CALCULUS PROBLEMS

Detailed Solutions To Exercises From The Textbook

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Contents

Chapter 1

Limits

1.1 Section 2.1

Problem 5

The deck of a bridge is suspended 80 meters above a river. If a pebble falls off the side of the bridge, the height, in meters, of the pebble above the water surface after t seconds is given by $y = 80 - 4.9t^2$.

(a) Find the average velocity of the pebble for the time period beginning when $t = 4$ and lasting

  (i) 0.1 seconds

  (ii) 0.05 seconds

  (iii) 0.01 seconds

(b) Estimate the instantaneous velocity of the pebble after 4 seconds.

For part (a), we shall use the formula which says that the average velocity of an object over a time interval is equal to the ratio between the change in position occurring in that time interval and the duration of the time interval. We shall denote average velocity with v for this part.

For part (i), we obtain

$$\begin{aligned}
v &= \frac{y(4+0.1)-y(4)}{0.1} = \frac{y(4.1)-y(4)}{0.1} = \\
&= \frac{80-4.9(4.1)^2-(80-4.9(4)^2)}{0.1} = \\
&= \frac{80-4.9(4.1)^2-80+4.9(4)^2}{0.1} = \\
&= \frac{4.9\cdot(4^2-4.1^2)}{0.1} = \frac{4.9\cdot(4-4.1)(4+4.1)}{0.1} = \\
&= \frac{4.9\cdot(-0.1)\cdot 8.1}{0.1} = -4.9\cdot 8.1 = -39.69
\end{aligned}$$

For part (ii), we obtain

$$\begin{aligned}
v &= \frac{y(4+0.05)-y(4)}{0.05} = \frac{y(4.05)-y(4)}{0.05} = \\
&= \frac{80-4.9(4.05)^2-(80-4.9(4)^2)}{0.05} = \\
&= \frac{80-4.9(4.05)^2-80+4.9(4)^2}{0.05} = \\
&= \frac{4.9\cdot(4^2-4.05^2)}{0.05} = \frac{4.9\cdot(4-4.05)(4+4.05)}{0.05} = \\
&= \frac{4.9\cdot(-0.05)\cdot 8.05}{0.05} = -4.9\cdot 8.05 = -39.445
\end{aligned}$$

For part (iii), we obtain

$$\begin{aligned}
v &= \frac{y(4+0.01)-y(4)}{0.01} = \frac{y(4.01)-y(4)}{0.01} = \\
&= \frac{80-4.9(4.01)^2-(80-4.9(4)^2)}{0.01} = \\
&= \frac{80-4.9(4.01)^2-80+4.9(4)^2}{0.01} = \\
&= \frac{4.9\cdot(4^2-4.01^2)}{0.01} = \frac{4.9\cdot(4-4.01)(4+4.01)}{0.01} = \\
&= \frac{4.9\cdot(-0.01)\cdot 8.01}{0.01} = -4.9\cdot 8.01 = -39.249
\end{aligned}$$

For part (b), we take the limit of the average velocity of the rock on the interval $[4, 4+h]$ as h goes to 0. We denote the instantaneous velocity at $t = 4$ with v.

$$\begin{aligned}
v &= \lim_{h\to 0}\frac{y(4+h)-y(4)}{h} = \lim_{h\to 0}\frac{80-4.9(4+h)^2-(80-4.9(4)^2)}{h} = \\
&= \lim_{h\to 0}\frac{80-4.9(4+h)^2-80+4.9(4)^2}{h} = \lim_{h\to 0}\frac{4.9\cdot(4^2-(4+h)^2)}{h} = \\
&= \lim_{h\to 0}\frac{4.9\cdot(4-(4+h))\cdot(4+h+4)}{h} = \lim_{h\to 0}\frac{4.9\cdot(-h)\cdot(8+h)}{h} = \\
&= \lim_{h\to 0}[-4.9\cdot(8+h)] = -4.9\cdot 8 = -39.2
\end{aligned}$$

Problem 6

If a rock is thrown upward on the planet Mars with a velocity of 10m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.

(a) Find the average velocity over the given time intervals:

- (i) $[1, 2]$
- (ii) $[1, 1.5]$
- (iii) $[1, 1.1]$

(iv) $[1, 1.01]$

(v) $[1, 1.001]$

(b) Estimate the instantaneous velocity when $t = 1$.

For part (a), we shall use the formula which says that the average velocity of an object over a time interval is equal to the ratio between the change in position occurring in that time interval and the duration of the time interval. We shall denote average velocity with v for this part.

For part (i), we obtain

$$v = \frac{y(2) - y(1)}{2 - 1} = \frac{10(2) - 1.86(2)^2 - (10(1) - 1.86(1)^2)}{1} =$$
$$= 20 - 1.86(4) - 10 + 1.86 = 10 - 1.86(3) = 10 - 5.58 = 4.42$$

For part (ii), we obtain

$$v = \frac{y(1.5) - y(1)}{1.5 - 1} = \frac{10(1.5) - 1.86(1.5)^2 - (10(1) - 1.86(1)^2)}{0.5} =$$
$$= 2 \cdot (15 - 1.86(2.25) - 10 + 1.86) = 2 \cdot (5 - 1.86(1.25)) =$$
$$= 2 \cdot (5 - 2.325) = 2 \cdot 2.675 = 5.35$$

For part (iii), we obtain

$$v = \frac{y(1.1) - y(1)}{1.1 - 1} = \frac{10(1.1) - 1.86(1.1)^2 - (10(1) - 1.86(1)^2)}{0.1} =$$
$$= 10 \cdot (11 - 1.86(1.21) - 10 + 1.86) = 10 \cdot (1 - 1.86(0.21)) =$$
$$= 10 \cdot (1 - 0.3906) = 10 \cdot 0.6094 = 6.094$$

For part (iv), we obtain

$$v = \frac{y(1.01) - y(1)}{1.01 - 1} = \frac{10(1.01) - 1.86(1.01)^2 - (10(1) - 1.86(1)^2)}{0.01} =$$
$$= 100 \cdot (10.1 - 1.86(1.0201) - 10 + 1.86) = 100 \cdot (0.1 - 1.86(0.0201)) =$$
$$= 100 \cdot (0.1 - 0.037386) = 100 \cdot 0.062614 = 6.2614$$

For part (v), we obtain

$$v = \frac{y(1.001) - y(1)}{1.001 - 1} = \frac{10(1.001) - 1.86(1.001)^2 - (10(1) - 1.86(1)^2)}{1.001 - 1} =$$
$$= 1000 \cdot (10.01 - 1.86(1.002001) - 10 + 1.86) = 1000 \cdot (0.01 - 1.86(0.002001)) =$$
$$= 1000 \cdot (0.01 - 0.00372186) = 1000 \cdot 0.00627814 = 6.27814$$

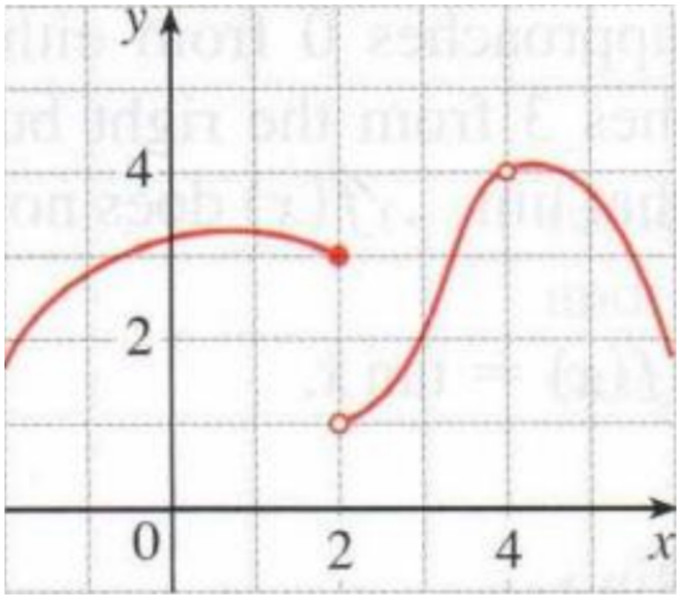
For part (b), we take the limit of the average velocity of the rock on the interval $[1, 1+h]$ as h goes to 0. We denote the instantaneous velocity at $t = 1$ with v.

$$v = \lim_{h \to 0} \frac{10(1+h) - 1.86(1+h)^2 - (10(1) - 1.86(1)^2)}{(1+h) - 1}$$
$$= \lim_{h \to 0} \frac{10(1) + 10h - 1.86(1^2 + 2h + h^2) - 10(1) + 1.86(1)^2}{h} =$$
$$= \lim_{h \to 0} \frac{10h - 1.86(1)^2 - 1.86(2h) - 1.86h^2 + 1.86(1)^2}{h} = \frac{10h - 3.72h - 1.86h^2}{h} =$$
$$= \lim_{h \to 0} \frac{6.28h - 1.86h^2}{h} = \lim_{h \to 0} [6.28 - 1.86h] = 6.28 - 1.86(0) = 6.28 - 0 = 6.28$$

1.2 Section 2.2

Problem 4

Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.



(a) $\lim_{x \to 2^-} f(x)$

(b) $\lim_{x \to 2^+} f(x)$

(c) $\lim_{x \to 2} f(x)$

(d) $f(2)$

(e) $\lim_{x \to 4} f(x)$

(f) $f(4)$

For part (a), $\lim_{x \to 2^-} f(x) = 3$

For part (b), $\lim_{x \to 2^+} f(x) = 1$

For part (c), the limit $\lim_{x \to 2} f(x)$ does not exist as its one-sided limits are not equal ($1 \neq 3$).

For part (d), $f(2) = 3$

For part (e), $\lim_{x \to 4} f(x) = 4$

For part (f), $f(4)$ is not defined according to the given graph.

Problem 5

For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

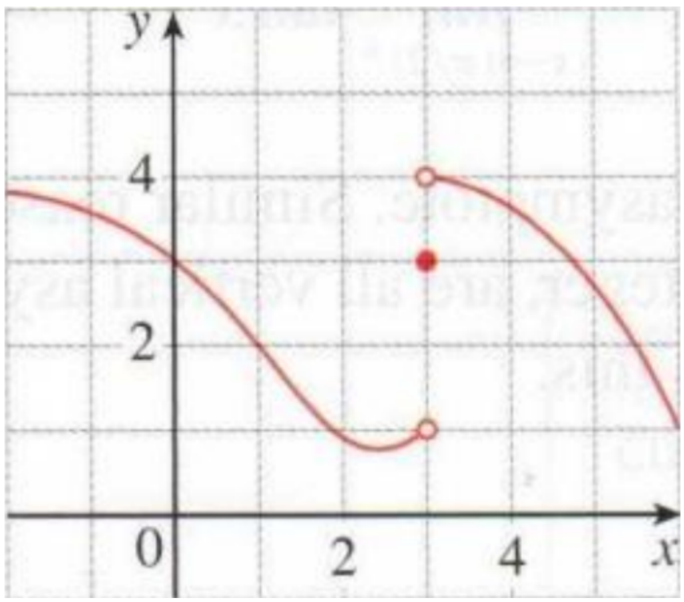
(a) $\lim_{x \to 1} f(x)$

(b) $\lim_{x \to 3^-} f(x)$

(c) $\lim_{x \to 3^+} f(x)$

(d) $\lim_{x \to 3} f(x)$

(e) $f(3)$



For part (a), $\lim_{x \to 1} f(x) = 2$

For part (b), $\lim_{x \to 3^-} f(x) = 1$

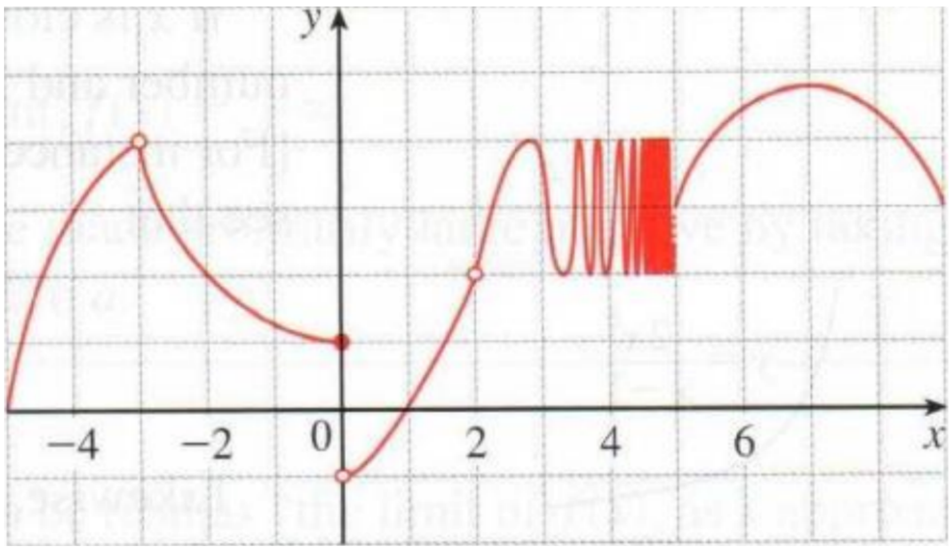
For part (c), $\lim_{x \to 3^+} f(x) = 4$

For part (d), the limit $\lim_{x \to 3} f(x)$ does not exist as its one-sided limits are not equal ($1 \neq 3$).

For part (e), $f(3) = 3$

Problem 6

For each function h whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.



(a) $\lim_{x \to -3^-} h(x)$

(b) $\lim_{x \to -3^+} h(x)$

(c) $\lim_{x \to -3} h(x)$

(d) $h(-3)$

(e) $\lim_{x \to 0^-} h(x)$

(f) $\lim_{x \to 0^+} h(x)$

(g) $\lim_{x \to 0} h(x)$

(h) $h(0)$

(i) $\lim_{x \to 2} h(x)$

(j) $h(2)$

(k) $\lim_{x \to 5^+} h(x)$

(l) $\lim_{x \to 5^-} h(x)$

The y-axis is assumed to have an equal proportion with the x-axis.
For part (a), $\lim_{x \to -3^-} h(x) = 4$
For part (b), $\lim_{x \to -3^+} h(x) = 4$
For part (c), $\lim_{x \to -3} h(x) = 4$
For part (d), $h(-3)$ is not defined according to the given graph.
For part (e), $\lim_{x \to 0^-} h(x) = 1$
For part (f), $\lim_{x \to 0^+} h(x) = -1$
For part (g), the limit $\lim_{x \to 0} h(x)$ does not exist as its one-sided limits are not equal.
For part (h), $h(0) = 1$
For part (i), $\lim_{x \to 2} h(x) = 2$

For part (j), $h(2)$ is not defined according to the given graph.

For part (k), $\lim_{x \to 5^+} h(x) = 3$

For part (l), $\lim_{x \to 5^-} h(x)$ does not exist as the graph of h at the left of $x = 5$ oscillates infinitely many times without decaying.

Problem 7

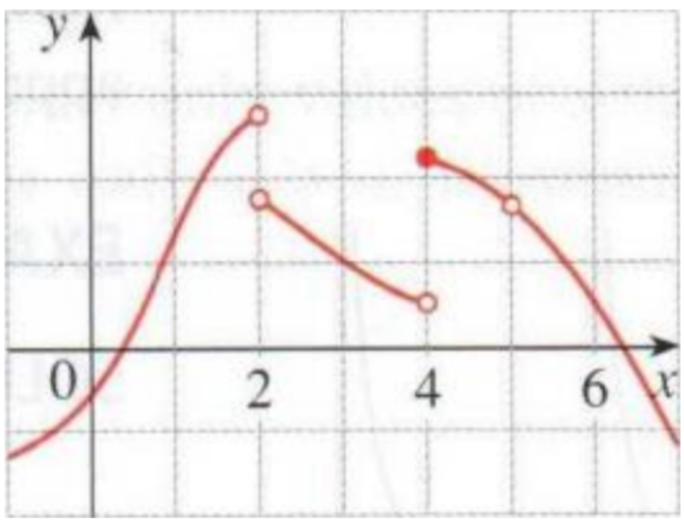
For the function g whose graph is shown, find a number a that satisfies the description.

(a) $\lim_{x \to a} g(x)$ does not exist but $g(a)$ is defined.

(b) $\lim_{x \to a} g(x)$ exists but $g(a)$ is not defined.

(c) $\lim_{x \to a^-} g(x)$ and $\lim_{x \to a^+} g(x)$ exist but $\lim_{x \to a} g(x)$ does not exist.

(d) $\lim_{x \to a^+} g(x) = g(a)$ but $\lim_{x \to a^-} g(x) \neq g(a)$.



For part (a), an instance of such a is $a = 4$ since the limit $\lim_{x \to 4} g(x)$ does not exist as its one-sided limits are not equal and $g(4)$ is defined.

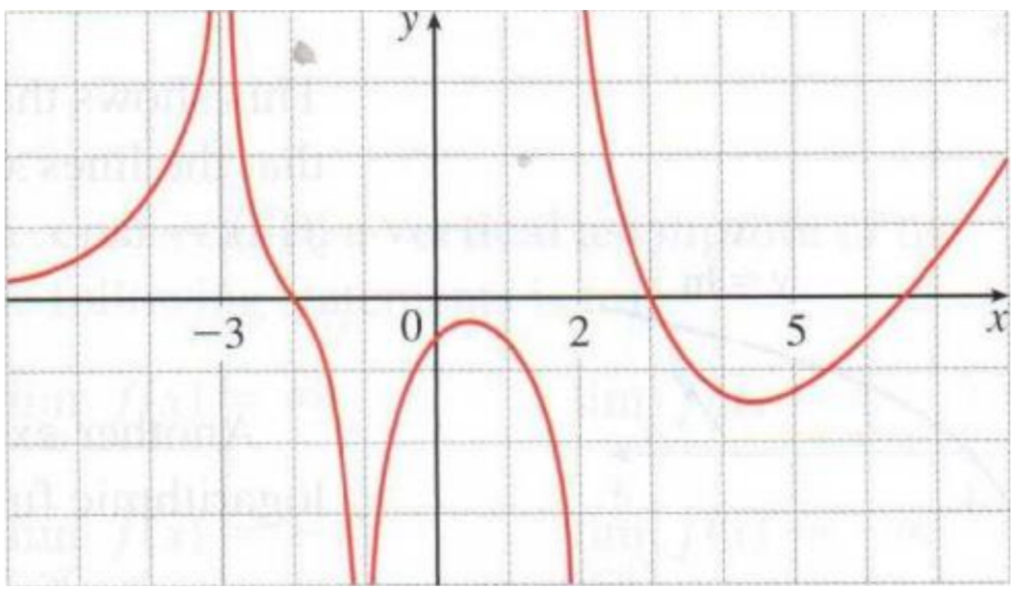
For part (b), an instance of such a is $a = 5$ since the limit $\lim_{x \to 5} g(x)$ exists and $g(5)$ is not defined.

For part (c), instances of such a are $a = 2$ and $a = 4$ since the one-sided limits $\lim_{x\to 2^-} g(x)$ and $\lim_{x\to 2^+} g(x)$ exist and are not equal, meaning $\lim_{x\to 2} g(x)$ does not exist and, similarly, the one-sided limits $\lim_{x\to 4^-} g(x)$ and $\lim_{x\to 4^+} g(x)$ exist and are not equal, meaning $\lim_{x\to 4} g(x)$ does not exist.

For part (d), an instance of such a is $a = 5$ since $\lim_{x\to 4^+} g(x) = g(4)$ and $\lim_{x\to 4^-} g(x) \neq g(4)$.

Problem 8

For the function A whose graph is shown, state the following.



1. $\lim_{x\to -3} A(x)$
2. $\lim_{x\to 2^-} A(x)$
3. $\lim_{x\to 2^+} A(x)$
4. $\lim_{x\to -1} A(x)$
5. The equations of the vertical asymptotes.

For part (a), the limit $\lim_{x\to -3} A(x)$ goes to positive infinity.

For part (b), the limit $\lim_{x\to 2^-} A(x)$ goes to negative infinity.

For part (c), the limit $\lim_{x\to 2^+} A(x)$ goes to positive infinity.

For part (d), the limit $\lim_{x\to -1} A(x)$ goes to negative infinity.

For part (e), the equations of the vertical asymptotes as given on the graph are $x = -3$, $x = -1$ and $x = 2$.

Problem 9

For the function f whose graph is shown, state the following.

(a) $\lim_{x\to -7} f(x)$

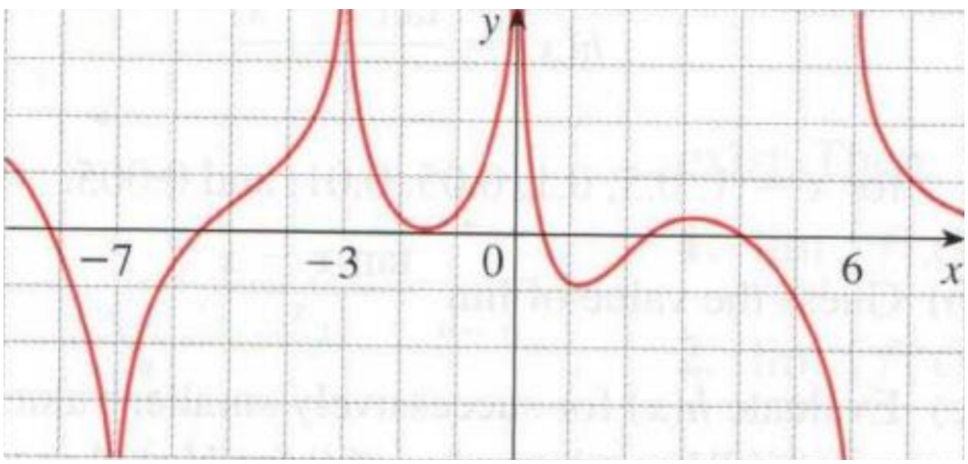
(b) $\lim_{x\to -3} f(x)$

(c) $\lim_{x\to 0} f(x)$

(d) $\lim_{x\to 6^-} f(x)$

(e) $\lim_{x\to 6^+} f(x)$

(f) The equations of the vertical asymptotes



For part (a), $\lim_{x\to -7} f(x) = -\infty$.

For part (b), $\lim_{x\to -3} f(x) = \infty$.

For part (c), $\lim_{x\to 0} f(x) = \infty$.

For part (d), $\lim_{x\to 6^-} f(x) = -\infty$.

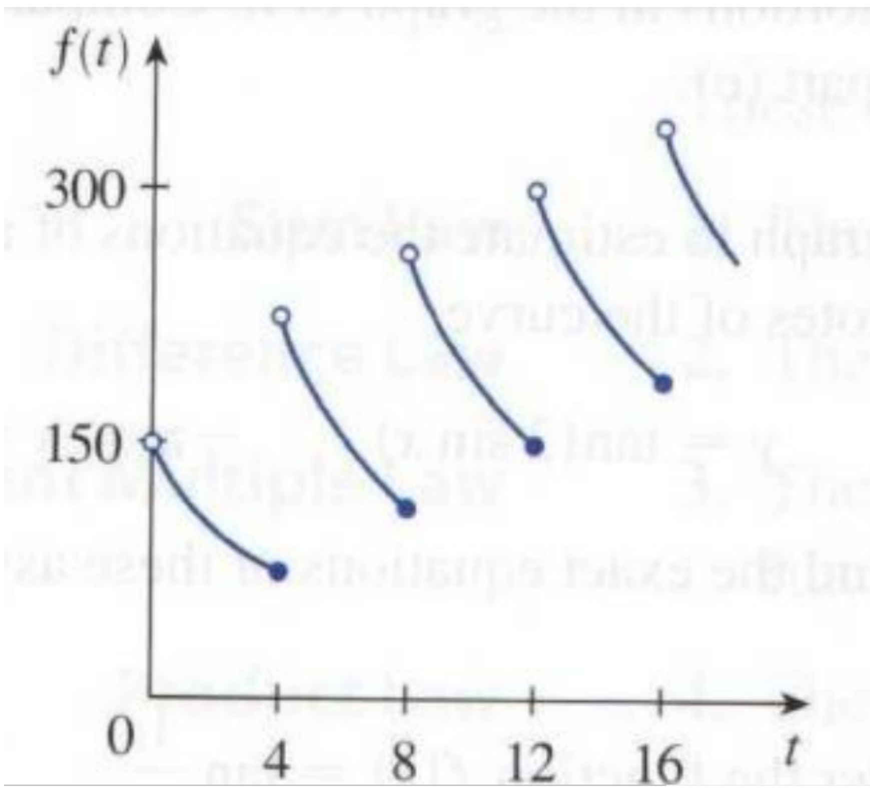
For part (e), $\lim_{x\to 6^+} f(x) = \infty$.

Problem 10

A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. Find

$$\lim_{t\to 12^-} f(t) \quad \text{and} \quad \lim_{t\to 12^+} f(t)$$

and explain the significance of these one-sided limits.



We have

$$\lim_{t\to 12^-} f(t) = 150$$

and

$$\lim_{t\to 12^+} f(t) = 300$$

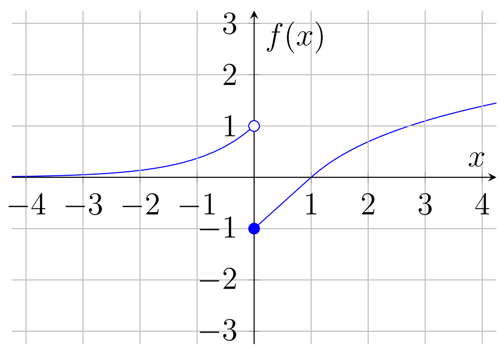
These one-sided limits present the amount of milligrams of the drug in the patient's body before and after an injection (which causes the jump discontinuity of the function as shown in the graph).

Problem 11

Sketch the graph of the function and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists.

$$f(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ x-1 & \text{if } 0 < x < 1 \\ \ln x & \text{if } x \geq 1 \end{cases}$$

The graph of $f(x)$ is given below.



The function $f(x)$ is equal to e^x on the open interval $(-\infty, 0)$ and e^x is a continuous function (implying its limit always exists in general), meaning $\lim_{x\to a} f(x)$ exists for all $a < 0$.

The limit $\lim_{x\to a} f(x)$ does not exist when $a = 0$ as $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x = e^0 = 1$ whereas $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x - 1 = 0 - 1 = -1$ and $1 \neq -1$, meaning the one-sided limits are not equal.

The function $f(x)$ is equal to $x-1$ on the open interval $(0, 1)$ and $x-1$ is a continuous function (implying its limit always exists in general), meaning $\lim_{x\to a} f(x)$ exists for all $0 < a < 1$.

The limit $\lim_{x\to a} f(x)$ exists when $a = 1$ as $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x - 1 = 1 - 1 = 0 = \ln 1 = \lim_{x\to 1^+} \ln x = \lim_{x\to 1^+} f(x)$ (i.e., both one-sided limits exists and are equal).

The function $f(x)$ is equal to $\ln x$ on the open interval $(1, \infty)$ and $\ln x$ is a continuous function (implying its limit always exists in general), meaning $\lim_{x\to a} f(x)$ exists for all $a > 1$.

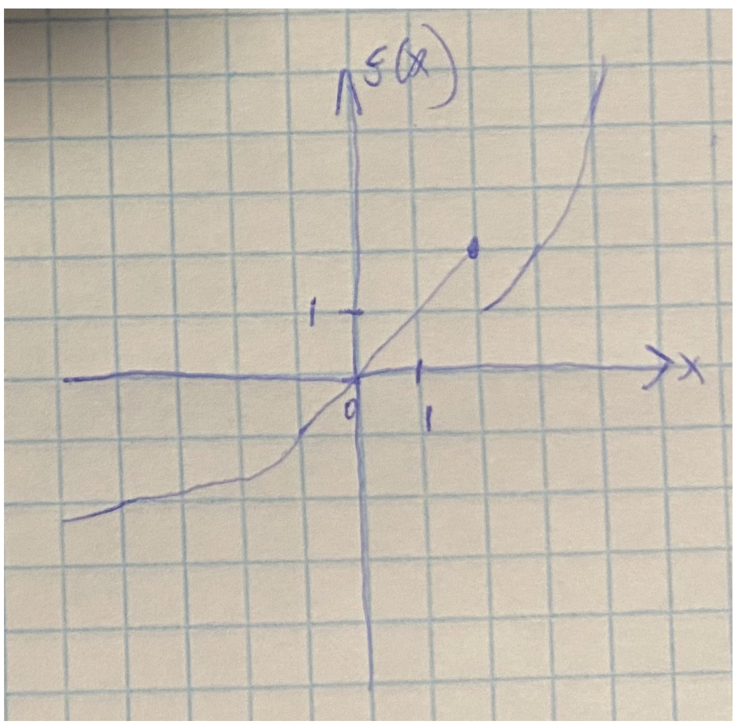
Finally, the set of values of a for which $\lim_{x \to a} f(x)$ exists is $\mathbb{R} \setminus \{0\}$.

Problem 12

Sketch the graph of the function and use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists.

$$f(x) = \begin{cases} \sqrt[3]{x} & \text{if } x \leq -1 \\ x & \text{if } -1 < x \leq 2 \\ (x-1)^2 & \text{if } x > 2 \end{cases}$$

A sketch of the graph of $f(x)$ is given below.



The function $f(x)$ is equal to $\sqrt[3]{x}$ on the open interval $(-\infty, -1)$ and $\sqrt[3]{x}$ is a continuous function (implying its limit always exists in general), meaning $\lim_{x \to a} f(x)$ exists for all $a < -1$.

The limit $\lim_{x\to a} f(x)$ exists when $a = -1$ as $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \sqrt[3]{x} = \sqrt[3]{-1} = -1 = \lim_{x\to 1^+} x = \lim_{x\to -1^+} f(x)$ (i.e., both one-sided limits exists and are equal).

The function $f(x)$ is equal to x on the open interval $(-1, 2)$ and x is a continuous function (implying its limit always exists in general), meaning $\lim_{x\to a} f(x)$ exists for all $-1 < a < 2$.

The limit $\lim_{x\to a} f(x)$ does not exist when $a = 2$ as $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x = 2$ whereas $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-1)^2 = (2-1)^2 = 1^2 = 1$ and $1 \neq -1$, meaning the one-sided limits are not equal.

The function $f(x)$ is equal to $(x-1)^2$ on the open interval $(2, \infty)$ and $(x-1)^2$ is a continuous function (implying its limit always exists in general), meaning $\lim_{x\to a} f(x)$ exists for all $a > 2$.

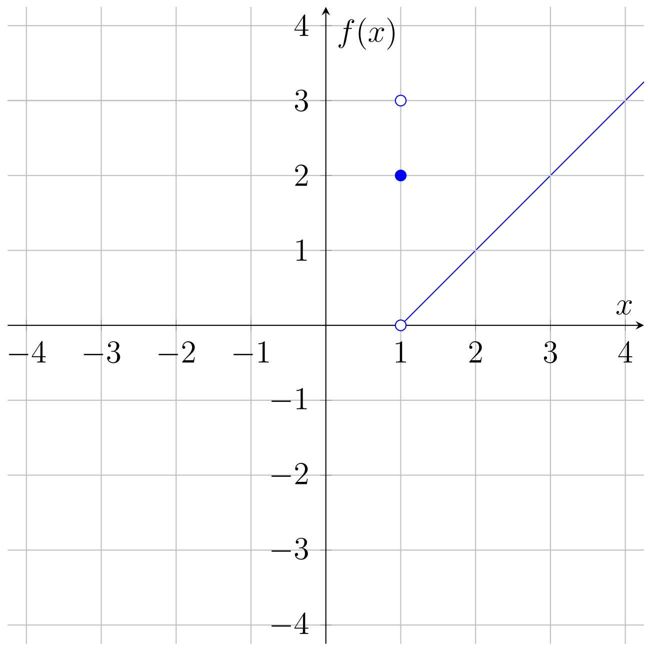
Finally, the set of values of a for which $\lim_{x\to a} f(x)$ exists is $\mathbb{R} \setminus \{2\}$.

Problem 15

Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$\lim_{x\to 1^-} f(x) = 3, \quad \lim_{x\to 1^+} f(x) = 0, \quad f(1) = 2$$

A sketched graph of a function f satisfying the given properties is given below.



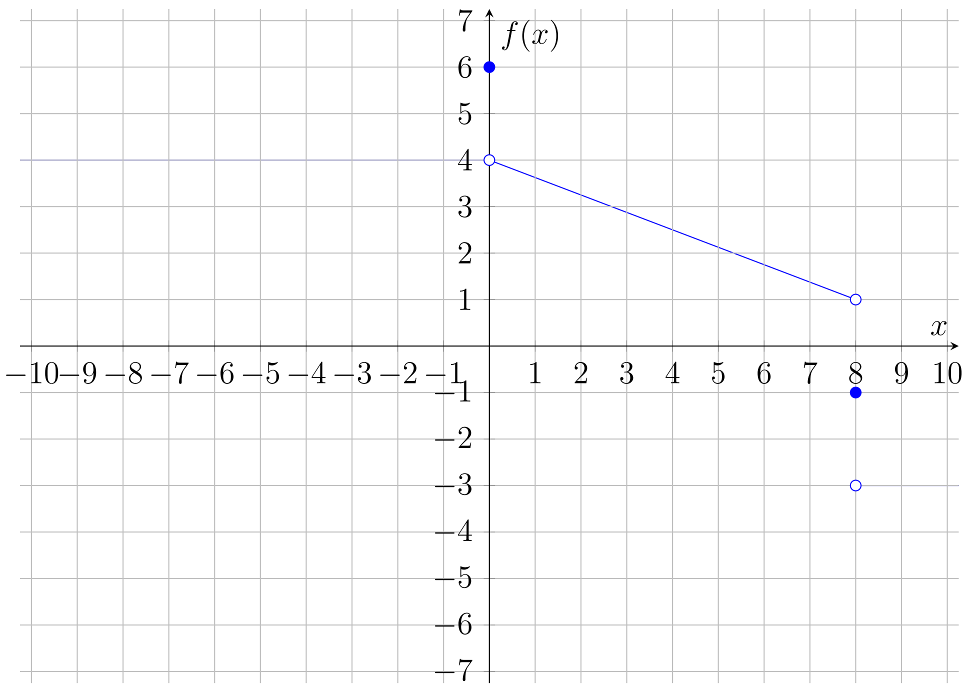
Problem 16

Sketch the graph of an example of a function f that satisfies all the given conditions.

$$\lim_{x\to 0} f(x) = 4, \quad \lim_{x\to 8^-} f(x) = 1, \quad \lim_{x\to 8^+} f(x) = -3,$$

$$f(0) = 6, \quad f(8) = -1$$

Here is an example of a function f that satisfies all the given conditions.



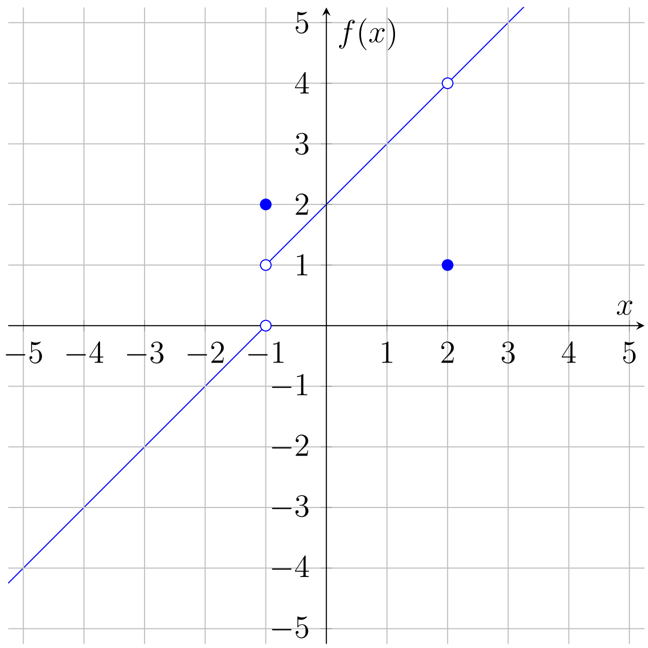
Problem 17

Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$\lim_{x\to -1^-} f(x) = 0, \quad \lim_{x\to -1^+} f(x) = 1, \lim_{x\to 2} f(x) = 3,$$

$$f(-1) = 2, \quad f(2) = 1$$

A sketched graph of a function f satisfying the given properties is given below.

