Course	AE 426 Project 3
Author(s)	Elijah Serena Ikemhe
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Instructor	Professor David Canales Garcia

Objectives
The purpose of this project is to study the properties and dynamics of a 3-1-3 motion with, and without torque. We explored quaternions, and DCM. However, the main focus is on the angular velocity, precession angle, and nutation angle.

Assumptions
$\hat{b}_i = \hat{e}_i$
Initial State(s) at t = 0
$\tilde{q} = [0\ 0\ 0\ 1]$
$w_1(0) = 1\frac{rad}{s}$, $w_2(0) = -1\frac{rad}{s}$, $w_3(0) = 2\frac{rad}{s}$
T = 80Nm ; A = 400 ; C = 100

Methodology
The kinematic equations of motion

From Euler EOMSs:
$M_1 = A\dot{w}_1 + \ (C-A)w_2w_3$; $M_2 = A\dot{w}_2 + \ (-C+A)w_1w_3$; $M_3 = C\dot{w}_3$
If $M_2 = \ M_3 = 0$ and $M_1 = T = 80$
Therefore,
$\dot{w}_1 = \frac{T}{A} + \frac{(A-C)}{A}w_2w_3$. At initial conditions, $\dot{w}_1 = \ -1.3\frac{rad}{s^2}$
$\dot{w}_2 = \frac{(C-A)}{A}w_1w_3$. At initial conditions, $\dot{w}_2 = \ -1.5\frac{rad}{s^2}$
$\dot{w}_3 = 0$. This indicates that w_3 is a constant.
Differentiating $\dot{w}_1$ & $\dot{w}_2$, we get:
$\ddot{w}_1 = \ \frac{(A-C)}{A}\dot{w}_2w_3$. Substituting the value of $\dot{w}_2$, $\ddot{w}_1 = \frac{-(C-A)^2}{A^2}w_1{w_3}^2$
$\ddot{w}_2 = \ \frac{(C-A)}{A}\dot{w}_1w_3$. Substituting the value of $\dot{w}_1$, $\ddot{w}_2 = \frac{(C-A)}{A^2}\ Tw_3 \ - \frac{(C-A)^2}{A^2}w_2{w_3}^2$

Solving Analytically:
Let $k = \frac{(C-A)^2}{A^2}{w_3}^2$. At *t = 0*, $k = \frac{9}{4}$
For $\ddot{w}_1$,
$\ddot{w}_1 + \ kw_1 = 0$
$w_1(t) = \ C_1 \cos kt + C_2 \sin kt$. At *t = 0*, $w_1 == 1 = C_1$
$\dot{w}_1(t) = \ -C_1 k \sin kt + \ C_2 k \cos kt$. At *t = 0*, $\dot{w}_1 == -1.3 = -C_2 * \frac{9}{4}$. $C_2 = \ \frac{26}{45}\ or\ 0.5\overline{7}$
Therefore, $w_1(t) = \cos\frac{9}{4}t + \ 0.5\overline{7}\sin\frac{9}{4}t$
For $\ddot{w}_2$,
$w_2(t) = \ w_{2h} + \ w_{2p}$. Starting with the particular solution, w_{2p}:
$w_{2p} = \frac{(C-A)}{A^2}\ Tw_3 - \ \frac{(C-A)^2}{A^2}C_5{w_3}^2 == 0$. Rearranging, $C_5 = \ \frac{T}{(C-A)w_3}$
Recall, w_3, is constant. Therefore, $w_{2p} = C_5 = \ -\frac{2}{15}$.
$w_2(t) = \ \ C_3 \cos kt + C_4 \sin kt - \frac{2}{15} == -1$. $C_3 = \ -1 + \frac{2}{15}$. Therefore, $C_3 \ = \frac{13}{15}$
$\dot{w}_2(t) = \ -C_3 k \sin kt + \ C_4 k \cos kt == \ -1.5$. $C_4 = \ -1.5 * \frac{4}{9} == \ -\frac{2}{3}$.
Therefore, $w_2(t) = \ \frac{13}{15}\cos\frac{9}{4}t \ \ -\frac{2}{3}\sin\frac{9}{4}t - \frac{2}{15}$.

Quaternions at *t = 0*
$\tilde{q} = [0\ 0\ 0\ 1]$
$\dot{\bar{q}} = \frac{1}{2} * (q_4\bar{w} + \ \bar{q} \times \bar{w}\ ) = -\frac{1}{2}(1[1 -1\ 2] + [\ 0\ 0\ 0] \ \times [1 -1\ 2]) = [\frac{1}{2} - \frac{1}{2}\ 1]$
$\dot{q}_4 = -\frac{1}{2}(\bar{w} * \bar{q}) = -\frac{1}{2} * ([1 -1\ 2] * [0\ 0\ 0]) = 0$
$\dot{\tilde{q}} = [\frac{1}{2} \ -\frac{1}{2}\ 1\ 0]$
To get the equations for precession rate and nutation rate (for ode to output precession, nutation angles):
3-1-3 rotation:

$$\text{DCM} = \begin{bmatrix} c1c3 - c2s1s3 & -c1s3 - c2c3s1 & s1s2 \\ c3s1 + c1c2s3 & c1c2c3 - s1s3 & -c1s2 \\ s2s3 & c3s2 & c2 \end{bmatrix}$$

