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Q1.

```
import numpy as np
import matplotlib.pyplot as plt

T = 1.0
N = 500
dt = T / N
t = np.linspace(0, T, N + 1)
W = np.zeros(N + 1)

W[1:] = np.cumsum(np.sqrt(dt) * np.random.randn(N))

plt.plot(t, W)
plt.title(’Single␣Path␣of␣Standard␣Brownian␣Motion’)
plt.xlabel(’t’)
plt.ylabel(’W(t)’)
plt.grid(True)
plt.show()
```

Q2.

```
alpha = 0.1
sigma = 0.2
S0 = 100
n_paths = 5

plt.figure(figsize=(10, 6))

for _ in range(n_paths):
    W = np.zeros(N + 1)
    W[1:] = np.cumsum(np.sqrt(dt) * np.random.randn(N))
    S = S0 * np.exp(sigma * W + (alpha - 0.5 * sigma ** 2) * t)
    plt.plot(t, S)

plt.title(’Geometric␣Brownian␣Motion␣-␣5␣Sample␣Paths’)
plt.xlabel(’t’)
plt.ylabel(’S(t)’)
plt.grid(True)
plt.show()
```

Q3.

Using Brownian motion:

$$W_t = W_s + (W_t - W_s), \quad \text{where } W_s \perp (W_t - W_s)$$

So,

$$\mathbb{E}[W_s W_t] = \mathbb{E}[W_s(W_s + (W_t - W_s))] = \mathbb{E}[W_s^2] + \mathbb{E}[W_s(W_t - W_s)] = s + 0 = \min(s, t)$$

Q4.

By Brownian motion properties:

- $W_t - W_s \sim \mathcal{N}(0, t - s)$
- Increments over non-overlap intrevals are independent.

Thus, for $0 \leq s < t$:

$$W_t - W_s \sim \mathcal{N}(0, t - s), \quad \text{and is independent of } W_s$$

Q5.

We decompose Wt:

$$W_t = W_s + (W_t - W_s), \quad \text{where } W_t - W_s \perp \mathcal{F}_s$$

Taking conditional expectation:

$$\mathbb{E}[W_t \mid \mathcal{F}_s] = \mathbb{E}[W_s + (W_t - W_s) \mid \mathcal{F}_s] = W_s + \mathbb{E}[W_t - W_s \mid \mathcal{F}_s] = W_s$$

Hence, W_t is a martingale.