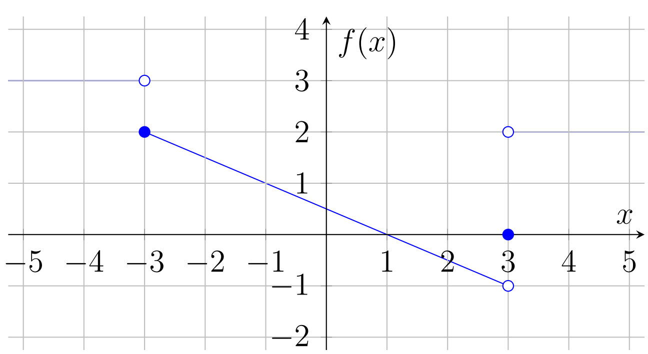


Problem 18

Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$\lim_{x\to -3^-} f(x) = 3, \quad \lim_{x\to -3^+} f(x) = 2, \quad \lim_{x\to 3^-} f(x) = -1,$$
$$\lim_{x\to 3^+} f(x) = 2, \quad f(-3) = 2, f(3) = 0$$

The graph of a function f that satisfies the given properties is given below.



Problem 29

Determine the infinite limit.

$$\lim_{x \to 5^+} \frac{x+1}{x-5}$$

The limit $\lim\limits_{x \to 5^+} \frac{x+1}{x-5}$ approaches positive infinity since $x + 1 \to 6^+ > 0$ as $x \to 5^+$ and $x - 5 \to 0^+$ as $x \to 0^+$, meaning the limit grows arbitrarily large and is positive as the numerator and the denominator are both always positive in the limit.

Problem 30

Determine the infinite limit.

$$\lim_{x \to 5^-} \frac{x+1}{x-5}$$

The limit $\lim\limits_{x \to 5^-} \frac{x+1}{x-5}$ goes to negative infinity as x is positive near 5, meaning $x + 1$ is also positive, and, since $x < 5$, we have that $x - 5$ is negative, meaning the fraction overall is negative and, since its absolute value grows arbitrarily large, it goes to negative infinity.

Problem 31

Determine the infinite limit.

$$\lim_{x \to 2} \frac{x^2}{(x-2)^2}$$

The limit $\lim\limits_{x \to 2} \frac{x^2}{(x-2)^2}$ approaches positive infinity since $x^2 \to 4 > 0$ as $x \to 2$ and $(x-2)^2 \to 0^+$ as $x \to 2$ (since $v^2 \geq 0$ for all real v), meaning the limit grows arbitrarily large and is positive as the numerator and the denominator are both always positive in the limit.

Problem 32

Determine the infinite limit.

$$\lim_{x\to 3^-} \frac{\sqrt{x}}{(x-3)^5}$$

The limit $\lim_{x\to 3^-} \frac{\sqrt{x}}{(x-3)^5}$ goes to negative infinity as x is positive near 3, meaning $\sqrt{x}$ is also positive, and, since $x < 3$, we have that $x - 3$ and $(x-3)^5$ are negative, meaning the fraction overall is negative and, since its absolute value grows arbitrarily large, it goes to negative infinity.

Problem 33

Determine the infinite limit.

$$\lim_{x\to 1^+} \ln\bigl(\sqrt{x} - 1\bigr)$$

$\sqrt{x} \to 1^+$ as $x \to 1^+$, meaning $\sqrt{x} - 1 \to 0^+$ as $x \to 1^+$. In general, whenever $t \to 0^+$, we have $\ln t \to -\infty$, meaning, since $\sqrt{x} - 1 \to 0^+$ as $x \to 1^+$, we have $\ln(\sqrt{x} - 1) \to -\infty$ as $x \to 1^+$.

Problem 34

Determine the infinite limit.

$$\lim_{x\to 0^+} \ln(\sin x)$$

As $x \to 0^+$, $\sin x$ approaches 0 from above. In general, ln goes to negative infinity given that its argument approaches 0 from above (positivity is required for the logarithm to be defined). Hence the limit $\lim_{x\to 0^+} \ln(\sin x)$ goes to negative infinity.

Problem 35

Determine the infinite limit.

$$\lim_{x\to(\pi/2)^+} \frac{1}{x}\sec x$$

Since $\sec x = \frac{1}{\cos x}$ and $\cos x \to 0^-$ as $x \to (\pi/2)^+$, we have $\sec x \to -\infty$ as $x \to (\pi/2)^+$. Since $\frac{1}{x}$ is nonzero on an open interval right to $\pi/2$, the limit $\lim\limits_{x\to(\pi/2)^+} \frac{1}{x}\sec x$ grows arbitrarily large and goes to negative infinity ($\frac{\pi}{2} > 0$).

Problem 36

Determine the infinite limit.

$$\lim_{x\to\pi^-} x\cot x$$

Since $\cot x = \frac{\cos x}{\sin x}$, $\cos x \to -1^+ < 0$ and $\sin x \to 0^+$ as $x \to \pi^-$, we have $\sec x \to -\infty$ as $x \to \pi^-$. Since x is nonzero on an open interval left to π, the limit $\lim\limits_{x\to\pi^-} x\cot x$ grows arbitrarily large and goe sto negative infinity ($\pi > 0$).

Problem 37

Determine the infinite limit.

$$\lim_{x\to 1} \frac{x^2+2x}{x^2-2x+1}$$

The limit $\lim\limits_{x\to 1} \frac{x^2+2x}{x^2-2x+1}$ goes to positive infinity since $x^2+2x \to 2 > 0$ as $x \to 1$ and $x^2-2x+1 \to 0^+$ as $x \to 1$ (since $x^2-2x+1 = (x-1)^2$ and $v^2 \geq 0$ for all real v), meaning the limit grows arbitrarily large and is positive as the numerator and denominator are both always positive in the limit.

Problem 38

Determine the infinite limit.

$$\lim_{x\to 3^-} \frac{x^2+4x}{x^2-2x-3}$$

The limit $\lim\limits_{x\to 3^-} \frac{x^2+4x}{x^2-2x-3}$ goes to negative infinity since $x^2+4x \to 21^- > 0$ as $x \to 3^-$ and $x^2-2x-3 \to 0^-$ as $x \to 3^-$ (since $x^2-2x-3 = (x-3)(x+1)$), meaning the limit grows arbitrarily large and is negative as the numerator is always positive and the denominator is always negative in the limit.

Problem 39

Determine the infinite limit.

$$\lim_{x\to 0} \left(\ln x^2 - x^{-2}\right)$$

The notation $\ln x^2$ is ambiguous, we assume that the author meant the expression $\ln\left(x^2\right)$ since having $(\ln x)^2$ in a limit as $x \to 0$ would also be nonexistent due to a more obvious reason (think what x approaching 0 from the left).

$x^2 \to 0^+$ as $x \to 0$ (since $v^2 \geq 0$ for all real v). In general, $\ln t \to -\infty$ as $t \to 0^+$. Meaning, since $x^2 \to 0^+$ as $x \to 0$, we have $\ln\left(x^2\right) \to -\infty$ as $x \to 0$. Also, $x^{-2} = \frac{1}{x^2} \to \infty$ as $x \to 0$ (also due to $x \to 0^+$ as $x \to 0$), meaning $-x^2 \to -\infty$ as $x \to 0$. Sum of limits diverging to the same infinity also diverges to the same infinity, thus $\ln\left(x^2\right) - x^{-2}$ goes to negative infinity as $x \to 0$.

Problem 40

Determine the infinite limit

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \ln x\right)$$

As already known, $\frac{1}{x}$ goes to positive infinity as $x \to 0^+$ whereas $\ln x$ goes to negative infinity as $x \to 0^+$. The latter means that $-\ln x$ goes to positive infinity as $x \to 0^+$. Obviously, a sum of functions that diverge to the same infinity at some point will also diverge to the same infinity at that point, meaning $\frac{1}{x} - \ln x$ goes to positive infinity as $x \to 0^+$.

Problem 41

Find the vertical asymptote of the function

$$f(x) = \frac{x-1}{2x+4}$$

For a vertical asymptote to occur at $x = a$, we would want $2x + 4 \to 0$ as $x \to a$. Clearly, the only such value of a is $a = -2$.

We now verify that a vertical asymptote indeed occurs at $x = -2$. $x-1 \to -3^+ < 0$ and $2x + 4 \to 0^+$ as $a = -2^+$, meaning $\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} \frac{x-1}{2x+4} = -\infty$ and a vertical asymptote indeed occurs at $x = -2$. No other vertical asymptote occurs as shown in the first two sentences.

Problem 42

(a) Find the vertical asymptotes of the function

$$y = \frac{x^2+1}{3x-2x^2}$$

(b) Confirm your answer to (a) by graphing the function.

For part (a), for a vertical asymptote to occur at $x = a$, we would want $3x - 2x^2 \to 0$ as $x \to a$. Clearly, the only such values of a are $a = 0$ and $a = \frac{3}{2}$.

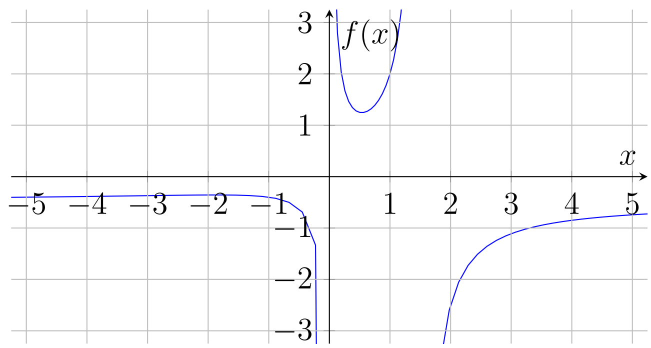
We now vertify that vertical asymptotes indeed occur at $x = 0$ and $x = -2$.

$x^2 + 1 \to 1^+ > 0$ and $3x - 2x^2 \to 0^+$ as $x \to 0^+$, meaning $\lim\limits_{x \to 0} y(x) = \lim\limits_{x \to 0} \frac{x^2+1}{3x-2x^2} = \infty$ and a vertical asymptote indeed occurs at $x = 0$.

Similarly, $x^2 + 1 \to \frac{13}{4} > 0$ and $3x - 2x^2 \to 0^-$ as $x \to \frac{3}{2}^+$, meaning $\lim_{x \to \frac{3}{2}^+} y(x) = \lim_{x \to \frac{3}{2}^+} \frac{x^2+1}{3x-2x^2} = -\infty$ and a vertical asymptote indeed occurs at $x = \frac{3}{2}$.

No other asymptote occurs as shown in the first two sentences.

For part (b), the graph of the function f is given below.



1.3 Section 2.3

Problem 5

Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

$$\lim_{v \to 2} (v^2 + 2v)(2v^3 - 5)$$

$$
\begin{aligned}
\lim_{v\to 2}(v^2+2v)(2v^3-5) &= \lim_{v\to 2}(v^2+2v)\cdot \lim_{v\to 2}(2v^3-5) && \text{(by Law 4)}\\
&= (\lim_{v\to 2} v^2 + \lim_{v\to 2} 2v)\cdot \lim_{v\to 2}(2v^3-5) && \text{(by Law 1)}\\
&= (2^2 + \lim_{v\to 2} 2v)\cdot \lim_{v\to 2}(2v^3-5) && \text{(by Law 10)}\\
&= (2^2 + 2\lim_{v\to 2} v)\cdot \lim_{v\to 2}(2v^3-5) && \text{(by Law 3)}\\
&= (2^2 + 2\cdot 2)\cdot \lim_{v\to 2}(2v^3-5) && \text{(by Law 9)}\\
&= (2^2 + 2\cdot 2)\cdot (\lim_{v\to 2} 2v^3 - \lim_{v\to 2} 5) && \text{(by Law 2)}\\
&= (2^2 + 2\cdot 2)\cdot (2\lim_{v\to 2} v^3 - \lim_{v\to 2} 5) && \text{(by Law 3)}\\
&= (2^2 + 2\cdot 2)\cdot (2(2)^3 - \lim_{v\to 2} 5) && \text{(by Law 10)}\\
&= (2^2 + 2\cdot 2)\cdot (2(2)^3 - 5) && \text{(by Law 8)}\\
&= 88
\end{aligned}
$$

Problem 6

Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

$$\lim_{t\to 7}\frac{3t^2+1}{t^2-5t+2}$$

$$
\begin{aligned}
\lim_{t\to 7}\frac{3t^2+1}{t^2-5t+2} &= \frac{\lim\limits_{t\to 7}[3t^2+1]}{\lim\limits_{t\to 7}[t^2-5t+2]} && \text{(by Law 5)}\\
&= \frac{\lim\limits_{t\to 7}3t^2+\lim\limits_{t\to 7}1}{\lim\limits_{t\to 7}[t^2-5t+2]} && \text{(by Law 1)}\\
&= \frac{\lim\limits_{t\to 7}3t^2+1}{\lim\limits_{t\to 7}[t^2-5t+2]} && \text{(by Law 8)}\\
&= \frac{3\lim\limits_{t\to 7}t^2+1}{\lim\limits_{t\to 7}[t^2-5t+2]} && \text{(by Law 3)}\\
&= \frac{3(7)^2+1}{\lim\limits_{t\to 7}[t^2-5t+2]} && \text{(by Law 10)}\\
&= \frac{3(7)^2+1}{\lim\limits_{t\to 7}[t^2-5t]+\lim\limits_{t\to 7}2} && \text{(by Law 1)}\\
&= \frac{3(7)^2+1}{\lim\limits_{t\to 7}[t^2-5t]+2} && \text{(by Law 8)}\\
&= \frac{3(7)^2+1}{\lim\limits_{t\to 7}t^2-\lim\limits_{t\to 7}5t+2} && \text{(by Law 2)}\\
&= \frac{3(7)^2+1}{7^2-\lim\limits_{t\to 7}5t+2} && \text{(by Law 10)}\\
&= \frac{3(7)^2+1}{7^2-5\lim\limits_{t\to 7}t+2} && \text{(by Law 3)}\\
&= \frac{3(7)^2+1}{7^2-5(7)+2} && \text{(by Law 9)}\\
&= \frac{37}{4}
\end{aligned}
$$

Problem 7

Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

$$\lim_{u\to -2} \sqrt{9-u^3+2u^2}$$

$$\begin{aligned}
\lim_{u\to -2} \sqrt{9-u^3+2u^2} &= \sqrt{\lim_{u\to -2}[9-u^3+2u^2]} && \text{(by Law 11)}\\
&= \sqrt{\lim_{u\to -2}[9-u^3] + \lim_{u\to -2} 2u^2} && \text{(by Law 1)}\\
&= \sqrt{\lim_{u\to -2} 9 - \lim_{u\to -2} u^3 + \lim_{u\to -2} 2u^2} && \text{(by Law 2)}\\
&= \sqrt{9 - \lim_{u\to -2} u^3 + \lim_{u\to -2} 2u^2} && \text{(by Law 8)}\\
&= \sqrt{9 - (-2)^3 + \lim_{u\to -2} 2u^2} && \text{(by Law 10)}\\
&= \sqrt{9 - (-2)^3 + 2\lim_{u\to -2} u^2} && \text{(by Law 3)}\\
&= \sqrt{9 - (-2)^3 + 2(-2)^2} && \text{(by Law 10)}\\
&= 5
\end{aligned}$$

Problem 8

Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

$$\lim_{x\to 3} \sqrt[3]{x+5}(2x^2-3x)$$

$$\begin{aligned}
\lim_{x\to 3} \sqrt[3]{x+5}(2x^2-3x) &= \lim_{x\to 3} \sqrt[3]{x+5} \cdot \lim_{x\to 3}(2x^2-3x) && \text{(by Law 4)} \\
&= \sqrt[3]{\lim_{x\to 3}[x+5]} \cdot \lim_{x\to 3}(2x^2-3x) && \text{(by Law 7)} \\
&= \sqrt[3]{\lim_{x\to 3} x + \lim_{x\to 3} 5} \cdot \lim_{x\to 3}(2x^2-3x) && \text{(by Law 1)} \\
&= \sqrt[3]{\lim_{x\to 3} x + 5} \cdot \lim_{x\to 3}(2x^2-3x) && \text{(by Law 8)} \\
&= \sqrt[3]{3+5} \cdot \lim_{x\to 3}(2x^2-3x) && \text{(by Law 9)} \\
&= \sqrt[3]{3+5} \cdot (\lim_{x\to 3} 2x^2 - \lim_{x\to 3} 3x) && \text{(by Law 2)} \\
&= \sqrt[3]{3+5} \cdot (\lim_{x\to 3} 2x^2 - 3\lim_{x\to 3} x) && \text{(by Law 3)} \\
&= \sqrt[3]{3+5} \cdot (\lim_{x\to 3} 2x^2 - 3(3)) && \text{(by Law 9)} \\
&= \sqrt[3]{3+5} \cdot (2\lim_{x\to 3} x^2 - 3(3)) && \text{(by Law 3)} \\
&= \sqrt[3]{3+5} \cdot (2(3)^2 - 3(3)) && \text{(by Law 10)} \\
&= 18
\end{aligned}$$

Problem 9

Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

$$\lim_{t\to -1} \left(\frac{2t^5 - t^4}{5t^2+4} \right)^3$$

$$
\begin{aligned}
\lim_{t\to-1}\left(\frac{2t^5-t^4}{5t^2+4}\right)^3 &= \left(\lim_{t\to-1}\frac{2t^5-t^4}{5t^2+4}\right)^3 && \text{(by Law 6)}\\
&= \left(\frac{\lim\limits_{t\to-1}[2t^5-t^4]}{\lim\limits_{t\to-1}[5t^2+4]}\right)^3 && \text{(by Law 5)}\\
&= \left(\frac{\lim\limits_{t\to-1}2t^5-\lim\limits_{t\to-1}t^4}{\lim\limits_{t\to-1}[5t^2+4]}\right)^3 && \text{(by Law 2)}\\
&= \left(\frac{2\lim\limits_{t\to-1}t^2-\lim\limits_{t\to-1}t^4}{\lim\limits_{t\to-1}[5t^2+4]}\right)^3 && \text{(by Law 3)}\\
&= \left(\frac{2(-1)^5-\lim\limits_{t\to-1}t^4}{\lim\limits_{t\to-1}[5t^2+4]}\right)^3 && \text{(by Law 10)}\\
&= \left(\frac{2(-1)^5-(-1)^4}{\lim\limits_{t\to-1}[5t^2+4]}\right)^3 && \text{(by Law 10)}\\
&= \left(\frac{2(-1)^5-(-1)^4}{\lim\limits_{t\to-1}5t^2+\lim\limits_{t\to-1}4}\right)^3 && \text{(by Law 1)}\\
&= \left(\frac{2(-1)^5-(-1)^4}{5\lim\limits_{t\to-1}t^2+\lim\limits_{t\to-1}4}\right)^3 && \text{(by Law 3)}\\
&= \left(\frac{2(-1)^5-(-1)^4}{5(-1)^2+\lim\limits_{t\to-1}4}\right)^3 && \text{(by Law 10)}\\
&= \left(\frac{2(-1)^5-(-1)^4}{5(-1)^2+4}\right)^3 && \text{(by Law 8)}\\
&= -\frac{1}{27}
\end{aligned}
$$

Problem 10

(a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2}(x + 3)$$

is correct.

For part (a), the equation is false for $x = 2$.

For part (b), generally, when considering a limit as $x \to a$, the value a itself is not considered, only the ones sufficiently close to it. Hence it is correct to replace $\frac{x^2+x-6}{x-2}$ with $x+3$ in the limit as $x \to 2$ since the case $x = 2$ is not considered.

Problem 11

Evaluate the limit, if it exists.

$$\lim_{x \to -2}(3x - 7)$$

We have the following

$$lim_{x \to -2}(3x - 7) = 3 \lim_{x \to -2} x - \lim_{x \to -2} 7 = 3(-2) - 7 = -13$$

Problem 12

Evaluate the limit, if it exists.

$$\lim_{x \to 6} \left(8 - \frac{1}{2}x\right) = \lim_{x \to 6} 8 - \frac{1}{2} \lim_{x \to 6} x = 8 - \frac{1}{2}(6) = 5$$

Problem 13

Evaluate the limit, if it exists.

$$\lim_{t\to 4} \frac{t^2 - 2t - 8}{t - 4}$$

We have the following.

$$\lim_{t\to 4} \frac{t^2 - 2t - 8}{t - 4} = \lim_{t\to 4} \frac{(t-4)(t+2)}{t - 4} = \lim_{t\to 4}[t+2] = 4 + 2 = 6$$

Problem 14

Evaluate the limit, if it exists.

$$\lim_{x\to -3} \frac{x^2 + 3x}{x^2 - x - 12}$$

We have the following.

$$\lim_{x\to -3} \frac{x^2 + 3x}{x^2 - x - 12} = \lim_{x\to -3} \frac{x(x+3)}{(x-4)(x+3)} = \lim_{x\to -3} \frac{x}{x-4} = \frac{-3}{-3-4} = \frac{3}{7}$$

Problem 15

Evaluate the limit, if it exists.

$$\lim_{x\to 2} \frac{x^2 + 5x + 4}{x - 2}$$

The limit does not exist as it grows arbitrarily large ($x^2 + 5x + 4 \to 18$ and $x - 2 \to 0$ as $x \to 2$).

Problem 16

Evaluate the limit, if it exists.

$$\lim_{x\to 4} \frac{x^2+3x}{x^2-x-12}$$

The limit does not exist as it grows arbitrarily large ($x^2+3x \to 38$ and $x^2-x-12 \to 0$ as $x \to 4$).

Problem 17

Evaluate the limit, if it exists.

$$\lim_{x\to -2} \frac{x^2-x-6}{3x^2+5x-2}$$

We have the following.

$$\lim_{x\to -2} \frac{x^2-x-6}{3x^2+5x-2} = \lim_{x\to -2} \frac{(x-3)(x+2)}{(3x-1)(x+2)} = \lim_{x\to -2} \frac{x-3}{3x-1} = \frac{-2-3}{3(-2)-1} = \frac{5}{7}$$

Problem 18

Evaluate the limit, if it exists.

$$\lim_{x\to -5} \frac{2x^2+9x-5}{x^2-25}$$

We have the following.

$$\begin{aligned}\lim_{x\to -5} \frac{2x^2+9x-5}{x^2-25} &= \frac{(2x-1)(x+5)}{(x-5)(x+5)} = \lim_{x\to -5} \frac{2x-1}{x-5} \\ &= \frac{2(-5)-1}{(-5)-5} = \frac{11}{10} = 1.1\end{aligned}$$

Problem 19

Evaluate the limit, if it exists.

$$\lim_{t\to 3}\frac{t^3-27}{t^2-9}$$

We have the following.

$$\lim_{t\to 3}\frac{t^3-27}{t^2-9}=\lim_{t\to 3}\frac{t^3-3^3}{t^2-3^2}=\lim_{t\to 3}\frac{(t-3)(t^2+3t+3^2)}{(t-3)(t+3)}=$$
$$=\lim_{t\to 3}\frac{t^2+3t+9}{t+3}=\frac{3^2+3(3)+9}{3+3}=4.5$$

Problem 20

Evaluate the limit, if it exists.

$$\lim_{u\to -1}\frac{u+1}{u^3+1}$$

We have the following.

$$\lim_{u\to -1}\frac{u+1}{u^3+1}=\lim_{u\to -1}\frac{u+1}{(u+1)(u^2-u+1)}=\lim_{u\to -1}\frac{1}{u^2-u+1}=\frac{1}{(-1)^2-(-1)+1}=\frac{1}{3}$$

Problem 21

Evaluate the limit, if it exists.

$$\lim_{h\to 0}\frac{(h-3)^2-9}{h}$$

We have the following.

$$\lim_{h\to 0}\frac{(h-3)^2-9}{h}=\lim_{h\to 0}\frac{(h-3)^2-3^2}{h}=\lim_{h\to 0}\frac{(h-3-3)(h-3+3)}{h}$$
$$=\lim_{h\to 0}\frac{(h-6)h}{h}=\lim_{h\to 0}[h-6]=0-6=-6$$

Problem 22

Evaluate the limit, if it exists.

$$\lim_{x\to 9} \frac{9-x}{3-\sqrt{x}}$$

We have the following.

$$\begin{aligned}\lim_{x\to 9} \frac{9-x}{3-\sqrt{x}} &= \lim_{x\to 9} \frac{(3-\sqrt{x})(3+\sqrt{x})}{3-\sqrt{x}} = \\ &= \lim_{x\to 9}[3+\sqrt{x}] = 3+\sqrt{9} = 6\end{aligned}$$

Problem 23

Evaluate the limit, if it exists.

$$\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h}$$

We have the following.

$$\begin{aligned}\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} &= \lim_{h\to 0} \frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)} = \lim_{h\to 0} \frac{\sqrt{9+h}^2-3^2}{h(\sqrt{9+h}+3)} = \\ &= \lim_{h\to 0} \frac{9+h-9}{h(\sqrt{9+h}+3)} = \lim_{h\to 0} \frac{h}{h(\sqrt{9+h}+3)} = \\ &= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{\sqrt{9+0}+3} = \frac{1}{6}\end{aligned}$$

Problem 24

Evaluate the limit, if it exists.

$$\lim_{x\to 2} \frac{2-x}{\sqrt{x+2}-2}$$

We have the following.

Problem 25

Evaluate the limit, if it exists.

$$\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}$$

We have the following.

$$\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}=\lim_{x\to 3}\frac{3-x}{(x-3)\cdot 3x}=-\lim_{x\to 3}\frac{1}{3x}=-\frac{1}{3\cdot 3}=-\frac{1}{9}$$

Problem 26

Evaluate the limit, if it exists.

$$\lim_{h\to 0}\frac{(-2+h)^{-1}+2^{-1}}{h}$$

We have the following.

$$\begin{aligned}\lim_{h\to 0}\frac{(-2+h)^{-1}+2^{-1}}{h}&=\lim_{h\to 0}\frac{2+(-2+h)}{2h(-2+h)}=\lim_{h\to 0}\frac{h}{2h(h-2)}\\&=\lim_{h\to 0}\frac{1}{2(h-2)}=\frac{1}{2(0-2)}=-\frac{1}{4}\end{aligned}$$

Problem 27

Evaluate the limit, if it exists.

$$\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}$$

We have the following.

$$\begin{aligned}
\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t} &= \lim_{t\to 0}\frac{(\sqrt{1+t}-\sqrt{1-t})(\sqrt{1+t}+\sqrt{1-t})}{t(\sqrt{1+t}+\sqrt{1-t})} = \\
&= \lim_{t\to 0}\frac{\sqrt{1+t}^2-\sqrt{1-t}^2}{t(\sqrt{1+t}+\sqrt{1-t})} = \lim_{t\to 0}\frac{1+t-(1-t)}{t(\sqrt{1+t}+\sqrt{1-t})} = \\
&= \lim_{t\to 0}\frac{2t}{t(\sqrt{1+t}+\sqrt{1-t})} = \lim_{t\to 0}\frac{2}{\sqrt{1+t}+\sqrt{1-t}} = \\
&= \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = 1
\end{aligned}$$

Problem 28

Evaluate the limit, if it exists.

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)$$

We have the following.

$$\begin{aligned}
\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right) &= \lim_{t\to 0}\frac{t+1-1}{t(t+1)} = \lim_{t\to 0}\frac{t}{t(t+1)} = \\
&= \lim_{t\to 0}\frac{1}{t+1} = \frac{1}{0+1} = 1
\end{aligned}$$

Problem 29

Evaluate the limit, if it exists.

$$\lim_{x\to 16}\frac{4-\sqrt{x}}{16x-x^2}$$

We have the following.

$$\lim_{x\to 16}\frac{4-\sqrt{x}}{16x-x^2}=\lim_{x\to 16}\frac{(4-\sqrt{x})(4+\sqrt{x})}{x(16-x)(4+\sqrt{x})}=\lim_{x\to 16}\frac{16-x}{x(16-x)(4+\sqrt{x})}$$
$$=\lim_{x\to 16}\frac{1}{x(4+\sqrt{x})}=\frac{1}{16(4+\sqrt{16})}=\frac{1}{128}$$

Problem 30

Evaluate the limit, if it exists.

$$\lim_{x\to 2}\frac{x^2-4x+4}{x^4-3x^2-4}$$

We have the following.

$$\lim_{x\to 2}\frac{x^2-4x+4}{x^4-3x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x^2-4)(x^2+1)}=\lim_{x\to 2}\frac{(x-2)(x-2)}{(x-2)(x+2)(x^2+1)}$$
$$=\lim_{x\to 2}\frac{x-2}{(x+2)(x^2+1)}=\frac{2-2}{(2+2)(2^2+1)}=0$$

Problem 31

Evaluate the limit, if it exists.

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)$$

We have the following.

$$\begin{aligned}
\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right) &= \lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t\to 0}\frac{(1-\sqrt{1+t})(1+\sqrt{1+t})}{t\sqrt{1+t}(1+\sqrt{1+t})} = \\
&= \lim_{t\to 0}\frac{1-(1+t)}{t\sqrt{1+t}(1+\sqrt{1+t})} \\
&= \lim_{t\to 0}\frac{-t}{t\sqrt{1+t}(1+\sqrt{1+t})} = \\
&= -\lim_{t\to 0}\frac{1}{\sqrt{1+t}(1+\sqrt{1+t})} \\
&= \frac{1}{\sqrt{1+0}(1+\sqrt{1+0})} = \frac{1}{2}
\end{aligned}$$

Problem 32

Evaluate the limit, if it exists.

$$\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}$$

We have the following.

$$\begin{aligned}
\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4} &= \lim_{x\to -4}\frac{(\sqrt{x^2+9}-5)(\sqrt{x^2+9}+5)}{(x+4)(\sqrt{x^2+9}+5)} \\
&= \lim_{x\to -4}\frac{x^2+9-25}{(x+4)(\sqrt{x^2+9}+5)} = \\
&= \lim_{x\to -4}\frac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)} \\
&= \lim_{x\to -4}\frac{(x+4)(x-4)}{(x+4)(\sqrt{x^2+9}+5)} = \\
&= \lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5} = \frac{-4-4}{\sqrt{(-4)^2+9}+5} = -\frac{4}{5}
\end{aligned}$$

Problem 33

Evaluate the limit, if it exists.

$$\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

We have the following.

$$\begin{aligned}
\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \\
&= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \\
&= \lim_{h\to 0} [3x^2 + 3xh + h^2] = 3x^2 + 3x(0) + 0^2 = 3x^2
\end{aligned}$$

Problem 34

Evaluate the limit, if it exists.

$$\lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

We have the following for $x \neq 0$.

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} &= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{(x-(x+h))(x+x+h)}{hx^2(x+h)^2} = \\
&= \lim_{h\to 0} \frac{-h(2x+h)}{hx^2(x+h)^2} = -\lim_{h\to 0} \frac{2x+h}{x^2(x+h)^2} = \\
&= -\frac{2x+0}{x^2(x+0)^2} = -\frac{2}{x^3}
\end{aligned}$$

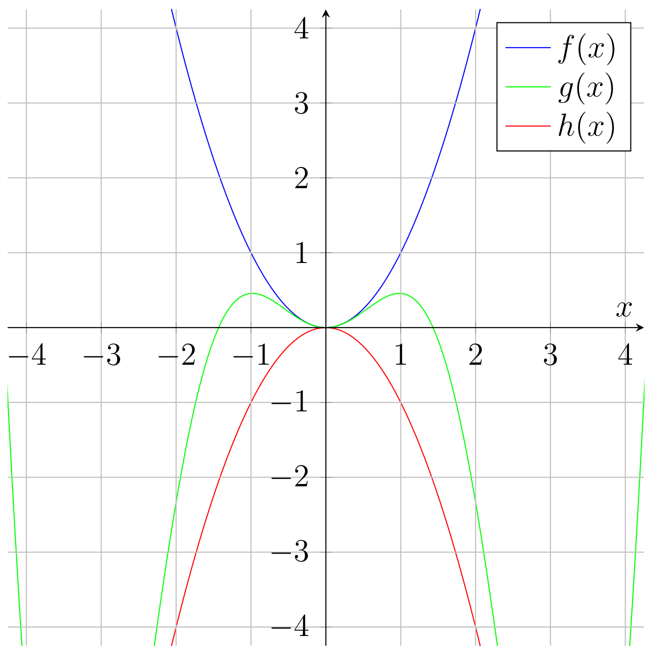
Problem 37

Use the Squeeze Theorem to show that

$$\lim_{x\to 0} x^2 \cos 20\pi x = 0$$

Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$ and $h(x) = x^2$ on the same screen For all x, $-1 \le \cos 20\pi x \le 1$ and $x^2 \ge 0$, meaning $-x^2 = x^2 \cdot (-1) \le x^2 \cos 20\pi x \le x^2 \cdot 1 = x^2$. Note that $\lim_{x\to 0}[-x^2] = 0 = \lim_{x\to 0} x^2$. By squeeze theore, we have $\lim_{x\to 0} x^2 \cos 20\pi x = 0$.

The graphs of $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$ and $h(x)$ on the same plane are given below.



Problem 38

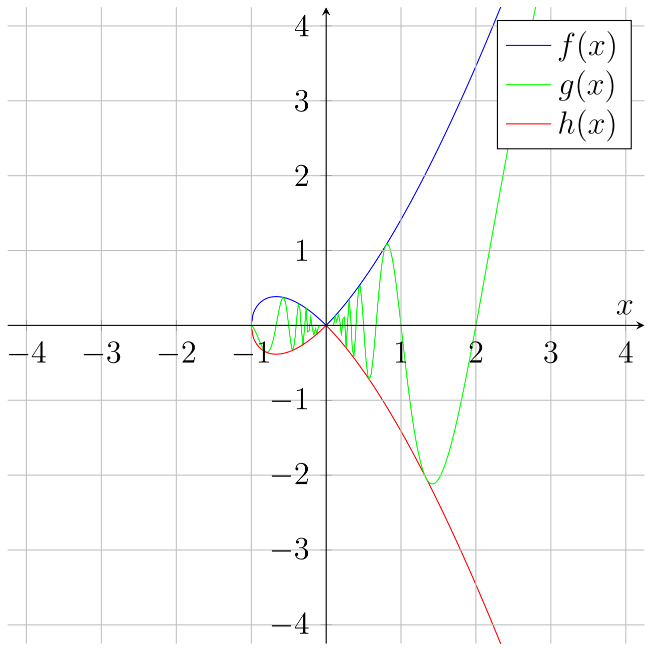
Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \sqrt{x^3 + x^2} \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g and h (in the notation of the Squeeze Theorem) on the same screen.

