Motion and Optical Flow

Introduction to Computational Photography:

EECS 395/495

Northwestern University

Motion and Optical Flow

Method to estimate apparent motion of scene objects from a sequence of images.

Topics:

(1) Motion Field

(2) Optical Flow

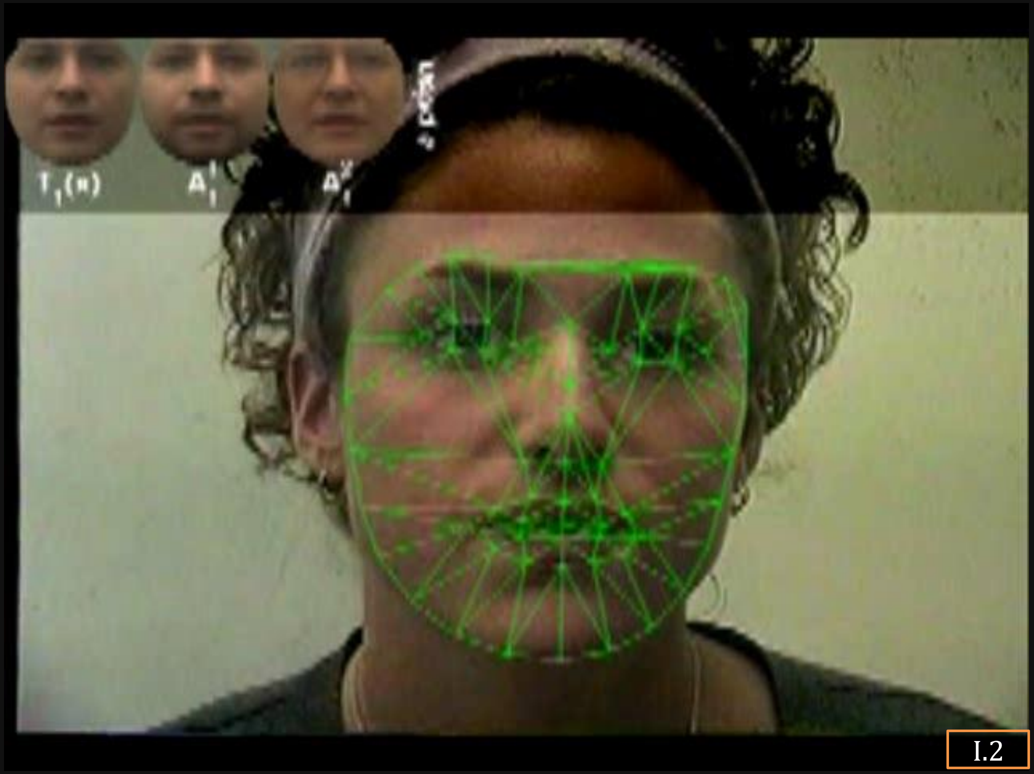
(3) Optical Flow Constraints

(4) Optical Flow Algorithms

Where is Motion Estimation Used?

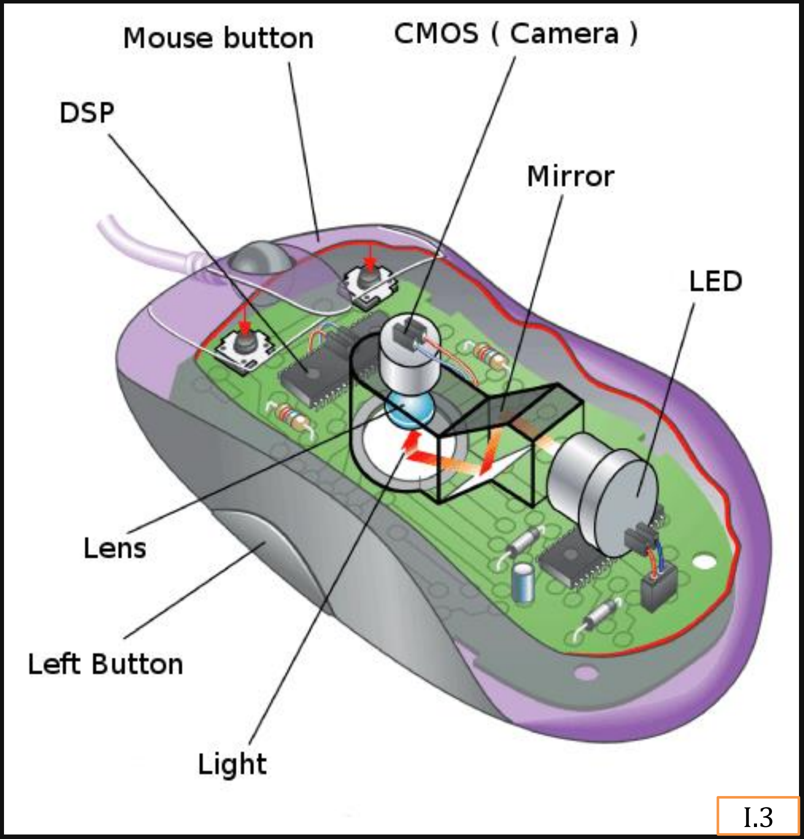
Finding Velocities of Vehicles

Where is Motion Estimation Used?



Tracking of Facial Features

Where is Motion Estimation Used?



Estimating Mouse Movements

Motion Field

Image velocity of a point that is moving in the scene

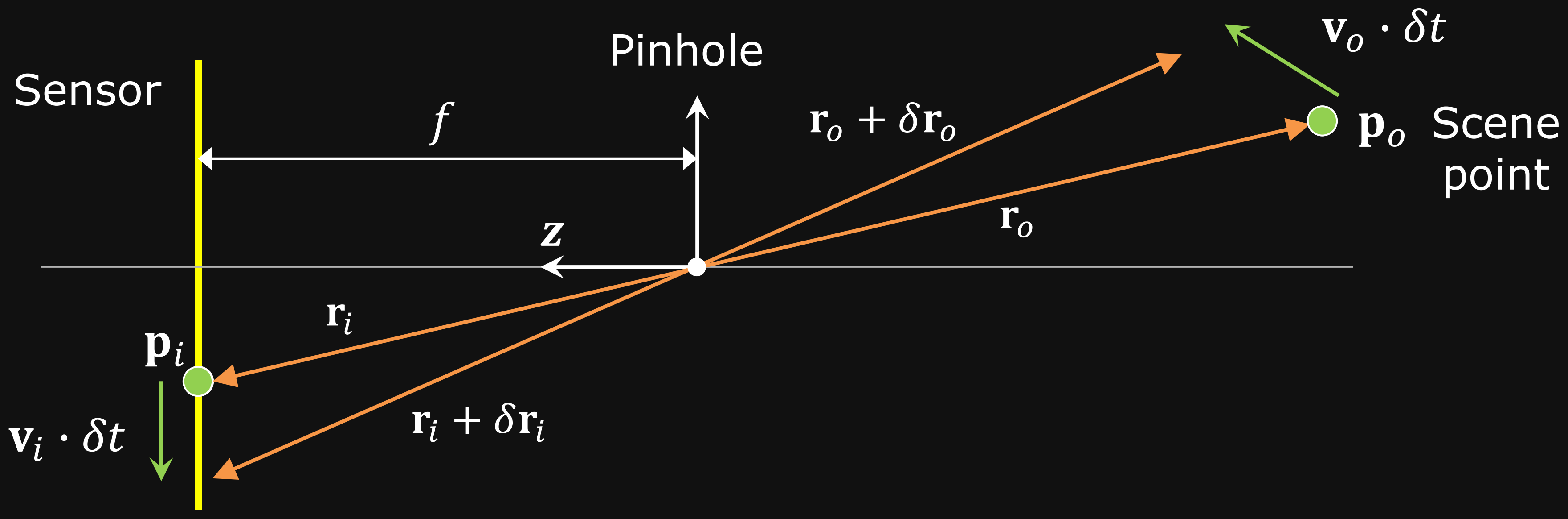


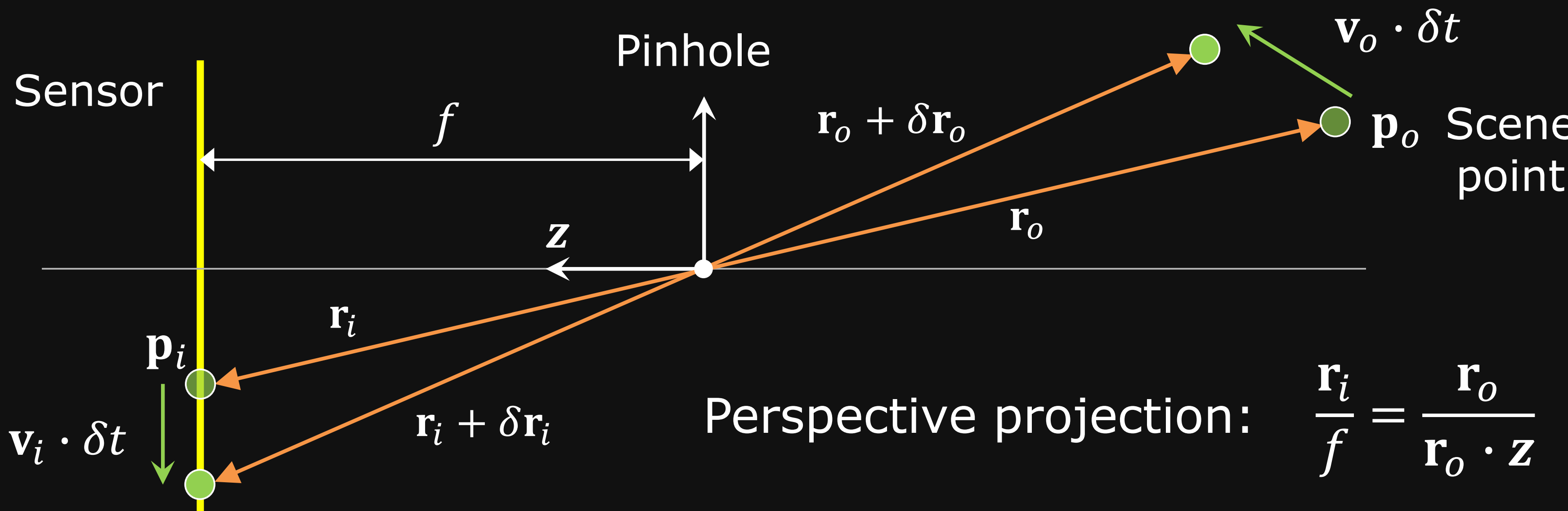
Image Point Velocity: $\mathbf{v}_i = \dfrac{d\mathbf{r}_i}{dt}$

