Casimir Effect and Black Hole Entropy

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Contents

Motivation

- Bekenstein, Hawking:

$$S_{BH} = \frac{A}{4 l_p^2}$$

A: the area of horizon.

- Boltzmann:

$$S = k_B \log \Omega$$

Ω: number of microstates.
$\Rightarrow$ Microscopic origin of black hole entropy?

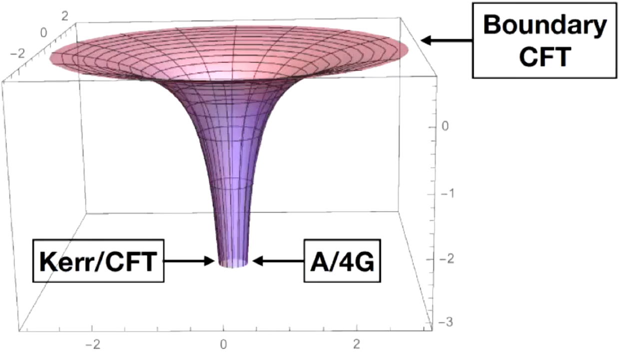
- Casimir energy (vacuum energy):

$$E_{vac}(\partial\Gamma) = E_0(\partial\Gamma) - E_0(0)$$

$\partial\Gamma$: an arbitrary boundary.
$\Rightarrow$ Arises from field quantization.

Introduction

- Different approaches:



- Information paradox; hints for quantum gravity.

Brief History

- The existence of a force between two polarizable atoms and between such an atom and a conducting plate. ('47 Casimir, Polder)
- Neutral perfectly conducting parallel plates placed in the vacuum attract each other. ('48 Casimir)

$$F = \hbar c \frac{\pi^2}{240} \frac{1}{a^4} = 0.013 \frac{1}{a_\mu^4} \mathrm{dyne/cm^2}$$

Although the effect is small, an experimental confirmation seems not unfeasable and might be of a certain interest.

Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken, Eindhoven.)

- Expereimental verification of this attraction. ('57 B.V.Deriagin and I.I.Abrikosova)

Brief History

- Investigations of the Casimir energy of the electromagnetic field under constraints.
  ('49 H.B.G.Casimir, J.Chim)
  $\Rightarrow$ the zero-point energy of the electromagnetic field can be usefully applied to explain van der Waals attraction,
  $\Rightarrow$ the existence of repulsive Casimir forces of electromagnetic origin seems to be contradictory.
- Presently, Casimir energies of quantized fields are studied in connection with a variety of problems, ranging from applications in particle physics, e.g. in QCD bag models, to gravitational physics, where its possible influence on the structure of space-time is studied.

Problems

- The evaluation of vacuum energies remains a problematic exercise, because the available methods, in most cases, only allow an approximate calculation.
  $\Rightarrow$ Mode summation method
  $\Rightarrow$ Local Green function method
- Correct results for the Casimir energy should be independent of the applied methods and the regularization scheme
- Energy density of the vacuum in cosmology

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G(\tilde{T}_{\mu\nu} - E g_{\mu\nu})$$

$$\lambda = 8\pi G E$$

$\Rightarrow$ Vacuum fluctuation produces way more!

Black Hole Thermodynamics

- Consider a Schwarzschild black hole:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

- Hawking radiation:('74 Hawking)
  Thermal spectrum of emitted particles

$$n_E \propto \Gamma_l(E)\left[e^{\frac{E}{T_H}} - 1\right]^{-1} \quad \text{with} \quad T_H \equiv \frac{1}{8\pi M}$$

- In Schwarzschild metric, we have BH entropy

$$A = 16\pi M^2 \quad \Rightarrow \quad \mathrm{d}M = \frac{1}{8\pi M}\mathrm{d}\frac{A}{4} \equiv T\mathrm{d}S$$