For all nonzero $-1 \le x \le 1$, $1 \le \sin\frac{\pi}{x} \le 1$ and $\sqrt{x^3+x^2} > 0$, meaning $-\sqrt{x^3+x^2} = \sqrt{x^3+x^2}\cdot(-1) \le \sqrt{x^3+x^2}\sin\frac{\pi}{x} \le \sqrt{x^3+x^2}\cdot 1 = \sqrt{x^3+x^2}$. Note that $\lim\limits_{x\to 0}[-\sqrt{x^3+x^2}] = 0 = \lim\limits_{x\to 0}\sqrt{x^3+x^2}$. By squeeze theorem, we have $\lim\limits_{x\to 0}\sqrt{x^3+x^2}\sin\frac{\pi}{x} = 0$.

In this case, we have chosen $f(x) = -\sqrt{x^3+x^2}$, $g(x) = \sqrt{x^3+x^2}\sin\frac{\pi}{x}$ and $h(x) = \sqrt{x^3+x^2}$. Their graphs on the same plane are given below.



Problem 39

If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$, find $\lim\limits_{x\to 4} f(x)$.

Note that $\lim\limits_{x\to 4}[4x-9] = 4\cdot 4 - 9 = 7 = 4^2 - 4\cdot 4 + 7 = \lim\limits_{x\to 4}[x^2-4x+7]$. By squeeze theorem, we have $\lim\limits_{x\to 4} f(x) = 7$.

Problem 40

If $2x \le g(x) \le x^4 - x^2 + 2$ for all x, evaluate $\lim\limits_{x\to 1} g(x)$.

Note that $\lim\limits_{x\to 1} 2x = 2 = \lim\limits_{x\to 1}[x^4 - x^2 + 2]$. By squeeze theorem, we have $\lim\limits_{x\to 1} g(x) = 2$.

Problem 41

Prove that $\lim\limits_{x\to 0} x^4 \cos\frac{2}{x} = 0$

For all $x \neq 0$, $-1 \leq \cos\left(\frac{2}{x}\right) \leq 1$ and $x^4 \geq 0$, meaning $-x^4 = x^4 \cdot (-1) \leq x^4 \cos\frac{2}{x} \leq x^4 \cdot 1 = x^4$. Note that $\lim\limits_{x\to 0}[-x^4] = -0^4 = 0 = 0^4 = \lim\limits_{x\to 0} x^4$. By squeeze theorem, we have $\lim\limits_{x\to 0} x^4 \cos\frac{2}{x} = 0$.

Problem 42

Prove that $\lim\limits_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$.

For all x, $-1 \leq \sin(\pi/x) \leq 1$, meaning $\frac{1}{e} = e^{-1} \leq e^{\sin(\pi/x)} \leq e^1 = e$ (since the exponential function base e is an increasing function) and $\frac{\sqrt{x}}{e} \leq \sqrt{x}e^{\sin(\pi/x)} \leq e\sqrt{x}$. Note that $\lim\limits_{x\to 0^+} \frac{\sqrt{x}}{e} = 0 = \lim\limits_{x\to 0^+} e\sqrt{x}$. By squeeze theorem, we have $\lim\limits_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$.

Problem 43

Find the limit, if it exists. If the limit does not exist, explain why.

$$\lim_{x\to -4}(|x+4| - 2x)$$

We begin by considering the one-sided limits.

$$\begin{aligned}
\lim_{x\to -4^-}(|x+4| - 2x) &= \lim_{x\to -4^-}(-(x+4) - 2x) = \lim_{x\to -4^-}(-x - 4 - 2x) = \\
&= \lim_{x\to -4^-}(-3x - 4) = -3(-4) - 4 = 8 \\
\lim_{x\to -4^+}(|x+4| - 2x) &= \lim_{x\to -4^+}(x + 4 - 2x) = \lim_{x\to -4^+}(4 - x) = 4 - (-4) = 8
\end{aligned}$$

The one-sided limits exist and are equal to 8, i.e., the limit $\lim_{x\to -4}(|x+4|-2x)$ indeed exists, namely, it equals 8.

Problem 44

Find the limit, if it exists. If the limit does not exist, explain why.

$$\lim_{x\to -4}\frac{|x+4|}{2x+8}$$

We begin by considering the one-sided limits.

$$\lim_{x\to -4^+}\frac{|x+4|}{2x+8}=\lim_{x\to -4^+}\frac{x+4}{2(x+4)}=\lim_{x\to -4^+}\frac{1}{2}=\frac{1}{2}$$
$$\lim_{x\to -4^-}\frac{|x+4|}{2x+8}=\lim_{x\to -4^-}\frac{-(x+4)}{2(x+4)}=\lim_{x\to -4^-}\left(-\frac{1}{2}\right)=-\frac{1}{2}$$

But $\frac{1}{2}\neq -\frac{1}{2}$, meaning the one-sided limits are not equal and hence the limit $\lim_{x\to -4}\frac{|x+4|}{2x+8}$ does not exist.

Problem 45

Find the limit, if it exists. If the limit does not exist, explain why.

$$\lim_{x\to 0.5^-}\frac{2x-1}{|2x^3-x^2|}$$

We have the following.

$$\begin{aligned}\lim_{x\to 0.5^-}\frac{2x-1}{|2x^3-x^2|}&=\lim_{x\to 0.5^-}\frac{2(x-0.5)}{|2x^2(x-0.5)|}=\lim_{x\to 0.5^-}\frac{x-0.5}{|2||x^2||x-0.5|}=\\&=\lim_{x\to 0.5^-}\frac{x-0.5}{2\cdot x^2\cdot(-(x-0.5))}=\\&=-\lim_{x\to 0.5^-}\frac{1}{2x^2}=-\frac{1}{2(0.5)^2}=-2\end{aligned}$$

Problem 46

Find the limit, if it exists. If the limit does not exist, explain why.

$$\lim_{x\to -2} \frac{2-|x|}{2+x}$$

We have the following.

$$\lim_{x\to -2} \frac{2-|x|}{2+x} = \lim_{x\to -2} \frac{2-(-x)}{2+x} = \lim_{x\to -2} 1 = 1$$

Problem 47

Find the limit, if it exists. If the limit does not exist, explain why.

$$\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right)$$

We have the following.

$$\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{-x}\right) = \lim_{x\to 0^-} \left(\frac{1}{x} + \frac{1}{x}\right) = 2 \lim_{x\to 0^-} \frac{1}{x}$$

The latter does not exist as the limit goes to negative infinity.

Problem 48

Find the limit, if it exists. If it does not exist, explain why.

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$$

We have the following.

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right) = \lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x\to 0^+} 0 = 0$$

Problem 49

The Signum Function The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.

(b) Find each of the following limits or explain why it does not exist.

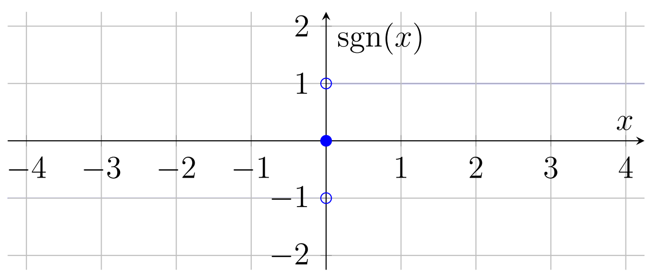
(i) $\lim_{x\to 0^+} \operatorname{sgn} x$

(ii) $\lim_{x\to 0^-} \operatorname{sgn} x$

(iii) $\lim_{x\to 0} \operatorname{sgn} x$

(iv) $\lim_{x\to 0} |\operatorname{sgn} x|$

For part (a), the graph of sgn is given below.



For part (i) of (b), $\lim_{x\to 0^+} \operatorname{sgn} x = \lim_{x\to 0^+} 1 = 1$.

For part (ii) of (b), $\lim_{x\to 0^-} \operatorname{sgn} x = \lim_{x\to 0^-} [-1] = -1$.

For part (iii) of (b), the limit $\lim_{x\to 0} \operatorname{sgn} x$ does not exist as its one-sided limits are not equal ($1 \neq -1$).

For part (iv) of (b), $|\operatorname{sgn} x| = 1$ for all nonzero x, meaning $\lim_{x\to 0} |\operatorname{sgn} x| = \lim_{x\to 0} 1 = 1$.

Problem 50

Let $g(x) = \operatorname{sgn}(\sin x)$.

(a) Find each of the following limit or explain why it does not exist.

(i) $\lim\limits_{x\to 0^+} g(x)$	(iii) $\lim\limits_{x\to 0} g(x)$	(v) $\lim\limits_{x\to \pi^-} g(x)$
(ii) $\lim\limits_{x\to 0^-} g(x)$	(iv) $\lim\limits_{x\to \pi^+} g(x)$	(vi) $\lim\limits_{x\to \pi} g(x)$

(b) For which values of a does $\lim\limits_{x\to a} g(x)$ not exist?

(c) Sketch the graph of g.

For part (i) of (a), $\sin x$ is positive for values of x sufficiently close to 0 on the right (namely, for $0 < x < \pi$), meaning $\operatorname{sgn}(\sin x) = 1$ for the same values of x and thus $\lim\limits_{x\to 0^+} g(x) = \lim\limits_{x\to 0^+} \operatorname{sgn}(\sin x) = \lim\limits_{x\to 0^+} 1 = 1$.

For part (ii) of (a), $\sin x$ is negative for values of x sufficiently close to 0 on the left (namely, for $-\pi < x < 0$), meaning $\operatorname{sgn}(\sin x) = -1$ for the same values of x and thus $\lim\limits_{x\to 0^-} g(x) = \lim\limits_{x\to 0^-} \operatorname{sgn}(\sin x) = \lim\limits_{x\to 0^-} [-1] = -1$.

For part (iii) of (a), the limit $\lim\limits_{x\to 0} g(x)$ does not exist as its one-sided limits are not equal ($1 \leq -1$).

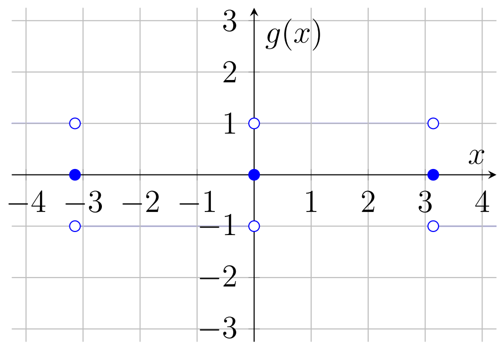
For part (iv) of (a), $\sin x$ is negative for values of x sufficiently close to π on the right (namely, for $\pi < x < 2\pi$), meaning $\operatorname{sgn}(\sin x) = -1$ for the same values of x and thus $\lim\limits_{x\to \pi^+} g(x) = \lim\limits_{x\to \pi^+} \operatorname{sgn}(\sin x) = \lim\limits_{x\to \pi^+} [-1] = -1$.

For part (v) of (a), $\sin x$ is positive for values of x sufficiently close to π on the left (namely, for $0 < x < \pi$), meaning $\operatorname{sgn}(\sin x) = 1$ for the same values of x and thus $\lim\limits_{x\to \pi^-} g(x) = \lim\limits_{x\to \pi^-} \operatorname{sgn}(\sin x) = \lim\limits_{x\to \pi^-} 1 = 1$.

For part (vi) of (a), the limit $\lim\limits_{x\to \pi} g(x)$ does not exist as its one-sided limits are not equal ($1 \neq -1$).

For part (b), the limit $\lim\limits_{x\to a} g(x)$ does not exist for values of a such that $\sin a = 0$, i.e., for all a which are integer multiples of π since for sufficiently close values of x on the right (namely, $a < x < a+\pi$) $\sin x$ has a different sign than for sufficiently close values of x on the left (namely, $a - \pi x < a$), meaning the one-sided limits equal 1 and -1. Since $1 \neq -1$, the limit $\lim\limits_{x\to a} g(x)$ indeed does not exist. For other values of a, $\sin a$ is either positive or negative. In either case, due to continuity of sin, for sufficiently close values x to a, $\sin x$ has the same sign as $\sin a$, meaning $\lim\limits_{x\to a} g(x)$ exists in that case.

For part (c), the graph of g is given below.



Problem 51

Let $g(x) = \frac{x^2+x-6}{|x-2|}$

(a) Find

(i) $\lim\limits_{x \to 2^+} g(x)$ (ii) $\lim\limits_{x \to 2^-} g(x)$

(b) Does $\lim\limits_{x \to 2} g(x)$ exist?

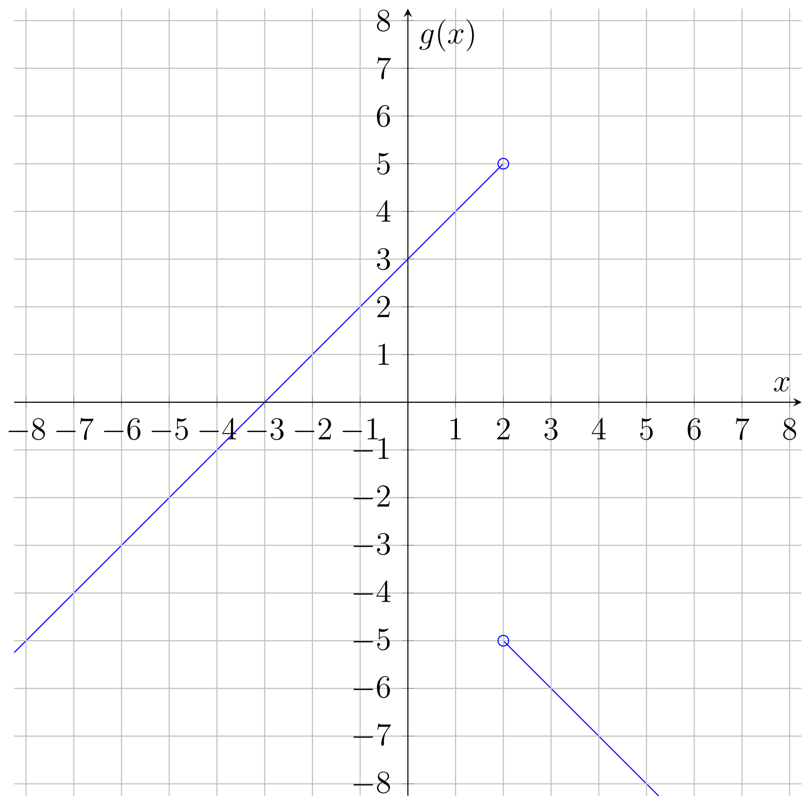
(c) Sketch the graph of g.

For part (i) of (a), $\lim\limits_{x \to 2^+} g(x) = \lim\limits_{x \to 2^+} \frac{x^2+x-6}{|x-2|} = \lim\limits_{x \to 2^+} |(x-2)(x+3)|x - 2 = \lim\limits_{x \to 2^+} [x+3] = 2+3 = 5.$

For part (ii) of (a), $\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^-} \frac{x^2+x-6}{|x-2|} = \lim\limits_{x \to 2^-} \frac{(x-2)(x+3)}{-(x-2)} = -\lim\limits_{x \to 2^-} [x+3] = -(2+3) = -5.$

For part (b), the limit $\lim\limits_{x \to 2} g(x)$ does not exist as its one-sided limits are not equal $(5 \neq -5)$.

For part (c), as implicitly observed in part (a), $g(x) = x+3$ for $x < 2$ and $g(x) = -(x+3)$ for $x > 2$. Obviously, $g(2)$ is undefined. A sketched graph of g is thus given below.



Problem 52

Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x-2)^2 & \text{if } x \geq 1 \end{cases}$$

(a) Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.

(b) Does $\lim_{x \to 1} f(x)$ exist.

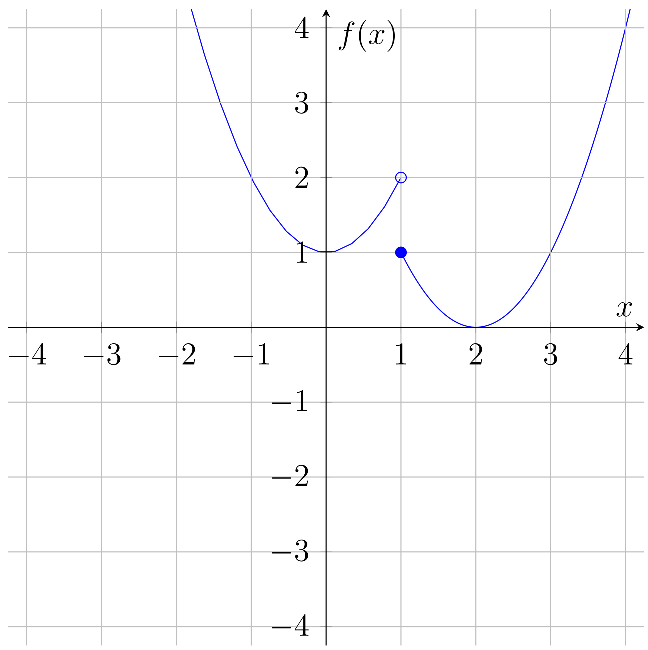
(c) Sketch the graph of f.

For part (a), we evaluate the one-sided limits as follows.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} [x^2+1] = 1^2+1 = 2$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-2)^2 = (1-2)^2 = 1$$

For part (b), the limit $\lim_{x\to 1} f(x)$ does not exist as its one-sided limits exist and are not equal $(2 \neq 1)$.

For part (c), a sketched graph of f is given below.



Problem 53

Let

$$B(t) = \begin{cases} 4 - \frac{1}{2}t & \text{if } x < 2 \\ \sqrt{t+c} & \text{if } t \geq 2 \end{cases}$$

Find the value of c so that $\lim_{t\to 2} B(t)$ exists.

For the limit $\lim_{t \to 2} B(t)$ to exist, we need its one-sided limits to exist and be equal. We first examine the existences. $\lim_{t \to 2^-} B(t) = \lim_{t \to 2^-} \left[4 - \frac{1}{2}t\right] = 4 - \frac{1}{2}(2) = 3$ and $\lim_{t \to 2^+} B(t) = \lim_{t \to 2^+} \sqrt{t+c} = \sqrt{2+c}$. Now, for the one sided limits to be equal, we need $3 = \sqrt{2+c}$, which implies $c = 7$.

We now verify that $\lim_{t \to 2} B(t)$ indeed exists given $c = 7$. Note that $\lim_{t \to 2^-} B(t) = \lim_{t \to 2^-} B(t) = 4 - \frac{1}{2}t = 4 - \frac{1}{2}(2) = 3 = \sqrt{2+7} = \sqrt{2+c} = \lim_{x \to 2^+} \sqrt{x+c} = \lim_{x \to 2^+} B(t)$. Since the one sided limits exist and are equal, the limit $\lim_{t \to 2} B(t)$ exists, meaning $c = 7$ is indeed a solution to the problem.

Problem 54

Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if} 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.

(i) $\lim_{x \to 1^-} g(x)$ (iii) $g(x)$ (v) $\lim_{x \to 2^+} g(x)$

(ii) $\lim_{x \to 1} g(x)$ (iv) $\lim_{x \to 2^-} g(x)$ (vi) $\lim_{x \to 2} g(x)$

(b) Sketch the graph of g.

For part (i) of (a), we have $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x = 1$.

For part (ii) of (a), we have $\lim_{x \to 1^-} g(x) = 1 = 2 - 1^2 = \lim_{x \to 1^+} [2 - x^2] = \lim_{x \to 1^+} g(x)$, meaning $\lim_{x \to 1} g(x) = 1$.

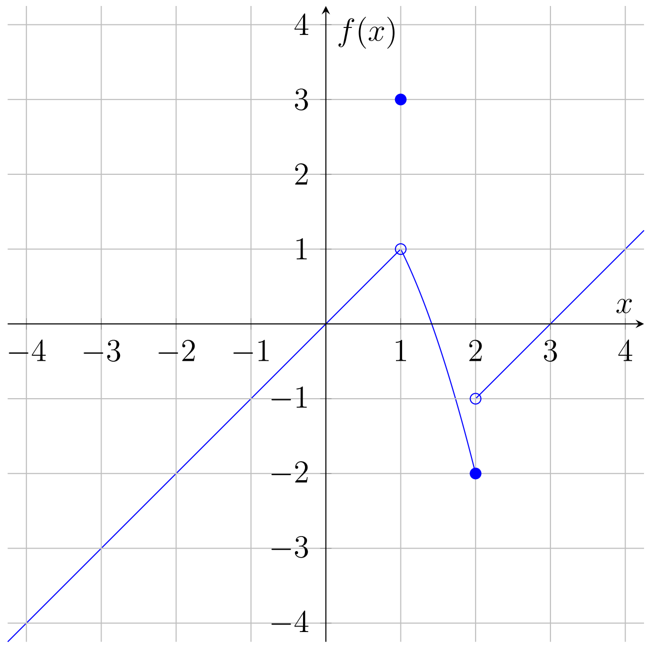
For part (iii) of (a), we have $g(1) = 3$.

For part (iv) of (a), we have $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} [2 - x^2] = 2 - 2^2 = -2$.

For part (v), of (a), we have $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} [x - 3] = 2 - 3 = -1$.

For part (vi) of (a), the limit $\lim_{x \to 2} g(x)$ does not exist as its one-sided limits are not equal ($-2 \neq -1$).

For part (b), the graph of g is given below.



Problem 55

(a) If the symbol $[\![\ ]\!]$ denotes the greatest integer function defined in Example 10, evaluate

(i) $\lim_{x\to -2^+} [\![x]\!]$ (ii) $\lim_{x\to -2} [\![x]\!]$ (iii) $\lim_{x\to -2.4} [\![x]\!]$

(b) If n is an integer, evaluate

(i) $\lim_{x\to n^-} [\![x]\!]$ (ii) $\lim_{x\to n^+} [\![x]\!]$

(c) For what values a does $\lim_{x\to a} [\![x]\!]$ exist?

For part (i) of (a), $[\![x]\!] = -2$ for values of x sufficiently close to -2 on the right (namely, $-2 < x < -1$), meaning $\lim_{x\to -2^+} [\![x]\!] = \lim_{x\to -2^+} [-2] = -2$.

For part (ii) of (a), $[\![x]\!] = -3$ for values of sufficiently close to -2 on the left (namely, $-3 < x < -2$), meaning $\lim_{x\to -2^-} [\![x]\!] = \lim_{x\to -2^-} [-3] = -3$. Thus the limit $\lim_{x\to -2} [\![x]\!]$ does not exist as its one-sided limits are not equal ($-2 \neq -3$).

For part (iii) of (a), $[\![x]\!] = -3$ for values of x sufficiently close to -2.4 (namely, $-2.8 < x < -2$), meaning $\lim_{x\to -2.4} [\![x]\!] = \lim_{x\to -2.4} [-3] = -3$.

For part (i) of (b), $[\![x]\!] = n - 1$ for values of x sufficiently close to n on the left (namely, $n - 1 < x < n$), meaning $\lim_{x\to n^-} [\![x]\!] = \lim_{x\to n^-} [n-1] = n - 1$.

For part (ii) of (b), $[\![x]\!] = n$ for values of x sufficiently close to n on the right (namely, $n < x < n+1$), meaning $\lim_{x\to n^+} [\![x]\!] = \lim_{x\to n^+} n = n$.

For part (c), we have shown in part (b) that a must not be an integer for $\lim_{x\to a} [\![x]\!]$ to exist. We now show that $\lim_{x\to a} [\![x]\!]$ exists for all non-integer values of a. $[\![x]\!] = [\![a]\!]$ for values of x sufficiently close to a (namely, $[\![a]\!] < x < [\![a]\!] + 1$), meaning $\lim_{x\to a} [\![x]\!] = \lim_{x\to a} [\![a]\!] = [\![a]\!]$.

Problem 56

Let $f(x) = [\![\cos x]\!]$, $-\pi \le x \le \pi$.

(a) Sketch the graph of f.

(b) Evaluate each limit, if it exists.

(i) $\lim_{x\to 0} f(x)$

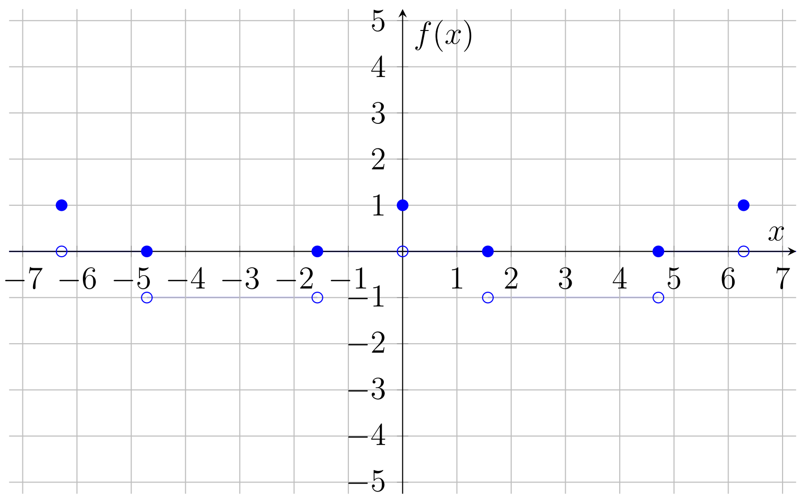
(ii) $\lim_{x\to(\pi/2)^-} f(x)$

(iii) $\lim_{x\to(\pi/2)^+} f(x)$

(iv) $\lim_{x\to\pi/2} f(x)$

(c) For what values of a does $\lim_{x\to a} f(x)$ exist?

For part (a), the graph of f is given below.



For part (i) of (b), $0 < \cos x < 1$ for values of x sufficiently close to 0 (namely, $-\frac{\pi}{2} < x < \frac{\pi}{2}$), meaning $[\![\cos x]\!] = 0$ for the same values of x and thus $\lim_{x\to 0} f(x) = \lim_{x\to 0}[\![\cos x]\!] = \lim_{x\to 0} 0 = 0$.

For part (ii) of (b), $0 < \cos x < 1$ for values of x sufficiently close to $\frac{\pi}{2}$ on the left (namely, $0 < x < \frac{\pi}{2}$), meaning $[\![\cos x]\!] = 0$ for the same values of x and thus $\lim_{x\to(\pi/2)^-} f(x) = \lim_{x\to(\pi/2)^-} [\![\cos x]\!] = \lim_{x\to(\pi/2)^-} 0 = 0$.

For part (iii) of (b), $-1 < \cos x < 0$ for values of x sufficiently close to $\frac{\pi}{2}$ on the right (namely, $\frac{\pi}{2} < x < \pi$), meaning $[\![\cos x]\!] = -1$ for the same values of x and thus $\lim_{x\to(\pi/2)^+} f(x) = \lim_{x\to(\pi/2)^+} [\![\cos x]\!] = \lim_{x\to(\pi/2)^+} [-1] = -1$.

For part (iv) of (b), the limit $\lim_{x\to\pi/2} f(x)$ does not exist as its one-sided limits are not equal ($0 \neq -1$).

For part (c), we claim that the limit $\lim_{x\to a} f(x)$ exists if and only if $\cos a \neq 0$. If $\cos a = 0$, then either $0 < \cos x < 1$ for values of x sufficiently close to a on the left (resp. right) and $-1 < \cos x < 0$ for values of x sufficiently close to a on the right (resp. left), meaning the one-sided limits are equal to 0 and -1. But $0 \neq -1$, meaning the limit $\lim_{x\to a} f(x) = \lim_{x\to a}[\![\cos x]\!]$ does not exist. If $\cos a \neq 0$, then either $0 < \cos a < 1$, $-1 \leq \cos a < 0$ or $\cos a = 1$. In the first two cases, due to continuity of cos, $[\![\cos x]\!] = [\![\cos a]\!]$ for values of x sufficiently close to a, meaning $\lim_{x\to a} f(x) = \lim_{x\to a}[\![\cos x]\!] = \lim_{x\to a}[\![\cos a]\!] = [\![\cos a]\!]$. In the third case, $[\![\cos x]\!] = 0$ for values of x sufficiently close to a (namely, $a - \frac{\pi}{2} < a < a + \frac{\pi}{2}$), meaning $\lim_{x\to a} f(x) = \lim_{x\to a}[\![\cos x]\!] = \lim_{x\to a} 0 = 0$.

Problem 57

If $f(x) = \llbracket x \rrbracket + \llbracket -x \rrbracket$, show that $\lim\limits_{x\to 2} f(x)$ exists but is not equal to $f(2)$.

We begin by considering the one-sided limits.

$\llbracket x \rrbracket = 2$ for values of x sufficiently close to 2 on the right (namely, $2 < x < 3$). For the same values of x, we have $\llbracket -x \rrbracket = -3$ (as $-3 < x < -2$). Hence we have $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (\llbracket x \rrbracket + \llbracket -x \rrbracket) = \lim\limits_{x\to 2^+} \llbracket x \rrbracket + \lim\limits_{x\to 2^+} \llbracket -x \rrbracket = \lim\limits_{x\to 2^+} 2 + \lim\limits_{x\to 2^+} [-3] = 2 + (-3) = -1$.

$\llbracket x \rrbracket = 1$ for values of x sufficiently close to 2 on the left (namely, $1 < x < 2$). For the same values of x, we have $\llbracket -x \rrbracket = -2$ (as $-2 < x < -1$). Hence we have $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} (\llbracket x \rrbracket + \llbracket -x \rrbracket) = \lim\limits_{x\to 2^-} \llbracket x \rrbracket + \lim\limits_{x\to 2^-} \llbracket -x \rrbracket = \lim\limits_{x\to 2^-} 1 + \lim\limits_{x\to 2^-} [-2] = 1 + (-2) = -1$.

The one-sided limits exist and are equal to -1, hence $\lim\limits_{x\to 2} f(x) = -1$.

$f(2) = \llbracket 2 \rrbracket + \llbracket -2 \rrbracket = 2 + (-2) = 0$ and $-1 \neq 0$. Thus $\lim\limits_{x\to 2} f(x)$ indeed exists and is not equal to $f(2)$.

Problem 58

In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim\limits_{v\to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

We have $\lim\limits_{v\to c^-} L = \lim\limits_{v\to c^-} L_0\sqrt{1 - \frac{v^2}{c^2}} = L_0\sqrt{1 - \lim\limits_{v\to c^-} \frac{v^2}{c^2}} = L_0\sqrt{1 - \frac{v^2}{v^2}} = L_0\sqrt{1-1} = 0$. This means that the faster the object moves (i.e., the closer is its speed to the speed of light), the shorter it appears to the observer. Here a left-hand limit is necessary as we require $1 - \frac{v^2}{c^2} \geq 0$ for its square root to exist, i.e., we require $v \leq c$ rather than $v \geq c$.

Problem 59

If p is a polynomial, show that $\lim_{x\to a} p(x) = p(a)$.

If p is the zero polynomial, then $\lim_{x\to a} p(x) = \lim_{x\to a} 0 = 0 = p(a)$. If p is a nonzero polynomial with degree n, then let $b_0, \ldots, b_n$ be the real numbers such that $p(x) = \sum_{i=0}^{n} b_i x^i$. Due to linearity of the limit operation (Limit Laws 1 and 3) and Limit Law 10, we have the following.

$$\lim_{x\to a} p(x) = \lim_{x\to a} \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} b_i \lim_{x\to a} x^i = \sum_{i=0}^{n} b_i a^i = p(a)$$

Problem 60

If r is a rational function, use Exercise 59 to show that $\lim_{x\to a} r(x) = r(a)$ for every number a in the domain of r.

Since r is a rational function, there exist polynomials p and q such that $r = p/q$. If a is in the domain of r, then $q(a) \neq 0$. Also, $\lim_{x\to a} q(x) = q(a)$ as shown in Exercise 59, meaning we can apply Limit Law 5. We have the following.

$$\lim_{x\to a} r(x) = \lim_{x\to a} \frac{p(x)}{q(x)} = \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} = \frac{p(a)}{q(a)} = r(a)$$

Problem 61

If $\lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$, find $\lim_{x\to 1} f(x)$.

Clearly, a limit of a fraction whose denominator approaches 0 exists if and only if the numerator also approaches 0. Since $\lim_{x\to 1} \frac{f(x)-8}{x-1}$ exists and

$x-1 \to 0$ as $x \to 1$, we therefore have $\lim_{x\to 1}[f(x)-8]=0$ and, by limit laws, $\lim_{x\to 1} f(x)=8$.

Alternatively (and more elegantly), one may proceed with the following reasoning:

$$\begin{aligned}\lim_{x\to 1} f(x) = \lim_{x\to 1} f(x) - 8 + 8 = \lim_{x\to 1}[f(x)-8]+8 = \\ = \lim_{x\to 1}\left[\frac{f(x)-8}{x-1}\cdot(x-1)\right]+8 = \\ = \lim_{x\to 1}\frac{f(x)-8}{x-1}\cdot\lim_{x\to 1}(x-1)+8 = 10\cdot 0+8 = 0+8 = 8\end{aligned}$$

Problem 62

If $\lim_{x\to 0}\frac{f(x)}{x^2}=5$, find the following limits,

1. $\lim_{x\to 0} f(x)$
2. $\lim_{x\to 0}\frac{f(x)}{x}$

For part (a), clearly, a limit of a fraction whose denominator approaches 0 exists if and only if the numerator also approaches 0. Since $\lim_{x\to 0}\frac{f(x)}{x^2}$ exists and $x^2 \to 0$ as $x \to 0$, we therefore have $\lim_{x\to 0} f(x)=0$.

Alternatively (and more elegantly), one may proceed with the following reasoning:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\frac{f(x)}{x^2}\cdot x^2\right] = \lim_{x\to 0}\frac{f(x)}{x^2}\cdot\lim_{x\to 0} x^2 = 5\cdot 0 = 0$$

For part (b), similarly, we rewrite $\frac{f(x)}{x^2}$ as $\frac{\frac{f(x)}{x}}{x}$ and, in the same manner as before, arrive at the conclusion $\lim_{x\to 0}\frac{f(x)}{x}=0$.

Again, one may proceed with the alternative approach:

$$\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\left[\frac{f(x)}{x^2}\cdot x\right] = \lim_{x\to 0}\frac{f(x)}{x^2}\cdot\lim_{x\to 0} x = 5\cdot 0 = 0$$

Problem 63

If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x) = 0$.

Clearly, for all real x, we have $0 \le f(x) \le x^2$. Note that $\lim_{x\to 0} 0 = 0 = 0^2 = \lim_{x\to 0} x^2$. By squeeze theorem, $\lim_{x\to 0} f(x) = 0$.

Problem 64

Show by means of an example that $\lim_{x\to a}[f(x)+g(x)]$ may exist though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

Let $f(x) = \frac{1}{x} = -g(x)$ for all nonzero x and $a = 0$. Then neither $\lim_{x\to a} f(x)$ not $\lim_{x\to a} g(x)$ exist, however, $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a}[f(x)-f(x)] = \lim_{x\to a} 0 = 0$.

Problem 65

Show by means of an example that $\lim_{x\to a}[f(x)g(x)]$ may exist though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

Let $f(x) = \text{sgn}(x) = g(-x)$ and $a = 0$. Then neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exist, however, $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} 0 = 0$ as $f(x)g(x) = 0$ for all nonzero x.

Problem 66

Evaluate $\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$.