$\bar{w} = \ w_p(c3\widehat{b_1} - s3\widehat{b_2}) + \ w_n(c3\widehat{b_1} - s3\widehat{b_2}) + w_s\ \widehat{b_3}$
$\bar{w} = (w_p c3 + w_n c3)\widehat{b_1} + (-s3w_p - s3w_n)\widehat{b_2} + w_s\ \widehat{b_3} == w_1\ \widehat{b_1} + \ w_2\ \widehat{b_2} + \ w_3\ \widehat{b_3}$
$w_n = \frac{1}{2} * (w_1 + \ w_2\ )$; $w_p = \ w_1 - (\frac{1}{2} * (w_1 + \ w_2\ ))$; $w_s = w_3 = 0$

Results & Discussion.
Analysis For Torque

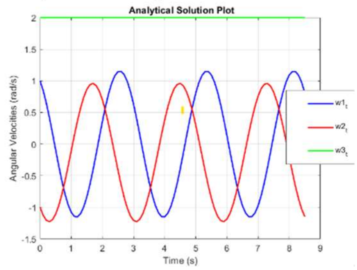

Figure 1: Plot of Angular Velocities obtained from Analytical solution.

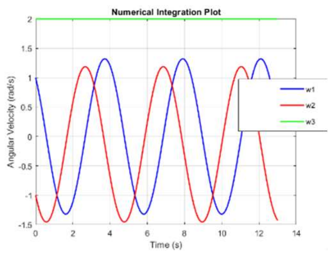

Figure 2: Plot of Angular Velocities obtained from Numerical Integration.

To determine the correct quadrant, for the angles:

A. Precession Normalization (p):
- Any angle is wrapped into [0∘,360∘) using MATLAB function mod(p, 360).
- Negative angles are corrected by adding 360∘.

B. Nutation Normalization (n):
1. First, all angles are wrapped into [0∘,360∘).
2. Any value greater than 180∘ is adjusted symmetrically to 360∘−n, which reflects the tilt symmetry.

As requested in the problem statement, Table 1 lists the quaternions, angular velocities, precession, nutation angle, and C12, C22, and C33 of the DCM.

Table 1: Properties for 1 second intervals until t = 12s.

T (s)	q_1	q_2	q_3	q_4	$w_1\ (\frac{rad}{s})$	$w_2\ (\frac{rad}{s})$	$w_3 (\frac{rad}{s})$	Precession(°)	Nutation(°)	C12	C22	C33
0	0.00	0.00	0.00	1.00	1.00	-1.00	2.00	0.00	0.00	0.00	1.00	1.00
1	0.06	-0.65	0.67	0.37	-0.79	-1.19	2.00	41.75	34.41	-0.57	0.11	0.16
2	-0.53	-0.75	0.12	-0.37	-1.11	0.58	2.00	17.74	78.22	0.88	0.41	-0.69
3	-0.76	-0.17	-0.62	0.09	0.64	1.03	2.00	339.69	57.11	0.37	-0.93	-0.22
4	-0.29	0.11	-0.20	0.93	1.20	-0.69	2.00	5.42	17.41	0.30	0.75	0.80
5	0.06	-0.37	0.70	0.61	-0.47	-1.37	2.00	54.20	40.00	-0.90	0.02	0.72
6	-0.33	-0.75	0.46	-0.32	-1.27	0.24	2.00	42.48	90.00	0.80	0.35	-0.36
7	-0.79	-0.38	-0.43	-0.21	0.29	1.16	2.00	359.13	81.58	0.43	-0.62	-0.55
8	-0.58	0.12	-0.36	0.72	1.31	-0.33	2.00	11.83	37.49	0.39	0.08	0.31
9	-0.05	-0.08	0.61	0.79	-0.10	-1.45	2.00	64.06	46.77	-0.95	0.26	0.98
10	-0.15	-0.63	0.75	-0.17	-1.32	-0.13	2.00	65.86	99.28	0.44	-0.16	0.17
11	-0.70	-0.56	-0.14	-0.43	-0.09	1.19	2.00	20.98	104.52	0.65	-0.01	-0.59
12	-0.79	0.02	-0.44	0.42	1.31	0.05	2.00	19.93	59.86	0.35	-0.64	-0.26

Precession and Nutation Plots. In Figure 3, and figure 4 below, we plot the angles of precession and nutation as a function of time.

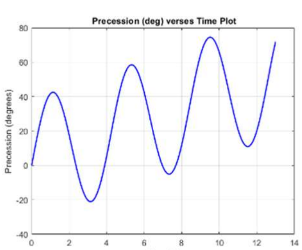

Figure 3: Plot of Precession as a function of time.

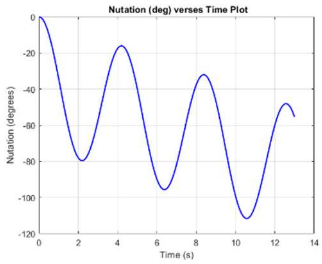

Figure 4: Plot of Nutation as a function of time.