(Motion Field)

Scene Point Velocity: $\mathbf{v}_o = \dfrac{d\mathbf{r}_o}{dt}$

Motion Field

Image velocity of a point that is moving in the scene



Perspective projection: $\dfrac{\mathbf{r}_i}{f} = \dfrac{\mathbf{r}_o}{\mathbf{r}_o \cdot \mathbf{z}}$

Image Point Velocity: $\mathbf{v}_i = \dfrac{d\mathbf{r}_i}{dt} = f\,\dfrac{(\mathbf{r}_o \cdot \mathbf{z})\mathbf{v}_0 - (\mathbf{v}_o \cdot \mathbf{z})\mathbf{r}_0}{(\mathbf{r}_o \cdot \mathbf{z})^2}$

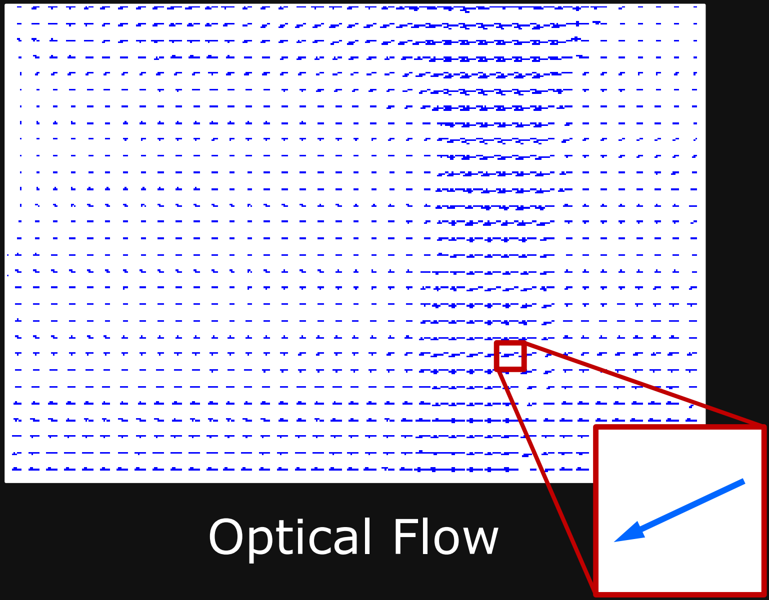
(Motion Field)

$$\boxed{\mathbf{v}_i = f\,\frac{(\mathbf{r}_o \times \mathbf{v}_0) \times \mathbf{z}}{(\mathbf{r}_o \cdot \mathbf{z})^2}}$$

Optical Flow

Motion of brightness patterns in the image

Image Sequence
(2 frames)

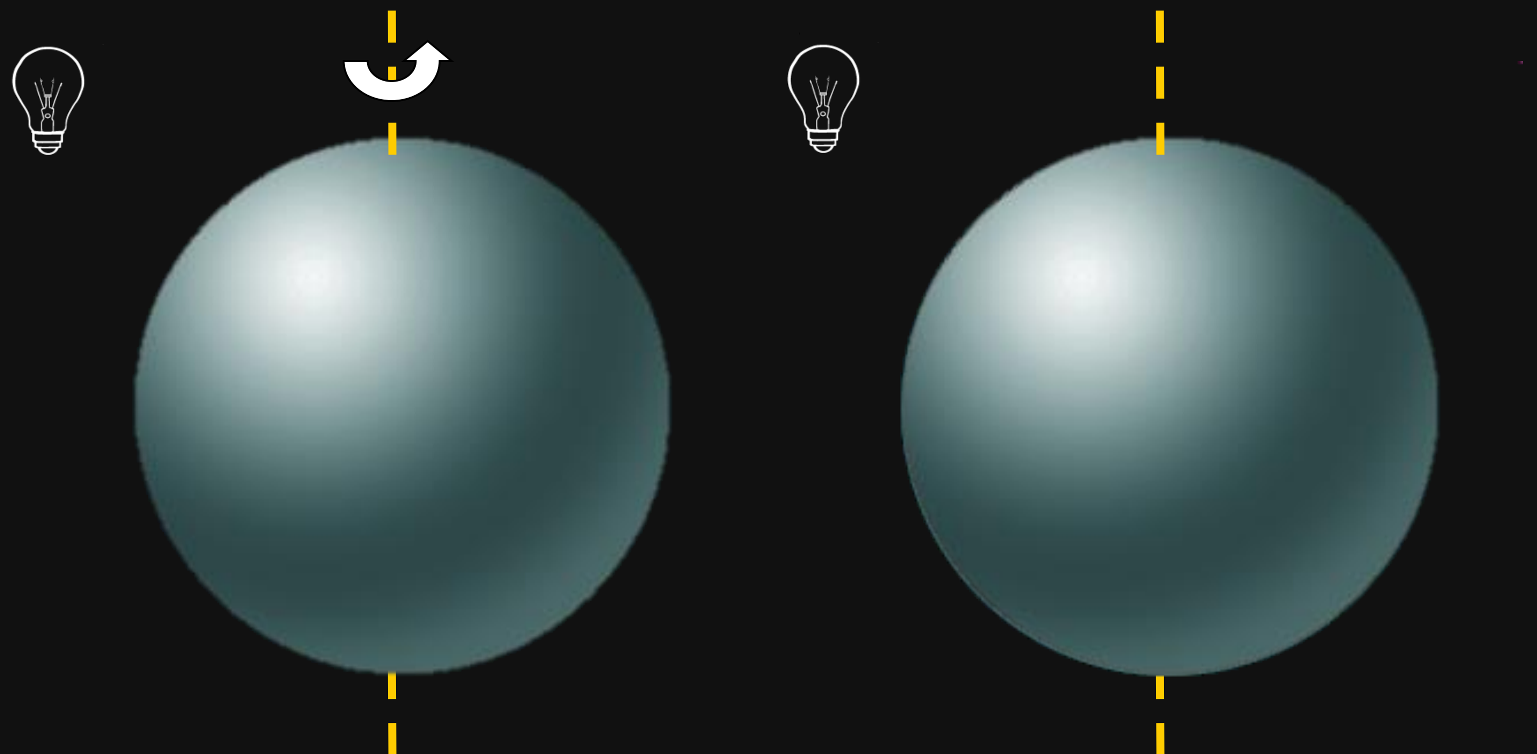


Optical Flow

Velocity of
brightness pattern

Ideally, Optical Flow = Motion Field

When is Optical Flow ≠ Motion Field?



Spinning Sphere
Stationary Light Source

Motion Field exists

But no Optical Flow

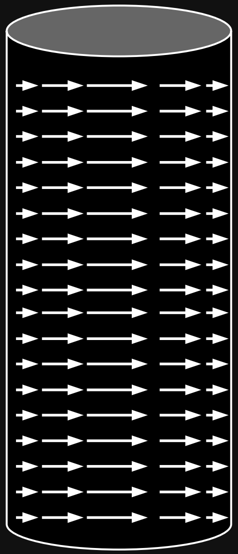
Stationary Sphere
Moving Light Source

No Motion Field exists

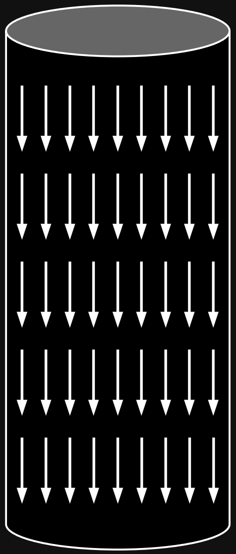
But There is Optical Flow

When is Optical Flow ≠ Motion Field?

