

FUNKSIYALARNI FORMULALAR KO`RINISHIDA IFODALASH

Reja:

Bul funksiyalar, ularning usullari. Bul funksiyalari soni. Bul algebrasi.

Ahamiyatli va ahamiyatsiz o'zgaruvchilar

Bul funksiyalarning formulalar orqali amalga oshirilishi

Ikkilamchi funksiyalar. Ikkilamchi prinsipi

Ma'lumki, mantiqiy amallar mulohazalar algebrasi nuqtai nazardan chinlik jadvallari bilan to'liq xarakterlanadi. Agarda funksiyaning jadval shaklda berilishini esga olsak, u vaqtda mulohazalar algebrasida ham funksiya tushunchasini aniqlashimiz mumkin.

Ta'rif. x_1, x_2, \dots, x_n mulohazalar algebrasining x_1, x_2, \dots, x_n argumentli $f(x_1, x_2, \dots, x_n)$ funksiyasi deb nol va bir qiymat qabul funksiyaga aytiladi va uning x_1, x_2, \dots, x_n argumentlari ham nol va bir qiymatlar qabul qilinadi.

Ta'rif. $F: \{0, 1\}^n \rightarrow \{0, 1\}$ funksiya mantiqiy algebraning funksiyasi yoki Bu funksiya to'plami P_n orqali belgilaymiz, ya'ni

Bir o'zgaruvchili funksiyalar 4 ta bo'lib, ular quyidagilar:

$f_0(x)=0$ – aynan nolga teng funksiya yoki aynan yolg'on funksiya

$f_1(x)=x$ – aynan funksiya
- inkor funksiya

$f_3(x)=1$ – aynan birga teng funksiya yoki aynan chin funksiya

| Argument | Bul funksiyalar | | | |
|----------|-----------------|----------|-----------------------|----------|
| x | 0 | x | $\bar{x}, \neg x, x'$ | 1 |
| | $f_0(x)$ | $f_1(x)$ | $f_2(x)$ | $f_3(x)$ |
| 1 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 |

Barcha ikki o'zgaruvchili funksiyalarni sanab o'tamiz.

| x | Y | 0 | \wedge | | | \downarrow | x | \oplus | \bar{x} | \leftrightarrow | y | \bar{y} | \vee | 1 | \rightarrow | | 1 |
|---|---|-------|----------|-------|-------|--------------|-------|----------|-----------|-------------------|-------|-----------|----------|----------|---------------|----------|----------|
| | | g_0 | g_1 | g_2 | g_3 | g_4 | g_5 | g_6 | g_7 | g_8 | g_9 | g_{10} | g_{11} | g_{12} | g_{13} | g_{14} | g_{15} |
| 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 |

Hammasi bo'lib 16 ta har xil ikki o'zgaruvchili funksiyalar mavjud. Ularning ko'pchiligi maxsus nomlanadi:

$g_1(x, y) = x \wedge y$ – konyunksiya

$g_4(x, y) = x \downarrow y$ - Pirs strelkasi

$g_6(x, y) = x \oplus y$ - 2 modul bo'yicha qo'shish yoki Jegalkin yig'indisi

Bul funksiyalarining qiymatlar jadvaliga chinlik jadvali deyiladi. Har qanday n o'lchovli $f(x_1, x_2, \dots, x_n)$ Bul funksiyani chinlik jadvali orqali berish mumkin:

| x | y | $x \oplus y$ |
|---|---|--------------|
| 1 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 0 | 0 | 0 |

$$x \oplus y = \overline{x \leftrightarrow y}$$

$$1 \oplus 1 = 0 = 2 \cdot 1 + 0$$

$$1 \oplus 0 = 1 = 2 \cdot 0 + 1$$

$$0 \oplus 1 = 1 = 2 \cdot 0 + 1$$

$$0 \oplus 0 = 0 = 2 \cdot 0 + 0$$

$$g_8(x, y) = x \leftrightarrow y - \text{ekvivalentlik}$$

$$g_{11}(x, y) = x \vee y - \text{dizyunksiya}$$

$$g_{12}(x, y) = x | y - \text{Sheffer shtrixi}$$

$$g_{13}(x, y) = x \rightarrow y - \text{implikatsiya}$$

Bul funksiyalarining qiymatlar jadvaliga chinlik jadvali deyiladi. Har qanday n o'lchovli $f(x_1, x_2, \dots, x_n)$ Bul funksiyani chinlik jadvali orqali berish mumkin:

| x_1 | x_2 | | x_n | $f(x_1, x_2, \dots, x_n)$ |
|-------|-------|-------|-------|---------------------------|
| 0 | 0 | | 0 | λ_1 |
| 1 | 0 | | 0 | λ_2 |
| 0 | 1 | | 0 | λ_3 |
| | | | | |
| 1 | 1 | | 1 | λ_{2^n} |

bu yerda $\lambda_i \in \{0,1\}, i=1,2,\dots,2^n$. Bu jadval 2^n ta satr bo'lib, ularga 2^{2^n} ta har xil ustunlar mos qo'yish mumkin. Lekin bunday har bir ustun biror n o'zgaruvchili Bul funksiyaga mos keladi. Shunday qilib, quyidagi teorema isbotlandi:

Teorema. N o'zgaruvchili har xil Bul funksiyalarining soni 2^{2^n} ga teng, ya'ni $|P_n|=2^{2^n}$

Bul algebrasi

Teorema. Konyunksiya ($x \wedge y$), dizyunksiya ($x \vee y$), inkor (\bar{x}) amallari va $0,1 \in M$ elementlari uchun quyidagi amallar:

| | | |
|--|---------------------------|--|
| $\overline{\bar{x}} = x$ | $x \wedge y = y \wedge x$ | $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ |
| $\overline{x \vee y} = \bar{x} \wedge \bar{y}$ | $x \vee y = y \vee x$ | $x \vee (y \vee z) = (x \vee y) \vee z$ |
| $\overline{x \wedge y} = \bar{x} \vee \bar{y}$ | $x \vee y = x$ | $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ |
| $x \wedge x = x$ | $1 \wedge x = x$ | $0 \vee x = x$ |

bajarilsa, bunday M to'plamga Bul algebrasi deyiladi.

Mulohazalar to'plami uchun konyunksiya (\wedge), dizyunksiya (\vee), inkor ($\neg, -$) amallari va $\{0,1\}$ elementlari aniqlangani uchun, bu to'plam Bul algebrasi bo'ladi.

Ta'rif. Agar o'zgaruvchining shunday $a_1, a_2, \dots, a_{i-1}, a_i, \dots, a_n$ qiymatlar majmuasi mavjud bo'lib,
 $f(a_1, a_2, \dots, a_{i-1}, 1, a_i, \dots, a_n) = f(a_1, a_2, \dots, a_{i-1}, 0, a_i, \dots, a_n)$ munosabat bajarilsa, u vaqtda x_i o'zgaruvchiga $f(x_1, x_2, \dots, x_n)$ funksiyaning nomuhim (sohta) o'zgaruvchisi, agar
 $f(a_1, a_2, \dots, a_{i-1}, 1, a_i, \dots, a_n) \neq f(a_1, a_2, \dots, a_{i-1}, 0, a_i, \dots, a_n)$ munosabat bajarilsa, u vaqtda x_i o'zgaruvchiga $f(x_1, x_2, \dots, x_n)$ funksiyaning muhim (sohta emas) o'zgaruvchisi deb ataladi.

Misol. $f(x, y) = x \vee (x \wedge y)$ funksiya da y o'zgaruvchi sohta bo'ladi.

Haqiqatdan,

$$x = 1, y = 0 \text{ da } f(1, 0) = 1 \vee (1 \wedge 0) = 1$$

$$x = 1, y = 1 \text{ da } f(1, 1) = 1 \vee (1 \wedge 1) = 1$$

$$\text{ya'ni } f(1, 0) = f(1, 1)$$

Misol. f_1, f_2 va f_3 funksiyalar quyidagi chinlik jadvali orqali berilgan bo'lsin:

| x | y | f_1 | f_2 | f_3 |
|-----|-----|-------|-------|-------|
| 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 |

Ko'rinib turibdiki, f_1 funksiya uchun x o'zgaruvchi muhim o'zgaruvchi, u esa nomuhim, f_2 uchun ikkala o'zgaruvchi ham nomuhim, f_3 uchun ikkala o'zgaruvchi ham muhim.

$\Phi = \{f_1, f_2, \dots, f_n\}$ Bul funksiyalar to'plami berilgan bo'lsin.

Ta'rif Φ to'plam ustida aniqlangan formula deb, $F(\Phi) = f(t_1, t_2, \dots, t_n)$ ifodaga aytiladi, bu yerda $f \in \Phi$ va $t_i \in \Phi$ ustidagi yoki o'zgaruvchi, yoki formula.