We have the following.

$$\begin{aligned}
\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} &= \lim_{x\to 2}\frac{(\sqrt{6-x}-2)(\sqrt{6-x}+2)(\sqrt{3-x}+1)}{(\sqrt{3-x}-1)(\sqrt{6-x}+2)(\sqrt{3-x}+1)} = \\
&= \lim_{x\to 2}\frac{(\sqrt{6-x}^2-2^2)(\sqrt{3-x}+1)}{(\sqrt{3-x}^2-1^2)(\sqrt{6-x}+2)} = \\
&= \lim_{x\to 2}\frac{(6-x-4)(\sqrt{3-x}+1)}{(3-x-1)(\sqrt{6-x}+2)} = \\
&= \lim_{x\to 2}\frac{(2-x)(\sqrt{3-x}+1)}{(2-x)(\sqrt{6-x}+2)} = \lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \\
&= \frac{\sqrt{3-2}+1}{\sqrt{6-2}+2} = \frac{1}{2}
\end{aligned}$$

Problem 67

Is there a number a such that

$$\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$$

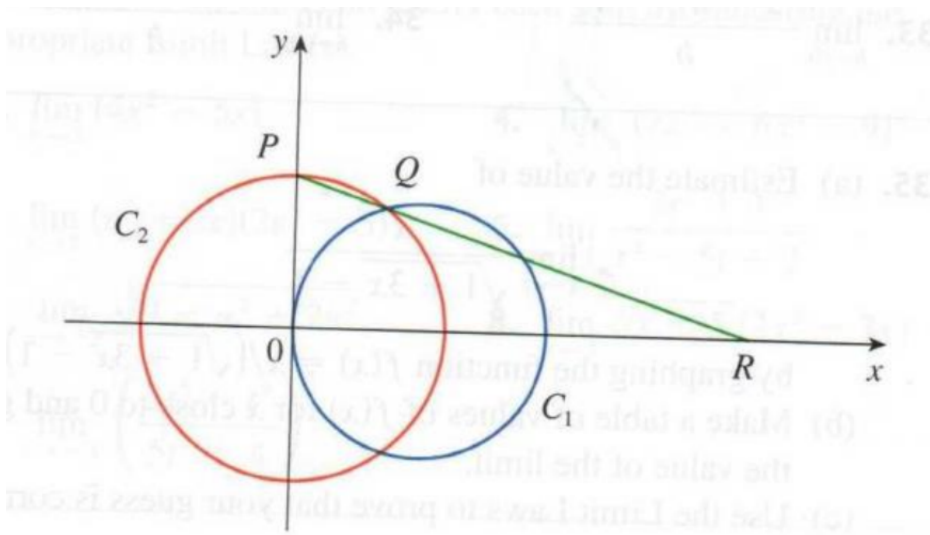
exists? If so, find the value of a and the value of the limit.

If $\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}$ exists, then, since $\lim_{x\to -2}[x^2+x-2]=0$, $\lim_{x\to -2}[3x^2+ax+a+3]=0$ (see part 1 of Exercise 62). The latter is equivalent to $3(-2)^2-2a+a+3=0$ (see Exercise 59) and $a=15$. We now verify that the limit exists for $a=15$.

$$\begin{aligned}
\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2} &= \lim_{x\to -2}\frac{3x^2+15a+18}{x^2+x-2} = \lim_{x\to -2}\frac{3(x+2)(x+3)}{(x+2)(x-1)} = \\
&= 3\lim_{x\to -2}\frac{x+3}{x-1} = 3\frac{-2+3}{-2-1} = -1
\end{aligned}$$

Problem 68

The figure shows a fixed circle C_1 with equation $(x-1)^2+y^2=1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0,r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



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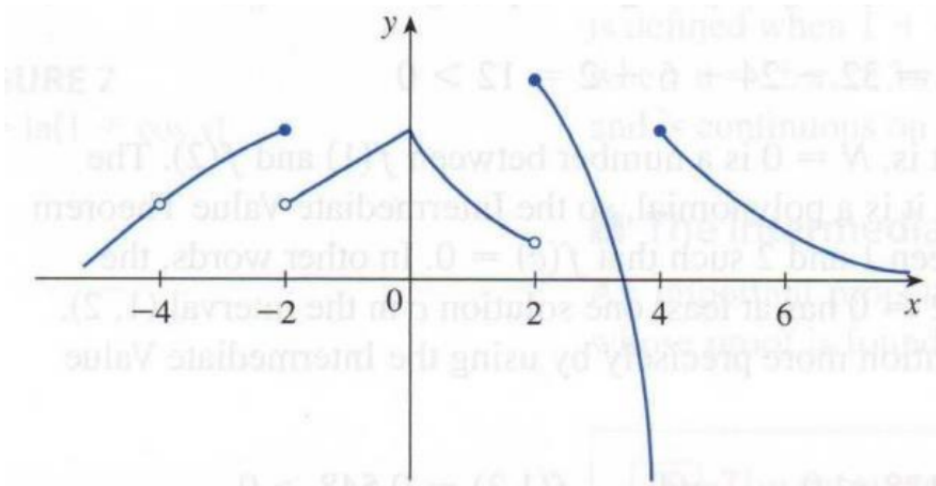
1.4 Section 2.5

Problem 4

From the given graph of g, state the numbers at which g is discontinuous and explain why.

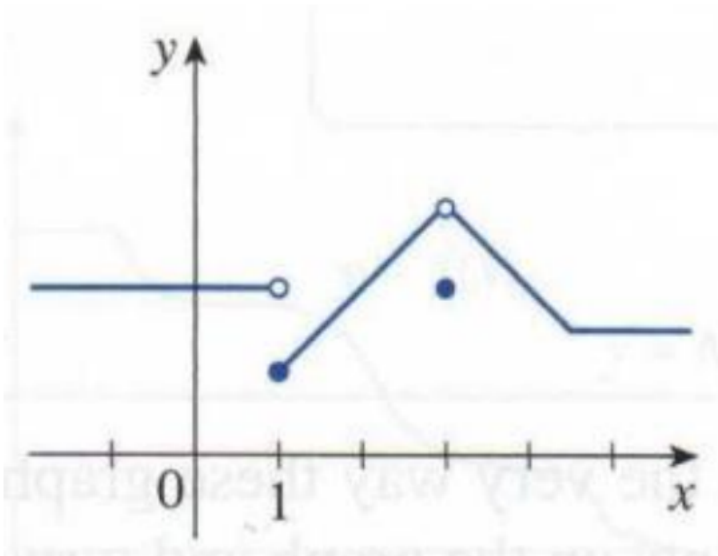


The function g is discontinuous on the points -4, -2 and 2 as it is not defined there. It is also discontinuous at $x = -2$, $x = 2$, $x = 4$ since its limits as there do not exist (the one-sided limits are not equal in each case). g is continuous everywhere else according to the graph.

Problem 5

The graph of a function f is given.

1. At what numbers a does $\lim_{x \to a} f(x)$ not exist?
2. At what numbers a is f not continuous?
3. At what numbers a does $\lim_{x \to a} f(x)$ exist but f is not continuous at a?



For part (a), the limit $\lim_{x \to a} f(x)$ does not exist when $a = 1$ (as its one-sided limits are not equal). The limit exists for all other values of a.

For part (b), the function f is not continuous at a when $a = 1$ (as the limit $\lim_{x \to a} f(x)$ does not exist) and also at $a = 3$ (as the limit $\lim_{x \to 3} g(x)$ exists, but is not equal to $g(3)$)

For part (c), the limit $\lim_{x \to a} f(x)$ exists and f is not continuous at a when $a = 3$ as discussed in the previous part.

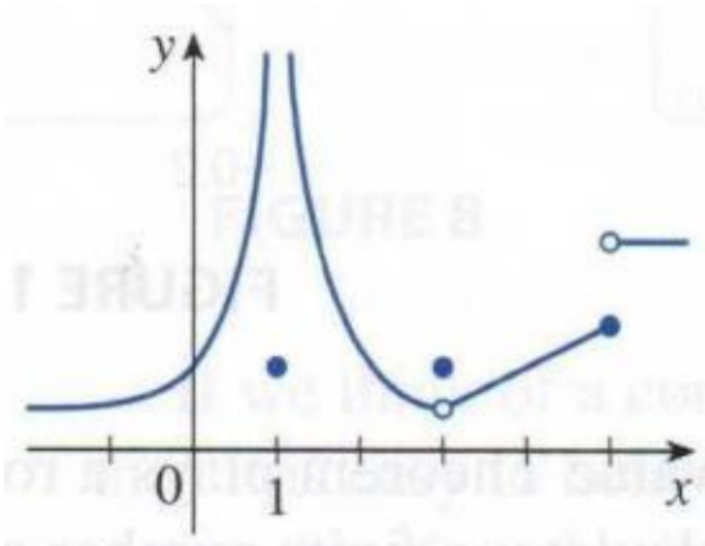
Problem 6

The graph of a function f is given.

(a) At what numbers a does $\lim_{x \to a} f(x)$ not exist?

(b) At what numbers a is f not continuous?

(c) At what numbers a does $\lim_{x \to a} f(x)$ exists but f is not continuous at a?



For part (a), the limit $\lim_{x\to a} f(x)$ does not exist when $a = 1$ (as the function diverges to positive infinity) and $a = 5$ (as its one-sided limits are not equal). The limit exists for all other values of a.

For part (b), the function f is not continuous at a when $a = 1$ and $a = 5$ (as the limits do not exist there) and also at $a = 3$ (as the limit $\lim_{x\to 3} g(x)$ exists, but is not equal to $g(3)$)

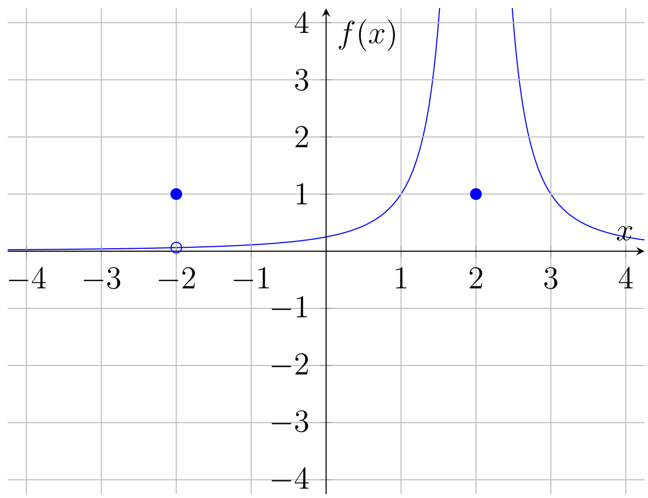
For part (c), the limit $\lim_{x\to a} f(x)$ exists and f is not continuous at a when $a = 3$ as discussed in the previous parts.

Problem 7

Sketch the graph of a function f that is defined on $\mathbb{R}$ and continuous except for the stated discontinuities.

Removable discontinuity at -2, infinity discontinuity at 2.

The graph of a function f satisfying the given conditions is given below.

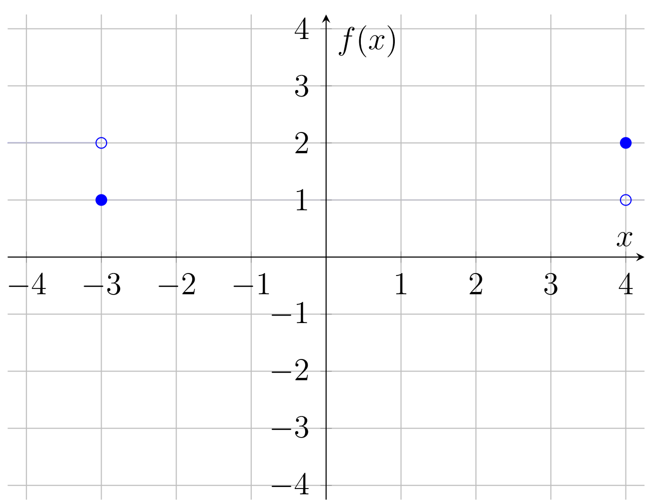


Problem 8

Sketch the graph of a function f that is defined on $\mathbb{R}$ and continuous except for the stated discontinuities.

Jump discontinuity at -3, removable discontinuity at 4.

The graph of a function f satisfying the given conditions is given below.

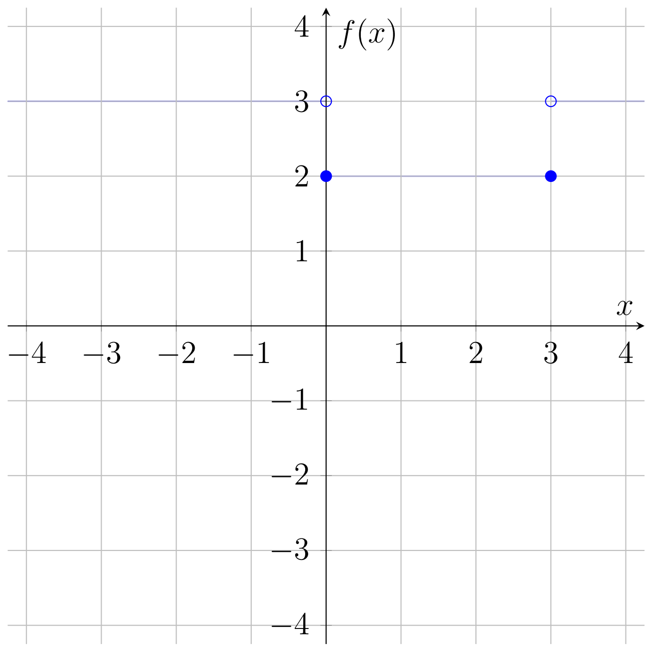


Problem 9

Sketch the graph of a function f that is defined on $\mathbb{R}$ and continuous except for the stated discontinuities.

Discontinuities at 0 and 3, but continuous from the right at 0 and from the left at 3.

The graph of a function f satisfying the given conditions is given below.

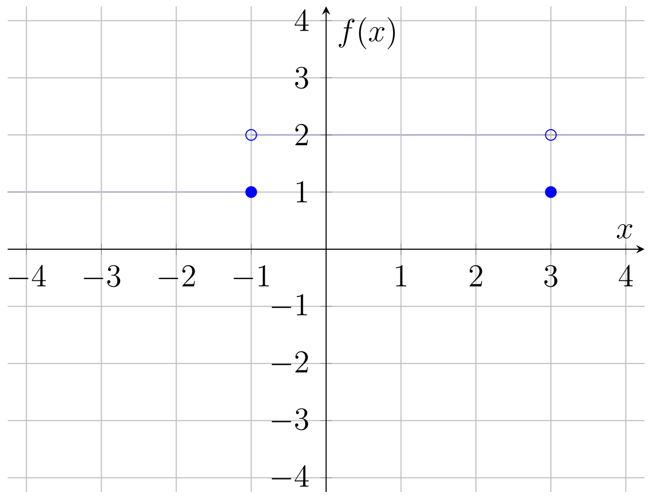


Problem 10

Sketch the graph of a function f that is defined on $\mathbb{R}$ and continuous excepy for the stated discontinuities.

Continuous only from the left at -1, not continuous from the left or right at 3.

The graph of a function f satisfying the given conditions is given below.



Problem 13

Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

$$f(x) = 3x^2 + (x+2)^5, \quad a = -1$$

We have the following.

$$\lim_{x \to a} f(x) = \lim_{x \to -1} \left[3x^2 + (x+2)^5\right] = 3 \lim_{x \to -1} x^2 + \left(\lim_{x \to -1} x + 2\right)^5 =$$
$$= 3(-1)^2 + (-1+2)^5 = f(-1) = f(a)$$

Problem 14

Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

$$g(t) = \frac{t^2 + 5t}{2t + 1}, \quad a = 2$$

We start by evaluating the limit of $g(t)$ as $t \to 2$.

$$\lim_{t\to 2} g(t) = \lim_{t\to 2} \frac{t^2+5t}{2t+1} = \frac{\lim\limits_{t\to 2}[t^2+5t]}{\lim\limits_{t\to 2}[2t+1]} =$$
$$= \frac{\lim\limits_{t\to 2} t^2 + 5\lim\limits_{t\to 2} t}{2\lim\limits_{t\to 2} t + \lim\limits_{t\to 2} 1} = \frac{2^2+5\cdot 2}{2\cdot 2+1} =$$
$$= \frac{4+10}{4+1} = \frac{14}{5}$$

Now, $g(2) = \frac{2^2+5\cdot 2}{2\cdot 2+1} = \frac{14}{5}$ as well. Hence $\lim\limits_{t\to a} g(t) = \lim\limits_{t\to 2} g(2) = \frac{14}{5} = g(2) = g(a)$, meaning g is indeed continuous at the given value of a.

Problem 15

Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

$$p(v) = 2\sqrt{3v^2+1}, \quad a = 1$$

We have the following.

$$\lim_{v\to a} p(v) = \lim_{v\to 1} 2\sqrt{3v^2+1} = 2\sqrt{\lim_{v\to 1}[3v^2+1]} = 2\sqrt{3(1)^2+1} = p(1) = p(a)$$

Problem 16

Use the definition of continuity and use the properties of limits to show that the function is continuous at the given number a.

$$f(r) = \sqrt[3]{4r^2-2r+7}, \quad a = -2$$

We have the following.

$$\lim_{r\to a} f(r) = \lim_{r\to -2} \sqrt[3]{4r^2-2r+7} = \sqrt[3]{4\lim_{r\to -2} r^2 - 2\lim_{r\to -2} r + \lim_{r\to -2}} =$$
$$= \sqrt[3]{4(-2)^2-2(-2)+7} = 3$$

Problem 17

Use the definition of continuity and use the properties of limits to show that the function is continuous on the given interval.

$$f(x) = x + \sqrt{x-4}, \quad [4, \infty)$$

Continuity at 4 here is assumed to mean right continuity at 4. We have $\lim_{x\to 4} f(x) = \lim_{x\to 4}[x + \sqrt{x-4}] = \lim_{x\to 4} x + \sqrt{\lim_{x\to 4}[x-4]} = 4 + \sqrt{4-4} = 0$

Let $a \in (4, \infty)$, we shall show that f is continuous at a. Since $a > 4$, $x-4$ approaches a positive value as $x \to a$ (as $\lim_{x\to a}[x-4] = a-4 > 4-4 = 0$), we hence may proceed as follows.

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}[x + \sqrt{x-4}] = \lim_{x\to a} x + \sqrt{\lim_{x\to a}[x-4]} = \\ &= a + \sqrt{a-4} = g(a)\end{aligned}$$

Problem 18

Use the definition of continuity and use the properties of limits to show that the function is continuous on the given interval.

$$g(x) = \frac{x-1}{3x+6}, \quad (-\infty, -2)$$

Let $a \in (-\infty, -2)$, we shall show that g is continuous at a. Since $a < 2$, $3x+6$ approaches a nonzero value as $x \to a$ (as $\lim_{x\to a}[3x+6] = 3a+6$ and $3a+6 < 3(-2)+6 = -6+6 = 0$), we hence may proceed as follows.

$$\begin{aligned}\lim_{x\to a} g(x) &= \lim_{x\to a} \frac{x-1}{3x+6} = \frac{\lim_{x\to a}[x-1]}{\lim_{x\to a}[3x+6]} = \\ &= \frac{\lim_{x\to a} x - \lim_{x\to a} 1}{3\lim_{x\to a} x + \lim_{x\to a} 6} = \frac{a-1}{3a+6} = g(a)\end{aligned}$$

Hence g is continuous on the open interval $(-\infty, -2)$.

Problem 19

Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

$$f(x) = \frac{1}{x+2} \quad a = -2$$

As $x \to -2^+$, $1 \to 1 > 0$ whereas $x + 2 \to 0^+$, meaning $\frac{1}{x+2} \to \infty$. Hence the function is not continuous at -2.

Problem 20

Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

$$f(x) = \begin{cases} \frac{1}{x+2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases} \quad a = -2$$

See the previous exercise.

Problem 21

Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

$$f(x) = \begin{cases} x+3 & \text{if } x \leq -1 \\ 2^x & \text{if } x > 1 \end{cases} \quad a = -1$$

We start by considering the one-sided limits.

$$\lim_{x \to a^+} f(x) = \lim_{x \to -1^+} 2^x = 2^{-1} = \frac{1}{2}$$
$$\lim_{x \to a^-} f(x) = \lim_{x \to -1^-} [x+3] = -1 + 3 = 2$$

Since $\frac{1}{2} \neq 2$, the one-sided limits are not equal, meaning the limit $\lim_{x \to a} f(x)$ does not exist and hence f is discontinuous at a.

Problem 22

Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

$$f(x) = \begin{cases} \frac{x^2-x}{x^2-1} & \text{if } x \neq 1 \\ 1 & \text{if } 1 \end{cases} \qquad a = 1$$

We have the following.

$$\begin{aligned} \lim_{x \to a} f(x) = \lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} &= \lim_{x \to 1} \frac{x(x-1)}{(x+1)(x-1)} = \\ &= \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{1+1} = \frac{1}{2} \neq 1 = f(1) = f(a) \end{aligned}$$

Problem 23

Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

$$f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases} \qquad a = 0$$

We begin by considering the one-sided limits.

$$\lim_{x \to a^+} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} [1 - x^2] = 1 - 0^2 = 1 \neq 0 = f(0) = f(a)$$

Right continuity at a does not hold, hence neither does continuity at a.

Problem 24

Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

$$f(x) = \begin{cases} \frac{2x^2-5x-3}{x-3} & \text{if } x \neq 33 \\ 6 & \text{if } x = 3 \end{cases} \qquad a = 3$$

We have the following.

$$\lim_{x\to a} f(x) = \lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{2x^2-5x-3}{x-3} = \lim_{x\to 3} \frac{(x-3)(2x+1)}{x-3} =$$
$$= \lim_{x\to 3}[2x+1] = 2(3)+1 = 7 \neq 6 = f(3) = f(a)$$

Problem 25

$$f(x) = \frac{x-3}{x^2-9}$$

(a) Show that f has a removable discontinuity at $x = 3$.

(b) Redefine $f(3)$ so that f is continuous at $x = 3$ (and thus the discontinuity is "removed")

We have the following.

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x-3}{x^2-9} = \lim_{x\to 3} \frac{x-3}{(x-3)(x+3)} = \lim_{x\to 3} \frac{1}{x+3} = \frac{1}{3+3} = \frac{1}{6}$$

But $f(3)$ is undefined, hence f has a removable discontinuity at $x = 3$.

In general, to "remove" a removable discontinuity of a real function g at $x = a$, $g(a)$ needs to be defined as $\lim_{x\to a} g(x)$. Hence, in this case, we need to define $f(3) = \lim_{x\to 3} f(x) = \frac{1}{6}$.

Problem 26

$$f(x) = \frac{x^2 - 7x + 12}{x - 3}$$

(a) Show that f has a removable discontinuity at $x = 3$.

(b) Redefine $f(3)$ so that f is continuous at $x = 3$ (and thus the discontinuity is "removed")

We have the following.

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2 - 7x + 12}{x - 3} = \lim_{x\to 3} \frac{(x-3)(x-4)}{x-3} = \lim_{x\to 3}[x-4] = 3-4 = -1$$

But $f(3)$ is undefined, hence f has a removable discontinuity at $x = 3$.

In general, to "remove" a removable discontinuity of a real function g at $x = a$, $g(a)$ needs to be defined as $\lim\limits_{x\to a} g(x)$. Hence, in this case, we need to define $f(3) = \lim\limits_{x\to 3} f(x) = -1$.

Problem 43

Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

$$f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \le x < 1 \\ 1/x & \text{if } x \ge 1 \end{cases}$$

$f(x) = x^2$ on the open interval $(-\infty, -1)$ and x^2, in general, is continuous, thus $f(x)$ is continuous for all $x < -1$.

$f(x)$ is discontinuous at $x = -1$ since $\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} x^2 = (-1)^2 = 1 \ne -1 = \lim_{x\to -1^+} x = \lim_{x\to -1^+} f(x)$. $f(x)$ is only right continuous at $x = -1$ as $f(-1) = (-1)^2 = 1 = \lim\limits_{x\to -1^-} f(x)$.

$f(x) = x$ on the open interval $(-1, 1)$ and x, in general, is continuous, thus $f(x)$ is continuous for all $-1 < x < 1$.

$f(x)$ is continuous at $x = 1$ as $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x = 1 = \frac{1}{1} = \lim\limits_{x\to 1^+} \frac{1}{x} = \lim\limits_{x\to 1^+} f(x)$.

$f(x) = \frac{1}{x}$ on the open interval $(1, \infty)$ and $\frac{1}{x}$, in general, is continuous for nonzero x, thus $f(x)$ is continuous for all $x > 1$.

Problem 48

Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$$

Let $f(x)$ be continuous everywhere, it then is continuous at $x = 2$ and $x = 3$, meaning the limits at those points exist. A limit exists if and only if its one-sided limits exist and are equal. We thus have

$$\begin{aligned} 4a - 2b + 3 = a(2)^2 - b(2) + 3 &= \lim_{x\to 2^+} [ax^2 - bx + 3] = \\ &= \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2^-} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^-} [x+2] = 2 + 2 = 4 \end{aligned}$$

Implying $4a - 2b = 1$.

We similarly have

$$\begin{aligned} 9a - 3b + 3 = a(3)^2 - b(3) + 3 &= \lim_{x\to 3^-} [ax^2 - bx + 3] = \\ &= \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} [2x - a + b] = \\ &= 2(3) - a + b = 6 - a + b \end{aligned}$$

Implying $10a - 4b = 3$.

Finally, we have

$$a = \frac{10a - 8a}{2} = \frac{10a - 4b - 8a + 4b}{2} = \frac{(10a - 4b) - 2(4a - 2b)}{2} = \frac{3 - 2(1)}{2} = \frac{1}{2}$$
$$b = \frac{2b}{2} = \frac{2b + 4a - 2b - 1}{2} = \frac{4a - 1}{2} = \frac{4\left(\frac{1}{2}\right) - 1}{2} = \frac{1}{2}$$

We now verify that the function f is indeed continuous with $a = b = \frac{1}{2}$. Clearly, on the open intervals $(-\infty, 2)$, $(2, 3)$ and $(3, \infty)$ the function is equal to continuous functions, so we examine continuity at $x = 2$ and $x = 3$ as follows.

$$\begin{aligned}\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} \frac{x^2 - 4}{x - 2} = 4 = \\ &= \frac{1}{2}(2)^2 - \frac{1}{2}(2) + 3 = \lim_{x\to 2^+}\left[\frac{1}{2}x^2 - \frac{1}{2}x + 3\right] = \lim_{x\to 2^+} f(x) \\ \Longrightarrow \lim_{x\to 2} f(x) &= 4 = \frac{1}{2}(2)^2 - \frac{1}{2}(2) + 3 = f(2)\end{aligned}$$

$$\begin{aligned}\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-}\left[\frac{1}{2}x^2 - \frac{1}{2}x + 3\right] = \frac{1}{2}(3)^2 - \frac{1}{2}(3) + 3 = \\ &= 6 = 2(3) - \frac{1}{2} + \frac{1}{2} = \lim_{x\to 3^+}\left[2x - \frac{1}{2} + \frac{1}{2}\right] = \lim_{x\to 3^+} f(x) \\ \Longrightarrow \lim_{x\to 3} f(x) &= 6 = \frac{1}{2}(3)^2 - \frac{1}{2}(3) + 3 = f(3)\end{aligned}$$

As desired.

Problem 55

Use the Intermediate Value Theorem to show that there is a solution of the given equation in the specified interval.

$$-x^3 + 4x + 1 = 0, \quad (-1, 0)$$

Let $f(x) = -x^3 + 4x + 1$. Clearly, f is continuous. We also have $f(-1) = -(-1)^3 + 4(-1) + 1 = -2$ and $f(0) = -0^3 + 4(0) + 1$, meaning $f(-1) = -2 < 0 < 1 = f(0)$. By intermediate value theorem, there exists $x \in (-1, 0)$ with $f(x) = 0$.

Problem 75

A woman leaves her house at 7:00 AM and takes her usual path to the top of a mountain, arriving at 7:00 PM. The following morning, she starts at 7:00 AM at the top and takes the same path back, arriving at her home at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the woman will cross at exactly the same time of day on both days.

Let $f(t)$ denote the distance between the woman and her home t hours after the (first) midnight of the first day and L be the length of the path. Clearly, f should be continuous. Consider the function $g(t) = f(24+t) - f(t)$ which should also be continuous. We have $g(7) = f(24+7) - f(7) = L - 0 = L$ and $g(12+7) = f(24+12+7) - f(12+7) = 0 - L$. Since L is defined to be positive (we assume that the mountain is not where the woman lives), we have $g(12+7) = -L < 0 < L = g(7)$. By intermediate value theorem, there exists $7 \leq t \leq 19$ with $g(t) = 0$, i.e., $f(24+t) - f(t) = 0$ and $f(24+t) = f(t)$, meaning there is a point on the path that the woman will cross at exactly the same time of day on both days.

1.5 Section 2.6

Problem 1

Explain in your own words the meaning of each of the following.

(a) $\lim\limits_{x \to \infty} f(x) = 5$ (b) $\lim\limits_{x \to -\infty} f(x) = 3$

For part (a), the meaning of $\lim_{x\to\infty} f(x) = 5$ is that the value of $f(x)$ gets arbitrarily close to 5 for sufficiently large x.

For part (b), the meaning of $\lim_{x\to-\infty} f(x) = 3$ is that the value of $f(x)$ gets arbitrarily close to 3 for negative x with sufficiently large absolute value.

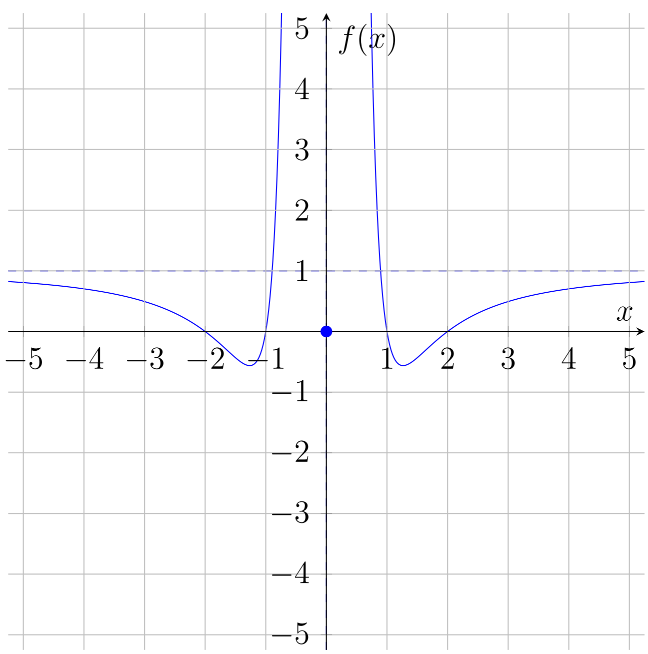
Problem 2

(a) Can the graph of $y = f(x)$ intersect a vertical asymptote?

Can it intersect a horizontal asymptote? Illustrate by sketching graphs.

(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch the graphs to illustrate the possibilities.

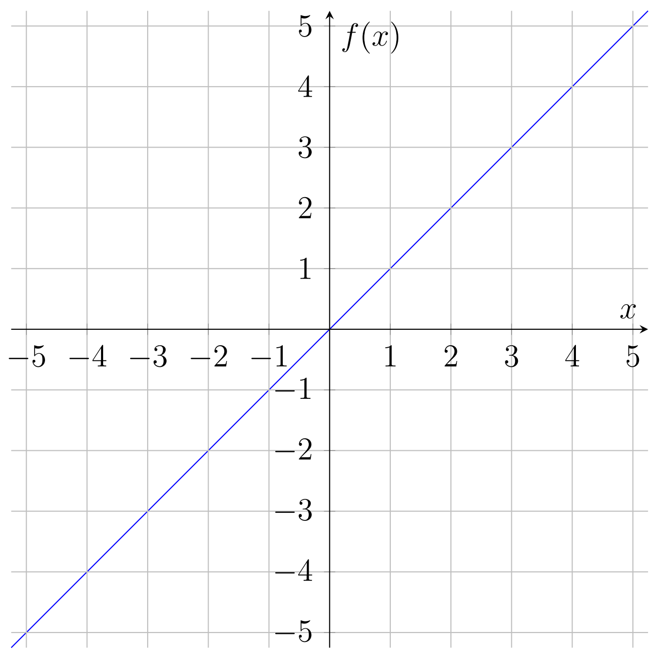
For part (a), the graph of a function $f(x)$ whose graph intersects its vertical and horizontal asymptotes is given below.



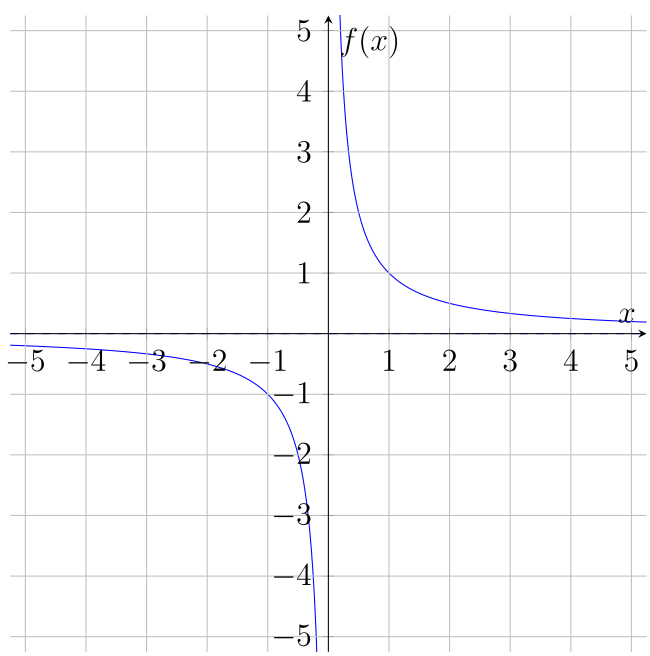
For part (b), the horizontal asymptotes of a function f are given by $y = \lim_{x\to\infty} f(x)$ if the limit $\lim_{x\to\infty} f(x)$ and $y = \lim_{x\to-\infty} f(x)$ if the limit $\lim_{x\to\infty} f(x)$

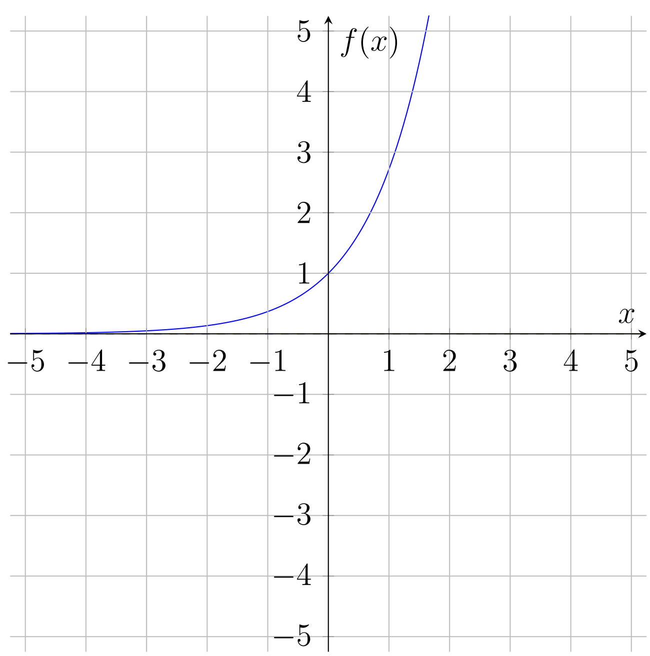
exists. Hence there are 3 possibilities regarding the number of horizontal asymptotes, it is either 0, 1 or 2.

Here is the graph of a function that has no horizontal asymptotes.