Barber Pole Illusion

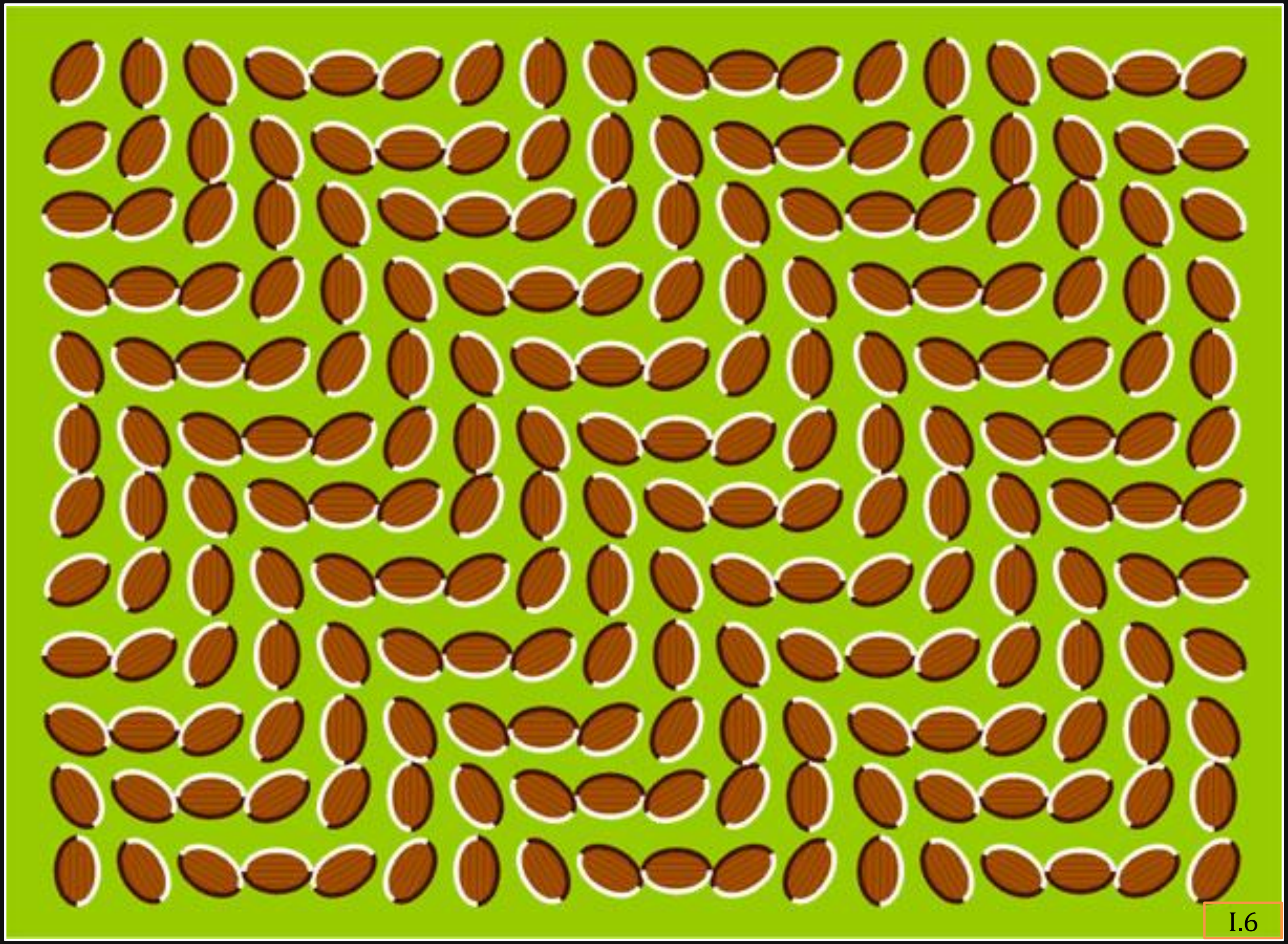


Motion Field



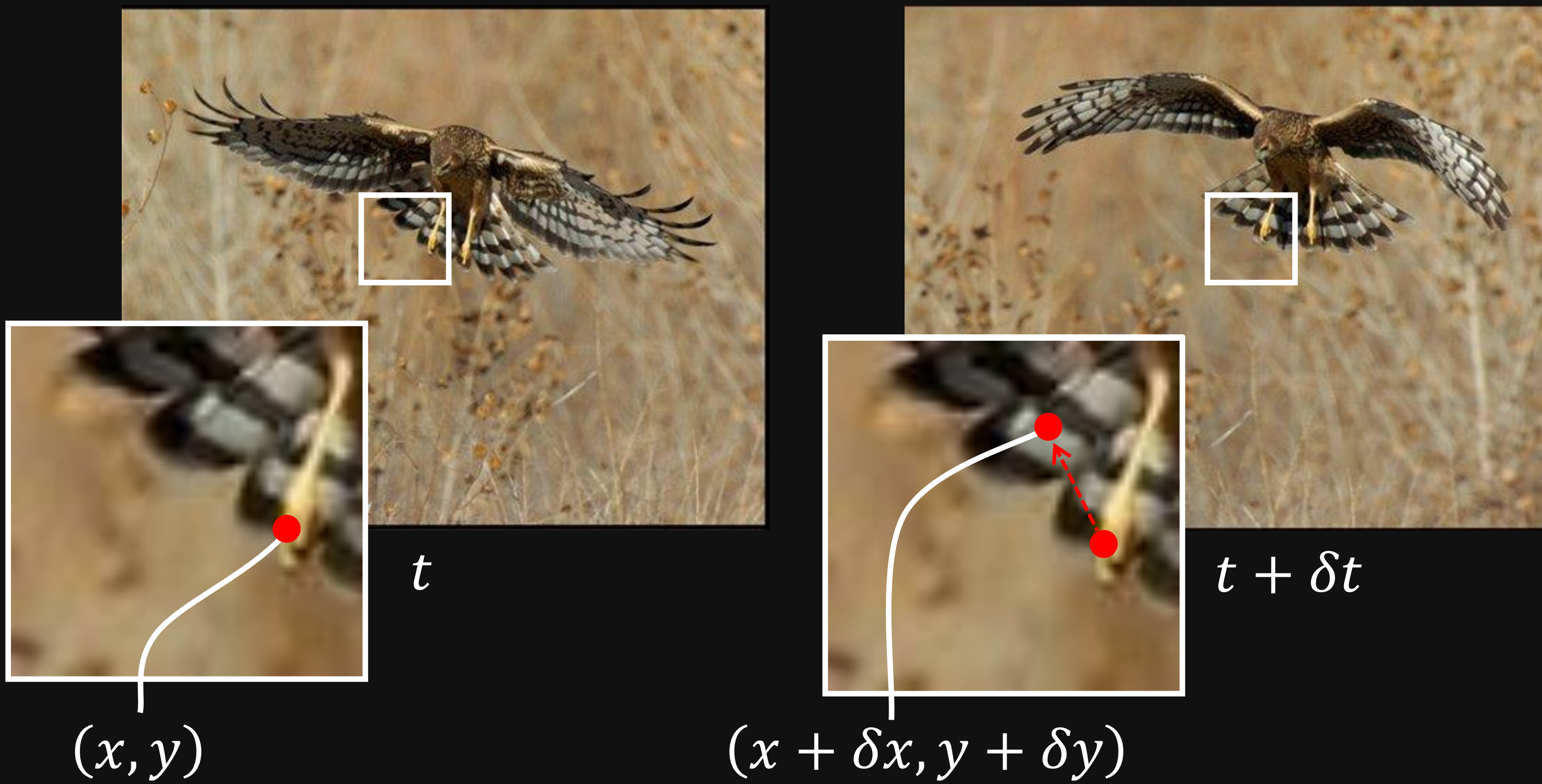
Optical Flow

Perceived Motion Without Motion



Donguri Wave Illusion

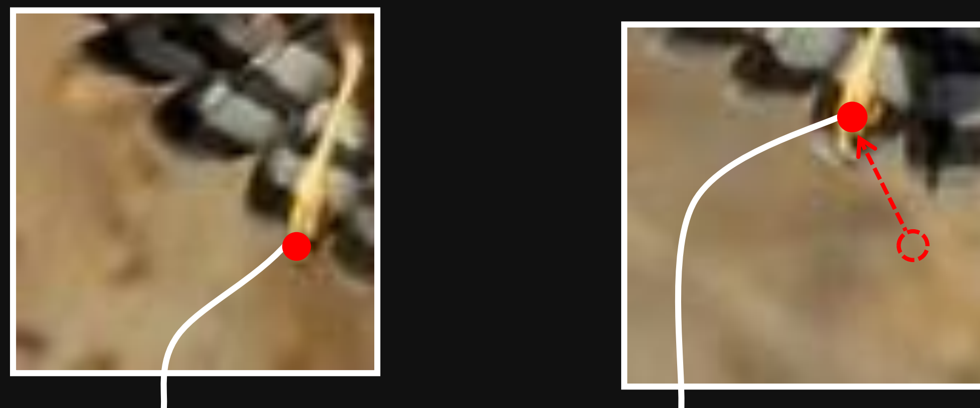
Optical Flow



Displacement: $(\delta x, \delta y)$

Optical Flow: $(u, v) = \left(\frac{\delta x}{\delta t}, \frac{\delta y}{\delta t}\right)$

