Fun with Logic

willie

Propositional logic

Each proposition represents a possible condition of the world that may be true or false

Operators:

Not	$\neg$	$\sim$
And	$\wedge$	
Or	$\vee$	
Implies	$\Rightarrow$	$\rightarrow$
Biconditionals	$\Leftrightarrow$	$\leftrightarrow$

In order of precedence

<u>Examples</u>

It is not raining

`¬raining`

It is raining and I have an umbrella

`raining ∧ umbrella`

It is raining or it is sunny

`raining ∨ sunny`

If it is raining, then I am wet

`raining ⇒ wet`

It is sunny if and only if it is not cloudy

`sunny ⇔ ¬cloudy`

Truth tables

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
false	*false*	*true*	*false*	*false*	*true*	*true*
false	*true*	*true*	*false*	*true*	*true*	*false*
true	*false*	*false*	*false*	*true*	*false*	*false*
true	*true*	*false*	*true*	*true*	*true*	*true*

$\neg (P \vee Q) \Rightarrow Q$

P	Q	$P \vee Q$	$\neg (P \vee Q)$	$\neg (P \vee Q) \Rightarrow Q$
F	F	F	T	F
F	T	T	F	T
T	F	T	F	T
T	T	T	F	T

Inference in Wumpus World

Enumerate all combinations of seven symbols (128 possibilities)

To see if KB ⊨ *α* , for all cases where KB is true, a should be true

Does KB ⊨ $P_{1,1}$?

Model Checking

Sound

Complete

Complexity $O(2^n)$

$B_{1,1}$	$B_{2,1}$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	$P_{2,2}$	$P_{3,1}$	R_1	R_2	R_3	R_4	R_5	KB
false	*false*	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*false*	*false*
false	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*false*	*true*	*false*	*false*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
false	*true*	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*false*	*true*	*true*	*false*
false	*true*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*true*	*true*	<u>*true*</u>
false	*true*	*false*	*false*	*false*	*true*	*false*	*true*	*true*	*true*	*true*	*true*	<u>*true*</u>
false	*true*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*true*	*true*	*true*	<u>*true*</u>
false	*true*	*false*	*false*	*true*	*false*	*false*	*true*	*false*	*false*	*true*	*true*	*false*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
true	*true*	*true*	*true*	*true*	*true*	*true*	*false*	*true*	*true*	*false*	*true*	*false*

Reasoning patterns

Modus Ponens

$$\frac{a \Rightarrow b,\ a}{b}$$

And Elimination

$$\frac{a \wedge b}{a}$$

Commutativity

De Morgan's Laws, etc.

a	b	a ⇒ b
F	F	T
F	T	T
T	F	F
T	T	T

a	b	a ∧ b
F	F	F
F	T	F
T	F	F
T	T	T

Logical equivalence

Two sentences are logically equivalent iff true in same models:

$\alpha \equiv \beta$ if and only if $\alpha \models \beta$ and $\beta \models \alpha$

$(\alpha \land \beta)$	$\equiv$	$(\beta \land \alpha)$	commutativity of $\land$
$(\alpha \lor \beta)$	$\equiv$	$(\beta \lor \alpha)$	commutativity of $\lor$
$((\alpha \land \beta) \land \gamma)$	$\equiv$	$(\alpha \land (\beta \land \gamma))$	associativity of $\land$
$((\alpha \lor \beta) \lor \gamma)$	$\equiv$	$(\alpha \lor (\beta \lor \gamma))$	associativity of $\lor$
$\neg(\neg\alpha)$	$\equiv$	α	double-negation elimination
$(\alpha \Rightarrow \beta)$	$\equiv$	$(\neg\beta \Rightarrow \neg\alpha)$	contraposition
$(\alpha \Rightarrow \beta)$	$\equiv$	$(\neg\alpha \lor \beta)$	implication elimination
$(\alpha \Leftrightarrow \beta)$	$\equiv$	$((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha))$	biconditional elimination
$\neg(\alpha \land \beta)$	$\equiv$	$(\neg\alpha \lor \neg\beta)$	De Morgan
$\neg(\alpha \lor \beta)$	$\equiv$	$(\neg\alpha \land \neg\beta)$	De Morgan
$(\alpha \land (\beta \lor \gamma))$	$\equiv$	$((\alpha \land \beta) \lor (\alpha \land \gamma))$	distributivity of $\land$ over $\lor$
$(\alpha \lor (\beta \land \gamma))$	$\equiv$	$((\alpha \lor \beta) \land (\alpha \lor \gamma))$	distributivity of $\lor$ over $\land$