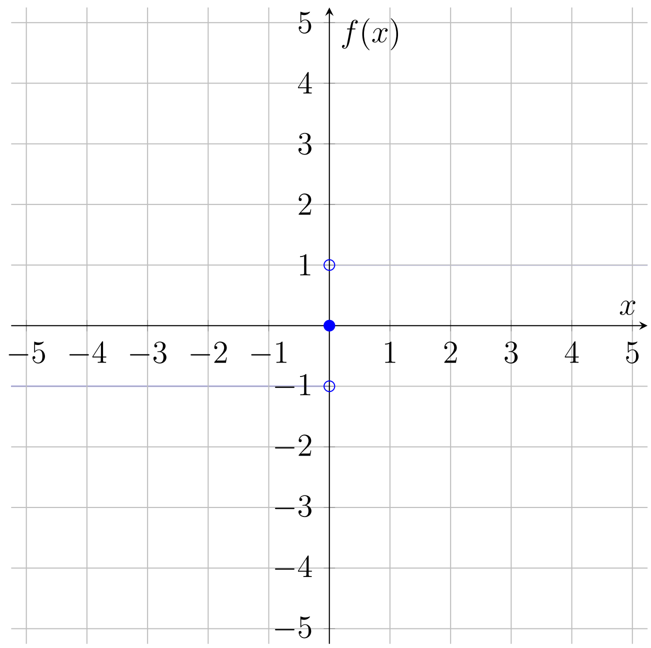


The graph of a function that has exactly one horizontal asymptote could either have both limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist and be equal, or have only one of them exist. Both cases are illustrated below.





The graph of a function with two distinct horizontal asymptotes is given below.



Problem 3

For the function f whose graph is given, state the following.

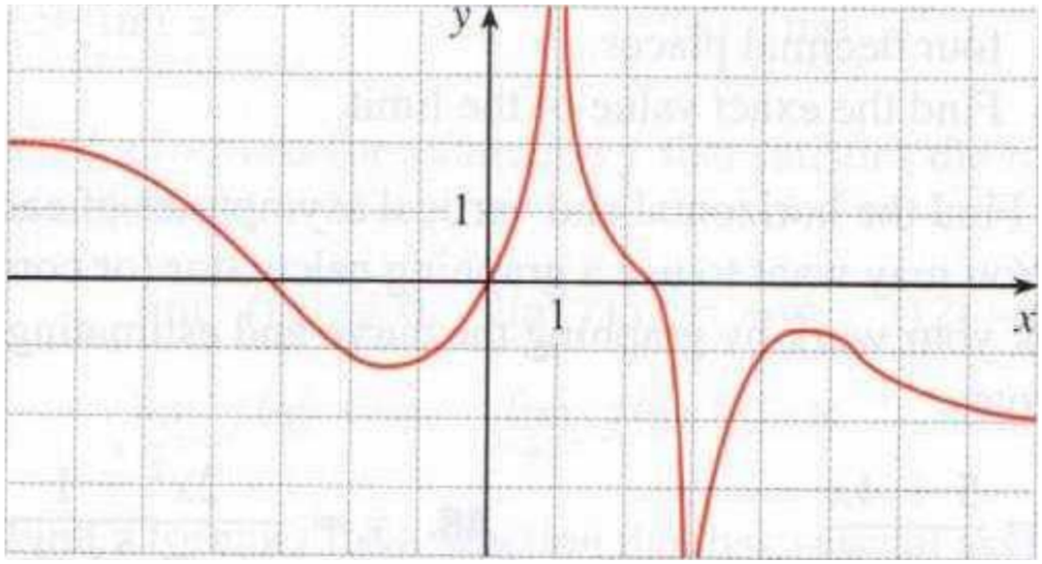
(a) $\lim_{x\to\infty} f(x)$

(b) $\lim_{x\to-\infty} f(x)$

(c) $\lim_{x\to 1} f(x)$

(d) $\lim_{x\to 3} f(x)$

(e) The equations of the asymptotes



For part (a), we have $\lim_{x\to\infty} f(x) = -2$

For part (b), we have $\lim_{x\to-\infty} f(x) = 2$

For part (c), we have $\lim_{x\to 1} f(x) = \infty$

For part (d), we have $\lim_{x\to 3} f(x) = -\infty$

For part (e), as shown in the previous parts, the equations of the asymptotes are $y = 2$, $y = -2$, $x = 1$ and $x = 3$.

Problem 4

For the function g whose graph is given, state the following.

(a) $\lim_{x\to\infty} g(x)$

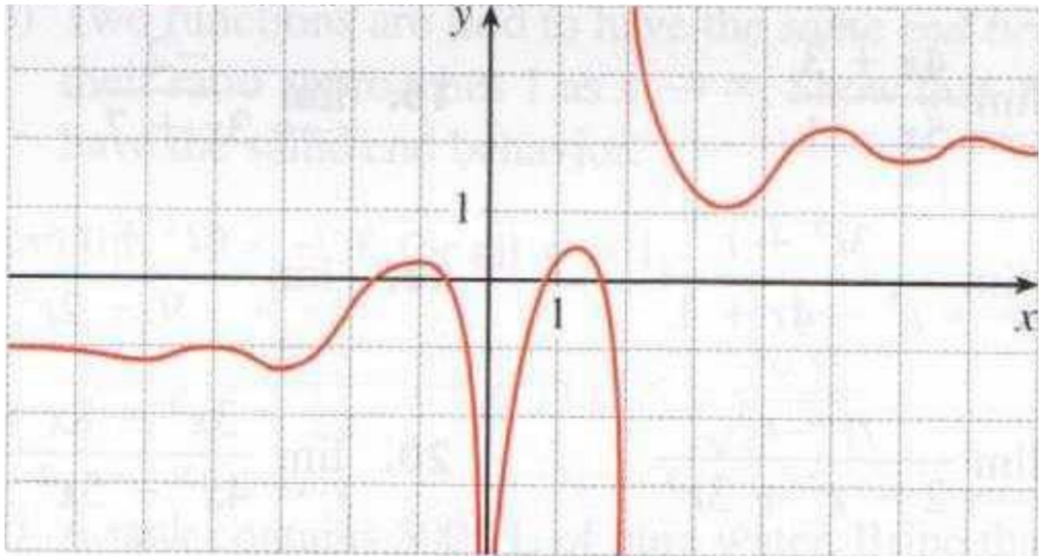
(b) $\lim_{x\to-\infty} g(x)$

(c) $\lim_{x\to 0} g(x)$

(d) $\lim_{x\to 2^-} g(x)$

(e) $\lim_{x\to 2^+} g(x)$

(f) The equations of the asymptotes



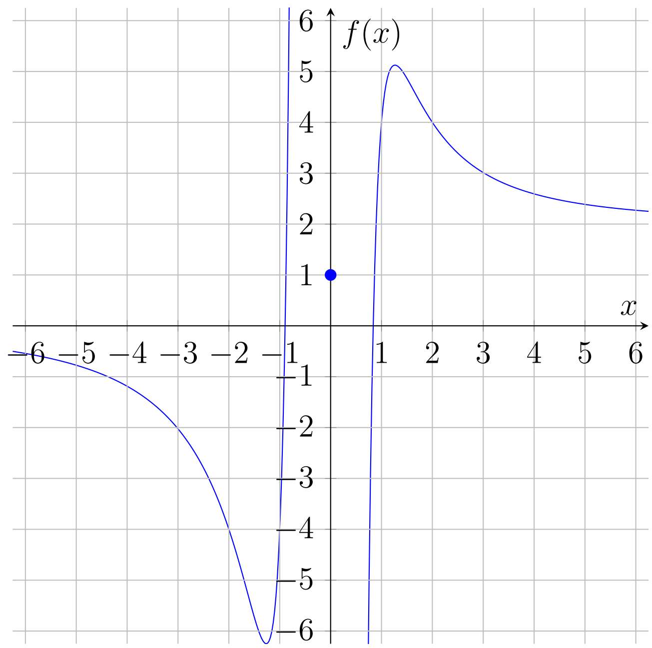
For part (a), we have $\lim_{x\to\infty} g(x) = 2$
For part (b), we have $\lim_{x\to-\infty} g(x) = -1$
For part (c), we have $\lim_{x\to 0} g(x) = -\infty$
For part (d), we have $\lim_{x\to 2^-} g(x) = -\infty$
For part (e), we have $\lim_{x\to 2^+} g(x) = \infty$
For part (f), as shown in the previous parts, the equations of the asymptotes are $y = 2$, $y = -1$, $x = 0$ and $x = 2$.

Problem 5

Sketch the graph of an example of a function f that satisfies the given conditions.

$$f(2) = 4, \quad f(-2) = -4, \quad \lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to\infty} f(x) = 2$$

The graph of a function f that satisfies the given conditions is given below.



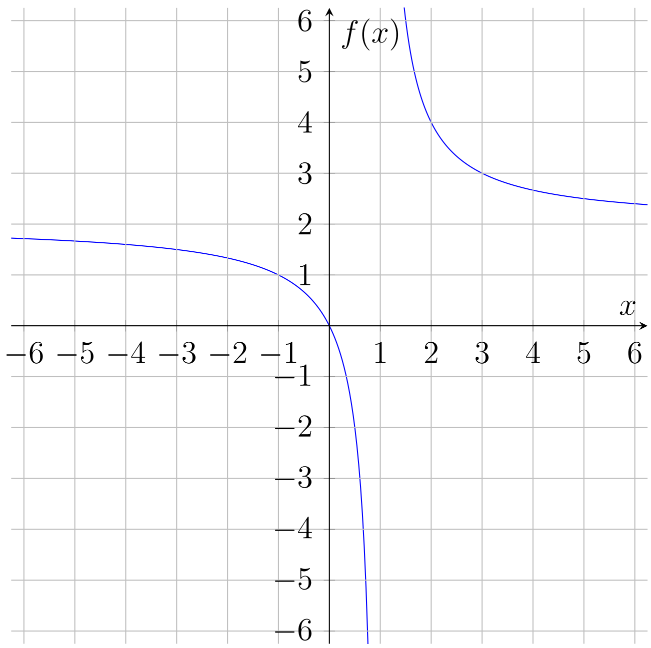
Problem 6

Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$f(0) = 0, \quad \lim_{x \to 1^-} f(x) = \infty, \quad \lim_{x \to 1^+} f(x) = -\infty,$$

$$\lim_{x \to -\infty} f(x) = -2, \quad \lim_{x \to \infty} f(x) = -2$$

The graph of a function f that satisfies the given conditions is given below. ...



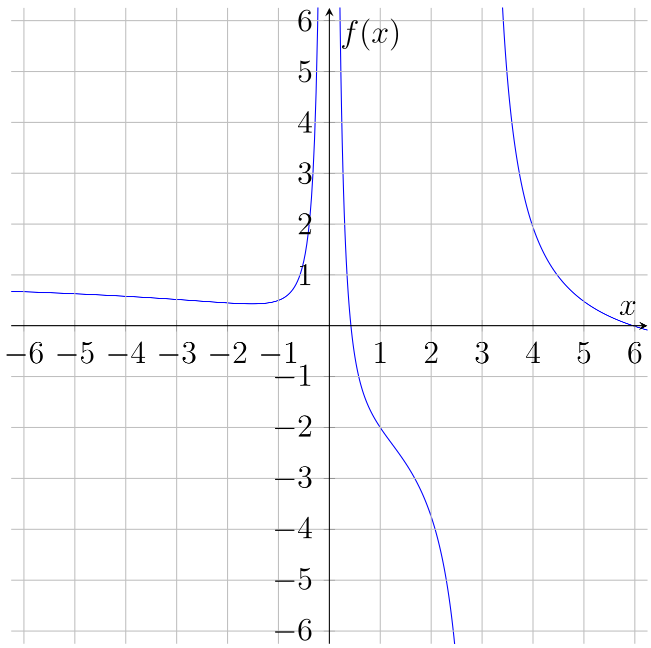
Problem 7

Sketch the graph of an example of a function f that satisfies the given conditions.

$$\lim_{x\to 0} f(x) = \infty, \quad \lim_{x\to 3^-} f(x) = -\infty, \quad , \lim_{x\to 3^+} f(x) = \infty,$$

$$\lim_{x\to -\infty} f(x) = 1, \lim_{x\to \infty} f(x) = -1$$

The graph of a function f that satisfies the given conditions is given below.



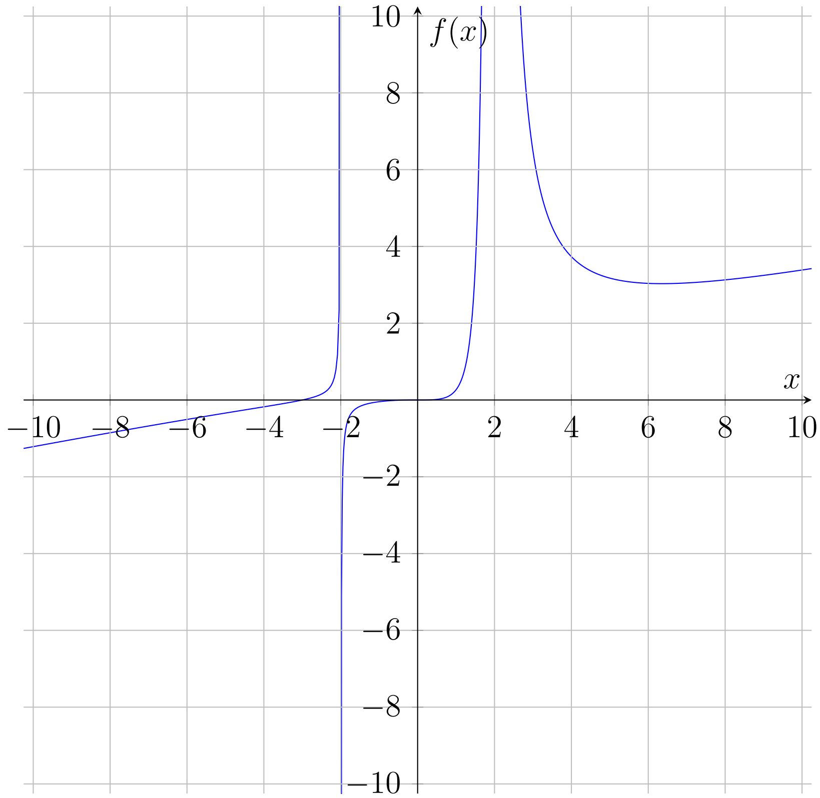
Problem 8

Sketch the graph of an example of a function f that satisfies the given conditions.

$$\lim_{x\to-\infty} f(x) = -\infty, \quad \lim_{x\to-2^-} f(x) = \infty, \quad \lim_{x\to-2^+} f(x) = -\infty$$

$$\lim_{x\to 2} f(x) = \infty, \quad \lim_{x\to\infty} = \infty$$

The graph of a function f that satisfies the given conditions is given below.

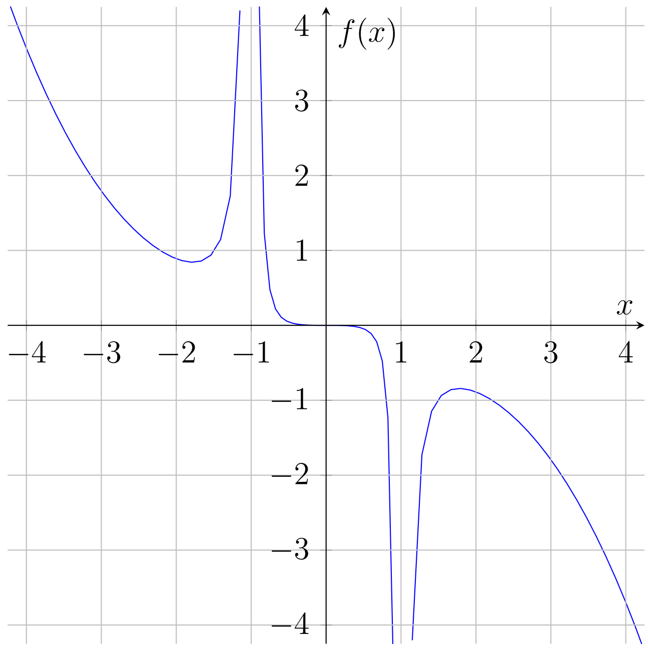


Problem 9

Sketch the graph of an example of a function f that satisfies the given conditions.

$$f(0) = 0, \quad \lim_{x \to 1} f(x) = -\infty, \quad \lim_{x \to \infty} f(x) = -\infty, \quad f \text{ is odd}$$

The graph of a function f that satisfies the given conditions is given below.



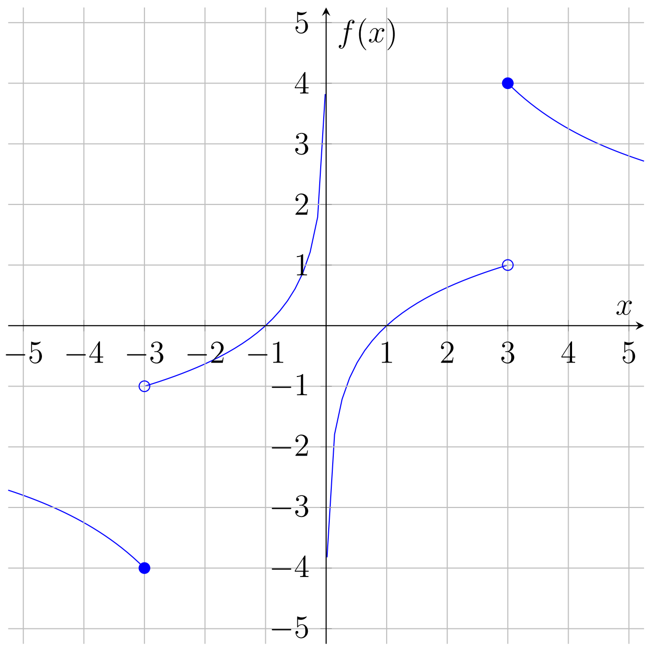
Problem 10

Sketch the graph of an example of a function f that satisfies the given conditions.

$$\lim_{x\to-\infty} f(x) = -1, \quad \lim_{x\to 0^-} f(x) = \infty, \quad \lim_{x\to 0^+} f(x) = -\infty,$$

$$\lim_{x\to 3^-} f(x) = 1, \quad f(3) = 4, \quad \lim_{x\to 3^+} f(x) = 4, \quad \lim_{x\to\infty} f(x) = 1$$

The graph of a function f that satisfies the given conditions is given below.



Problem 13

Evaluate the limit and justify each step by indicating the appropriate properties of limits.

$$\lim_{x\to\infty} \frac{2x^2-7}{5x^2+x-3}$$

We have the following.

$$\lim_{x\to\infty} \frac{2x^2-7}{5x^2+x-3} = \lim_{x\to\infty} \frac{2-\frac{7}{x^2}}{5+\frac{1}{x}-\frac{3}{x^2}} = \frac{2}{5}$$

Problem 14

Evaluate the limit and justify each step by indicating the appropriate properties of limits.

$$\lim_{x\to\infty} \sqrt{\frac{9x^3+8x-4}{3-5x+x^3}}$$

We begin by investigating the limit of the expression under the root since if it is positive or approaches 0 only from the right, then the given limit also exists and is equal to the square root of the limit of the expression under the root. We have the following.

$$\lim_{x\to\infty} \frac{9x^3+8x-4}{x^3-5x+3} = \lim_{x\to\infty} \frac{9-\frac{8}{x^2}-\frac{4}{x^3}}{1-\frac{5}{x^2}+\frac{3}{x^3}} = \frac{9}{1} = 9$$

Since $9 > 0$, we have

$$\lim_{x\to\infty} \sqrt{\frac{9x^3+8x-4}{3-5x+x^3}} = \sqrt{\lim_{x\to\infty} \frac{9x^3+8x-4}{x^3-5x+3}} = \sqrt{9} = 3$$

Problem 15

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{4x+3}{5x-1}$$

We have the following.

$$\lim_{x\to\infty} \frac{4x+3}{5x-1} = \lim_{x\to\infty} \frac{4+\frac{3}{x}}{5-\frac{1}{x}} = \frac{4}{5}$$

Problem 16

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{-2}{3x+7}$$

We have the following.

$$\lim_{x\to\infty} \frac{-2}{3x+7} = 0$$

Problem 17

Find the limit or show that it does not exist.

$$\lim_{t\to-\infty} \frac{3t^2+t}{t^3-4t+1}$$

We have the following.

$$\lim_{x\to-\infty} \frac{3+\frac{1}{t}}{t-\frac{4}{t}+\frac{1}{t^2}} = 0$$

Problem 18

Find the limit or show that it does not exist.

$$\lim_{t\to-\infty} \frac{6t^2+t-5}{9-2t^2}$$

We have the following.

$$\lim_{t\to-\infty} \frac{6t^2+t-5}{9-2t^2} = \lim_{t\to-\infty} \frac{6-\frac{1}{t}-\frac{5}{t}}{\frac{9}{t^2}-2} = \frac{6}{-2} = -3$$

Problem 19

Find the limit or show that it does not exist.

$$\lim_{r\to\infty} \frac{r-r^3}{2-r^2+3r^3}$$

We have the following.

$$\lim_{r\to\infty} \frac{r-r^3}{2-r^2+3r^3} = \lim_{r\to\infty} \frac{\frac{1}{r^2}-1}{\frac{2}{r^3}-\frac{1}{r}+3} = -\frac{1}{3}$$

Problem 20

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{3x^3 - 8x + 2}{4x^3 - 5x^2 - 2}$$

We have the following.

$$\lim_{x\to\infty} \frac{3x^3 - 8x + 2}{4x^3 - 5x^2 - 2} = \lim_{x\to\infty} \frac{3 - \frac{8}{x^2} + \frac{2}{x^3}}{4 - \frac{5}{x} - \frac{2}{x^3}} = \frac{3}{4}$$

Problem 21

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{4 - \sqrt{x}}{2 + \sqrt{x}}$$

We have the following.

$$\lim_{x\to\infty} \frac{4 - \sqrt{x}}{2 + \sqrt{x}} = \lim_{x\to\infty} \frac{\frac{4}{\sqrt{x}} - 1}{\frac{2}{\sqrt{x}} + 1} = \frac{-1}{1} = -1$$

Problem 22

Find the limit or show that it does not exist.

$$\lim_{u\to-\infty} \frac{(u^2+1)(2u^2-1)}{(u^2+2)^2}$$

We have the following.

$$\lim_{u\to-\infty} \frac{(u^2+1)(2u^2-1)}{(u^2+2)^2} = \lim_{u\to-\infty} \frac{\frac{u^2+1}{u^2}\frac{2u^2-1}{u^2}}{\frac{(u^2+2)^2}{(u^2)^2}} = \lim_{u\to-\infty} \frac{\left(1+\frac{1}{u^2}\right)\left(2-\frac{1}{u^2}\right)}{\left(1+\frac{2}{u^2}\right)^2} = \frac{1\cdot 2}{1} = 2$$

Problem 23

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{\sqrt{x+3x^2}}{4x-1}$$

We have the following.

$$\lim_{x\to\infty} \frac{\sqrt{x+3x^2}}{4x-1} = \lim_{x\to\infty} \frac{\frac{\sqrt{x+3x^2}}{\sqrt{x^2}}}{\frac{4x-1}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x}+3}}{4-\frac{1}{x}} = \frac{\sqrt{3}}{4}$$

Problem 24

Find the limit or show that it does not exist.

$$\lim_{t\to\infty} \frac{t+3}{\sqrt{2t^2-1}}$$

We have the following.

$$\lim_{t\to\infty} \frac{t+3}{\sqrt{2t^2-1}} = \lim_{t\to\infty} \frac{\frac{t+3}{t}}{\frac{\sqrt{2t^2-1}}{\sqrt{t^2}}} = \lim_{t\to\infty} \frac{1+\frac{3}{t}}{\sqrt{2-\frac{1}{t^2}}} = \frac{1}{\sqrt{2}}$$

Problem 25

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$$

We have the following.

$$\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to\infty} \frac{\frac{\sqrt{1+4x^6}}{x^6}}{\frac{2-x^3}{x^3}} = \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} = \frac{\sqrt{4}}{-1} = -2$$

Problem 26

Find the limit or show that it does not exist.

$$\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}}{2-x^3}$$

We have the following.

$$\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}}{2-x^3}=\lim_{x\to-\infty}\frac{\frac{\sqrt{1+4x^6}}{-\sqrt{x^6}}}{\frac{2-x^3}{x^3}}=-\lim_{x\to-\infty}\frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1}=-\frac{\sqrt{4}}{-1}=2$$

Problem 27

Find the limit or show that it does not exist.

$$\lim_{x\to-\infty}\frac{2x^5-x}{x^4+3}$$

The given limit does not exist since it diverges to negative infinity as shown below.

$$\lim_{x\to-\infty}\frac{2x^5-x}{x^4+3}=\lim_{x\to-\infty}\frac{2x-\frac{1}{x^3}}{1+\frac{1}{x^4}}=-\infty$$

Problem 28

Find the limit or show that it does not exist.

$$\lim_{q\to\infty}\frac{q^3+6q-4}{4q^2-3q+3}$$

The limit does not exist since it diverges to infinity as shown below.

$$\lim_{q\to\infty}\frac{q^3+6q-4}{4q^2-3q+3}=\lim_{q\to\infty}\frac{q+\frac{6}{q}-\frac{4}{q^2}}{4-\frac{3}{q}+\frac{3}{q^2}}=\infty$$

Problem 29

Find the limit or show that it does not exist.

$$\lim_{t\to\infty}(\sqrt{25t^2+2}-5t)$$

We have the following.

$$\lim_{t\to\infty}(\sqrt{25t^2+2}-5t) = \lim_{t\to\infty}\frac{(\sqrt{25t^2+2}-5t)(\sqrt{25t^2+2}+5t)}{\sqrt{25t^2+2}+5t} = \lim_{t\to\infty}\frac{2}{\sqrt{25t^2+2}+5t} = 0$$

Problem 30

Find the limit or show that it does not exist.

$$\lim_{x\to-\infty}(\sqrt{4x^2+3x}+2x)$$

The given limit does not exist since it diverges to infinity as shown below

$$\lim_{x\to-\infty}(\sqrt{4x^2+3x}+2x) = \lim_{x\to-\infty}\frac{(\sqrt{4x^2+3x}+2x)(\sqrt{4x^2+3x}-2x)}{\sqrt{4x^2+3x}-2x} = \lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+3x}-2x} =$$

$$= \lim_{x\to-\infty}\frac{3}{\frac{\sqrt{4x^2+3x}}{-\sqrt{x^2}}-\frac{2x}{x}} = -\lim_{x\to-\infty}\frac{3}{\sqrt{4+\frac{3}{x}}-2} = \infty$$

Problem 31

Find the limit or show that it does not exist.

$$\lim_{x\to\infty}(\sqrt{x^2+ax}-\sqrt{x^2+bx})$$

We have the following.

$$lim_{x\to\infty}(\sqrt{x^2+ax}-\sqrt{x^2+bx}) = \lim_{x\to\infty}\frac{(\sqrt{x^2+ax}-\sqrt{x^2+bx})(\sqrt{x^2+ax}+\sqrt{x^2+bx})}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} = \lim_{x\to\infty}\frac{}{\sqrt{x^2}}$$

$$= \lim_{x\to\infty}\frac{a-b}{\frac{\sqrt{x^2+ax}}{\sqrt{x^2}}+\frac{\sqrt{x^2+bx}}{\sqrt{x^2}}} = \lim_{x\to\infty}\frac{a-b}{\sqrt{1+\frac{a}{x}}+\sqrt{1+\frac{b}{x}}} =$$

$$= \frac{a-b}{1+1} = \frac{a-b}{2}$$

Problem 32

Find the limit or show that it does not exist.

$$\lim_{x\to\infty}(x-\sqrt{x})$$

The limit does not exist as it diverges to infinity as shown below.

$$\lim_{x\to\infty}(x-\sqrt{x}) = \lim_{x\to\infty}\sqrt{x}(\sqrt{x}-1) = \infty$$

Problem 33

Find the limit or show that it does not exist.

$$\lim_{x\to-\infty}(x^2+2x^7)$$

The given limit does not exist as it diverges to negative infinity as shown below.

$$\lim_{x\to-\infty}(x^2+2x^7) = \lim_{x\to-\infty} x^2(1+2x^5) = -\infty$$

Problem 34

Find the limit or show that it does not exist.

$$\lim_{x\to\infty}(e^{-x}+2\cos 3x)$$

By contradiction, assume that the given limit exists. We then have the following.

$$\lim_{x\to\infty} 2\cos 3x = \lim_{x\to\infty}(e^{-x}+2\cos 3x-e^{-x}) = \lim_{x\to\infty}(e^{-x}+2\cos 3x)-\lim_{x\to\infty} e^{-x} = \lim_{x\to\infty}(e^{-x}+2\cos 3x)$$

Which would imply the limit of $\lim_{x\to\infty}\cos 3x$ to exist, which is false. Hence the limit $\lim_{x\to\infty}(e^{-x}+2\cos 3x)$ does not exist.

Problem 35

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} (e^{-2x} \cos x)$$

For all x, we have $-1 \leq \cos x \leq 1$, meaning $-e^{-2x} = e^{-2x} \cdot (-1) \leq e^{-2x} \cos x \leq e^{-2x} \cdot 1 = e^{-2x}$. Note that $\lim\limits_{x\to\infty} [-e^{-2x}] = 0 = \lim\limits_{x\to\infty} e^{-2x}$. By squeeze theorem, we have $\lim\limits_{x\to\infty} e^{-2x} \cos x$.

Problem 36

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{\sin^2 x}{x^2+1}$$

For all x, we have $0 \leq \sin^2 x \leq 1$, meaning $-\frac{1}{x^2+1} \leq \frac{\sin^2 x}{x^2+1} \leq \frac{1}{x^2+1}$. Note that $\lim\limits_{x\to\infty} \left[-\frac{1}{x^2+1}\right] = 0 = \lim\limits_{x\to\infty} \frac{1}{x^2+1}$. By squeeze theorem, we have $\lim\limits_{x\to\infty} \frac{\sin^2 x}{x^2+1}$.

Problem 37

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{1-e^x}{1+2e^x}$$

We have the following.

$$\lim_{x\to\infty} \frac{1-e^x}{1+2e^x} = \lim_{x\to\infty} \frac{\frac{1-e^x}{e^x}}{\frac{1+2e^x}{e^x}} = \lim_{x\to\infty} \frac{\frac{1}{e^x}-1}{\frac{1}{e^x}+2} = -\frac{1}{2}$$

Problem 38

Find the limit or show that it does not exist.

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$$

We have the following.

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x\to\infty} \frac{e^{-3x}(e^{3x} - e^{-3x})}{e^{-3x}(e^{3x} + e^{-3x})} = \lim_{x\to\infty} \frac{1 - e^{-6x}}{1 + e^{-6x}} = \frac{1}{1} = 1$$

Problem 39

Find the limit or show that it does not exist.

$$\lim_{x\to(\pi/2)^+} e^{\sec x}$$

We have $\cos x \to 0^-$ as $x \to (\pi/2)^+$, meaning $\sec x \to -\infty$ as $x \to (\pi/2)^+$. In general, $e^t \to 0$ given $t \to -\infty$. Since we have $\sec x \to -\infty$ as $x \to (\pi/2)^+$, we also have $e^{\sec x} \to 0$ as $x \to (\pi/2)^+$.

Problem 40

Find the limit or show that it does not exist.

$$\lim_{x\to 0^+} \tan^{-1}(\ln x)$$

In general, $\tan^{-1} t \to -\frac{\pi}{2}$ given $t \to -\infty$. Since we have $\ln x \to -\infty$ as $x \to 0^+$, we also have $\tan^{-1}(\ln x) \to \frac{\pi}{2}$.

Problem 41

Find the limit or show that it does not exist.

$$\lim_{x\to\infty}\left[\ln\left(1+x^2\right)-\ln(1+x)\right]$$

Note the following.

$$\frac{1+x^2}{1+x}=e^{\ln\left(\frac{1+x^2}{1+x}\right)}=e^{\ln\left(1+x^2\right)-\ln(1+x)}$$

By contradiction, assume that the given limit exists. Then it would imply the limit $\lim\limits_{x\to\infty}\frac{1+x^2}{1+x}$ to exist. The latter is false. Hence the limit $\lim\limits_{x\to\infty}[\ln(1+x^2)-\ln(1+x)]$ does not exist. In fact, it diverges to infinity since $\lim\limits_{x\to\infty}\frac{1+x^2}{1+x}=\infty$ and, in general, $\ln t\to\infty$ given $t\to\infty$.

Problem 42

Find the limit or show that it does not exist.

$$\lim_{x\to\infty}[\ln(2+x)-\ln(1+x)]$$

We have the following due to continuity of ln.

$$\begin{aligned}\lim_{x\to\infty}[\ln(2+x)-\ln(1+x)]&=\lim_{x\to\infty}\ln\left(\frac{2+x}{1+x}\right)=\ln\left(\lim_{x\to\infty}\frac{2+x}{1+x}\right)=\\&=\ln\left(\lim_{x\to\infty}\frac{\frac{2+x}{x}}{\frac{1+x}{x}}\right)=\ln\left(\lim_{x\to\infty}\frac{\frac{2}{x}+1}{\frac{1}{x}+1}\right)=\\&=\ln\left(\frac{1}{1}\right)=\ln 1=0\end{aligned}$$

Problem 43

(a) For $f(x)=\frac{x}{\ln x}$ find each of the following limits.

(i) $\lim\limits_{x \to 0^+} f(x)$ (ii) $\lim\limits_{x \to 1^-} f(x)$ (iii) $\lim\limits_{x \to 1^+} f(x)$

(b) Use a table of values to estimate $\lim\limits_{x \to \infty} f(x)$.

(c) Use the information from parts (a) and (b) to make a rough sketch of the graph of f.

For part (i) of (a), we have the following.

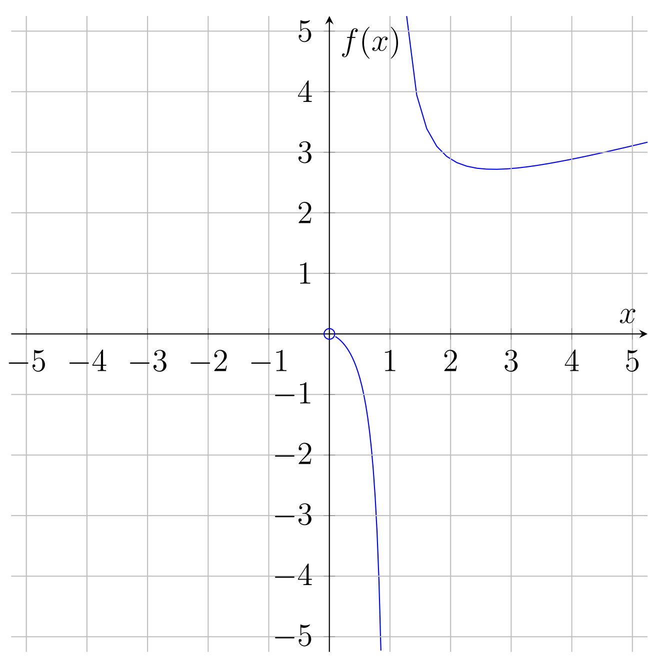
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{\ln x} = \lim_{x \to 0^+} \left[x \cdot \frac{1}{\ln x} \right] = \lim_{x \to 0^+} x \cdot \lim_{x \to 0^+} \frac{1}{\ln x} = 0 \cdot 0 = 0$$

For part (ii) of (a), we have $x \to 1^- > 0$ and $\ln x \to 0^- < 0$ as $x \to 1^-$, meaning $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \frac{x}{\ln x} = -\infty$.

For part (iii) of (a), we have $x \to 1^+ > 0$ and $\ln x \to 0^+ > 0$ as $x \to 1^+$, meaning $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \frac{x}{\ln x} = \infty$.