Optical Flow Constraints



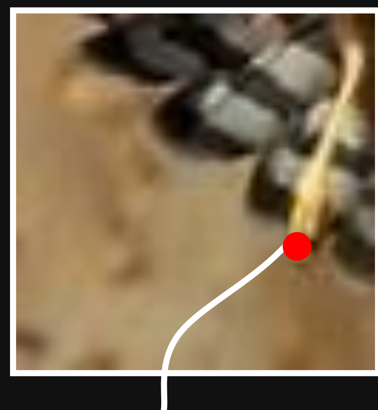
$E(x, y, t)$ $\quad$ $E(x + \delta x, y + \delta y, t + \delta t)$

Assumption #1:

Brightness of an image point remains constant over time

$$\boxed{E(x + \delta x, y + \delta y, t + \delta t) = E(x, y, t)}$$

Optical Flow Constraints



$E(x, y, t)$

$E(x + \delta x, y + \delta y, t + \delta t)$

Assumption #2:

Displacement $(\delta x, \delta y)$ and time step δt are small

Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

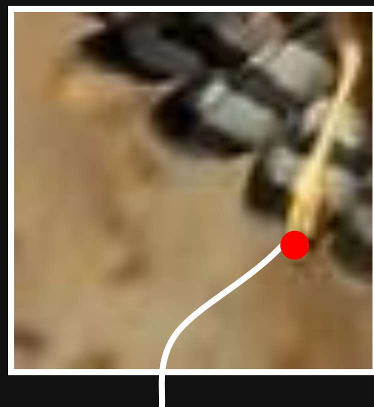
If δx is small:

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \boxed{O(\delta x^2)} \rightarrow \text{Almost Zero}$$

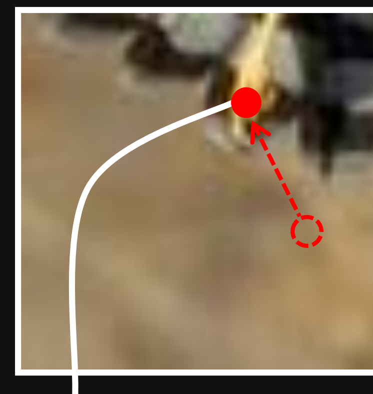
For a function of three variables with small $\delta x, \delta y, \delta t$:

$$f(x+\delta x, y+\delta y, t+\delta t) \approx f(x,y,t) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial t}\delta t$$

Optical Flow Constraints



$E(x, y, t)$



$E(x + \delta x, y + \delta y, t + \delta t)$

Assumption #2:

Displacement and time step are small

$$E(x + \delta x, y + \delta y, t + \delta t) = E(x, y, t) + \frac{\partial E}{\partial x}\delta x + \frac{\partial E}{\partial y}\delta y + \frac{\partial E}{\partial t}\delta t$$

$$\boxed{E(x + \delta x, y + \delta y, t + \delta t) = E(x, y, t) + E_x \delta x + E_y \delta y + E_t \delta t}$$

Optical Flow Constraint Equation

$$E(x+\delta x, y+\delta y, t+\delta t) = E(x, y, t) \qquad \text{------------ (1)}$$

$$E(x+\delta x, y+\delta y, t+\delta t) = E(x, y, t) + E_x\delta x + E_y\delta y + E_t\delta t$$

$$\text{------------ (2)}$$

Subtract (1) from (2): $E_x\delta x + E_y\delta y + E_t\delta t = 0$

Divide by δt and take limit as $\delta t \to 0$: $E_x\dfrac{\partial x}{\partial t} + E_y\dfrac{\partial y}{\partial t} + E_t = 0$

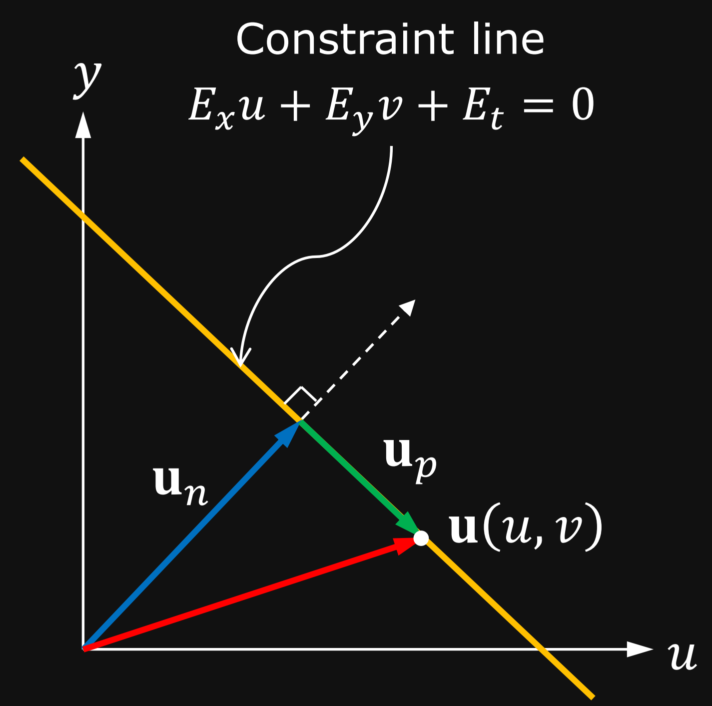
Constraint Equation: $\boxed{E_x u + E_y v + E_t = 0}$

(E_x, E_y, E_t) can be easily computed from two frames (later).

Geometric Interpretation

For any point (x, y) in the image, its optical flow (u, v) lies on the line:

$$E_x u + E_y v + E_t = 0$$



Optical Flow can be split into two components.

$$\mathbf{u} = \mathbf{u}_n + \mathbf{u}_p$$

$\mathbf{u}_n$: Normal Flow

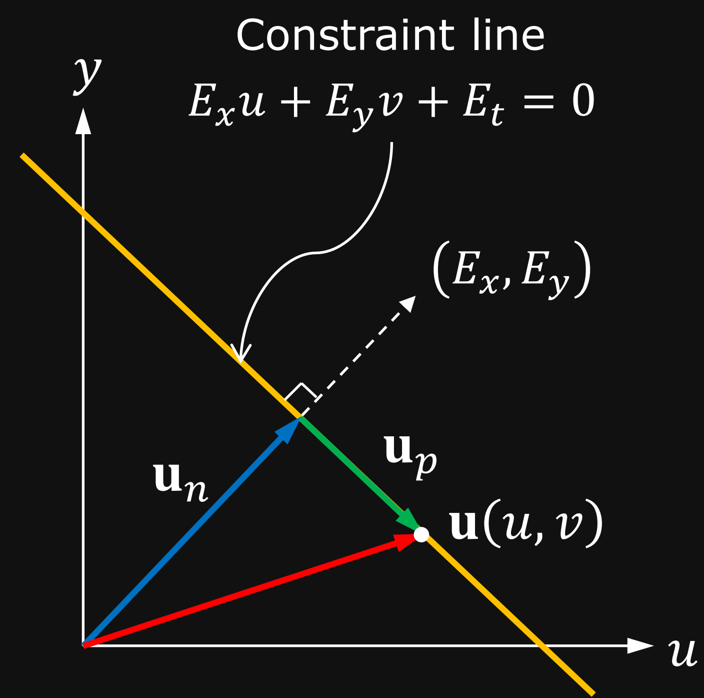
$\mathbf{u}_p$: Parallel Flow

Normal Flow

Direction of Normal Flow:

Unit vector perpendicular to the constraint line:

$$\hat{\mathbf{u}}_n = \frac{(E_x, E_y)}{\sqrt{E_x^2 + E_y^2}}$$



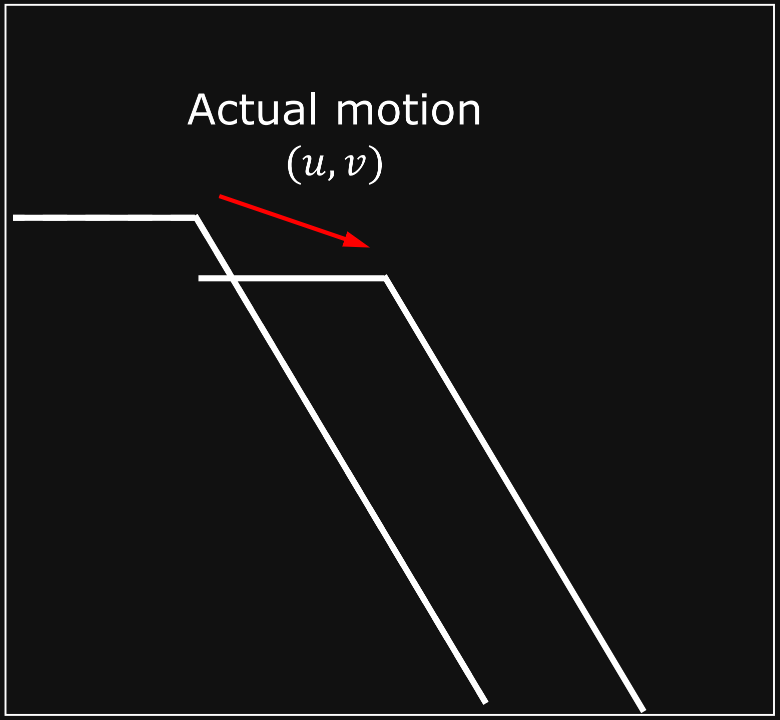
Magnitude of Normal Flow:

Distance of origin from the constraint line:

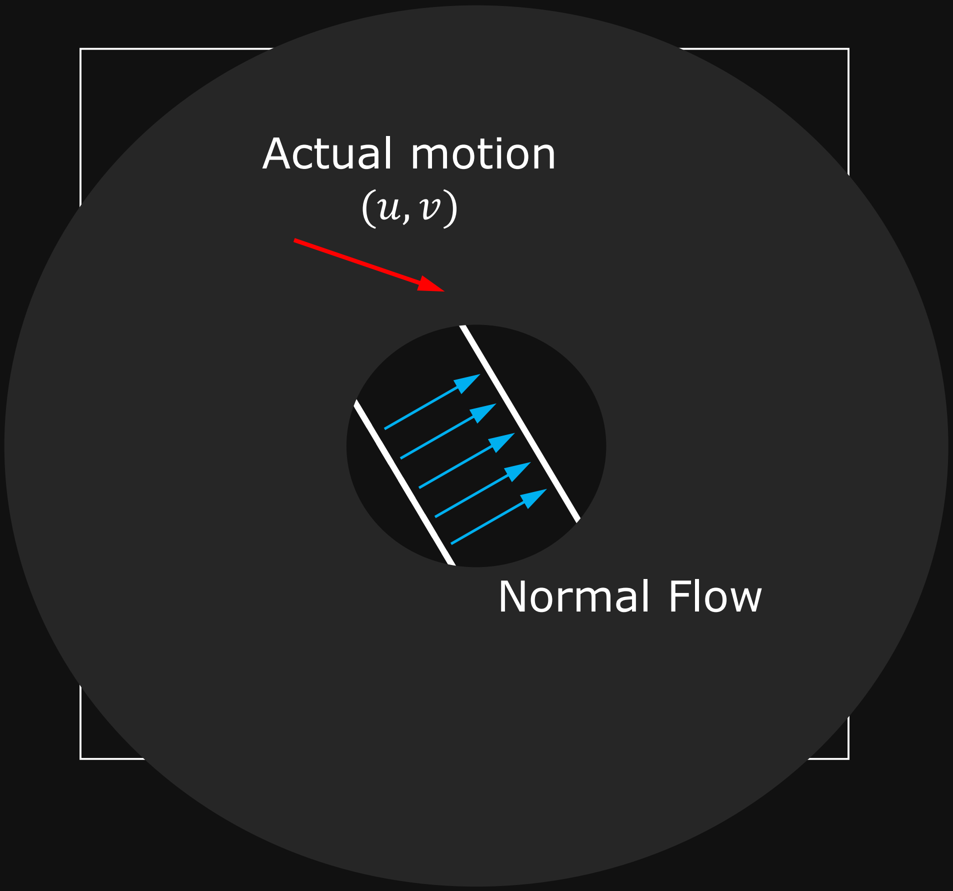
$$|\mathbf{u}_n| = \frac{|E_t|}{\sqrt{E_x^2 + E_y^2}}$$

$$\boxed{\mathbf{u}_n = \frac{|E_t|}{(E_x^2 + E_y^2)}(E_x, E_y)}$$

Aperture Problem



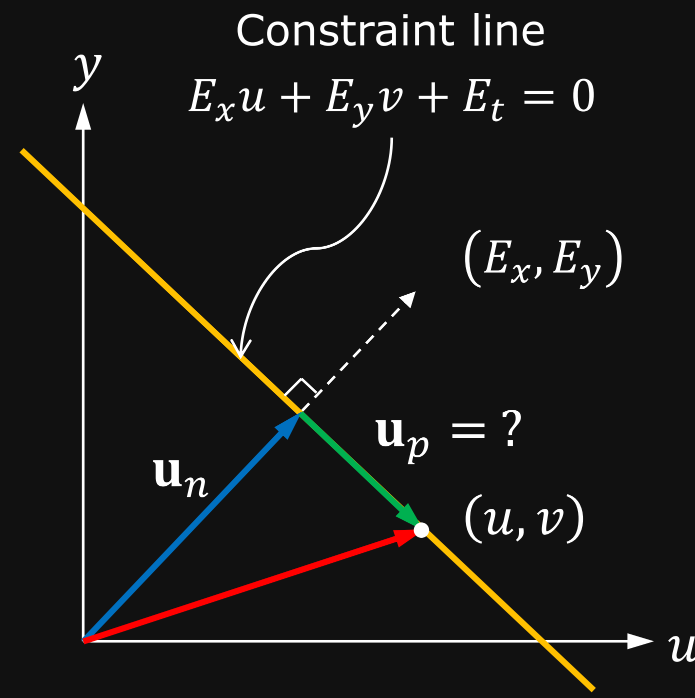
Aperture Problem



Locally, we can only determine normal flow!

Optical Flow is Under Constrained

We cannot determine the optical flow component $\mathbf{u}_p$ parallel to the constraint line.



Finding optical flow (u, v) is under-constrained just like finding gradient (p, q) is under-constrained in shape from shading.

We need additional assumptions

Optical Flow Constraint

Requirement: Optical Flow must satisfy the constraint equation: $E_x u + E_y v + E_t = 0$

Minimize:

$$e_c = \iint_{\text{Image}} \left(E_x u + E_y v + E_t\right)^2 dx\, dy$$

Aim: Penalize errors/departure from the constraint equation.

Smoothness Constraint

Assumption: Motion field and hence optical flow (u, v) varies "smoothly" in an image.

Minimize:

$$e_s = \iint_{\text{Image}} \left(u_x^2 + u_y^2\right) + \left(v_x^2 + v_y^2\right) dx\, dy$$

where: $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $v_x = \frac{\partial v}{\partial x}$ and $v_y = \frac{\partial v}{\partial y}$

Aim: Penalize rapid changes in u and v.

Computing Optical Flow

Find optical flow (u, v) at each pixel that minimizes:

$$\boxed{e = e_s + \lambda e_c}$$

where:

e_s: Smoothness Error

e_c: Optical Flow Error

λ: Weighting factor

Numerical Optical Flow Algorithm

Optical Flow Constraint Error at point (i, j)

$$e_{c_{i,j}} = \left(E_{x_{i,j}} u_{i,j} + E_{y_{i,j}} v_{i,j} + E_{t_{i,j}}\right)^2$$

Smoothness Error at point (i, j)

$$e_{s_{i,j}} = \frac{1}{4}\left(\left(u_{i+1,j} - u_{i,j}\right)^2 + \left(u_{i,j+1} - u_{i,j}\right)^2$$

Find $\left(u_{i,j}, v_{i,j}\right)$ for all (i, j) that minimize:

$$\boxed{e = \sum_i \sum_j \left(e_{s_{i,j}} + \lambda\, e_{c_{i,j}}\right)}$$

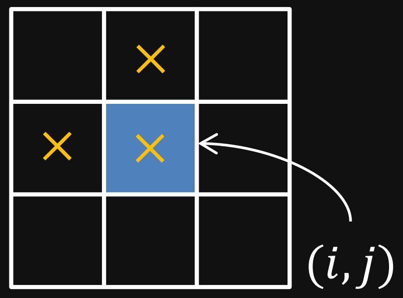
[Horn 1981]

Numerical Optical Flow Algorithm

If $(u_{k,l}, v_{k,l})$ minimizes e, then $\dfrac{\partial e}{\partial u_{k,l}} = 0$ and $\dfrac{\partial e}{\partial v_{k,l}} = 0$

Given an image of size $N \times N$, there are $2N^2$ unknowns.
(N^2 $u_{i,j}$'s and N^2 $v_{i,j}$'s)

However, note that each $u_{i,j}$ and $v_{i,j}$ appears in the terms of 3 pixels.



[Horn 1981]

Numerical Optical Flow Algorithm

If $(u_{k,l}, v_{k,l})$ minimizes e, then $\frac{\partial e}{\partial u_{k,l}} = 0$ and $\frac{\partial e}{\partial v_{k,l}} = 0$

Therefore:

Eq 1: $\frac{\partial e}{\partial u_{k,l}} = 2(u_{k,l} - \bar{u}_{k,l}) - 2E_{x_{k,l}}\left(E_{x_{k,l}}u_{k,l} + E_{y_{k,l}}v_{k,l} + E_{t_{k,l}}\right) = 0$

Eq 2: $\frac{\partial e}{\partial v_{k,l}} = 2(v_{k,l} - \bar{v}_{k,l}) - 2E_{y_{k,l}}\left(E_{x_{k,l}}u_{k,l} + E_{y_{k,l}}v_{k,l} + E_{t_{k,l}}\right) = 0$

where $\bar{u}_{i,j}$ and $\bar{v}_{i,j}$ are local averages.

$$\bar{u}_{i,j} = \tfrac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right)$$

$$\bar{v}_{i,j} = \tfrac{1}{4}\left(v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1}\right)$$

[Horn 1981]

Numerical Optical Flow Algorithm

If $(u_{k,l}, v_{k,l})$ minimizes e, then $\frac{\partial e}{\partial u_{k,l}} = 0$ and $\frac{\partial e}{\partial v_{k,l}} = 0$

Therefore:

Eq 1: $$\frac{\partial e}{\partial u_{k,l}} = 2(u_{k,l} - \bar{u}_{k,l}) - 2E_{x_{k,l}}\left(E_{x_{k,l}}u_{k,l} + E_{y_{k,l}}v_{k,l} + E_{t_{k,l}}\right) = 0$$

