = \frac{3f(x-h) + f(x+3h) - 4f(x)}{6h^2} - \frac{h}{12}\left[9f'''(\xi_2) - f'''(\xi_1)\right].$$

(b) Use the three-point centred difference formula for the second derivative to approximate $f''(1)$, where $f(x) = x^{-1}$, for $h = 0.1$, 0.01 and 0.001. Furthermore, determine the approximation error. Use an accuracy of 6 decimal digits for the final answers of the derivative values only. (7)

Solution

By letting $D(x_0, h)$ denote the approximation of the derivative at x_0 with step size h, and $E(x_0, h) := |f''(x_0) - D(x_0, h)|$ denote the error at x_0, we have the following data. Furthermore, $f''(x_0) = 2x_0^{-3}$, so $f''(1) = 2$.

h	$D(1,h)$	$E(1,h)$
0.1	2.020202	0.020202
0.01	2.000200	0.000200
0.001	2.000002	0.000002

END OF QUESTION PAPER

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APPLIED MATHEMATICS

Introduction to Differential Equations
APM2B

Test 2 V2: 07/10/2021

Duration: 80 minutes + 20 minutes download/upload time **Marks:** 36
Assessor: Dr F. Chirove and Mr J. Homann
Moderator: Prof E. Momoniat

Surname: ______________________ **Assigned Number:** _______________

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Question 1 (15 marks)

(a) Consider the task of approximating $\int_0^1 e^{-3x^2}dx$ using the Composite Trapezoidal rule. How large should n and h be chosen in order to ensure that the error is at most 0.001? (9)

Solution: (a) $f'(x) = -6e^{-3x^2}x$ $f''(x) = 6e^{-3x^2}(6x^2-1)$ $f'''(x) = -108e^{-3x^2}x(2x^2-1)$
$f'''(x) = 0$ *at* $x = 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ *and only* $x = 0, \frac{1}{\sqrt{2}} \in [0,1]$
$|f''(0)| = 6,\ |f''(\frac{1}{\sqrt{2}})| = 2.6776,\ |f''(1)| = 1.4936$ *and so*
$M = max\{6, 2.6776, 1.4936\} = 6$
$\frac{h^2(b-a)M}{12} \leq 0.001 \implies h^2 \leq 2\times 0.001 = 0.002$
$\implies h \leq 0.04472 \implies n \geq 22.3607$
Choose $n = 23$ *and* $h = 0.04348$ 1/2 1/2

(b) The composite trapezoidal rule is applied with $h = 0.2$ to approximate the integral

$$I = \int_{-1}^{0} x(x^2-1)dx.$$

i. Complete the table below leaving your solutions correct to four decimal places. (3)

x_i	-1	-0.8	-0.6	-0.4	-0.2	0
$f(x_i)$	A=0.0000	B=2880	C=3840	D=0.3360	E=0.1920	F=0.0000

1/2 1/2 1/2 1/2 1/2 1/2

ii. Find the approximate value of I on $[-1,0]$ using the composite trapezoidal rule. (3)

Solution:
$I \approx \frac{h}{2}[A+E+2(B+C+D)] = \frac{0.2}{2}(2(0.2880+0.3840+0.3360+0.1920)+0+0) =$
0.2400

Question 2 (10 marks)

(a) Find an approximate value for the integral $\int_0^1 \cos^2 x\ dx$ using the Composite Simpson's rule with $N = 2$ correct to four decimal points. (6)

Solution:
$a = 0\ b = 1,\ N = 2$ *and* $h = \frac{b-a}{2N} = 1/4$

x_i	*0*	*0.25*	*0.5*	*0.75*	*1*
$f(x_i)$	*1.0000*	*0.9388*	*0.7702*	*0.5354*	*0.2919*

2½

$\int_0^1 \cos^2 x\ dx \approx \frac{0.25}{3}[1.0000+0.2919+2(0.7702)+4(0.9388+0.5354)]$
$= 0.7274$ 1/2

(b) Use the answer to part (a) and an appropriate trigonometrical identity to deduce an approximate value for $\int_0^1 (1-\cos(2x))\ dx$. (4)

Solution: $1 - \cos(2x) = 2 - 2\cos^2 x$ *and so*
$\int_0^1 (1 - \cos(2x))\, dx = \int_0^1 (2 - 2\cos^2 x)\, dx = 2 - 2(0.7274) = 0.5452$

Question 3 (11 marks)

(a) Develop a first-order method for approximating $f''(x)$ which uses the data $f(x-3h), f(x)$ and $f(x+h)$. (4)

Solution

We have

$$f(x-3h) = f(x) - 3hf'(x) + \frac{9h^2}{2}f''(x) - \frac{9h^3}{2}f'''(\xi_1), \quad \xi_1 \in [x-3h, x]$$

and

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(\xi_2), \quad \xi_2 \in [x, x+h]$$

Hence,

$$3f(x+h) + f(x-3h) = 4f(x) + 6h^2 f''(x) + \frac{h^3}{2}\left[f'''(\xi_2) - 9f'''(\xi_1)\right].$$

Thus,

$$f''(x) = \frac{3f(x+h) + f(x-3h) - 4f(x)}{6h^2} + \frac{h}{12}\left[f'''(\xi_2) - 9f'''(\xi_1)\right].$$

(b) Use the three-point centred difference formula for the second derivative to approximate $f''(1)$, where $f(x) = x^{-2}$, for $h = 0.1$, 0.01 and 0.001. Furthermore, determine the approximation error. Use an accuracy of 6 decimal digits for the final answers of the derivative values only. (7)