For part (b), no.

For part (c), the graph of f is given below.



Problem 44

(a) For $f(x) = \frac{2}{x} - \frac{1}{\ln x}$ find each of the following limits.

(i) $\lim_{x \to \infty} f(x)$ (iii) $\lim_{x \to 1^-} f(x)$

(ii) $\lim_{x \to 0^+} f(x)$ (iv) $\lim_{x \to 1^+} f(x)$

(b) Use the information from part (a) to make a rough sketch the of the graph of f.

For part (i) of (a), we have the following.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left[\frac{2}{x} - \frac{1}{\ln x} \right] = \lim_{x \to \infty} \frac{2}{x} - \lim_{x \to \infty} \frac{1}{\ln x} = 0 - 0 = 0$$

For part (ii) of (a), by contradiction, assume that the given limit exists. We then have the following.

$$\lim_{x \to 0^+} \frac{2}{x} = \lim_{x \to 0^+} \left[\frac{1}{\ln x} \frac{2}{x} - \frac{1}{\ln x} \right] = \lim_{x \to 0^+} \frac{1}{\ln x} + \lim_{x \to 0^+} \left[\frac{2}{x} - \frac{1}{\ln x} \right]$$

Which would imply the limit $\lim_{x \to 0^+} \frac{2}{x}$ to exist. The latter is false. Hence the limit $\lim_{x \to 0^+} \left[\frac{2}{x} - \frac{1}{\ln x} \right]$ does not exist. In fact, it diverges to infinity since $\frac{2}{x} \to \infty$ and $\frac{1}{\ln x} \to 0$ as $x \to 0^+$.

For part (iii) of (a), by contradiction, assume that the given limit exists. We then have the following.

$$\lim_{x \to 1^-} \left[-\frac{1}{\ln x} \right] = \lim_{x \to 1^-} \left[\frac{1}{x} - \frac{1}{\ln x} - \frac{1}{x} \right] = \lim_{x \to 1^-} \left[\frac{1}{x} - \frac{1}{\ln x} \right] - \lim_{x \to 1^-} \frac{1}{x} = \lim_{x \to 1^+} \left[\frac{1}{x} - \frac{1}{\ln x} \right] - 1$$

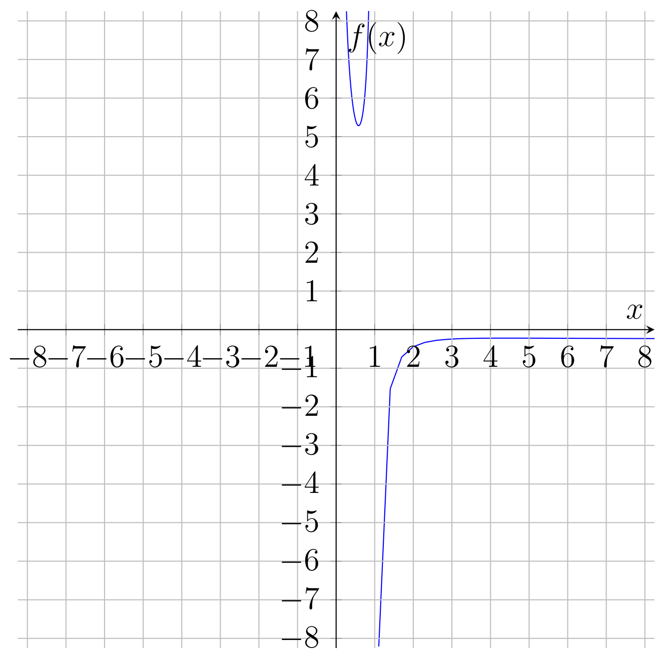
Which would imply the limit $\lim_{x \to 1^+} \left[-\frac{1}{\ln x} \right]$ to exist. The latter is false. Hence the limit $\lim_{x \to 1^-} \left[\frac{2}{x} - \frac{1}{\ln x} \right]$ does not exist. In fact, it diverges to infinity since $\frac{2}{x} \to 2$ and $\ln x \to 0^-$ as $x \to 1^-$, meaning $-\frac{1}{\ln x} \to \infty$ as $x \to 1^-$.

For part (iv) of (a), by contradiction, assume that the given limit exists. We then have the following.

$$\lim_{x \to 1^+} \left[-\frac{1}{\ln x} \right] = \lim_{x \to 1^+} \left[\frac{1}{x} - \frac{1}{\ln x} - \frac{1}{x} \right] = \lim_{x \to 1^+} \left[\frac{1}{x} - \frac{1}{\ln x} \right] - \lim_{x \to 1^+} \frac{1}{x} = \lim_{x \to 1^+} \left[\frac{1}{x} - \frac{1}{\ln x} \right] - 1$$

Which would imply the limit $\lim_{x\to 1^+}\left[-\frac{1}{\ln x}\right]$ to exist. The latter is false. Hence the limit $\lim_{x\to 1^-}\left[\frac{2}{x}-\frac{1}{\ln x}\right]$ does not exist. In fact, it diverges to negative infinity since $\frac{2}{x}\to 2$ and $\ln x\to 0^+$ as $x\to 1^+$, meaning $-\frac{1}{\ln x}\to -\infty$ as $x\to 1^+$.

For part (b), the graph of f is given below.



Problem 47

Find the horizontal and vertical asymptotes of each curve. You may want to use a graphing calculator (or computer) to check your work by graphing the curve and estimating the asymptotes.

$$y=\frac{5+4x}{x+3}$$

We begin by testing the limits at both infinities.

$$\lim_{x\to\pm\infty}\frac{5+4x}{x+3}=\lim_{x\to\pm\infty}\frac{\frac{5+4x}{x}}{\frac{x+3}{x}}=\lim_{x\to\pm\infty}\frac{\frac{5}{x}+4}{1+\frac{3}{x}}=\frac{4}{1}=4$$

Hence $y = 4$ is the horizontal asymptote of the given curve.

For a vertical asymptote of the curve of a rational function to occur at $x = a$ for some a, its denominator must approach 0 as $x \to a$, we hence shall solve for values of a such that $\lim\limits_{x \to a}[x + 3] = 0$. We have the following.

$$a = a + 3 - 3 = \lim_{x \to a}[x + 3] - 3 = 0 - 3 = -3$$

We now verify that a vertical asymptote indeed occurs at $x = -3$. We have $\lim_{x \to -3^+}[x+3] = -3^+ + 3 = 0^+$ whereas $\lim_{x \to -3}[5+4x] = 5+4(-3^+) = 5 - 12^+ = -7^- < 0$, meaning $\lim\limits_{x \to -3} \frac{5+4x}{x+3} = -\infty$ as desired.

Problem 48

Find the horizontal and vertical asymptotes of each curve. You may want to use a graphing calculator (or computer) to check your work by graphing the curve and estimating the asymptotes.

$$y = \frac{2x^2 + 1}{3x^2 + 2x - 1}$$

We begin by testing the limits at both infinities.

$$\lim_{x \to \pm\infty} \frac{2x^2 + 1}{3x^2 + 2x - 1} = \lim_{x \to \pm\infty} \frac{\frac{2x^2+1}{x^2}}{\frac{3x^2+2x-1}{x^2}} = \lim_{x \to \pm\infty} \frac{2 + \frac{1}{x^2}}{3 + \frac{2}{x} - \frac{1}{x^2}} = \frac{2}{3}$$

Hence $y = \frac{2}{3}$ is the horizontal asymptote of the given curve.

For a vertical asymptote of the curve of a rational function to occur at $x = a$ for some a, its denominator must approach 0 as $x \to a$, we hence shall solve for the values of a such that $\lim\limits_{x \to a}[3x^2 + 2x - 1] = 0$. We have the following.

$$(a - (-1))\left(a - \frac{1}{3}\right) = \frac{3a^2 + 2a - 1}{3} = \frac{\lim_{x \to a}[3x^2 + 2x - 1]}{3} = \frac{0}{3} = 0$$

$$\implies \begin{cases} a = -1 \\ a = \frac{1}{3} \end{cases}$$

We now verify that vertical asymptotes indeed occur at $x = -1$ and $x = \frac{1}{3}$. We have $\lim_{x\to -1}[3x^2 + 2x - 1] = 3(-1)^2 + 2(-1) - 1 = 0$ whereas $\lim_{x\to -1}[2x^2 + 1] = 2(-1)^2 + 1 = 3 \neq 0$, meaning the limit $\lim_{x\to -1} \frac{2x^2+1}{3x^2+2x-1}$ grows arbitrarily large as desired. We have $\lim_{x\to \frac{1}{3}}[3x^2+2x-1] = 3\left(\frac{1}{3}\right)+2\left(\frac{1}{3}\right)-1 = 0$ whereas $\lim_{x\to \frac{1}{3}}[2x^2 + 1] = 2\left(\frac{1}{3}\right)^2 + 1 = \frac{11}{9} \neq 0$, meaning the limit $\lim_{x\to \frac{1}{3}} \frac{2x^2+1}{3x^2+2x-1}$ grows arbitrarily large as desired.

Problem 49

Find the horizontal and vertical asymptotes of each curve. You may want to use a graphing calculator (or computer) to check your work by graphing the curve and estimating the asymptotes.

$$y = \frac{2x^2 + x - 1}{x^2 + x - 2}$$

We begin by testing the limits at both infinities.

$$\lim_{x\to\pm\infty} \frac{2x^2 + x - 1}{x^2 + x - 2} = \lim_{x\to\pm\infty} \frac{\frac{2x^2+x-1}{x^2}}{\frac{x^2+x-2}{x^2}} = \lim_{x\to\pm\infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}} = \frac{2}{1} = 2$$

Hence $y = 2$ is the horizontal asymptote.

For a vertical asymptote of the curve of a rational function to occur at $x = a$ for some a, its denominator must approach 0 as $x \to a$, we hence shall solve for the values of a such that $\lim_{x\to a}[x^2+x-2] = 0$. We have the following.

$$(a - (-2))(a - 1) = a^2 + a - 2 = \lim_{x\to a}[x^2 + x - 2] = 0$$
$$\implies \begin{cases} a = -2 \\ a = 1 \end{cases}$$

We now verify that vertical asymptotes indeed occur at $x = -2$ and $x = 1$. We have $\lim_{x\to -2}[x^2 + x - 2] = (-2)^2 + (-2) - 2 = 0$ whereas $\lim_{x\to -2}[2x^2 +$

$x-1] = 2(-2)^2 + (-2) - 1 = 5 \neq 0$, meaning the limit $\lim\limits_{x\to -2} \frac{2x^2+x-1}{x^2+x-2}$ grows arbitrarily large as desired. We have $\lim\limits_{x\to 1}[x^2+x-2] = 1^2+1-2 = 0$ whereas $\lim\limits_{x\to 1}[2x^2+x-1] = 2(1)^2+1-1 = 2 \neq 0$, meaning the limit $\lim\limits_{x\to 1} \frac{2x^2+x-1}{x^2+x-2}$ grows arbitrarily large as desired.

Problem 50

Find the horizontal and vertical asymptotes of each curve. You may want to use a graphing calculator (or computer) to check your work by graphing the curve and estimating the asymptotes.

$$y = \frac{1+x^4}{x^2-x^4}$$

We begin by testing the limits at both infinities.

$$\lim_{x\to\pm\infty} \frac{1+x^4}{x^2-x^4} = \lim_{x\to\pm\infty} \frac{\frac{1+x^4}{x^4}}{\frac{x^2-x^4}{x^4}} = \lim_{x\to\pm\infty} \frac{\frac{1}{x^4}+1}{\frac{1}{x^2}-1} = \frac{1}{-1} = -1$$

Hence $y = -1$ is the horizontal asymptote.

For a vertical asymptote of the curve of a rational function to occur at $x = a$ for some a, its denominator must approach 0 as $x \to a$, we hence shall solve for the values of a such that $\lim\limits_{x\to a}[x^2 - x^4] = 0$. We have the following.

$$a^2(1-a)(1+a) = a^2 - a^4 = \lim_{x\to a}[a^2 - a^4] = 0$$
$$\implies \begin{cases} a = 0 \\ a = 1 \\ a = -1 \end{cases}$$

We now verify that vertical asymptotes indeed occur at $x = 0$, $x = 1$ and $x = -1$. For each $a \in \{-1, 0, 1\}$, we have $\lim_{x\to a}[x^2 - x^4] = 0$ whereas $\lim_{x\to a}[1 + x^4] = 1 + a^4 \neq 0$ since $1 + a^4 \geq 1$ for all $a \in \mathbb{R}$, meaning the limit $\lim\limits_{x\to a} \frac{1+x^4}{x^2-x^4}$ grows arbitrarily large for all $a \in \{-1, 0, 1\}$ as desired.

Problem 51

Find the horizontal and vertical asymptotes of each curve. You may want to use a graphing calculator (or computer) to check your work by graphing the curve and estimating the asymptotes.

$$y = \frac{x^2 - x}{x^2 - 6x + 5}$$

We begin by testing the limits at both infinities.

$$\lim_{x\to\pm\infty} \frac{x^2 - x}{x^2 - 6x + 5} = \lim_{x\to\pm\infty} \frac{\frac{x^2-x}{x^2}}{\frac{x^2-6x+5}{x^2}} = \lim_{x\to\pm\infty} \frac{1 - \frac{1}{x}}{1 - \frac{6}{x} + \frac{5}{x^2}} = \frac{1}{1} = 1$$

Hence $y = 1$ is the horizontal asymptote.

For a vertical asymptote of the curve of a rational function to occur at $x = a$ for some a, its denominator must approach 0 as $x \to a$, we hence shall solve for the values of a such that $\lim_{x\to a}[x^2 - 6x + 5] = 0$. We have the following.

$$(a-1)(a-5) = a^2 - 6a + 5 = \lim_{x\to a}[x^2 - 6x + 5] = 0$$
$$\Longrightarrow \begin{cases} a = 1 \\ a = 5 \end{cases}$$

We now check whether vertical asymptotes indeed occur at $x = 1$ and $x = 5$. We have the following.

$$\lim_{x\to 1} \frac{x^2 - x}{x^2 - 6x + 5} = \lim_{x\to 1 \frac{x(x-1)}{(x-1)(x-5)}} = \lim_{x\to 1} \frac{x}{x-5} = \frac{1}{1-4} = \frac{1}{4}$$

Meaning a vertical asymptote does not occur at $x = 1$. We have $\lim_{x\to 5}[x^2 - 6x + 5] = 5^2 - 6(5) + 5 = 0$ whereas $\lim_{x\to 5}[x^2 - x] = 5^2 - 5 = 20 \neq 0$, meaning the limit $\lim_{x\to 5} \frac{x^2-x}{x^2-6x+5}$ grows arbitrarily large as desired.

Problem 52

Find the horizontal and vertical asymptotes of each curve. You may want to use a graphing calculator (or computer) to check your work by graphing the curve and estimating the asymptotes.

$$y = \frac{2e^x}{e^x - 5}$$

We begin by testing the limits at both infinities.

$$\lim_{x\to\infty} \frac{2e^x}{e^x - 5} = \lim_{x\to\infty} \frac{\frac{2e^x}{e^x}}{\frac{e^x-5}{e^x}} = \lim_{x\to\infty} \frac{2}{1 - \frac{5}{e^x}} = \frac{2}{1} = 1$$
$$\lim_{x\to-\infty} \frac{2e^x}{e^x - 5} = \frac{\lim_{x\to-\infty} 2e^x}{\lim_{x\to-\infty}[e^x - 5]} = \frac{0}{-5} = 0$$

Hence $y = 1$ and $y = 0$ are the vertical asymptotes.

For a vertical asymptote of the curve of a rational function to occur at $x = a$ for some a, its denominator must approach 0 as $x \to a$, we hence shall solve for the values of a such that $\lim_{x\to a}[e^x - 5] = 0$. We have the following.

$$a = \ln(e^a) = \ln(5 + e^a - 5) = \ln\left(5 + \lim_{x\to a}[e^x - 5]\right) = \ln(5 + 0) = \ln 5$$

We now verify that a vertical asymptote indeed occurs at $x = \ln 5$. We have $\lim_{x\to\ln 5}[e^x - 5] = e^{\ln 5} - 5 = 0$ whereas $\lim_{x\to\ln 5} 2e^x = 2e^{\ln 5} = 10 \neq 0$, meaning the limit $\lim_{x\to\ln 5} \frac{2e^x}{e^x-5}$ grows arbitrarily large as desired.

Problem 55

Let P and Q be polynomials. Find

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

Let P and Q be polynomials with degrees n and m respectively, i.e., let $P(x) = \sum_{i=0}^{n} a_i x^i$ and $Q(x) = \sum_{i=0}^{m} b_i x^i$ for some finite sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_m$.

For part (a), $n < m$. We have the following.

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i} = \lim_{x\to\infty} \frac{\frac{\sum_{i=0}^{n} a_i x^i}{x^m}}{\frac{\sum_{i=0}^{m} b_i x^i}{x^m}} =$$
$$= \lim_{x\to\infty} \frac{\sum_{i=0}^{n} a_i x^{i-m}}{\sum_{i=0}^{m} b_i x^{i-m}}$$

Since $n < m$ and $0 \le i \le n$ in the sum in the numerator, we have $i - m < 0$, meaning $x^{i-m} \to 0$ as $x \to 0$ as $x \to \infty$. In the sum in the denominator, we have $0 \le i \ne m$, meaning for $0 \le i < m$ we similarly have $x^{i-m} \to 0$ as $x \to \infty$ and $b_m x^{m-m} = b_m \to b_m$ as $x \to \infty$. Hence the numerator approaches 0 and the denominator approaches $b_m \ne 0$, meaning the entire fraction approaches 0, i.e., $\lim_{x\to\infty} \frac{P(x)}{Q(x)} = 0$.

For part (b), $m < n$. Note, as implicitly shown in part (a), that $\frac{P(x)}{Q(x)} = \frac{\sum_{i=0}^{n} a_i x^{i-m}}{\sum_{i=0}^{m} b_m x^{i-m}}$. Since the leading coefficient in the numerator is $a_n x^{n-m}$ and $m < n$, we have that the numerator grows arbitrarily large whereas the denominator approaches $b_m \ne 0$. Hence the limit $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$ does not exist as it diverges to either positive or negative infinity depending on the sign of a_n and b_m.

Problem 57

Find a formula for a function f that satisfies the following conditions:

$$\lim_{x\to\pm\infty} f(x) = 0, \quad \lim_{x\to 0} f(x) = -\infty, \quad f(2) = 0,$$
$$\lim_{x\to 3^-} f(x) = \infty, \quad \lim_{x\to 3^+} f(x) = -\infty$$

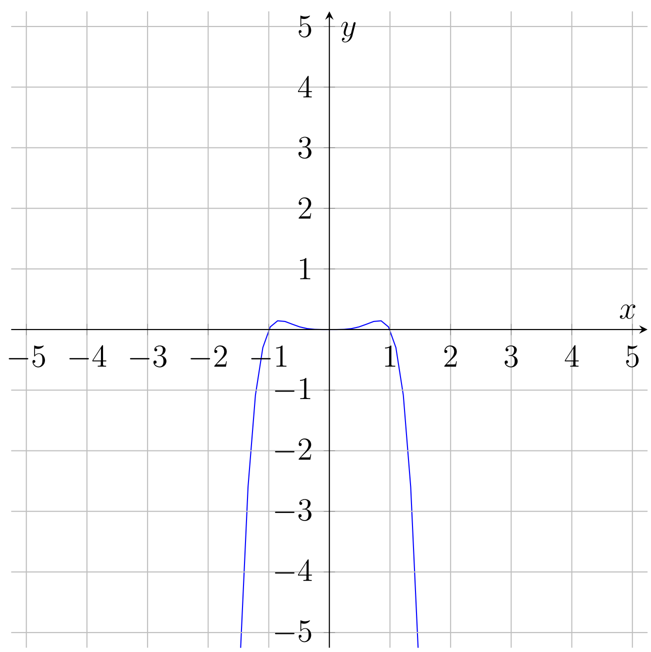
An example of such function f is $f(x) = \frac{x-2}{x^2(3-x)}$ as one may verify.

Problem 61

Find the limits as $x \to \infty$ and $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 12.

$$y = x^4 - x^6$$

For the limits at infinities, we have $x^4 - x^6 = x^6\left(\frac{1}{x^2} - 1\right)$, $\frac{1}{x^2} \to 0$ and $x^6 \to \infty$ as $x \to \pm\infty$, meaning $x^6\left(\frac{1}{x^2} - 1\right) \to -\infty$ as $x \to \infty$ (since $\frac{1}{x^2} - 1 \to -1 < 0$). The graph of the curve $u = x^4 - x^6$ is given below.



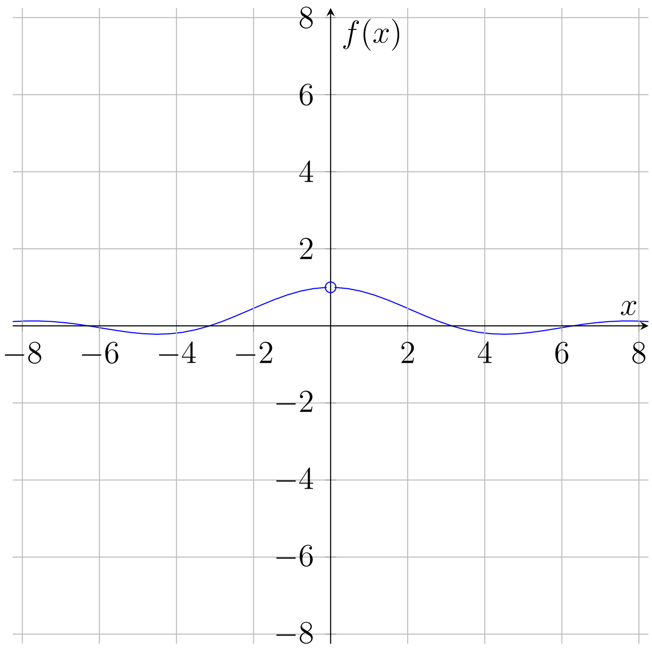
Problem 65

(a) Use the Squeeze Theorem to evaluate $\lim_{x\to\infty} \frac{\sin x}{x}$.

(b) Graph $f(x) = (\sin x)/x$. How many times does the graph cross the asymptote?

For part (a), we have $-1 \leq \sin x \leq 1$ for all x, meaning $-\frac{1}{x} = \frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$ for all $x \neq 0$. We also have $\lim\limits_{x\to\infty} \left[-\frac{1}{x}\right] = 0 = \lim\limits_{x\to\infty} \frac{1}{x}$. By squeeze theorem, we have $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0$

For part (b), the graph of $f(x) = \frac{\sin x}{x}$ is given below.



The graph crosses the asymptote infinitely many times as we have $\frac{\sin x}{x} = 0$ for all $x = n\pi$ where n is a positive integer.

Chapter 2

Derivatives

2.1 Section 2.7

Problem 1

A curve has equation $y = f(x)$.

(a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.

(b) Write an expression for the slope of the tangent line at P.

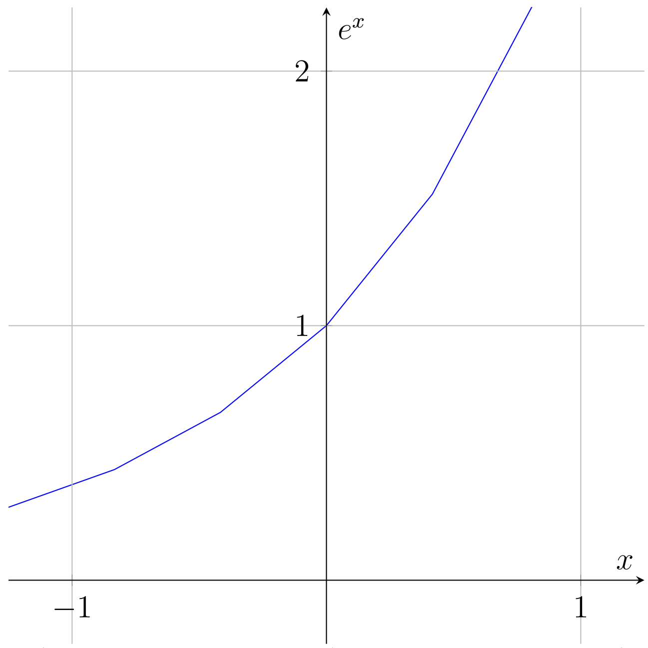
For part (a), assuming f is defined at x, the slope of the secant line through the point P and Q is $\frac{f(x)-f(3)}{x-3}$.

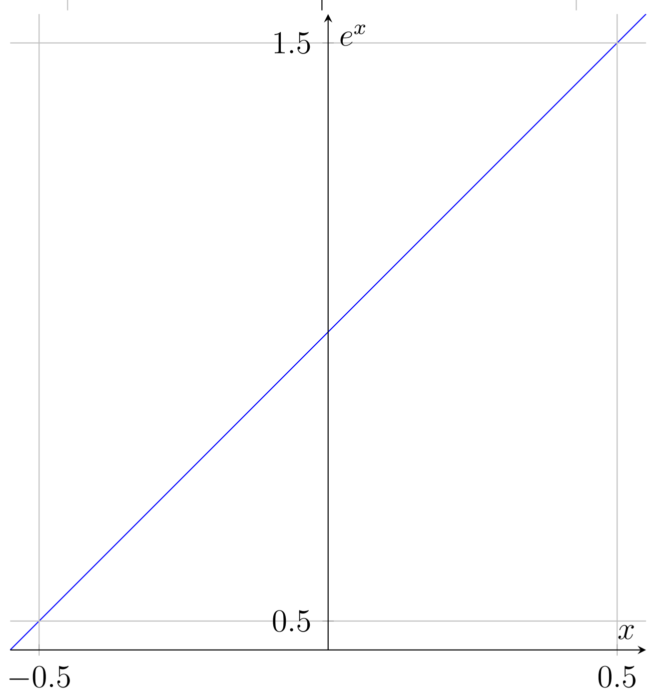
For part (b), assuming the tangent line at P exists, it is $\lim\limits_{x\to 3} \frac{f(x)-f(3)}{x-3}$.

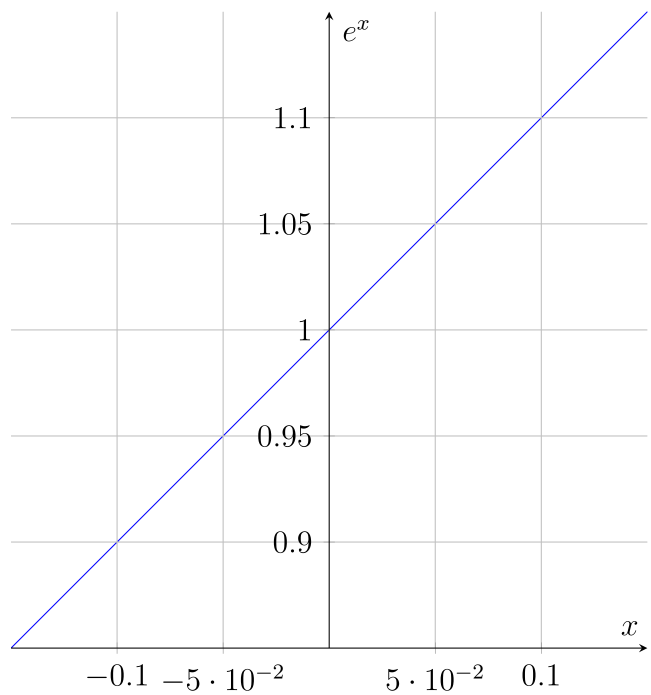
Problem 2

Graph the curve $y = e^x$ in the viewing rectangles $[-1, 1]$ by $[0, 2]$, $[-0.5, 0.5]$ by $[0.5, 1.5]$, and $[-0.1, 0.1]$ by $[0.9, 1.1]$. What do you notice about the curve as you zoom in towards the point $(0, 1)$?

The graphs of e^x in the given rectangles are given below.


e^x
2
1
x
-1
1


e^x
1.5
0.5
x
-0.5
0.5



The further we zoom into the point $(0, 1)$, the more the observable curve resembles a straight line.

Problem 3

(a) Find the slope of the tangent line to the parabola $y = x^2 + 3x$ at the point $(-1, -2)$

(i) using Definition 1 (ii) using Definition 2

(b) Find an equation of the tangent line in part (a).

(c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point (-1, -2) until the parabola and the tangent line are indistinguishable.

For part (i) of (a), we have the following.

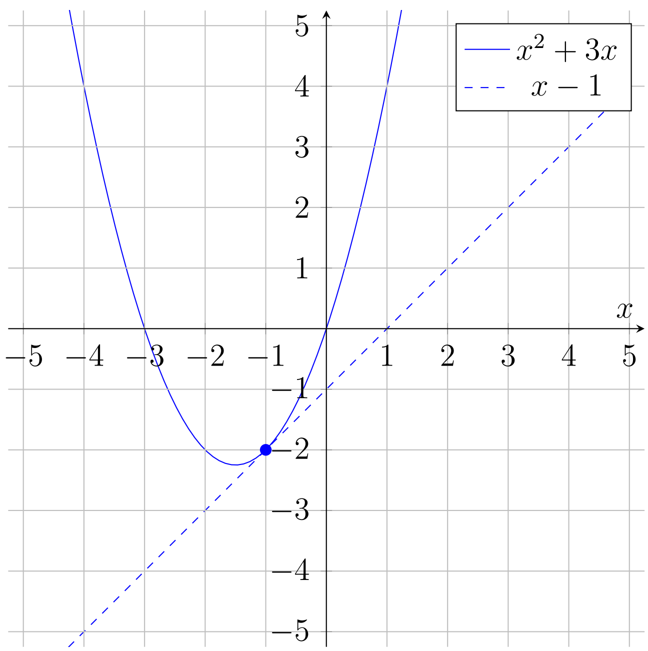
$$\begin{aligned}
\lim_{x\to-1}\frac{x^2+3x-((-1)^2+3(-1))}{x-(-1)} &= \lim_{x\to-1}\frac{x^2-1+3x+3}{x+1} = \\
&= \lim_{x\to-1}\frac{(x-1)(x+1)+3(x+1)}{x+1} = \\
&= \lim_{x\to-1}\frac{(x-1+3)(x+1)}{x+1} = \\
&= \lim_{x\to-1}[x+2] = (-1)+2 = 1
\end{aligned}$$

For part (ii) of (a), we have the following.

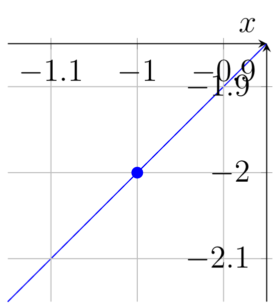
$$\begin{aligned}
&\lim_{h\to0}\frac{(-1+h)^2+3(-1+h)-((-1)^2+3(-1))}{h} = \\
=&\lim_{h\to0}\frac{(-1+h)^2-1+3(-1+h)+3}{h} = \\
=&\lim_{h\to0}\frac{(-1+h-1)(-1+h+1)+3(-1+h+1)}{h} = \\
=&\lim_{h\to0}\frac{h(h-2)+3h}{h} = \lim_{h\to0}[h-2+3] = 0-2+3 = 1
\end{aligned}$$

For part (b), in general, we have that the tangent line to a point $(a, f(a))$ on the curve $y = f(x)$ (assuming it exists) is given by $y = f'(a)(x-a)+f(a)$. In this case, we have $a = -1$, $f(a) = f(-1) = -2$, $f'(a) = f'(-1) = 1$ (as shown in part (a)). Hence the tangent line is $y = 1(x-(-1)) + (-2) = x+1-2 = x-1$.

For part (c), graph of the parabola and the tangent line are given below.



Once we zoom towards the point $(-1, -2)$, we see the following:



Problem 4

(a) Find the slope of the tangent line to the curve $y = x^3 + 1$ at the point $(1, 2)$

  (i) using Definition 1 (ii) using Definition 2

(b) Find an equation of the tangent line in part (a).

(c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(1, 2)$ until the curve and the line appear to coincide.

For part (i) of (a), we have the following.

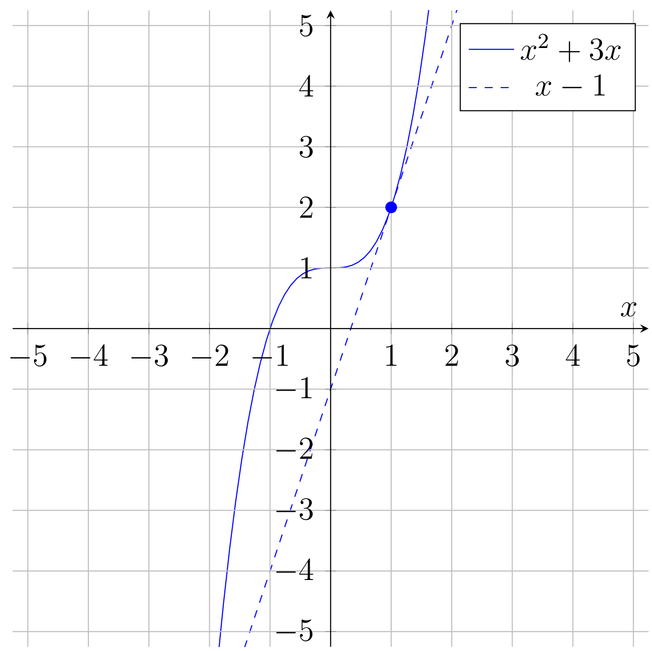
$$\lim_{x\to 1}\frac{x^3+1-(1^3+1)}{x-1} = \lim_{x\to 1}\frac{x^3-1+1-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1} =$$
$$= \lim_{x\to 1}[x^2+x+1] = 1^2+1+1 = 3$$

For part (ii) of (a), we have the following.

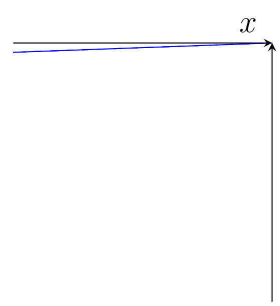
$$\lim_{h\to 0}\frac{(1+h)^3+1-(1^3-1)}{h} = \lim_{h\to 0}\frac{(1+h)^3-1+1-1}{h} = \lim_{h\to 0}\frac{1+3h+3h^2+h-1}{h} =$$
$$= \lim_{h\to 0}\frac{3h+3h^2+h^3}{h} = \lim_{h\to 0}[3+3h+h^2] = 3$$

For part (b), in general, we have that the tangent line to a point $(a, f(a))$ on the curve $y = f(x)$ (assuming it exists) is given by $y = f'(a)(x-a)+f(a)$. In this case, we have $a = 1$, $f(a) = f(1) = 2$, $f'(a) = f'(1) = 3$ (as shown in part (a)). Hence the tangent line is $y = 3(x-1)+2 = 3x-1$.

For part (c), graph of the parabola and the tangent line are given below.



Once we zoom towards the point $(1, 2)$, we see the following:



Problem 5

Find an equation of the tangent line to the curve at the given point.

$$y = 2x^2 - 5x + 1, \quad (3, 4)$$

We verify that the point $(3, 4)$ indeed lies on the given curve. $y(3) = 2(3)^2 - 5(3) + 1 = 4$.