Eq 2: $$\frac{\partial e}{\partial v_{k,l}} = 2(v_{k,l} - \bar{v}_{k,l}) - 2E_{y_{k,l}}\left(E_{x_{k,l}}u_{k,l} + E_{y_{k,l}}v_{k,l} + E_{t_{k,l}}\right) = 0$$

Moving all $u_{k,l}$'s and $v_{k,l}$'s to one side, we get…

[Horn 1981]

Iterative Solution of (u, v)

Update Rule: (No Need to Memorize)

$$u_{k,l}^{(n+1)} = \bar{u}_{k,l}^{(n)} + \frac{E_{x_{k,l}}\,\bar{u}_{k,l}^{(n)} + E_{y_{k,l}}\,\bar{v}_{k,l}^{(n)} + E_{t_{k,l}}}{1 + \lambda\left[\left(E_{x_{k,l}}\right)^2 + \left(E_{y_{k,l}}\right)^2\right]} E_{x_{k,l}}$$

$$v_{k,l}^{(n+1)} = \bar{v}_{k,l}^{(n)} + \frac{E_{x_{k,l}}\,\bar{u}_{k,l}^{(n)} + E_{y_{k,l}}\,\bar{v}_{k,l}^{(n)} + E_{t_{k,l}}}{1 + \lambda\left[\left(E_{x_{k,l}}\right)^2 + \left(E_{y_{k,l}}\right)^2\right]} E_{y_{k,l}}$$

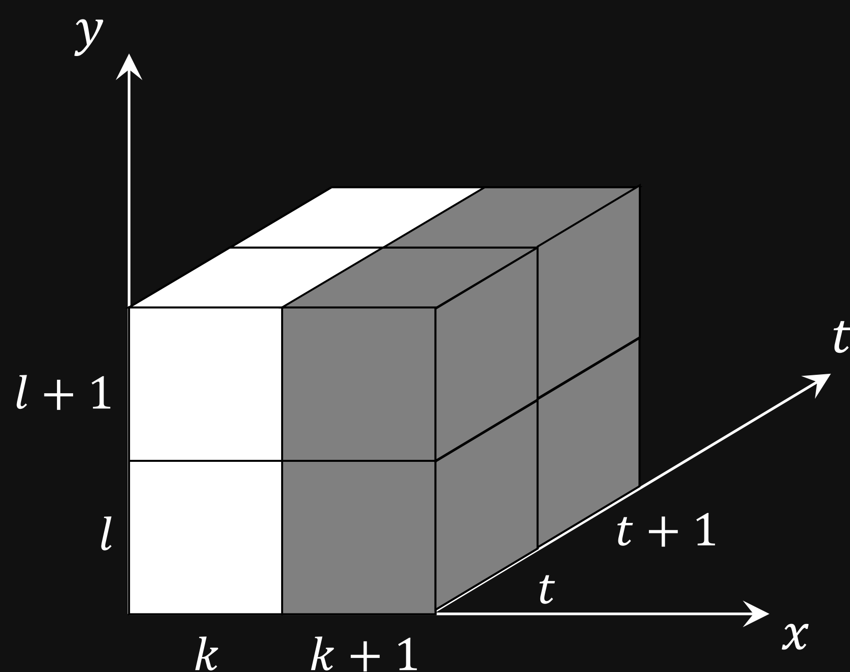
where: n is iteration index

Initialize $u_{k,l}^{(0)} = 0$ and $v_{k,l}^{(0)} = 0$

$E_{\mathrm{x}_{k,l}}$, $E_{\mathrm{y}_{k,l}}$, $E_{\mathrm{t}_{k,l}}$ and λ are known.

[Horn 1981]

Finding Partial Derivatives E_x, E_y, E_t



$$E_{x_{k,l}} = \frac{1}{4}\left(E_{k+1,l,t} + E_{k+1,l,t+1} + E_{k+1,l+1,t} + E_{k+1,l+1,t+1}\right)$$
$$-\frac{1}{4}\left(E_{k,l,t} + E_{k,l,t+1} + E_{k,l+1,t} + E_{k,l+1,t+1}\right)$$

Similarly find $E_{\mathrm{y}_{k,l}}$ and $E_{\mathrm{t}_{k,l}}$

Results: Rotating Ball

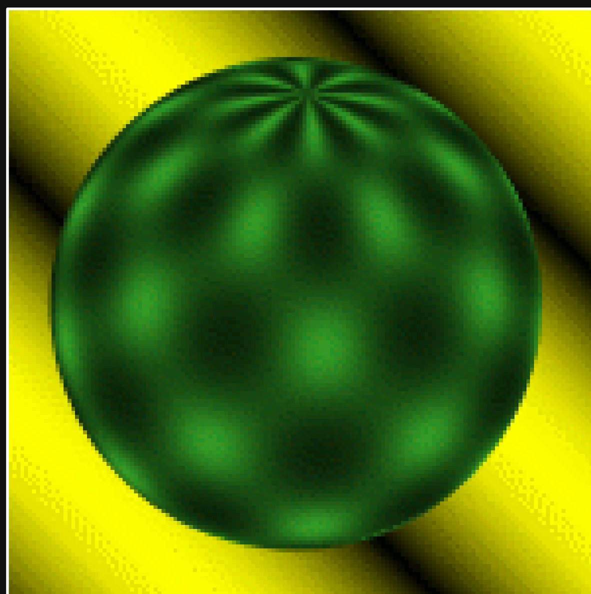
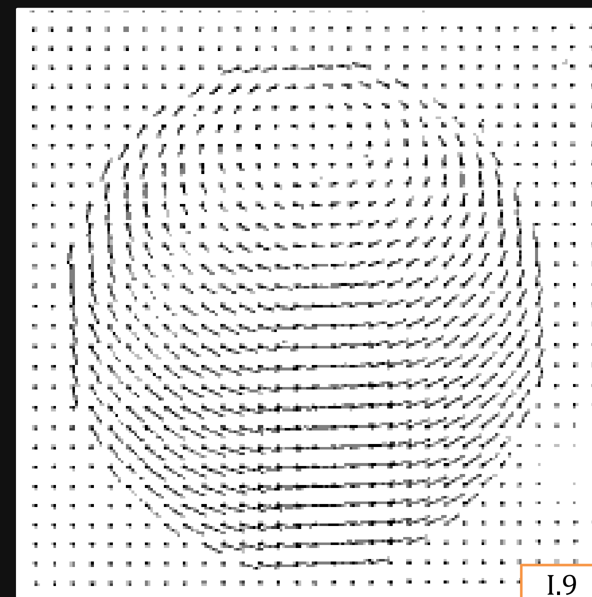


Image Sequence

(2 Frames)



Optical Flow

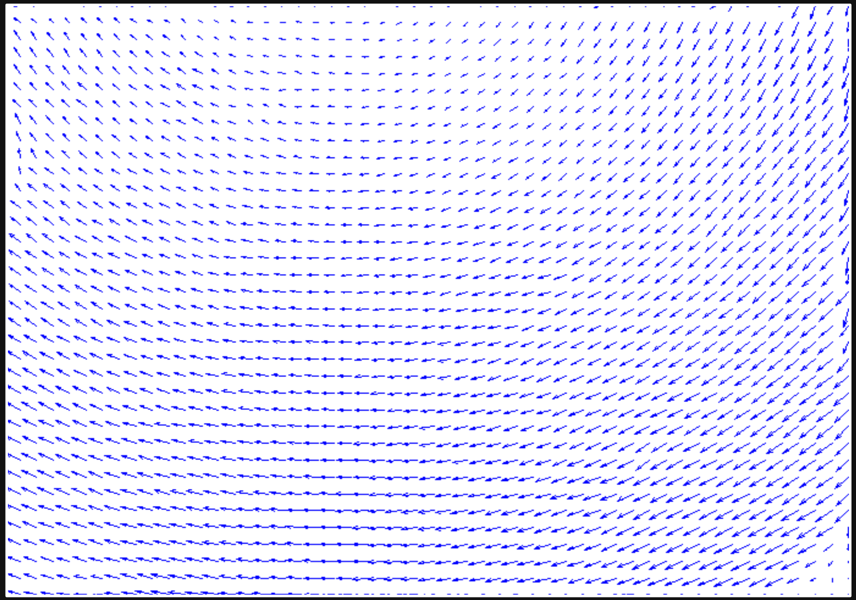
(Iteration 16)

(Final Result)