Solution

By letting $D(x_0, h)$ denote the approximation of the derivative at x_0 with step size h, and $E(x_0, h) := |f''(x_0) - D(x_0, h)|$ denote the error at x_0, we have the following data. Furthermore, $f''(x_0) = 6x_0^{-4}$, so $f''(2) = \frac{3}{8}$.

h	$D(1, h)$	$E(1, h)$
0.1	6.101418	0.101418
0.01	6.001000	0.001000
0.001	6.000010	0.000010

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APPLIED MATHEMATICS

Introduction to Differential Equations
APPM2B

Test 2 V3: 07/10/2021

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Moderator: Prof E. Momoniat

Surname: ______________________ **Assigned Number:** ______________

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Question 1 (15 marks)

(a) Consider the task of approximating $\int_0^1 e^{-4x^2}dx$ using the Composite Trapezoidal rule. How large should n and h be chosen in order to ensure that the error is at most 0.001? (9)

Solution: (a) $f'(x) = -8e^{-4x^2}x$, $f''(x) = 8e^{-4x^2}(8x^2-1)$ $f'''(x) = -64e^{-4x^2}x\,(8x^2-3)$

$f'''(x) = 0$ at $x = 0, -\frac{1}{2}\sqrt{\frac{3}{2}}, \frac{1}{2}\sqrt{\frac{3}{2}}$ and only $x = 0, \frac{1}{2}\sqrt{\frac{3}{2}} \in [0,1]$

$|f''(0)| = 8,\ |f''(\frac{1}{2}\sqrt{\frac{3}{2}})| = 3.5701,\ |f''(1)| = 1.0257$ and so

$M = max\{8, 3.5701, 1.0257\} = 8$

$\frac{h^2(b-a)M}{12} \le 0.001 \implies h^2 \le \frac{3}{2} \times 0.001 = 0.0015$

$\implies h \le 0.03873 \implies n \ge 25.8199$

Choose $n = 26$ and $h = 0.0385$ 1/2 1/2

(b) The composite trapezoidal rule is applied with $h = 0.2$ to approximate the integral

$$I = \int_0^1 x(1+x^2)dx.$$

i. Complete the table below leaving your solutions correct to four decimal places. (3)

x_i	0	0.2	0.4	0.6	0.8	1
$f(x_i)$	A=0.0000	B=0.2080	C=0.4640	D=0.8160	E=1.3120	F=2.0000

1/2 1/2 1/2 1/2 1/2 1/2

ii. Find the approximate value of I on $[0,1]$ using the composite trapezoidal rule. (3)

Solution:
$I \approx \frac{h}{2}[A+E+2(B+C+D)] = \frac{0.2}{2}(2(0.2080 + 0.4640 + 0.8160 + 1.3120) + 2+0) = 0.7600$

Question 2 (10 marks)

(a) Find an approximate value for the integral $\int_0^1 \sin^2 x\ dx$ using the Composite Simpson's rule with $N = 2$ correct to four decimal points. (6)

Solution:
$a = 0\ b = 1,\ N = 2$ and $h = \frac{b-a}{2N} = 1/4$

x_i	0	0.25	0.5	0.75	1
$f(x_i)$	0.0000	0.06121	0.2298	0.4646	0.7081

3 1 2 1/2

$\int_0^1 \cos^2 x\ dx \approx \frac{0.25}{3}[0 + 0.7081 + 2(0.2298) + 4(0.06121 + 0.4646)]$
$= 0.2726$ 1/2

(b) Use the solution to part (a) and an appropriate trigonometrical identity to deduce an approximate value for $\int_0^1 \cos^2 x\ dx$. (4)

Solution: $\cos^2 x = 1 - \sin^2 x$ *and so* $\displaystyle\int_0^1 \cos^2 x\ dx = \int_0^1 (1 - \sin^2 x)\ dx = \int_0^1\ dx - \int_0^1 \sin^2 x\ dx = 1 - 0.2726 =$ 0.7274

Question 3 (11 marks)

(a) Develop a first-order method for approximating $f''(x)$ which uses the data $f(x-2h), f(x)$ and $f(x+3h)$. (4)

Solution

We have

$$f(x-2h) = f(x) - 2hf'(x) + \frac{4h^2}{2}f''(x) - \frac{8h^3}{6}f'''(\xi_1), \quad \xi_1 \in [x-3h, x]$$

and

$$f(x+3h) = f(x) + 3hf'(x) + \frac{9h^2}{2}f''(x) + \frac{9h^3}{2}f'''(\xi_2), \quad \xi_2 \in [x, x+h]$$

Hence,

$$3f(x-2h) + 2f(x+3h) = 5f(x) + 15h^2 f''(x) + \frac{h^3}{2}\left[-8f'''(\xi_1) + 18f'''(\xi_2)\right].$$

Thus,

$$f''(x) = \frac{3f(x-2h) + 2f(x+2h) - 5f(x)}{15h^2} + \frac{h}{30}\left[-8f'''(\xi_1) + 18f'''(\xi_2)\right].$$

(b) Use the three-point centred difference formula for the second derivative to approximate $f''(1)$, where $f(x) = x^{-5}$, for $h = 0.1$, 0.01 and 0.001. Furthermore, determine the approximation error. Use an accuracy of 6 decimal digits for the final answers of the derivative values only. (7)

Solution

By letting $D(x_0, h)$ denote the approximation of the derivative at x_0 with step size h, and $E(x_0, h) := |f''(x_0) - D(x_0, h)|$ denote the error at x_0, we have the following data. Furthermore, $f''(x_0) = 30x_0^{-7}$, so $f''(2) = \frac{10}{729}$.

h	$D(1,h)$	$E(1,h)$
0.1	31.443010	1.44301
0.01	30.014004	0.014004
0.001	30.000140	0.000140

END OF QUESTION PAPER