We start by finding the slope of the tangent line at $(3, 4)$.

$$\lim_{x\to 3} \frac{2x^2 - 5x + 1 - 4}{x - 3} = \lim_{x\to 3} \frac{2x^2 - 5x - 3}{x - 3} = \lim_{x\to 3} \frac{(2x + 1)(x - 3)}{x - 3} =$$
$$= \lim_{x\to 3} [2x + 1] = 2(3) + 1 = 7$$

We thus obtain the equation of the tangent line.

$$y = f'(a)(x - a) + f(a) \overset{a=3}{\iff} y = 7(x - 3) + 4$$
$$\iff y = 7x - 17$$

Problem 6

Find an equation of the tangent line to the curve at the given point.

$$y = x^2 - 2x^3, \quad (1, -1)$$

We verify that the point $(1, -1)$ indeed lies on the given curve. $y(1) = 1^2 - 2(1)^3 = -1$.

We start by finding the slope of the tangent line at $(1, -1)$.

$$\lim_{x\to 1} \frac{x^2 - 2x^3 - (-1)}{x - 1} = \lim_{x\to 1} \frac{x^2 - 2x^3 + 1}{x - 1} = \lim_{x\to 1} \frac{x^2 - 1 - 2x^3 + 2}{x - 1} =$$
$$= \lim_{x\to 1} \frac{(x - 1)(x + 1) - 2(x^3 - 1)}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x + 1) - 2(x - 1)(x^2 + x + 1)}{x - 1}$$
$$= \lim_{x\to 1} \frac{(x - 1)(x + 1 - 2(x^2 + x + 1))}{x - 1} = \lim_{x\to 1} [x + 1 - 2(x^2 + x + 1)] = 2 - 2(3)$$

We thus obtain the equation of the tangent line.

$$y = f'(a)(x - a) + f(a) \overset{a=1}{\iff} y = -4(x - 1) + (-1)$$
$$\iff y = 3 - 4x$$

Problem 7

Find an equation of the tangent line to the curve at the given point.

$$y = \frac{x+2}{x-3}, \quad (2, -4)$$

We verify that the point $(2, -4)$ indeed lies on the given curve. $y(2) = \frac{2+2}{2-3} = \frac{4}{-1} = -4$.

We start by finding the slope of the tangent line at $(2, -4)$.

$$\begin{aligned}\lim_{x\to 2} \frac{\frac{x+2}{x-3} - (-4)}{x-2} &= \lim_{x\to 2} \frac{(x+2)+4(x-3)}{(x-2)(x-3)} = \lim_{x\to 2} \frac{x+2+4x-12}{(x-2)(x-3)} = \\ &= \lim_{x\to 2} \frac{5x-10}{(x-2)(x-3)} = \lim_{x\to 2} \frac{5(x-2)}{(x-2)(x-3)} = \\ &= \lim_{x\to 2} \frac{5}{x-3} = \frac{5}{2-3} = -5\end{aligned}$$

We thus obtain the equation of the tangent line.

$$\begin{aligned} y = f'(a)(x-a) + f(a) &\overset{x=2}{\iff} y = -5(x-2) + (-4) \\ &\iff y = 6 - 5x \end{aligned}$$

Problem 8

Find an equation of the tangent line to the curve at the given point.

$$y = \sqrt{1-3x}, \quad (-1, 2)$$

We verify that the point $(-1, 2)$ indeed lies on the given curve. $y(-1) = \sqrt{1-3(-1)} = 2$.

We start by finding the slope of the tangent line at $(-1, 2)$.

$$\lim_{x \to -1} \frac{\sqrt{1-3x}-2}{x-(-1)} = \lim_{x \to -1} \frac{(\sqrt{1-3x}-2)(\sqrt{1-3x}+2)}{(x+1)(\sqrt{1-3x}+2)} = \lim_{x \to -1} \frac{\sqrt{1-3x}^2 - 2^2}{(x+1)(\sqrt{1-3x}+2)} =$$

$$= \lim_{x \to -1} \frac{}{(x}$$

$$= \lim_{x \to -1} \frac{}{(x}$$

$$= \frac{-}{\sqrt{1-3}}$$

We thus obtain the equation of the tangent line.

$$\begin{aligned} y = f'(a)(x-a) + f(a) &\overset{a=-1}{\iff} y = -\frac{3}{4}(x-(-1)) + 2 \\ &\iff y = \frac{5-3x}{4} \end{aligned}$$

Problem 9

(a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.

(b) Find equations of the tangent lines at the points $(1, 5)$ and $(2, 3)$.

(c) Graph the curve and both tangent lines on a common screen.

For part (a), we have the following.

$$\begin{aligned} \lim_{x \to a} \frac{3 + 4x^2 - 2x^3 - (3 + 4a^2 - 2a^3)}{x-a} &= \lim_{x \to a} \frac{3 - 3 + 4x^2 - 4a^2 - 2x^3 + 2a^3}{x-a} = \lim_{x \to a} \frac{4(x^2-a^2) - 2(x^3}{x-a} \\ &= \lim_{x \to a} \frac{4(x-a)(x+a) - 2(x-a)(x^2+ax+a^2)}{x-a} = \lim_{x \to a} \frac{(x-a)}{} \\ &= \lim_{x \to a} [4x + 4a - 2(x^2 + ax + a^2)] = 8a - 6a^2 \end{aligned}$$

For part (b), we verify that the points $(1, 5)$ and $(2, 3)$ indeed lie on the given curve.

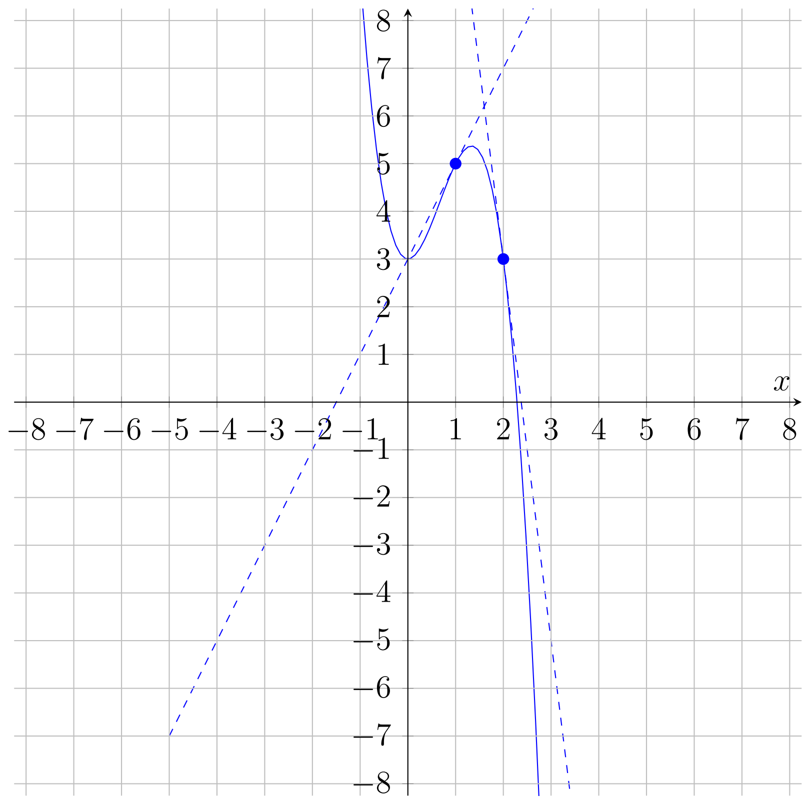
$$y(1) = 3 + 4(1)^2 - 2(1)^3 = 5$$
$$y(2) = 3 + 4(2)^2 - 2(2)^3 = 3$$

We thus use the formula derived in part (a) and obtain the tangent lines at $(1, 5)$ and $(2, 3)$ respectively.

$$\begin{aligned} y = f'(a)(x - a) + f(a) &\overset{a=1}{\iff} y = (8(1) - 6(1)^2)(x - 1) + 5 \\ &\iff y = 2x + 3 \end{aligned}$$

$$\begin{aligned} y = f'(a)(x - a) + f(a) &\overset{a=2}{\iff} y = (8(2) - 6(2)^2)(x - 2) + 3 \\ &\iff y = 19 - 8x \end{aligned}$$

For part (c), the graph of the given curve and both tangent lines is given below.



Problem 10

(a) Find the slope of the tangent to the curve $y = 2\sqrt{x}$ at the point where $x = a$.

(b) Find equations of the tangent lines at the points $(1, 2)$ and $(9, 6)$.

(c) Graph the curve and both tangents on the same screen.

For part (a), we have the following.

$$\lim_{x\to a} \frac{2\sqrt{x} - 2\sqrt{a}}{x - a} = \lim_{x\to a} \frac{2(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{(x - a)(\sqrt{x} + \sqrt{a})} = \lim_{x\to a} \frac{2(x - a)}{(x - a)(\sqrt{x} + \sqrt{a})} =$$
$$= \lim_{x\to a} \frac{2}{\sqrt{x} + \sqrt{a}} = \frac{2}{\sqrt{a} + \sqrt{a}} = \frac{1}{\sqrt{a}}$$

For part (b), we verify that the points $(1, 2)$ and $(9, 6)$ lie on the given curve.

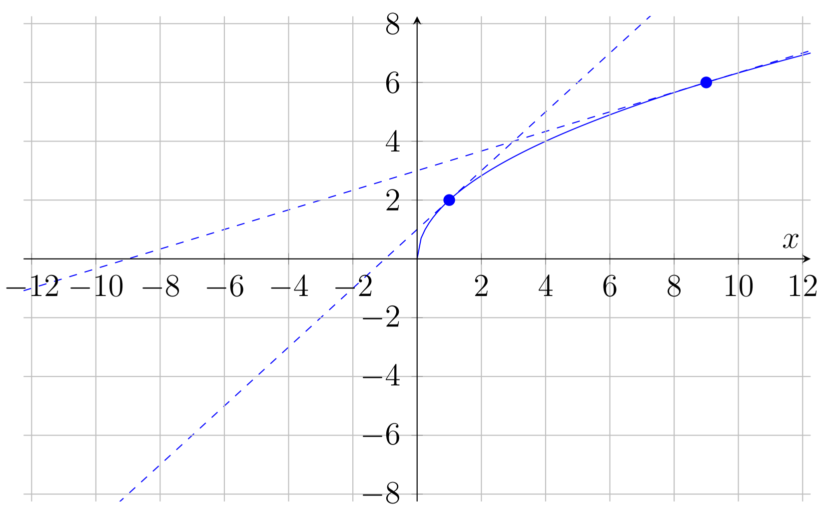
$$y(1) = 2\sqrt{1} = 2$$
$$y(9) = 2\sqrt{9} = 6$$

We thus use the formula derived in part (a) and obtain the tangent lines at $(1, 2)$ and $(9, 6)$ respectively.

$$\begin{aligned} y = f'(a)(x-a) + f(a) &\overset{a=1}{\iff} y = \frac{1}{\sqrt{1}}(x-1) + 2 \\ &\iff y = x + 1 \end{aligned}$$

$$\begin{aligned} y = f'(a)(x-a) + f(a) &\overset{a=9}{\iff} y = \frac{1}{\sqrt{9}}(x-9) + 6 \\ &\iff y = \frac{x}{3} + 3 \end{aligned}$$

For part (c), the graph of the given curve and both tangent lines is given below.



Problem 11

A cliff diver plunges from a height of 30 m above the water surface. The distance the diver falls in t seconds is given by the function $d(t) = 4.9t^2$ m.

(a) After how many seconds will the diver hit the water?

(b) With what velocity does the diver hit the water?

For part (a), we would require the distance that the diver had fallen to be exactly 30 meters, i.e., the diver would hit the water after t seconds given by $d(t) = 30$. We solve for t.

$$\begin{aligned} d(t) &= 30 \\ 4.9t^2 &= 30 \\ t^2 &= \frac{30}{4.9} = \frac{300}{49} \\ t &= \sqrt{\frac{300}{49}} = \frac{10\sqrt{3}}{7} && (t \geq 0) \end{aligned}$$

Hence the diver will hit the water $\frac{10\sqrt{3}}{7}$ seconds after plunging.

For part (b), we shall find the derivative of $d(t)$ at $t = \frac{10\sqrt{3}}{7}$ as follows.

$$\begin{aligned} \lim_{t \to \frac{10\sqrt{3}}{7}} \frac{d(t) - d\left(\frac{10\sqrt{3}}{7}\right)}{t - \frac{10\sqrt{3}}{7}} &= \lim_{t \to \frac{10\sqrt{3}}{7}} \frac{4.9t^2 - 4.9\left(\frac{10\sqrt{3}}{7}\right)^2}{t - \frac{10\sqrt{3}}{7}} = \\ &= \lim_{t \to \frac{10\sqrt{3}}{7}} \frac{4.9\left(t^2 - \left(\frac{10\sqrt{3}}{7}\right)^2\right)}{t - \frac{10\sqrt{3}}{7}} = 4.9 \lim_{t \to \frac{10\sqrt{3}}{7}} \frac{\left(t - \frac{10\sqrt{3}}{7}\right)\left(t + \frac{10\sqrt{3}}{7}\right)}{t - \frac{10\sqrt{3}}{7}} = \\ &= 4.9 \lim_{t \to \frac{10\sqrt{3}}{7}} \left(t + \frac{10\sqrt{3}}{7}\right) = 4.9\left(\frac{10\sqrt{3}}{7} + \frac{10\sqrt{3}}{7}\right) = 14\sqrt{3} \end{aligned}$$

Problem 12

If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 8.6t^2$.

(a) Find the velocity of the rock after one second.

(b) Find the velocity of the rock at $x = a$.

(c) When will the rock hit the surface?

(d) With what velocity will the rock hit the surface?

For part (a), we calculate the derivative of H at $t = 1$ as follows.

$$\begin{aligned}
\lim_{t\to 1} \frac{H(t) - H(1)}{t-1} &= \lim_{t\to 1} \frac{10t - 8.6t^2 - (10(1) - 8.6(1)^2)}{t-1} = \lim_{t\to 1} \frac{10t - 10 - 8.6t^2 + 8.6}{t-1} = \\
&= \lim_{t\to 1} \frac{10(t-1) - 8.6(t^2-1)}{t-1} = \lim_{t\to 1} \frac{10(t-1) - 8.6(t-1)(t+1)}{t-1} = \\
&= \lim_{t\to 1} \frac{(t-1)(10 - 8.6t - 8.6)}{t-1} = \lim_{t\to 1}[1.4 - 8.6t] = \\
&= 1.4 - 8.6(1) = -7.2
\end{aligned}$$

For part (b), we have the following.

$$\begin{aligned}
\lim_{t\to a} \frac{H(t) - H(a)}{t-a} &= \lim_{t\to a} \frac{10t - 8.6t^2 - (10a - 8.6a^2)}{x-a} = \lim_{t\to a} \frac{10t - 10a - 8.6t^2 + 8.6a^2}{t-a} = \\
&= \lim_{t\to a} \frac{10(t-a) - 8.6(t^2-a^2)}{t-a} = \lim_{t\to a} \frac{10(t-a) - 8.6(t-a)(t+a)}{t-a} = \\
&= \lim_{t\to a} \frac{(t-a)(10 - 8.6t - 8.6a)}{t-a} = \lim_{t\to a}[10 - 8.6t - 8.6a] = \\
&= 10 - 8.6a - 8.6a = 10 - 17.2a
\end{aligned}$$

For part (c), we would require the height of the rock to be 0 after t seconds for some $t > 0$, i.e., the rock would hit the ground after t seconds given by

$H(t) = 0$ with $t > 0$. We solve for t.

$$\begin{aligned}
H(t) &= 0 \\
10t - 8.6t^2 &= 0 \\
t(10 - 8.6t) &= 0 \\
10 - 8.6t &= 0 && (t \neq 0) \\
10 &= 8.6t \\
t &= \frac{10}{8.6} = \frac{50}{43}
\end{aligned}$$

For part (d), the velocity which the rock will hit the group with is the derivative of H at t whenever t is the amount of time it took the rock to fall to the ground. We use the results of parts (b) and (c) and obtain that that velocity is $10 - 17.2\left(\frac{50}{43}\right) = -10$.

Problem 13

The displacement (in meters) of a particle moving in a straight line given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

We shall find the derivative of s at $t = a$ for nonzero a (since s is undefined at $t = 0$) and then simply plug in $a = 1$, $a = 2$ and $a = 3$ to find the velocities of the particle at times $t = 1$, $t = 2$ and $t = 3$ respectively. We have the following.

$$\begin{aligned}
\lim_{t\to a} \frac{s(t) - s(a)}{t - a} &= \lim_{t\to a} \frac{\frac{1}{t^2} - \frac{1}{a^2}}{t - a} = \lim_{t\to a} \frac{a^2 - t^2}{(t-a)t^2a^2} = \\
&= \lim_{t\to a} \frac{(a-t)(a+t)}{(t-a)t^2a^2} = -\lim_{t\to a} \frac{a+t}{t^2a^2} = \\
&= -\frac{a+a}{a^2a^2} = -\frac{2}{a^3}
\end{aligned}$$

Thus the velocities of the particle at times $t = 1$, $t = 2$ and $t = 3$ are $-\frac{2}{1^3} = -2$, $-\frac{2}{2^3} = -\frac{1}{4}$ and $-\frac{2}{3^3} = -\frac{2}{27}$ respectively.

Problem 14

The displacement (in meters) of a particle moving in a straight line is given by $s = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.

(a) Find the average velocity over each time interval:

(i) $[4, 8]$

(ii) $[6, 8]$

(iii) $[8, 10]$

(iv) $[6, 12]$

(b) Find the instantaneous velocity when $t = 8$.

(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a). Then draw the tangent line whose slope is the instantaneous velocity in part (b).

For part (a), we shall use the formula which says that the average velocity of an object over a time interval is equal to the ratio between the change in position occurring in that time interval and the duration of the time interval. We shall denote average velocity with v for this part.

For part (i), we have the following.

$$\begin{aligned}\frac{s(8) - s(4)}{8 - 4} &= \frac{\frac{1}{2}(8)^2 - 6(8) + 23 - (\frac{1}{2}(4)^2 - 6(4) + 23)}{4} = \\ &= \frac{\frac{1}{2}(8)^2 - \frac{1}{2}(4)^2 - 6(8) + 6(4) + 23 - 23}{4} = \\ &= \frac{\frac{1}{2}(8^2 - 4^2) - 6(8 - 4)}{4} = \frac{\frac{1}{2}(4)(12) - 6(4)}{4} = \\ &= 6 - 6 = 0\end{aligned}$$

For part (ii), we have the following.

$$\begin{aligned}\frac{s(8)-s(6)}{8-6} &= \frac{\frac{1}{2}(8)^2-6(8)+23-(\frac{1}{2}(6)^2-6(6)+23)}{2} = \\ &= \frac{\frac{1}{2}(8)^2-\frac{1}{2}(6)^2-6(8)+6(6)+23-23}{2} = \\ &= \frac{\frac{1}{2}(14)(2)-6(2)}{2} = 7-6 = 1\end{aligned}$$

For part (iii), we have the following.

$$\begin{aligned}\frac{s(10)-s(8)}{10-8} &= \frac{\frac{1}{2}(10)^2-6(10)+23-(\frac{1}{2}(8)^2-6(8)+23)}{2} = \\ &= \frac{\frac{1}{2}(10)^2-\frac{1}{2}(8)^2-6(10)+6(8)+23-23}{2} = \\ &= \frac{\frac{1}{2}(18)(2)-6(2)}{2} = \frac{1}{2}(18)-6 = \\ &= 9-6 = 3\end{aligned}$$

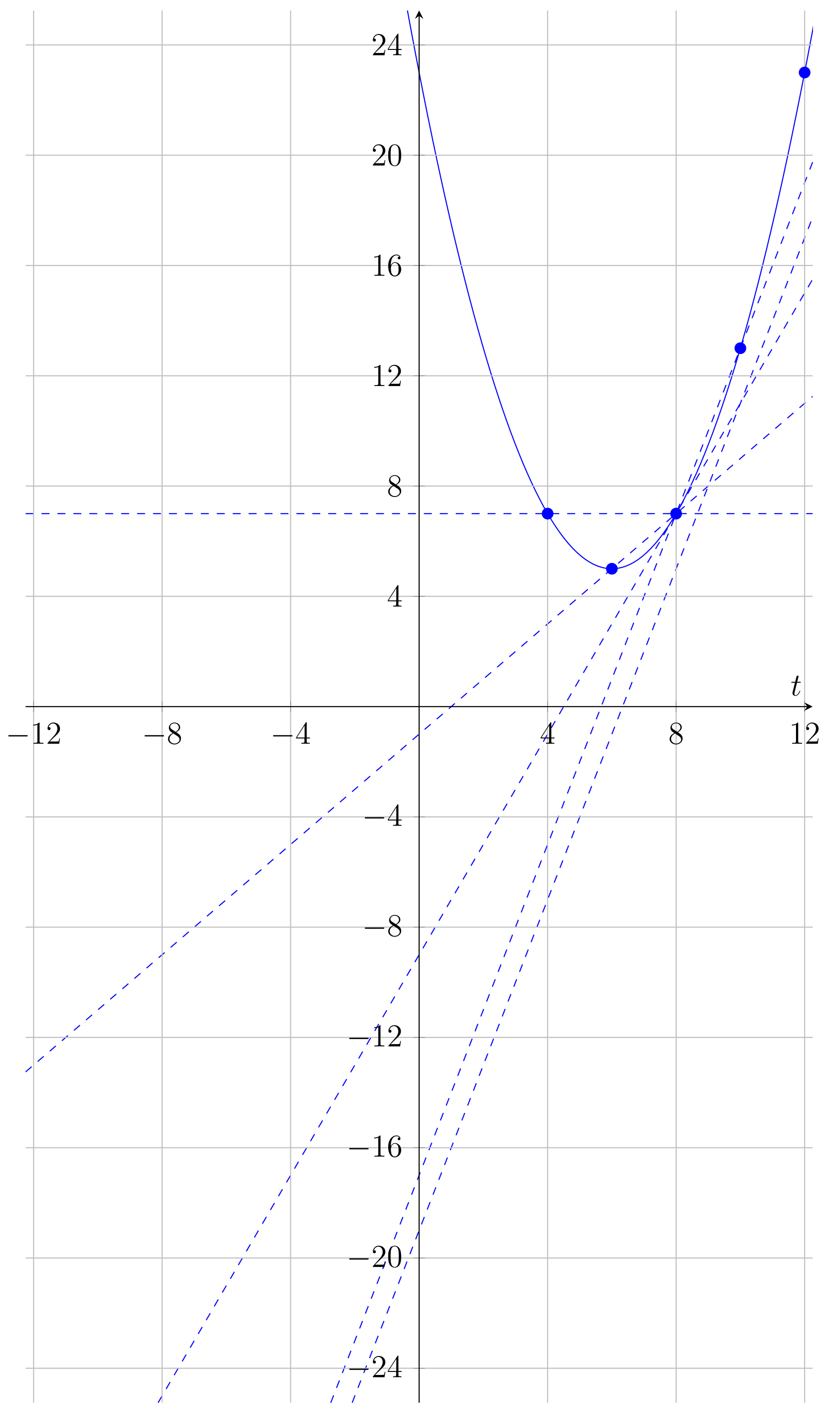
For part (iv), we have the following.

$$\begin{aligned}\frac{s(12)-s(6)}{12-6} &= \frac{\frac{1}{2}(12)^2-6(12)+12-(\frac{1}{2}(6)^2-6(6)+23)}{6} = \\ &= \frac{\frac{1}{2}(12)^2-\frac{1}{2}(6)^2-6(12)+6(6)+23-23}{6} = \\ &= \frac{\frac{1}{2}(18)(6)-6(6)}{6} = \frac{1}{2}(18)-6 = \\ &= 9-6 = 3\end{aligned}$$

For part (b), we have the following.

$$\begin{aligned}
\lim_{t\to 8}\frac{s(t)-s(8)}{t-8} &= \lim_{t\to 8}\frac{\frac{1}{2}t^2-6t+23-(\frac{1}{2}(8)^2-6(8)+23)}{t-8} = \\
&= \lim_{t\to 8}\frac{\frac{1}{2}t^2-\frac{1}{2}(8)^2-6t+6(8)+23-23}{t-8} = \\
&= \lim_{t\to 8}\frac{\frac{1}{2}(t^2-8^2)-6(t-8)}{t-8} = \\
&= \lim_{t\to 8}\frac{\frac{1}{2}(t-8)(t+8)-6(t-8)}{t-8} = \lim_{t\to 8}[\frac{1}{2}(t+8)-6] \\
&= \frac{1}{2}(8+8)-6 = 2
\end{aligned}$$

For part (c), secant lines whose slopes are the average velocities in part (a) can be given by $y = 0\cdot(t-8)+s(8) = \frac{1}{2}(8)^2-6(8)+23 = \frac{1}{2}(64)-48+23 = 32-48+23 = 55-48 = 7$, $y = 1\cdot(t-8)+s(8) = t-1$, $y = 3(t-8)+s(8) = 3t-17$ and $y = 3(t-8)+s(6) = 3t-24+\frac{1}{2}(6)^2-6(6)+23 = 3t-1+\frac{1}{2}(36)-36 = 3t-1+18-36 = 3t-1-18 = 3t-19$. The tangent line whose slope is the instantaneous velocity in part (b) is given by $y = s'(8)(t-8)+s(8) = 2(t-8)+7 = 2t-9$. The graph of s, the secant lines and the tangent line is given below.

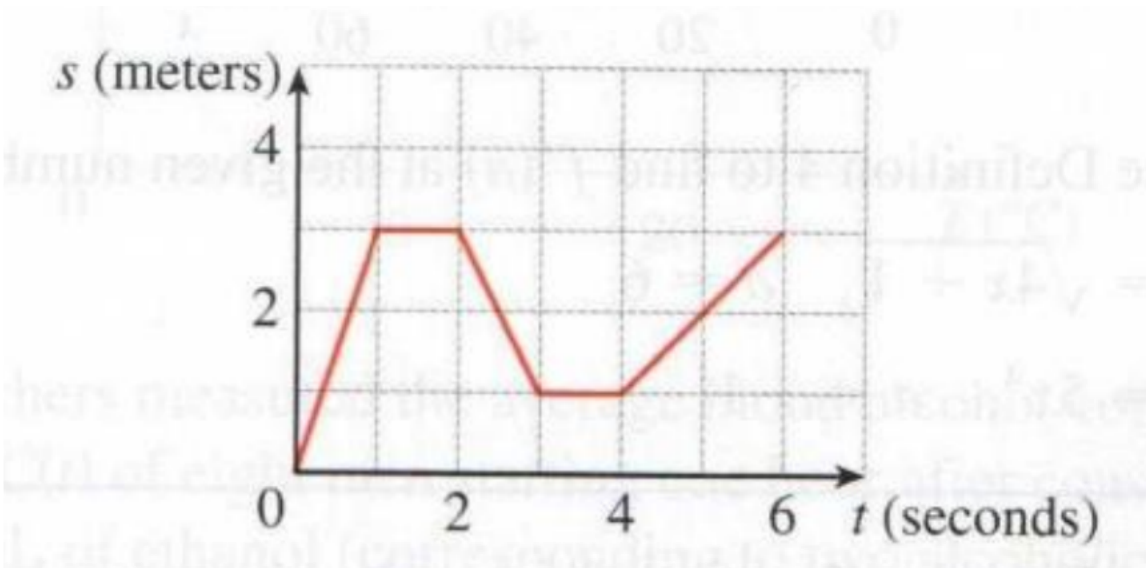


24
20
16
12
8
4
t
−12
−8
−4
4
8
12
−4
−8
−12
−16
−20
−24

Problem 15

(a) A particle starts moving to the right along a horizontal line; the graph of its position function is shown in the figure. When is the particle moving to the right? Moving to the left? Standing still?

(b) Draw a graph of the velocity function.



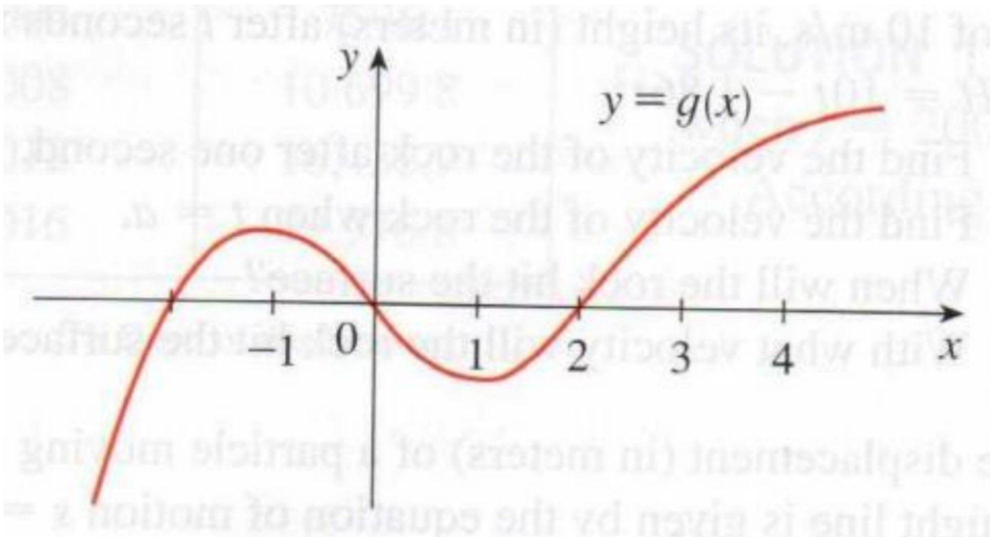
For part (a), since we know that the particle initially is moving to the right, clearly, the positive vertical direction in the graph of s indicates moving to the right and the negative vertical direction thus indicates moving to the left. Hence the particle is moving to the right on the open intervals $(0, 1)$ and $(4, 6)$, moving to the left on the open interval $(2, 4)$ and standing still on the open intervals $(1, 2)$ and $(3, 4)$.

For part (b), ...

Problem 17

For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \quad g'(-2) \quad g'(0) \quad g'(2) \quad g'(4)$$



Clearly, the function g is decreasing at the given points only at $x = 0$, meaning $0 > g'(0)$. Clearly, the curve is steeper at $x = -2$ than at $x = 2$, meaning $g'(-2) > g'(2)$ and, finally, the curve is steeper at $x = 2$ than at $x = 4$, meaning $g'(2) > g'(4)$. We thus have $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

Problem 19

Use Definition 4 to find $f'(a)$ at the given number a.

$$f(x) = \sqrt{4x+1}, \quad a = 6$$

We have the following.

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{f(6+h)-f(6)}{h} = \lim_{h\to 0}\frac{\sqrt{4(6+h)+1}-\sqrt{4(6)+1}}{h} = \\
&= \lim_{h\to 0}\frac{(\sqrt{4(6+h)+1}-\sqrt{4(6)+1})(\sqrt{4(6+h)+1}+\sqrt{4(6)+1})}{h(\sqrt{4(6+h)+1}+\sqrt{4(6)+1})} = \\
&= \lim_{h\to 0}\frac{\sqrt{4(6)+4h+1}^2-\sqrt{4(6)+1}^2}{h(\sqrt{4(6+h)+1}+\sqrt{4(6)+1})} = \\
&= \lim_{h\to 0}\frac{4h+4(6)+1-(4(6)+1)}{h(\sqrt{4(6+h)+1}+\sqrt{4(6)+1})} \\
&= \lim_{h\to 0}\frac{4h}{h(\sqrt{4(6+h)+1}+\sqrt{4(6)+1})} = \\
&= \lim_{h\to 0}\frac{4}{\sqrt{4(6+h)+1}+\sqrt{4(6)+1}} \\
&= \frac{4}{\sqrt{4(6+0)+1}+\sqrt{4(6)+1}} = \frac{2}{5}
\end{aligned}
$$

Hence $f'(a) = \lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h} = \frac{2}{5}$.

Problem 20

Use Definition 4 to find $f'(a)$ at the given number a.

$$f(x) = 5x^4, \quad a = -1$$

We have the following.

$$\begin{aligned}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0}\frac{5(h-1)^4-5(-1)^4}{h} = \\ &= \lim_{h\to 0}\frac{5((h-1)^4-1^4)}{h} = \\ &= \lim_{h\to 0}\frac{5((h-1)^2-1^2)((h-1)^2+1^2)}{h} = \\ &= \lim_{h\to 0}\frac{5((h-1)-1)((h-1)+1)((h-1)^2+1)}{h} = \\ &= \lim_{h\to 0}\frac{5(h-2)h((h-1)^2+1)}{h} = \\ &= \lim_{h\to 0} 5(h-2)((h-1)^2+1) = 5(0-2)((0-1)^2+1) = -20\end{aligned}$$

Hence $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = -20$.

Problem 21

Use Equation 5 to find $f'(a)$ at the given number a.

$$f(x) = \frac{x^2}{x+6}, \quad a = 3$$

We have the following.

$$\begin{aligned}f'(a) = f'(3) = \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} &= \lim_{x\to 3}\frac{\frac{x^2}{x+6}-\frac{3^2}{3+6}}{x-3} = \lim_{x\to 3}\frac{x^2-(x+6)}{(x-3)(x+6)} = \\ = \lim_{x\to 3}\frac{x^2-x-6}{(x-3)(x+6)} &= \lim_{x\to 3}\frac{(x-3)(x+2)}{(x-3)(x+6)} = \\ = \lim_{x\to 3}\frac{x+2}{x+6} = \frac{3+2}{3+6} = \frac{5}{9}\end{aligned}$$

Problem 22

Use Equation 5 to find $f'(a)$ at the given number a.

$$f(x) = \frac{1}{\sqrt{2x+2}}, \quad a = 1$$

We have the following.

$$\begin{aligned}
f'(a) = f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} &= \lim_{x\to 1}\frac{\sqrt{2x+2}-\sqrt{2(1)+2}}{x-1} = \\
&= \lim_{x\to 1}\frac{(\sqrt{2x+2}-2)(\sqrt{2x+2}+2)}{(x-1)(\sqrt{2x+2}+2)} = \\
&= \lim_{x\to 1}\frac{\sqrt{2x+2}^2-2^2}{(x-1)(\sqrt{2x+2}+2)} = \\
&= \lim_{x\to 1}\frac{2x+2-4}{(x-1)(\sqrt{2x+2}+2)} = \\
&= \lim_{x\to 1}\frac{2x-2}{(x-1)(\sqrt{2x+2}+2)} = \\
&= \lim_{x\to 1}\frac{2(x-1)}{(x-1)(\sqrt{2x+2}+2)} = \\
&= \lim_{x\to 1}\frac{2}{\sqrt{2x+2}+2} = \frac{2}{\sqrt{2(1)+2}+2} = \frac{1}{2}
\end{aligned}$$

Problem 23

Find $f'(a)$.

$$f(x) = 2x^2 - 5x + 3$$

We have the following.

$$
\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{2x^2-5x+3-(2a^2-5a+3)}{x-a} = \\
&= \lim_{x\to a}\frac{2x^2-2a^2-5x+5a+3-3}{x-a} = \\
&= \lim_{x\to a}\frac{2(x^2-a^2)-5(x-a)}{x-a} = \\
&= \lim_{x\to a}\frac{2(x-a)(x+a)-5(x-a)}{x-a} = \\
&= \lim_{x\to a}[2x+2a-5] = 2a+2a-5 = 10a-5
\end{aligned}
$$

Problem 24

Find $f'(a)$.

$$f(t) = t^3 - 3t$$

We have the following.

$$
\begin{aligned}
f'(a) &= \lim_{t\to a}\frac{f(t)-f(a)}{t-a} = \lim_{t\to a}\frac{t^3-3t-(a^3-3a)}{t-a} = \\
&= \lim_{t\to a}\frac{t^3-a^3-3t+3a}{t-a} = \\
&= \lim_{t\to a}\frac{(t-a)(t^2+at+a^2)-3(t-a)}{t-a} = \\
&= \lim_{t\to a}[(t^2+at+a^2)-3] = 3a^2-3
\end{aligned}
$$

Problem 25

Find $f'(a)$.

$$f(t) = \frac{1}{t^2+1}$$

We have the following.

$$
\begin{aligned}
f'(a) &= \lim_{t\to a}\frac{f(t)-f(a)}{t-a} = \lim_{t\to a}\frac{\frac{1}{t^2+1}-\frac{1}{a^2+1}}{t-a} = \\
&= \lim_{t\to a}\frac{(a^2+1)-(t^2+1)}{(t-a)(t^2+1)(a^2+1)} = \\
&= \lim_{t\to a}\frac{a^2-t^2+1-1}{(t-a)(t^2+1)(a^2+1)} = \\
&= \lim_{t\to a}\frac{(a-t)(a+t)}{(t-a)(t^2+1)(a^2+1)} = \\
&= -\lim_{t\to a}\frac{a+t}{(t^2+1)(a^2+1)} = -\frac{2a}{(a^2+1)^2}
\end{aligned}
$$

Problem 26

Find $f'(a)$.

$$
f(x) = \frac{x}{1-4x}
$$

We have the following.

$$
\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{\frac{x}{1-4x}-\frac{a}{1-4a}}{x-a} = \\
&= \lim_{x\to a}\frac{x(1-4a)-a(1-4x)}{(x-a)(1-4x)(1-4a)} = \\
&= \lim_{x\to a}\frac{x-4ax-a+4ax}{(x-a)(1-4x)(1-4a)} = \\
&= \lim_{x\to a}\frac{x-a}{(x-a)(1-4x)(1-4a)} = \\
&= \lim_{x\to a}\frac{1}{(1-4x)(1-4a)} = \frac{1}{(1-4a)^2}
\end{aligned}
$$

Problem 27

Find an equation of the tangent line to the graph of $y = B(x)$ at $x = 6$ if $B(6) = 0$ and $B'(6) = -\frac{1}{2}$.