Results: Rotating Camera

Image Sequence

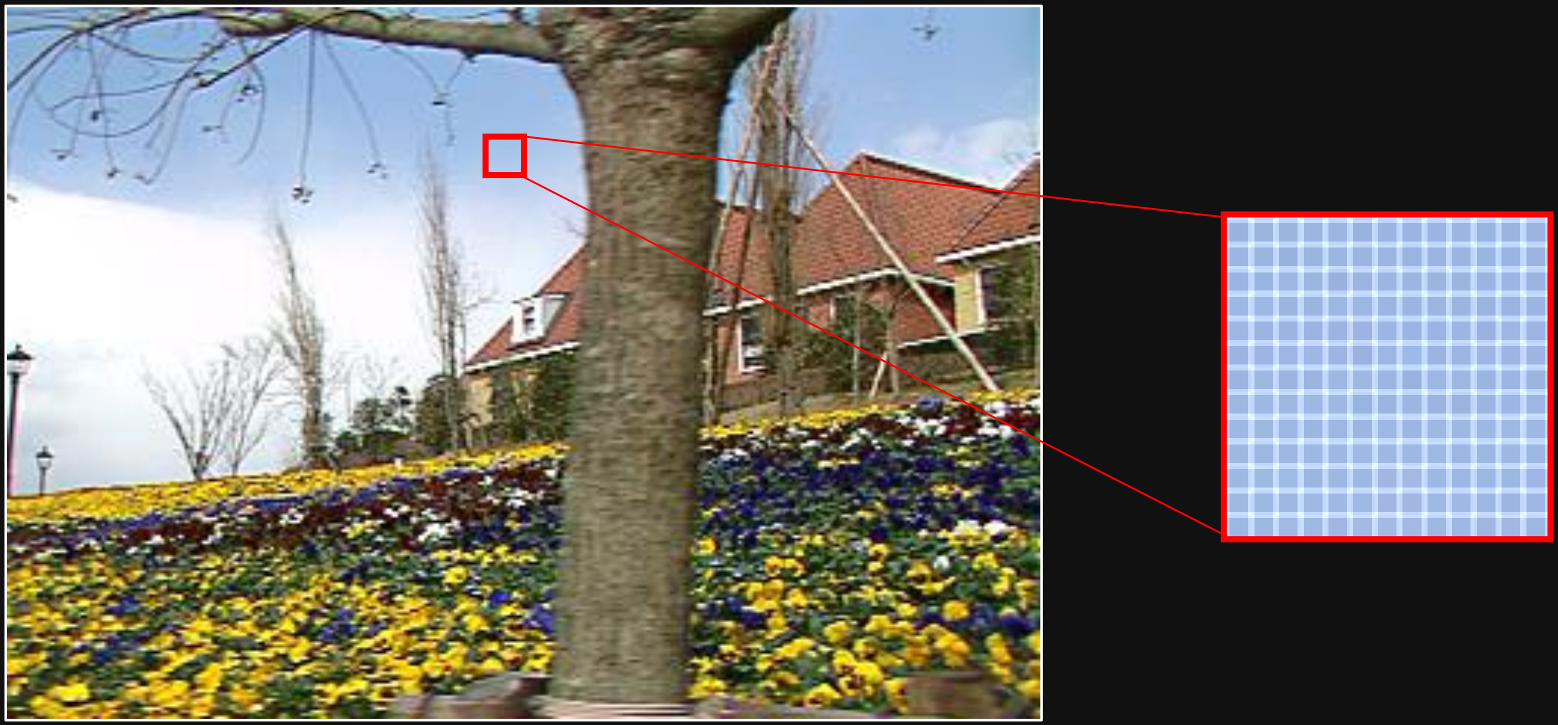
(2 Frames)



Optical Flow

(Final Result)

Remarks: Low Texture Region (Bad)



Small gradient magnitude, can’t compute flow

Remarks: Edges (Aperture Problem)

Large gradient magnitude, but constant along the edge

Remarks: High Texture Region (Good)

Large and diverse gradient magnitudes

What if we have large motion?

Taylor Series Expansion:

$$E(x + \delta x, y + \delta y, t + \delta t) =$$

$$E(x, y, t) + E_x \delta x + E_y \delta y + E_t \delta t$$

$$\boxed{+O(\delta x^2, \delta y^2, \delta t^2)}$$

Higher order terms cannot be ignored

Constraint Equation is Invalid!

$$E_x u + E_y v + E_t \neq 0$$

Large Motion: Solution 1

Determine Flow using Template Matching

Image E_1 at time t

Image E_2 at time $t + \delta t$

For each template window T in image E_1,
find the corresponding match in image E_2.

Similarity Metrics for Template Matching

Find pixel $(k, l) \in W$ with Minimum Sum of Absolute Differences:

$$SAD(k,l) = \sum_{(i,j)\in T} |E_1(i,j) - E_2(i+k, j+l)|$$

Find pixel $(k, l) \in W$ with Minimum Sum of Squared Differences:

$$SSD(k,l) = \sum_{(i,j)\in T} |E_1(i,j) - E_2(i+k, j+l)|^2$$

Find pixel $(k, l) \in W$ with Minimum Normalized Cross-Correlation:

$$NCC(k,l) = \frac{\sum_{(i,j)\in T} |E_1(i,j) - E_2(i+k, j+l)|^2}{\sqrt{\sum_{(i,j)\in T} |E_1(i,j)|^2 \sum_{(i,j)\in T} |E_2(i+k, j+l)|^2}}$$

(See Image Processing Lecture 1)

Large Motion: Solution 1

Determine Flow using Template Matching

Template Window T

Image E_1 at time t

Search Window W

Image E_2 at time $t + \delta t$

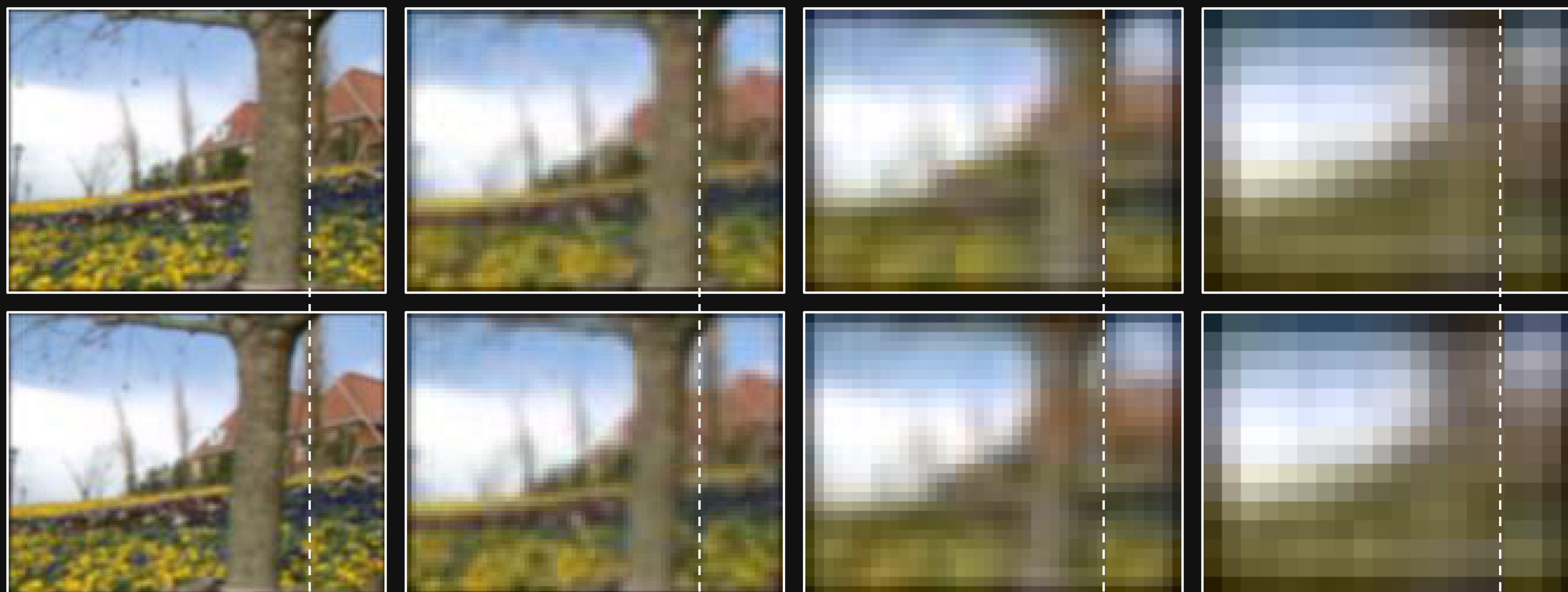
Template matching is slow
when search window W is large.