Logical equivalences

Implication elimination

$$a \Rightarrow b \equiv (\neg a \vee b)$$

De Morgan

$$\neg(a \wedge b) \equiv (\neg a \vee \neg b)$$

P	Q	P ⇒ Q	¬ P v Q
False	False	True	True
False	True	True	True
True	False	False	False
True	True	True	True

P	Q	¬ (P ∧ Q)	(¬ P v ¬Q)
False	False	True	True
False	True	True	True
True	False	True	True
True	True	False	False

Proofs

Applying a sequence of rules is called a proof

Equivalent to searching for a solution

Monotonicity: if KB $\models$ a then {KB, b} $\models$ a

The proof of a sentence *a* from a set of sentences *KB* is the derivation of *a* by applying a series of sound inference rules

KB

1. Battery-OK $\land$ Bulbs-OK $\Rightarrow$ Headlights-Work
2. Battery-OK $\land$ Starter-OK $\land$ $\neg$Empty-Gas-Tank $\Rightarrow$ Engine-Starts
3. Engine-Starts $\land$ $\neg$Flat-Tire $\Rightarrow$ Car-OK
4. Headlights-Work
5. Battery-OK
6. Starter-OK
7. $\neg$Empty-Gas-Tank
8. $\neg$Car-OK

KB $\models$ Flat-Tire

$(\alpha \land \beta) \equiv (\beta \land \alpha)$	commutativity of $\land$
$(\alpha \lor \beta) \equiv (\beta \lor \alpha)$	commutativity of $\lor$
$((\alpha \land \beta) \land \gamma) \equiv (\alpha \land (\beta \land \gamma))$	associativity of $\land$
$((\alpha \lor \beta) \lor \gamma) \equiv (\alpha \lor (\beta \lor \gamma))$	associativity of $\lor$
$\neg(\neg\alpha) \equiv \alpha$	double-negation elimination
$(\alpha \Rightarrow \beta) \equiv (\neg\beta \Rightarrow \neg\alpha)$	contraposition
$(\alpha \Rightarrow \beta) \equiv (\neg\alpha \lor \beta)$	implication elimination
$(\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha))$	biconditional elimination
$\neg(\alpha \land \beta) \equiv (\neg\alpha \lor \neg\beta)$	De Morgan
$\neg(\alpha \lor \beta) \equiv (\neg\alpha \land \neg\beta)$	De Morgan
$(\alpha \land (\beta \lor \gamma)) \equiv ((\alpha \land \beta) \lor (\alpha \land \gamma))$	distributivity of $\land$ over $\lor$
$(\alpha \lor (\beta \land \gamma)) \equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma))$	distributivity of $\lor$ over $\land$

KB

1. Battery-OK ∧ Bulbs-OK ⇒ Headlights-Work
2. Battery-OK ∧ Starter-OK ∧ ¬Empty-Gas-Tank ⇒ Engine-Starts
3. Engine-Starts ∧ ¬Flat-Tire ⇒ Car-OK
4. Headlights-Work
5. Battery-OK
6. Starter-OK
7. ¬Empty-Gas-Tank
8. ¬Car-OK
9. Battery-OK ∧ Starter-OK ← (5+6)
10. Battery-OK ∧ Starter-OK ∧ ¬Empty-Gas-Tank ← (9+7)
11. Engine-Starts ← (2+10)
12. Engine-Starts ⇒ Flat-Tire ← (3+8)
13. Flat-Tire ← (11+12)

KB ⊨ Flat-Tire

Resolution

Complete and sound inference algorithm

$$\frac{a \lor b \;, \quad \neg a \lor c}{b \lor c}$$

Only works on disjunction of literals

Convert to conjunctive normal form (CNF)

$(L_{11} \lor L_{12} \lor .. \lor L_{1i}) \land (L_{21} \lor L_{22} \lor .. \lor L_{2j}) \land .. \land (L_{n1} \lor L_{n2} \lor .. \lor L_{nk})$