The tangent line is given by $y = B'(6)(x-6)+B(6) = -\frac{1}{2}(x-6)+0 = \frac{6-x}{2}$.

Problem 28

Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

The tangent line is given by $y = g'(5)(x-5) + g(5) = 4(x-5) + (-3) = 4x - 23$.

Problem 29

If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point $(1, 2)$.

We verify that the point $(1, 2)$ indeed lies on the curve $y = 3x^2 - x^3$.

$$y(1) = 3(1)^2 - (1)^3 = 2$$

For $f'(1)$, consider the following.

$$\begin{aligned}
\lim_{x\to 1} \frac{3x^2 - x^3 - 2}{x-1} &= \lim_{x\to 1} \frac{3x^2 - 3 - x^3 + 1}{x-1} = \lim_{x\to 1} \frac{3(x^2-1) - (x^3-1)}{x-1} = \\
&= \lim_{x\to 1} \frac{3(x-1)(x+1) - (x-1)(x^2+x+1)}{x-1} = \\
&= \lim_{x\to 1}[3x + 3 - (x^2 + x + 1)] = \\
&= 3 + 3 - (1^2 + 1 + 1) = 3
\end{aligned}$$

Hence the equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point $(1, 2)$ is $y = f'(1)(x-1) + f(1) = 3(x-1) + 2 = 3x - 1$.

Problem 30

If $g(x) = x^4 - 2$, find $g'(1)$ and use it to find an equation to the tangent line to the curve $y = x^4 - 2$ at the point $(1, -1)$.

We verify that the point $(1, -1)$ indeed lies on the given curve. We have $y(1) = 1^4 - 2 = -1$.

We now find $g'(1)$ as follows.

$$\begin{aligned} g'(1) &= \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1} \frac{x^4 - 2 - (-1)}{x - 1} = \\ &= \lim_{x\to 1} \frac{(x^2 - 1)(x^2 + 1)}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x + 1)(x^2 + 1)}{x - 1} = \\ &= \lim_{x\to 1} (x + 1)(x^2 + 1) = (1 + 1)(1^2 + 1) = 4 \end{aligned}$$

Hence the equation of the tangent line at $x = 1$ is given by $y = g'(1)(x - 1) + g(1) = 4(x - 1) + (-1) = 4x - 5$.

Problem 31

(a) If $F(x) = 5x/(1 + x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1 + x^2)$ at the point $(2, 2)$.

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

For part (a), we verify that the point $(2, 2)$ indeed lies on the given curve. $y(2) = 5(2)/(1 + 2^2) = 2$.

We now find $F'(2)$ as follows.

$$\begin{aligned}
F'(2) &= \lim_{x\to 2}\frac{F(x)-F(2)}{x-2} = \lim_{x\to 2}\frac{\frac{5x}{1+x^2}-2}{x-2} = \\
&= \lim_{x\to 2}\frac{5x-2(1+x^2)}{(x-2)(1+x^2)} = \lim_{x\to 2}\frac{-2x^2+5x-2}{(x-2)(1+x^2)} = \\
&= \lim_{x\to 2}\frac{-(x-2)(2x-1)}{(x-2)(1+x^2)} = -\lim_{x\to 2}\frac{2x-1}{1+x^2} = \\
&= -\frac{2(2)-1}{1+2^2} = -\frac{3}{5}
\end{aligned}$$

Hence the equation of the tangent line is $y = F'(2)(x-2)+F(2) = -\frac{3}{5}(x-2)+2 = \frac{16-3x}{5}$.

For part (b), the graph of the given curve and the tangent line is given below.

Problem 32

Problem 33

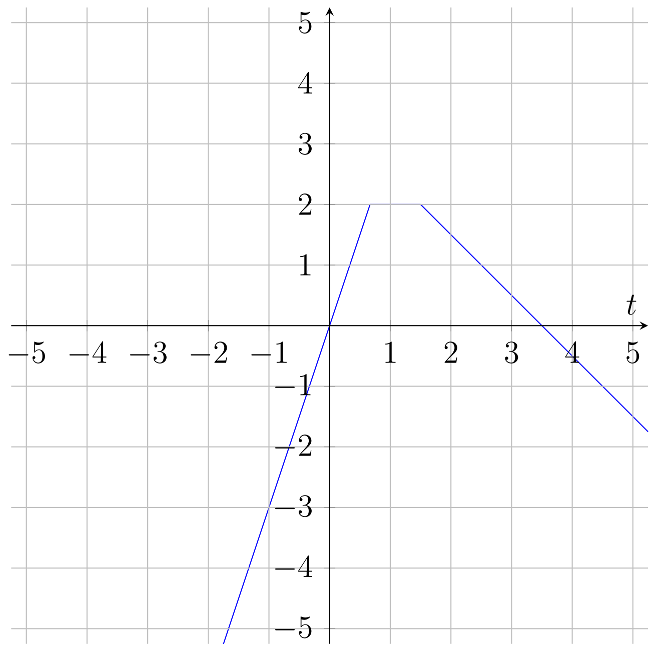
If an equation of the tangent line to the curve $y = f(x)$ at the point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

By definition, the tangent line to the curve at a point should intersect that point, meaning, in this case $4(2)-5 = f(2)$ and $f(2) = 3$. By definition, $f'(2)$ is the slope of the tangent line at $(2, f(2))$, meaning, in this case, it's 4.

Problem 39

Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

The graph of a function f that satisfies the given conditions is given below.



Problem 43

Each limit represents the derivative of some function f at some number a. State such an f and a in each case

$$\lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h}$$

An example of such f and a would be $f(x) = \sqrt{x}$ and $a = 9$.

Problem 49

The cost (in dollars) of producing x units of a certain commodity is $C(x) = 500 + 10x + 0.05x^2$.

(a) Find the average rate of change of C with respect to x when the production level is changed

  (i) from $x = 100$ to $x = 105$

  (ii) from $x = 100$ to $x = 101$

(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 3.7.)

For part (i) of (a), we have the following.

$$\begin{aligned}
\frac{C(105)-C(100)}{105-100} &= \frac{500+10(105)+0.05(105)^2-(500+10(100)+0.05(100)^2)}{105-100} = \\
&= \frac{500-500+10(105)-10(100)+0.05(105)^2-0.05(100)^2}{105-100} = \\
&= \frac{10(105-100)+0.05(105^2-100^2)}{5} = \\
&= \frac{10(105-100)+0.05(105-100)(105+100)}{105-100} = \\
&= \frac{10+0.05(205)}{=} 20.25
\end{aligned}$$

For part (ii) of (a), we have the following.

$$\begin{aligned}
\frac{C(101)-C(100)}{101-100} &= \frac{500+10(101)+0.05(101)^2-(500+10(100)+0.05(100)^2)}{101-100} = \\
&= \frac{500-500+10(101)-10(100)+0.05(101)^2-0.05(100)^2}{101-100} = \\
&= \frac{10(101-100)+0.05(101^2-100^2)}{5} = \\
&= \frac{10(101-100)+0.05(101-100)(101+100)}{101-100} = \\
&= 10+0.05(201) = 20.05
\end{aligned}$$

For part (b), we have the following.

$$\begin{aligned}
\lim_{x\to 100}\frac{C(x)-C(100)}{x-100} &= \lim_{x\to 100}\frac{500+10x+0.05x^2-(500+10(100)+0.05(100)^2)}{x-100} = \\
&= \lim_{x\to 100}\frac{500-500+10x-10(100)+0.05x^2-0.05(100)^2}{x-100} = \\
&= \lim_{x\to 100}\frac{10(x-100)+0.05(x^2-100^2)}{x-100} = \\
&= \lim_{x\to 100}\frac{10(x-100)+0.05(x-100)(x+100)}{x-100} = \\
&= \lim_{x\to 100}[10+0.05(x+100)] = 10+0.05(100+100) = 20
\end{aligned}$$

Problem 57

Determine whether $f'(0)$ exists.

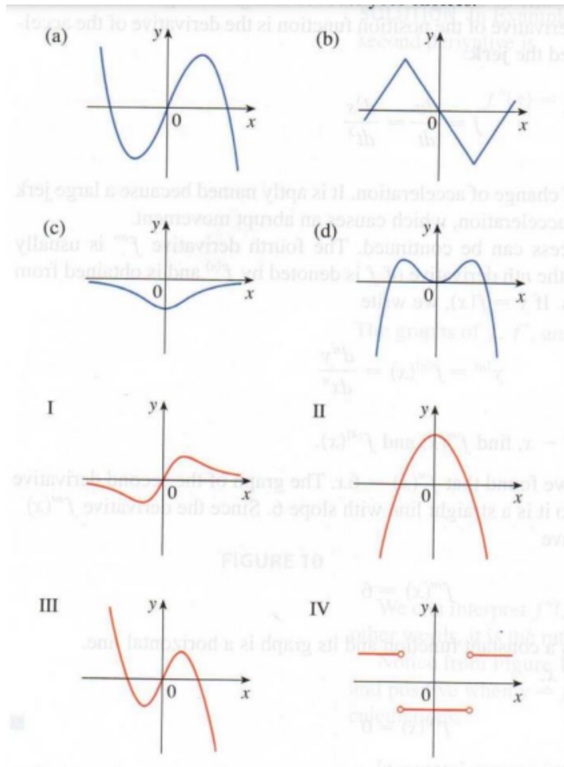
$$f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Note that $\frac{f(x)-f(0)}{x-0} = \frac{x\sin\frac{1}{x}}{x} = \sin\frac{1}{x}$ and, as already known, the function $\sin\frac{1}{x}$ has no limit as $x \to 0$, meaning neither does $\frac{f(x)-f(0)}{x-0}$, i.e., $f'(0)$ doesn't exist.

2.2 Section 2.8

Problem 3

Match the graph of each function in (a)-(d) with the graph of its derivative in I-IV. Give reasons for your choices.



We shall assume that for each function whose graph is in (a)-(d) there is exactly one graph in I-IV that matches its derivative.

For the function with graph (a), the function is increasing at $x = 0$, i.e., its derivative is positive at $x = 0$. The only graph that satisfies the latter is II.

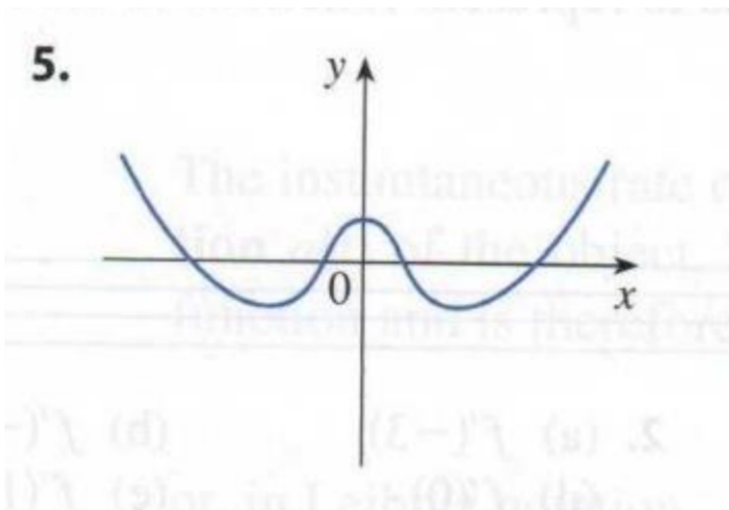
For the function with graph (b), the function everywhere is either not differentiable or has a constant derivative on some open interval, i.e., its derivative everywhere is either undefined or is constant on some open interval; The only graph that satisfies the latter in I-IV is IV.

For the function with graph (c), the function the function is decreasing for negative values of x and increasing for positive values of x, i.e., its derivative is negative for $x < 0$ and $x > 0$; The only graph that satisfies the latter in I-IV is I.

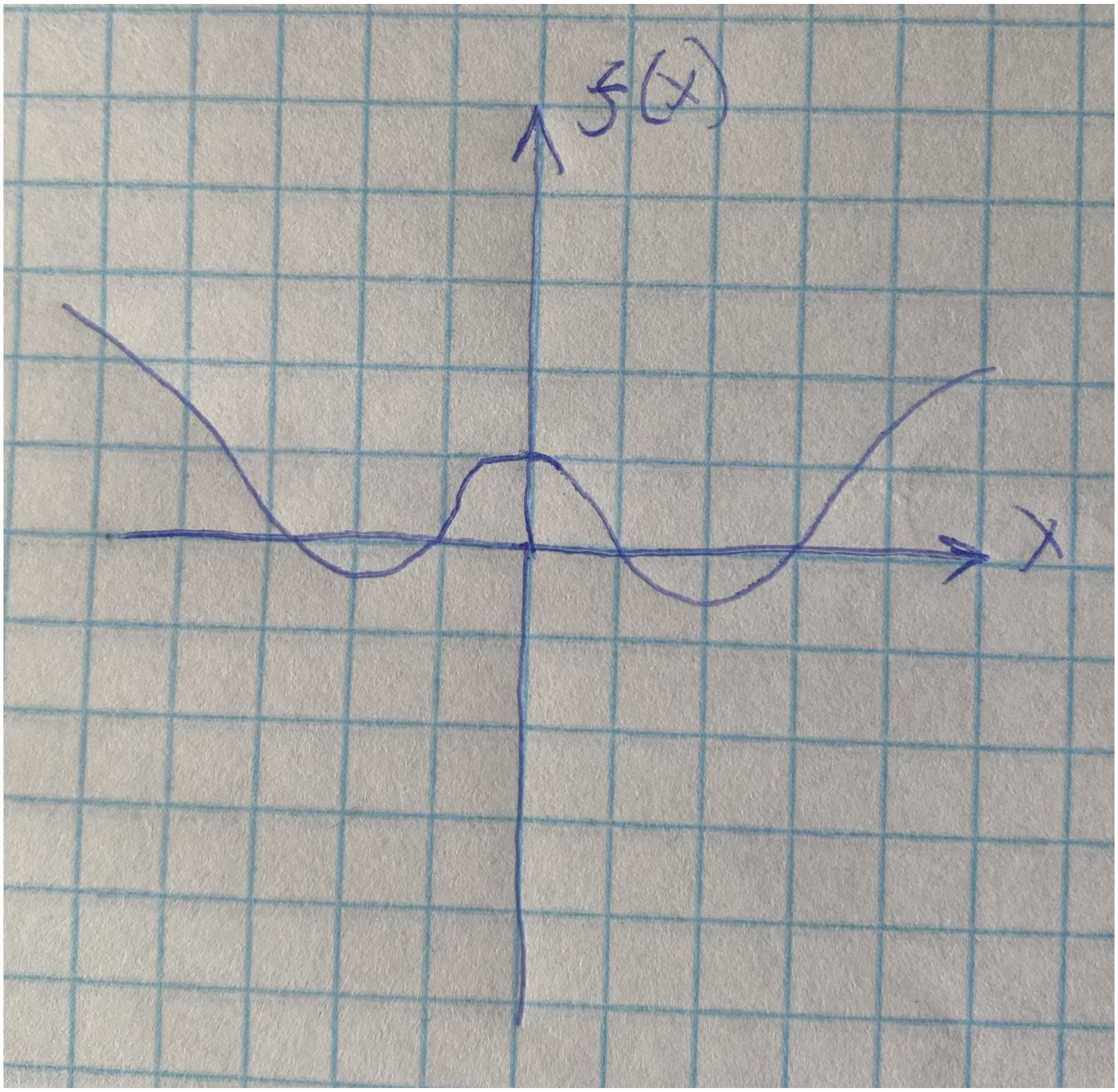
For the function with graph (d), the function has a horizontal tangent line at 3 points, i.e., its derivative function has 3 intersections with the x-axis. The only graph that satisfies the latter is III.

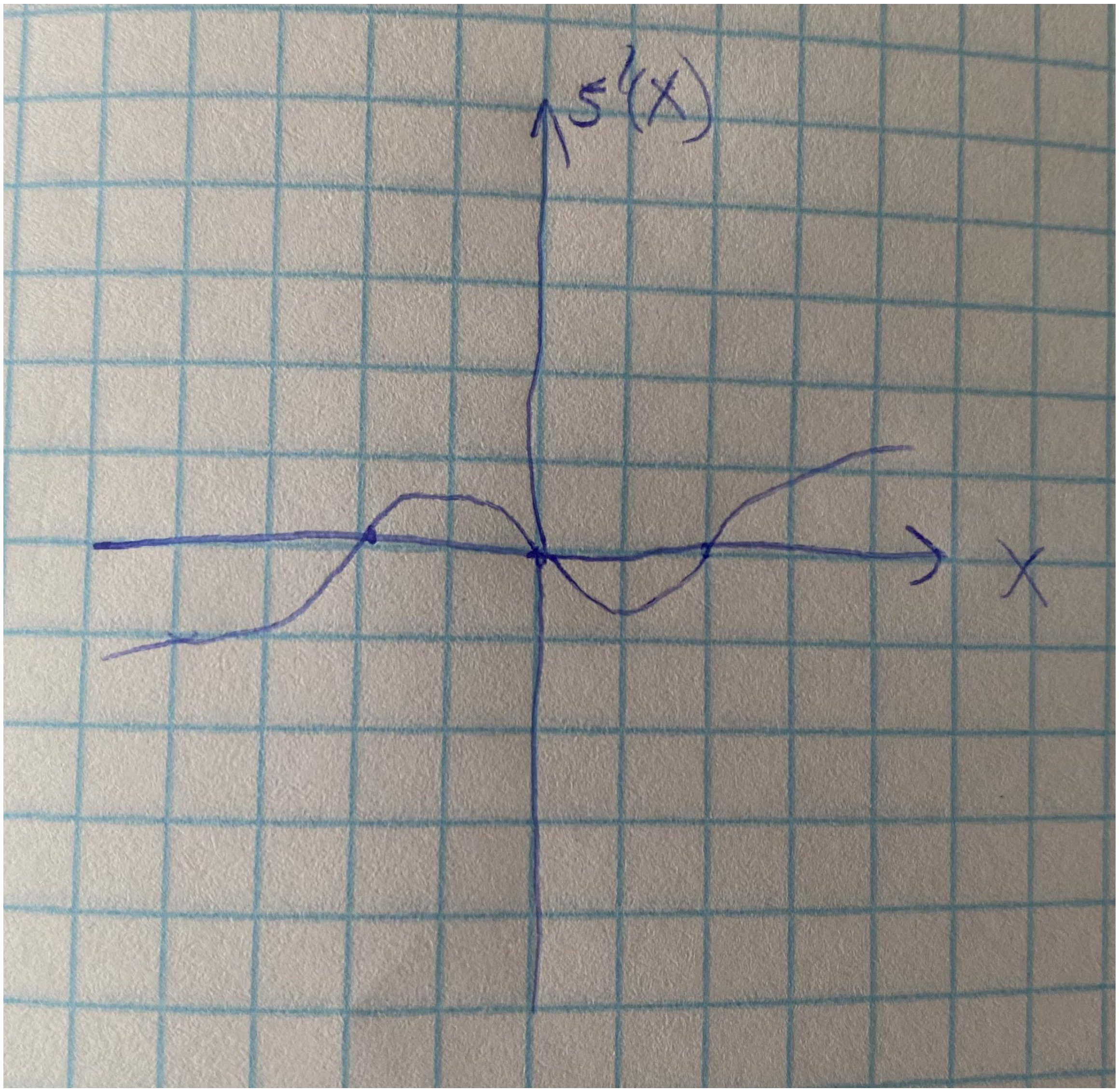
Problem 5

Trace of copy the graph of the given function f. (Assume that the axis have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.



The sketched graphs of f and f' are given below.


f(x)
x

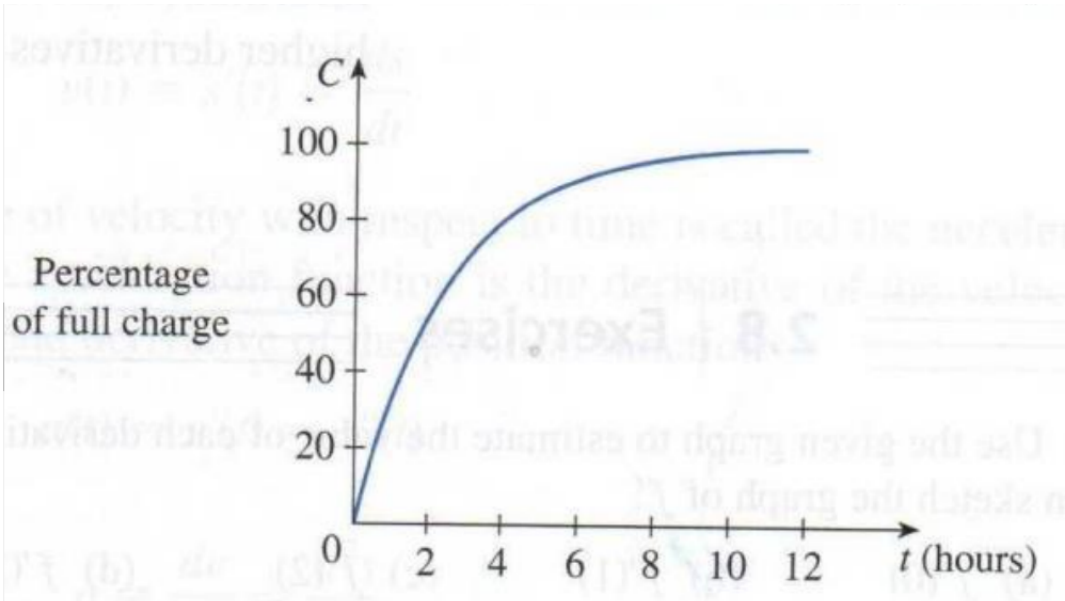


Problem 13

A rechargeable battery is plugged into a charger. The graph shows $C(t)$, the percentage of full capacity that the battery reaches as a function of time t elapsed (in hours).

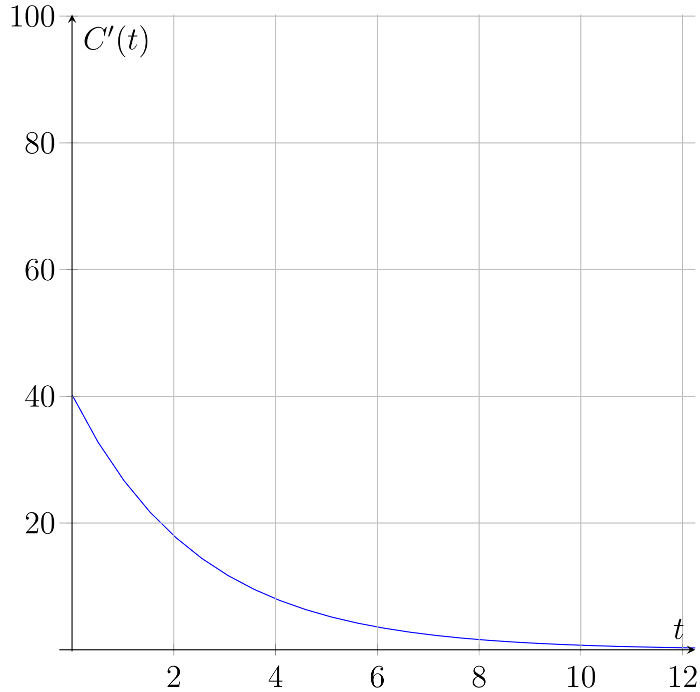
(a) What is the meaning of the derivative $C'(t)$?

(b) Sketch the graph of $C'(t)$. What does the graph tell you?



For part (a), the derivative $C'(t)$ is the instantaneous rate of the percentage as a function of time t elapsed in hours.

For part (b), the graph of $C'(t)$ is given below.



The graph tells us that the value of C has the initial rate of change of approximately 40 percent per hour, only for it to then slowly start approaching 0.

Problem 21

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = 3x - 8$$

The maximal domain of f in $\mathbb{R}$ is $\mathbb{R}$ itself. Let $x \in \mathbb{R}$, then we have the following.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)-8-(3x-8)}{h} = \\ &= \lim_{h\to 0} \frac{3x+3h-3x-8+9}{h} = \lim_{h\to 0} \frac{3h}{h} = \\ &= \lim_{h\to 0} 3 = 3 \end{aligned}$$

Hence the domain of the derivative is exactly the domain of f.

Problem 27

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = \frac{1}{x^2-4}$$

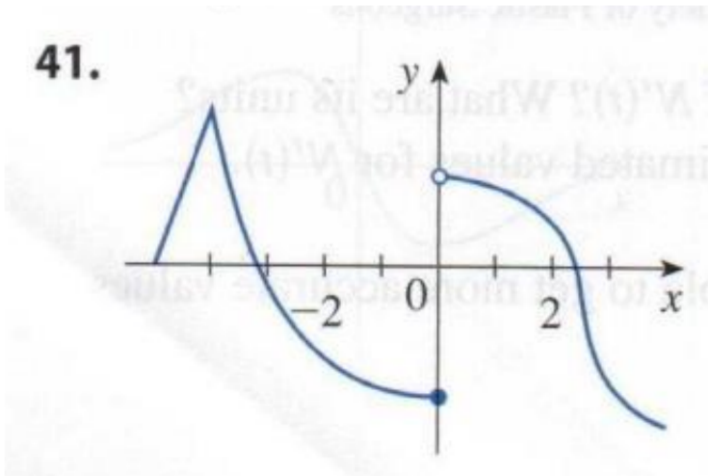
The maximal domain of f in $\mathbb{R}$ is $\mathbb{R} \setminus \{-2, 2\}$. Let $x \in \mathbb{R} \setminus \{-2, 2\}$, we then have the following.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2-4} - \frac{1}{x^2-4}}{h} = \\ &= \lim_{h\to 0} \frac{(x^2-4)-((x+h)^2-4)}{h((x+h)^2-4)(x^2-4)} = \\ &= \lim_{h\to 0} \frac{x^2-4-x^2-2xh-h^2+4}{h((x+h)^2-4)(x^2-4)} = \\ &= \lim_{h\to 0} \frac{-2xh}{h((x+h)^2-4)(x^2-4)} = \\ &= \lim_{h\to 0} \frac{-2x}{((x+h)^2-4)(x^2-4)} = \frac{-2x}{(x^2-4)^2} \end{aligned}$$

Hence the domain of the derivative is exactly the domain of f.

Problem 41

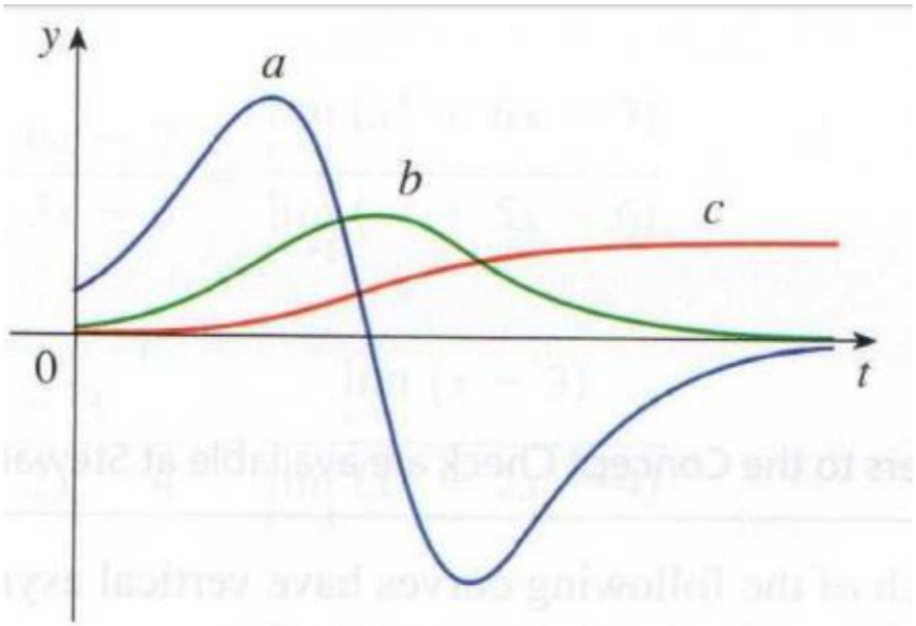
The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



The function f is not differentiable at $x = -4$ due to the sharp point and $x = 0$ due to the jump discontinuity occurring there.

Problem 51

The figure shows the graphs of three functions. One is the position of the car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your reasoning.



The function a has horizontal tangent lines at two points, meaning its derivative would have two intersections with the t-axis. None of the functions satisfy the latter, thus a must be the acceleration. The function b has horizontal tangent line only once, meaning its derivative would have only one intersection with the t-axis. Only c satisfies the latter, meaning the derivative of b is acceleration, i.e., b must be the velocity. Hence c is the position function of the car.

Problem 57

Let $f(x) = \sqrt[3]{x}$.

(a) If $a \neq 0$, use Equation 2.7.5 to find $f'(a)$.

(b) Show that $f'(0)$ does not exist.

(c) Show that $\sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f. See Figure 1.2.13.)

For part (a), we have the following.

$$
\begin{aligned}
f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a} = \\
&= \lim_{x\to a}\frac{(\sqrt[3]{x}-\sqrt[3]{a})(\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2)}{(x-a)(\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2)} = \\
&= \lim_{x\to a}\frac{\sqrt[3]{x}^3-\sqrt[3]{a}^3}{(x-a)(\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2)} = \\
&= \lim_{x\to a}\frac{x-a}{(x-a)(\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2)} = \\
&= \lim_{x\to a}\frac{1}{\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2} = \\
&= \frac{1}{\sqrt[3]{a}^2+\sqrt[3]{a}\sqrt[3]{a}+\sqrt[3]{a}^2} = \frac{1}{3\sqrt[3]{a}^2}
\end{aligned}
$$

For part (b), as implicitly shown in part (a), we have $\frac{f(x)-f(a)}{x-a} = \frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2}$ with $x \neq a$, we hence have $\frac{f(x)-0}{x-0} = \frac{1}{\sqrt[3]{x}^2}$ and the the limit of the latter as $x \to 0$ does not exist as goes to ∞ since $\lim_{x\to 0}\sqrt[3]{x}^2 = \sqrt[3]{0}^2 = 0$.

For part (c), verify that $(0,0)$ indeed lies on the curve $y = \sqrt[3]{x}$ and see part (b).

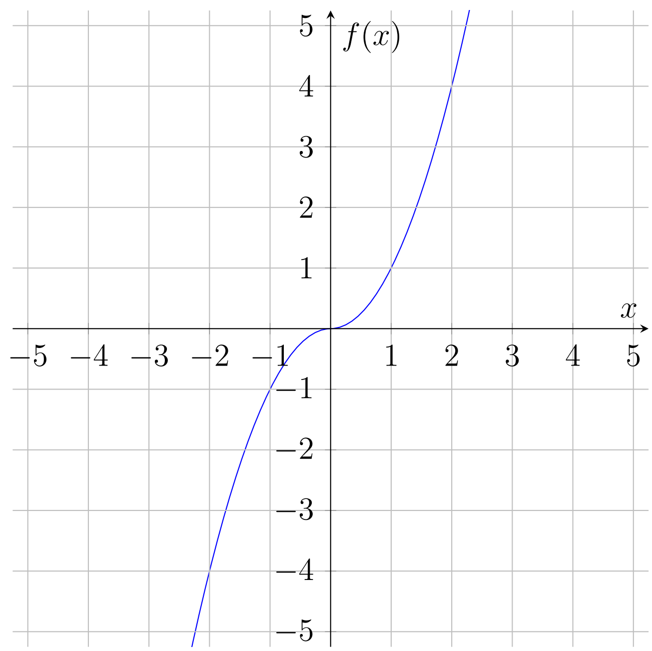
Problem 61

(a) Sketch the graph of the function $f(x) = x|x|$

(b) For what values of x is f differentiable?

(c) Find a formula for f'.

For part (a), the graph of the function $f(x) = x|x|$ is given below.



For part (b), we start by showing that f is differentiable for all $x > 0$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)|(x+h)| - x|x|}{h} = \\
&= \lim_{h\to 0} \frac{(x+h)(x+h) - xx}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \\
&= \lim_{h\to 0} \frac{(x+h-x)(x+h+x)}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \\
&= \lim_{h\to 0} \frac{2x+h}{=} 2x + 0 = 2x
\end{aligned}
$$

Now we show that f is differentiable for all $x < 0$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)|x+h| - x|x|}{h} = \\
&= \lim_{h\to 0} \frac{(x+h)(-(x+h)) - x(-x)}{h} = -\lim_{h\to 0} \frac{(x+h)(x+h) - xx}{h} \overset{\text{part b}}{=} -2x
\end{aligned}
$$

Finally, we show that f is differentiable at $x = 0$ by considering its one-sided limits.

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h|h| - 0}{h} =$$
$$= \lim_{h\to 0} \frac{h|h|}{h} = \lim_{h\to 0} |h| = 0$$

Thus f is differentiable for all values of x.

For part (c), we have $f(x) = x^2 \operatorname{sgn}(x)$ (see part (b)).

Problem 63

Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x. Prove each of the following.

(a) The derivative of an even function is an odd function.

(b) The derivative of an odd function is an even function.

For this problem, we are going to use the fact that $\lim_{h\to 0} F(h) = \lim_{h\to 0} F(-h)$ given that one of the limits exists.

For part (a), let f be an even function and be differentiable at x. We then have the following.

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} =$$
$$= \lim_{h\to 0} \frac{f(-x-h) - f(-x)}{-h} = -\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = -f'(x)$$

For part (b), let f be an odd function and differentiable x. We then have the following.

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} \frac{f(-x-h) - f(-x)}{-h} =$$
$$= \lim_{h\to 0} \frac{-(-f(x+h) + f(x))}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$