Large Motion: Solution 2

Coarse-to-fine estimation of optical flow

t

$t + \delta t$



Resolution $N \times N$

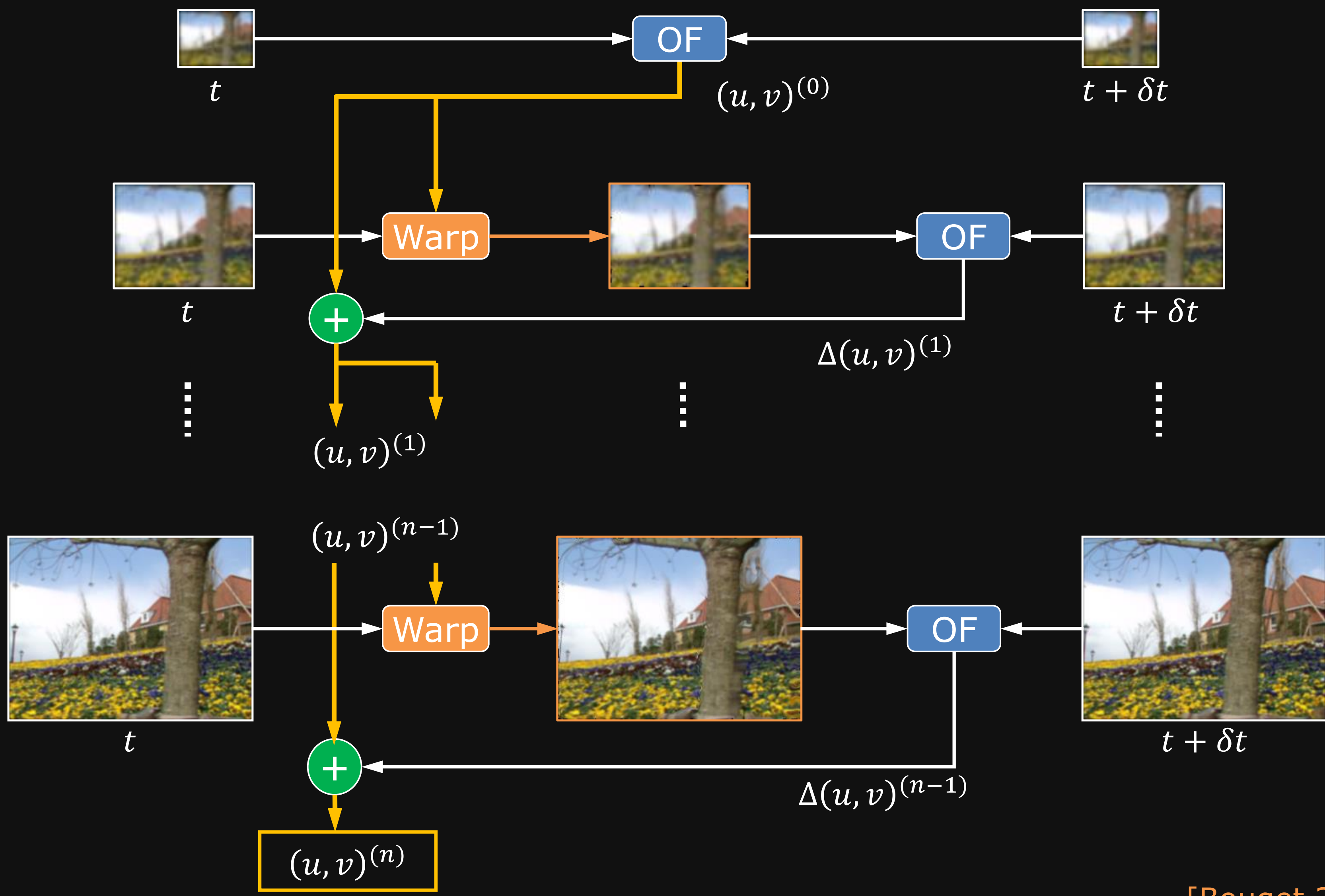
Resolution $\frac{N}{2} \times \frac{N}{2}$

Resolution $\frac{N}{4} \times \frac{N}{4}$

Resolution $\frac{N}{8} \times \frac{N}{8}$

At lowest resolution, motion ≤ 1 pixel

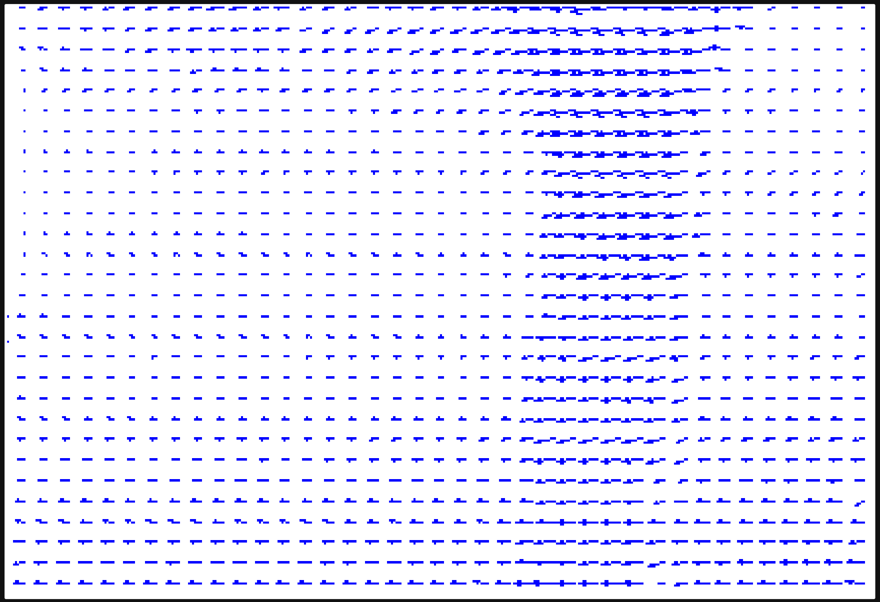
Large Motion: Solution 2



[Bouget 2000]

Result: Large Motion

Image Sequence

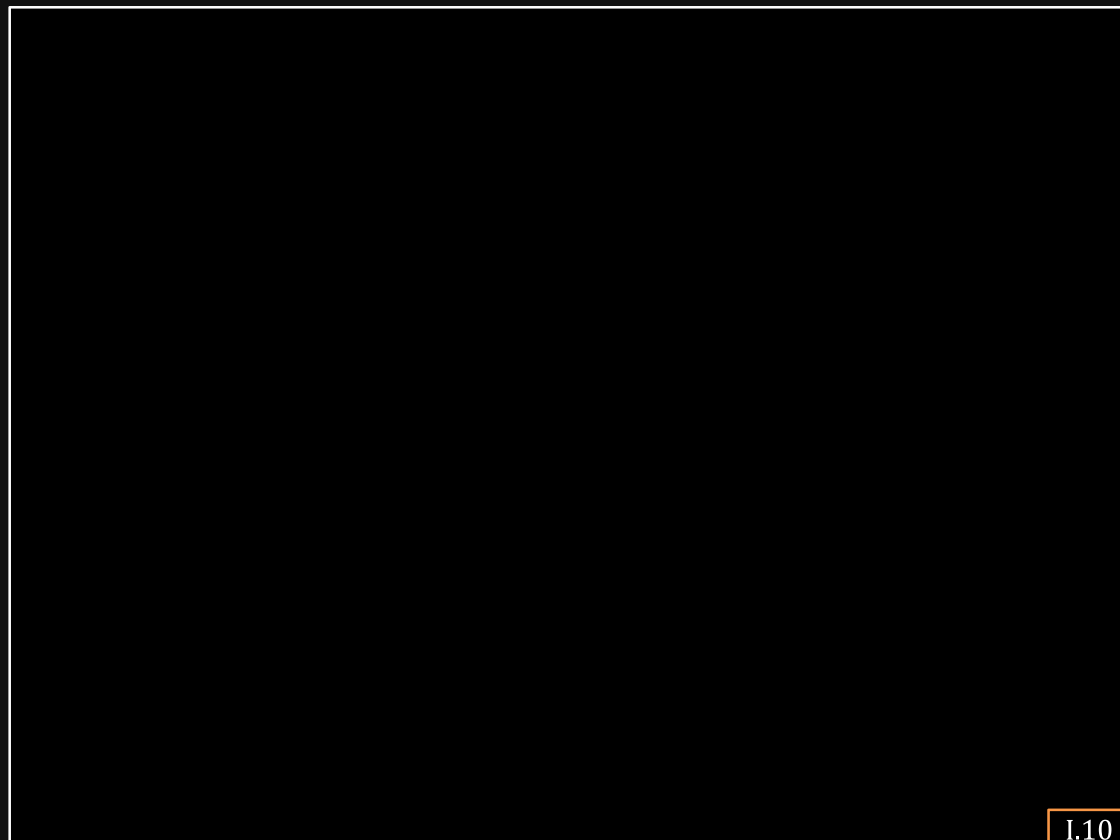


Optical Flow

Result: Optical Flow on Video

Optical flow computed on video frames

Application: Video Retiming



Optical Flow is used to determine the intermediate frames when producing slow motion effects.

Application: Image Stabilization

Optical Flow is used to stabilize camera shake.

References

[Barron 2005] J. L. Barron, D. J. Fleet, and S. Beauchemin, "Performance of optical flow techniques". IJCV, 2005.

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[Bouget 2000] J. Y. Bouguet, "Pyramidal Implementation of the Lucas Kanade Feature Tracker", Intel Corporation 2000.

[Brox 2004] T. Brox, A. Bruhn, N. Papenberg, and J.Weickert, "High accuracy optical flow estimation based on a theory for warping." ECCV, 2004.

[Horn 1981] B. K. P. Horn and B. G. Schunck, "Determining optical flow." Artificial Intelligence, 1981.

[Lucas 1981] B. D. Lucas and T. Kanade, "An iterative image registration technique with an application to stereo vision". Proceedings of Imaging understanding workshop, 1981.

Image Credits

I.1	http://www.youtube.com/watch?v=49BQVszT5mI
I.2	Simon Baker et. al
I.3	http://www.petervaldivia.com/eso/computers/images/optical-mouse.png
I.4	http://www.greenbang.com/wp-content/uploads/2009/02/green-pcb.jpg
I.5	http://mybirdie.ca/files/barbershop.gif
I.6	http://www.ritsumei.ac.jp/~akitaoka/wave-e.html
I.7	B. K. P. Horn, Robot Vision
I.8	Frames from http://vimeo.com/30779242
I.9	http://www.youtube.com/watch?v=JlLkkom6tWwt
I.10	http://vimeo.com/9608102
I.11	http://www.youtube.com/watch?v=2AA6NI7BWkst