Φ to'plam bazis, f tashqi funksiya, t_i lar esa qism formulalar deyiladi. Har qanday F formulaga bir qiymatli biror f Bul funksiya mos keladi. Bu holda F formula f funksiyaning ifodalaydi deyiladi va $f = \text{func } F$ ko'rinishida belgilanadi.

Bazis funksiyalarini chinlik jadvalini bilgan holda, bu formula ifodalaydigan funksiyaning chinlik jadvalini hisoblashimiz mumkin.

Misol. $\Phi = \{\wedge, \rightarrow\}$ va $F = (x \wedge y) \rightarrow x$

| x | y | $x \wedge y$ | $F = (x \wedge y) \rightarrow x$ |
|-----|-----|--------------|----------------------------------|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 |

□

F formulaga mos keluvchi f funksiyani Φ dan olingan funksiyalarning superpozitsiyasi, f funksiyani Φ dan hosil qilinish jarayonini superpozitsiya amali deb ataymiz.

Misol. $f(x_1, x_2, x_3) = ((x_1 \wedge x_2) \vee x_1) \rightarrow x_3$ formula berilgan bo'lsin.

$((x_1 \wedge x_2) \vee x_1) \rightarrow x_3$ formula uchta qadamda ko'riladi. Haqiqatdan, biz quyidagi uchta qism formulalarga ega bo'lamiz:

$(x_1 \wedge x_2)$, $((x_1 \wedge x_2) \vee x_1)$, $((x_1 \wedge x_2) \vee x_1) \rightarrow x_3$

| x_1 | x_2 | x_3 | $x_1 \wedge x_2$ | $(x_1 \wedge x_2) \vee x_1$ | $((x_1 \wedge x_2) \vee x_1) \rightarrow x_3$ |
|-------|-------|-------|------------------|-----------------------------|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 | 1 |

□

Biz yuqorida ko'rdikki, Φ to'plamdan hosil qilingan har bir formulaga mantiq algebrasining formulasi mos keladi, biroq har xil formulalarga teng funksiyalar mos kelishi mumkin.

Ta'rif Bitta Bul funksiyasini ifodalovchi formulalar ekvivalent deyiladi, ya'ni

$$F_1 = F_2 \stackrel{\square}{\Leftrightarrow} \text{func} F_1 = \text{func} F_2$$

Teorema. Ixtiyoriy f, g, h Bul funksiyalar uchun quyidagi ekvivalentliklar o'rinli:

1. $\bar{\bar{f}} = f$
2. Konyunksiya, dizyunksiya va ikki modul bo'yicha qo'shishning idempotentligi:
 $f \wedge f = f, \quad f \vee f = f, \quad f \oplus f = f$
3. Konyunksiya, dizyunksiya va ikki modul bo'yicha qo'shishning kommunikativligi:
 $f \wedge g = g \wedge f, \quad f \vee g = g \vee f, \quad f \oplus g = g \oplus f$
4. Konyunksiya, dizyunksiya va ikki modul bo'yicha qo'shishning assotsiativligi:
 $f \wedge (g \wedge h) = (f \wedge g) \wedge h, \quad f \vee (g \vee h) = (f \vee g) \vee h, \quad f \oplus (g \oplus h) = (f \oplus g) \oplus h$

5. Distributivlik qonunlari:

$$f \wedge (g \vee h) = (f \wedge g) \vee (f \wedge h), \quad f \vee (g \wedge h) = (f \vee g) \wedge (f \vee h), \quad (f \wedge g) \oplus (f \wedge h)$$

6. Yutish qonuni:

$$f \wedge (f \vee g) = f, \quad f \vee (f \wedge g) = f$$

7. De Morgan qonuni:

$$8. \overline{f \vee g} = \bar{f} \wedge \bar{g}, \quad \overline{f \wedge g} = \bar{f} \vee \bar{g},$$

$$9. f \vee \bar{f} = 1, \quad f \wedge \bar{f} = 0$$

$$10. f \rightarrow g = \bar{g} \rightarrow \bar{f}$$

11. Implikatsiyani yo'qotish qonuni:

$$f \leftrightarrow g = \bar{f} \vee g$$

12. Ekvivalentlikni yo'qotish qoidasi:

$$f \leftrightarrow g = (f \rightarrow g) \wedge (g \rightarrow f)$$

$$13. \bar{f} = f | f = f \downarrow f = f \oplus 1$$

$$14. f | g = \overline{(f \wedge g)}, \quad f \downarrow g = \overline{f \vee g}$$