Resolution

To show that KB $\models$ a

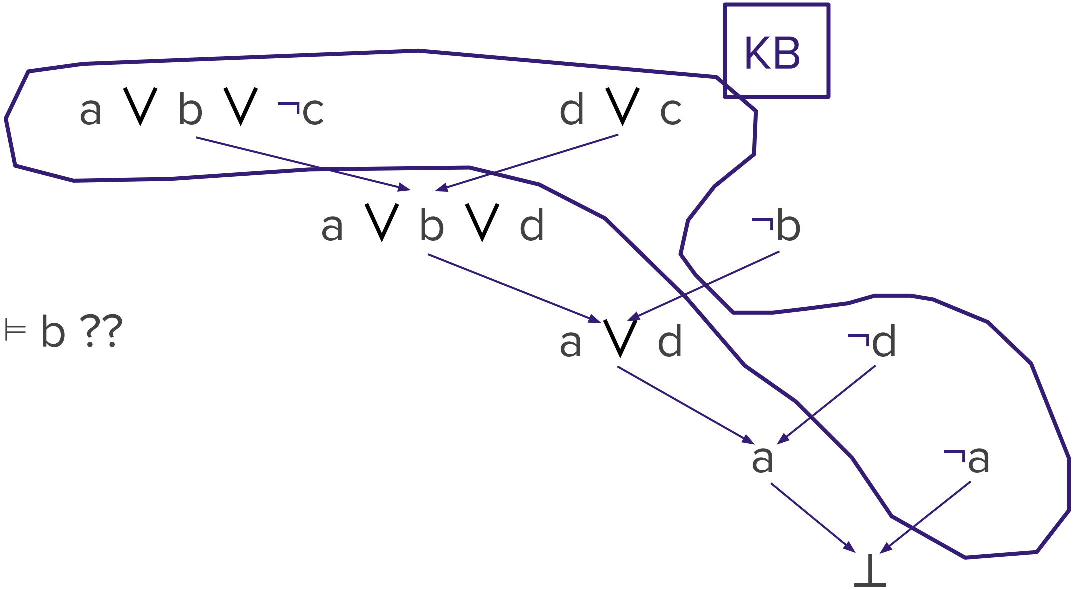
We show that {KB, ¬a} is unsatisfiable

Every pair that contains complementary literals is resolved

Continue until there are no new clauses KB $\not\models$ a

Or we derive a contradiction (from {a , ¬a}) KB $\models$ a

Example



KB ⊨ b ??

First-order logic

Types of symbols

- Objects
- Properties
- Relations
- Functions

Atomic sentences state facts: Brother (richard,john)

Complex sentences: $\neg$King(richard) $\Rightarrow$ King(john)

Universal quantifiers: $\forall$ x King(x) $\Rightarrow$ Person(x)

Existential quantifiers: $\exists$ x Crown(x) $\land$ OnHead(x,john)

Function: bro(john)=richard

First order logic sentences

What is the interpretation for the following?

$King(richard) \lor King(john)$

$\neg Brother(LeftLeg(richard),john)$

$\forall\ x \forall y\ Brother(x,y) \Rightarrow Sibling(x,y)$

$In(Paris,France) \land In(Marseilles,France)$

$\forall c\ Country(c) \land Border(c,Ecuador) \Rightarrow In(c,SouthAmerica)$

$\exists c\ Country(c) \land Border(c,Spain) \land Border(c,Italy)$

Representation in FOL

Raj has only two brothers, Jose and German:

No region in South America borders any region in Europe

No two adjacent countries have the same map color

Representation in FOL

Raj has only two brothers, Jose and German:

> $Brother(Jose, Raj) \land Brother(German, Raj) \land \neg(Jose = German) \land \forall x\ Brother(x, Raj) \Rightarrow (x = Jose \lor x = German)$

No region in South America borders any region in Europe

> $\forall c,d\ In(c, SouthAmerica) \land In(d, Europe) \Rightarrow \neg Border(c,d)$

No two adjacent countries have the same map color

> $\forall x,y\ Country(x) \land Country(y) \land Border(x,y) \Rightarrow \neg(Color(x) = Color(y)) \land \neg(x=y)$