As we can observe from the figure 4 above, the nutation is initially at zero, and it returns to zero indicates a periodic motion typical of rotational dynamics in a 3-1-3 rotation sequence. This periodic motion can be attributed to the following reasons below:
1. Cyclic Nature of Rotational Dynamics
2. Absence of External Disturbances
3. Mathematical Modeling
4. Coupled Motion with Precession

Performing analysis for a condition of No Torque

Table 2: Properties for 1 second intervals until t = 12s at zero Torque.

T (s)	q_1	q_2	q_3	q_4	$w_1\ (\frac{rad}{s})$	$w_2\ (\frac{rad}{s})$	$w_3 (\frac{rad}{s})$	Precession(°)	Nutation(°)	C12	C22	C33
0	0.00	0.00	0.00	1.00	1.00	-1.00	2.00	0.00	0.00	0.00	1.00	1.00
1	0.02	-0.64	0.66	0.38	-0.93	-1.07	2.00	38.10	35.50	-0.54	0.11	0.17
2	-0.62	-0.71	0.09	-0.33	-1.13	0.85	2.00	5.39	76.01	0.94	0.22	-0.77
3	-0.73	-0.08	-0.65	0.19	0.77	1.19	2.00	322.66	46.25	0.36	-0.91	-0.08
4	-0.11	0.08	-0.19	0.97	1.24	-0.68	2.00	349.33	1.52	0.35	0.91	0.96
5	0.09	-0.53	0.63	0.56	-0.59	-1.28	2.00	35.83	24.96	-0.81	0.20	0.42
6	-0.50	-0.77	0.27	-0.27	-1.32	0.50	2.00	15.74	73.00	0.92	0.35	-0.70
7	-0.79	-0.20	-0.59	0.02	0.40	1.36	2.00	326.40	56.36	0.33	-0.92	-0.31
8	-0.23	0.13	-0.36	0.90	1.38	-0.31	2.00	339.50	5.96	0.58	0.64	0.86
9	0.13	-0.40	0.54	0.73	-0.21	-1.40	2.00	30.70	15.47	-0.89	0.39	0.64
10	-0.37	-0.80	0.43	-0.17	-1.41	0.11	2.00	24.84	67.22	0.74	0.35	-0.56
11	-0.80	-0.33	-0.47	-0.13	0.01	1.41	2.00	332.81	65.03	0.40	-0.75	-0.51
12	-0.36	0.14	-0.50	0.77	1.41	0.09	2.00	331.31	12.97	0.67	0.23	0.70

In Figure 5 below, we can observe that the peaks for w_1 & w_2 in Numerical integration methods are equal. However, this is not the case for figure 1 & 2, where it can be visually concluded that $w_1 > w_2$.

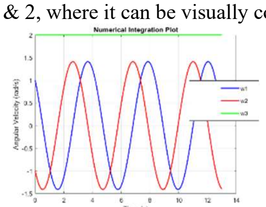

Figure 5: Plot of Angular Velocities obtained from Numerical Integration at zero Torque.

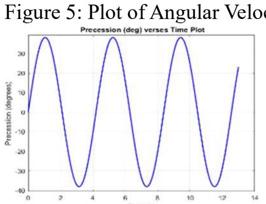

Figure 6: Plot of Precession as a function of time at zero Torque.

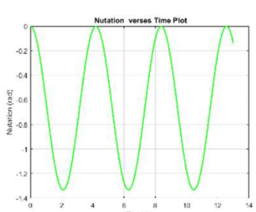

Figure 7: Plot of Nutation as a function of time at zero Torque.

DCM Elements Plot

Observed from the plot in Figure 8 and Figure 9
The plots are out of the page. The curve is a closed, looping pattern with some variation in shape and tightness. Upon further analysis, the curves suggest a damped periodic motion, common in systems experiencing oscillations. This observation is in line, because we know that the system is experiencing a torque.

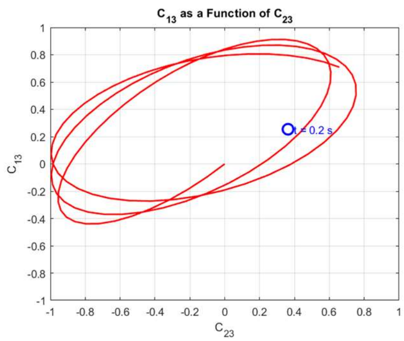

Figure 8: 2D Plot of c13 verses c23.

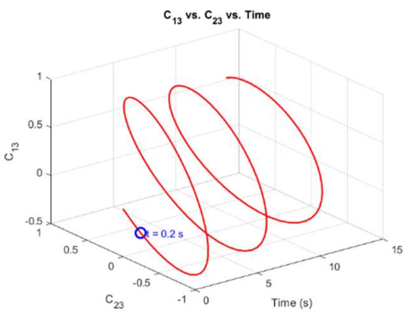

Figure 9: 3D Plot of c13 verses c23 verses time.