$$15. f \vee g = \overline{(f | f) | (g | g)}, \quad f \wedge g = (f \downarrow f) \downarrow (g \downarrow g), \quad f \rightarrow g = f | (g | g)$$

$$16. f \oplus g = \bar{f} \leftrightarrow g$$

Ta'rif. $f(x_1, x_2, \dots, x_n) \in P_n$ bul funksiya bo'lsin, unda $f^*(x_1, x_2, \dots, x_n) = \overline{f^*(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)}$ funksiya, f bul funksiya ga ikkilamchi bo'lgan funksiya deyiladi.

Bu ta'rifdan bevosita, ixtiyoriy f bul funksiya uchun $f^{**}=f$ ekanligi kelib chiqadi. Haqiqatdan,

$$f^{**} = (f^*)^* = \overline{(f^*(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n))^*} = \overline{f^*(\bar{\bar{x}}_1, \bar{\bar{x}}_2, \dots, \bar{\bar{x}}_n)} = f(x_1, x_2, \dots, x_n) = f.$$

Misol. a) $f = x \vee g$, $f^* = ?$ b) $g = x$, $g^* = ?$ c) $h = \bar{x}$, $h^* = ?$

Yechish.

$$a) f^* = \overline{\bar{x} \vee \bar{g}} = \bar{\bar{x}} \wedge \bar{\bar{g}} = x \wedge g;$$

$$b) g^* = \overline{(\bar{x})} = x = g, \quad g^* = g$$

$$c) h^* = \overline{(\bar{\bar{x}})} = \bar{x} = h$$

Ta'rif. Agar $f^*=f$ bo'lsa, f funksiya o'z-o'ziga ikkilamchi deyiladi.

Yuqoridagi misoldan ko'rinadiki, inkor va aynan funksiya o'z-o'ziga ikkilamchi, dizyunksiya funksiya o'z-o'ziga ikkilamchi emas.

Teorema. Agar $\varphi(x_1, x_2, \dots, x_n)$ bul funksiya

$f(f_1(x_1, x_2, \dots, x_n), f_2(x_1, x_2, \dots, x_n), \dots, f_n(x_1, x_2, \dots, x_n))$ formula ko'rinishida ifodalangan bo'lsa, bu yerda f_1, f_2, \dots, f_n lar bul funksiylar, unda

$f^*(f_1^*(x_1, x_2, \dots, x_n), f_2^*(x_1, x_2, \dots, x_n), \dots, f_n^*(x_1, x_2, \dots, x_n))$ formula $\varphi^*(x_1, x_2, \dots, x_n)$ funksiyaning ifodalaydi.

Isbot.

$$\begin{aligned}\varphi^*(x_1, x_2, \dots, x_n) &= \bar{\varphi}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = \\ &= \text{funcf} \left(f_1(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n), \dots, f_n(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \right) = \\ &= \text{funcf} \left(\bar{\bar{f}}_1(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n), \dots, \bar{\bar{f}}_n(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \right) = \\ &= \text{funcf} \left(\bar{f}_1^*(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n), \dots, \bar{f}_n^*(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \right) = \\ &= \text{funcf}^* \left(f_1^*(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n), \dots, f_n^*(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \right).\end{aligned}$$

Teorema isbotlandi.

Keyingi teorema “ikkilamchi prinsipli” deb nomlanadi va matematik induksiya usuli bilan isbotlanadi. Bunda induksiya o'tishlar yuqoridagi isbotlangan teorema asosida amalga oshiriladi.

Teorema. (Ikkilamchi prinsipli)

$\Phi = \{f_1, f_2, \dots, f_n\}$ va $\Phi^* = \{f_1^*, f_2^*, \dots, f_m^*\}$ - bazislar bo'lsin. U holda, agar F formula Φ bazisda f funksiyani ifodalasa, unda F formuladan f_i ni uni ikkilamchi f_i^* funksiyaga almashtirish natijasida hosil qilingan F^* formula Φ bazisda f^* funksiyani ifodalaydi, ya'ni

$$f = func[\Phi]u \Rightarrow f^* = funcF^*[\Phi^*], \text{ bu yerda } F^*[\Phi^*] = F[\Phi]\{f_i^*|f_i\}_{i=1}^m$$