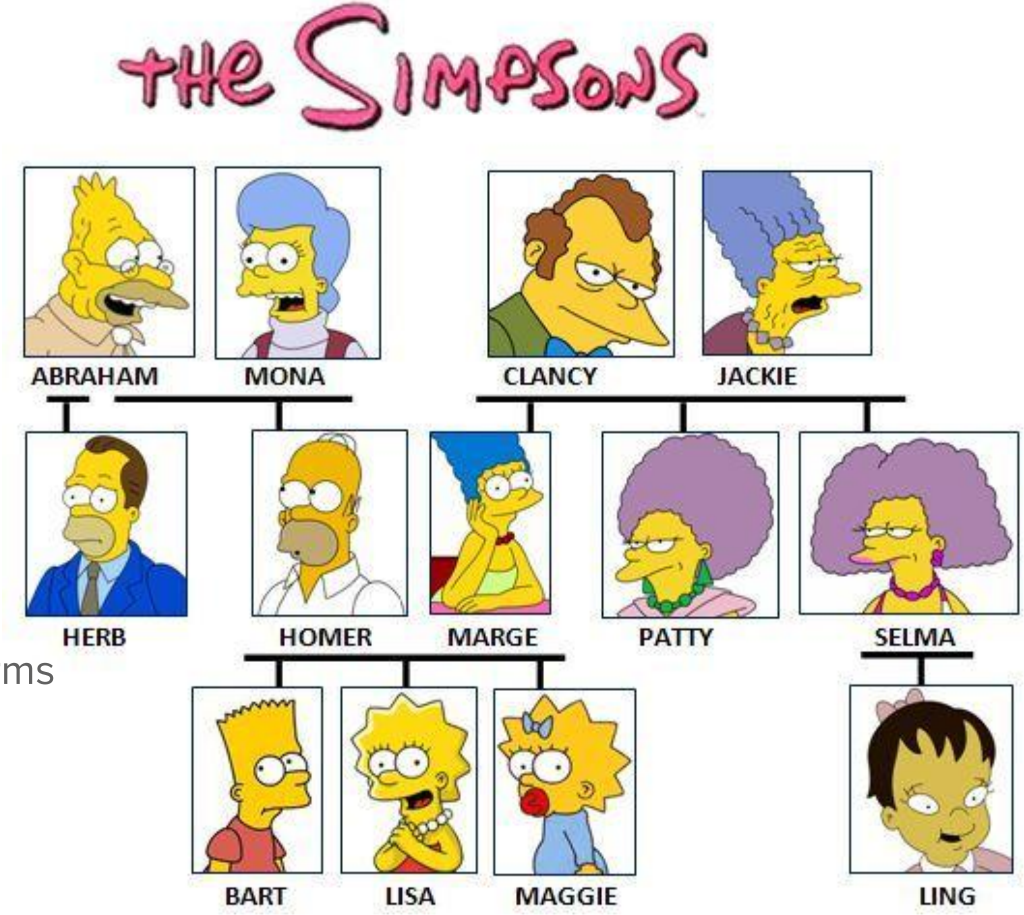
Kinship

Using properties, relations, and functions; express

- Homer likes donuts
- Lisa is smart
- Marge has 3 children
- Bart is male

Using quantifiers, define the following in terms of parent(x,y), male(x), and female(x)

- grandparent(x,y)
- sibling(x,y)
- aunt(x,y)
- cousin(x,y)



Kinship

Using quantifiers, define the following in terms of parent(x,y), male(x), and female(x)

- grandparent(x,y)

  Ax,y Ep grandparent(x,y) iff parent(x,p) ^ parent(p,y)

- sibling(x,y)

  Ax,y Ep sibling(x,y) iff parent(p,x) ^ parent(p,y) ^ -(x = y)

- aunt(x,y)

  Ax,y Ep aunt(x,y) iff parent(p,y) ^ sibling(x,p) ^ female(x)

- cousin(x,y)

  Ax,y Ep,q cousin(x,y) iff parent(p,x) ^ parent(q,y) ^ sibling(p,q)

Is Colonel West a criminal?

“The Law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, and enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American”

1: $\forall x,y,z$ American(x) $\land$ Weapon(y) $\land$ Sells(x, y, z) $\land$ Hostile(z) $\Rightarrow$ Criminal(x)

2: $\exists x$ Owns(Nono, x)

3: $\exists x$ Missile(x)

4: $\forall x$ Missile(x) $\Rightarrow$ Weapon(x)

5: $\forall x$ Missile(x) $\land$ Owns(Nono, x) $\Rightarrow$ Sells(West, x, Nono)

6: $\forall x$ Enemy(x, America) $\Rightarrow$ Hostile(x)

7: American(West)

8: Enemy(Nono, America)

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Sherlock Robison & Dr. Wilson

The Case of the Missing Pencil