The first law of of thermodynamics (generalized to Kerr-Newman metric):

$$\mathrm{d}M = T\mathrm{d}S + \Phi_I \mathrm{d}Q_I + \Omega_i \mathrm{d}J_i$$

Tree-level BH Thermodynamics

- Euclidean action of a 2D dilaton gravity:
  ('96 Frolov, Israel, Solodukhin)

$$
\begin{aligned}
W_{cl} = &-\frac{1}{4G}\int_{M^2}[r^2R + 2(\nabla r)^2 + 2U(r)]\sqrt{\gamma}\,\mathrm{d}^2z \\
&-\frac{1}{2G}\int_{\partial M^2} r^2 k\,\mathrm{d}z - \frac{\pi r_+^2}{G}(1-\alpha)
\end{aligned}
$$

- We are free to define other thermal quantities with the action

$$
F = \frac{1}{2\pi\beta}W_{cl}, \quad S = (\beta\partial_\beta - 1)W_{cl}, \quad E = \frac{1}{2\pi}\partial_\beta W_{cl}
$$

- Extremal black hole (defined as $T_H = 0$):

$$
W_{ext} = 2\pi\beta E, \quad S_{ext} = 0
$$

AdS/CFT

- Maldacena('97)

  4D U(N) $\mathcal{N}=4$ SYM $\Longleftrightarrow$ IIB string theory on $AdS_5 \times S^5$

- Witten; Gubster, Klebanov, Polyakov ('98)

$$Z_{CFT} = Z_{AdS},$$

$$N^2 = \frac{\pi}{2}\frac{L^3}{G_5}, \quad \lambda \equiv g_{YM}^2 N = \left(\frac{L}{l_s}\right)^2$$

- Compute Z_{CFT} from the field side $\Rightarrow$ Z_{AdS}
  BPS black hole free energy $\Rightarrow$ $F = \log Z_{AdS}$
  BPS black hole entropy $\Rightarrow$ $S \simeq F$
- In the presence of chemical potentials

$$Z(\Delta,\omega) = \sum_{Q,J} \Omega(Q,J) e^{Q\Delta} e^{J\omega}, \;\; S_{BH} = \log \Omega(Q,J)$$

Brief History

- 5d asymptotically flat BPS black hole:
  ('96 Strominger, Vafa)
  Type II string theory compactified on $K_3 \times S^1$
  $\Rightarrow$ BPS black hole in 5d flat spacetime
  $\Rightarrow$ take nBPS limit to study AdS_3 black hole
- Black Hole Entropy from Near–Horizon Microstates (apply Cardy formula to near-horizon AdS_3 BPS black hole)
  ('97 Strominger)
- Works have been done on asymptotically flat black holes
- Black hole microstates in AdS_4 from supersymmetric localization:
  ('15 Benini, Hristov, Zaffaroni)
  $\Rightarrow$ Topologically twisted index of ABJM theory on $S^1 \times S^2$
  $\Rightarrow$ AdS_4 magnetically charged BPS black hole entropy

AdS_5: Difficulty

- $\mathcal{N} = 4$ SYM theory partition function on $S^1 \times S^3$:

$$Z(\beta, \Delta_I, \omega_i) = \mathrm{Tr}\left[e^{-\beta E} e^{\sum_{I=1}^{3} \Delta_I Q_I} e^{-\sum_{i=1}^{2} \omega_i J_i}\right]$$

  Different boundary conditions for fermion and boson along S^1
  $\Rightarrow$ break SUSY!

- $\mathcal{N} = 4$ SYM superconformal index on $S^1 \times S^3$: ('07 Kinney, Maldacena, Minwalla, Raju)

$$\mathcal{I}(\beta, \Delta_I, \omega_i) = \mathrm{Tr}\left[(-1)^F e^{-\beta E} e^{\sum_{I=1}^{3} \Delta_I Q_I} e^{-\sum_{i=1}^{2} \omega_i J_i}\right]$$

$$\mathcal{I} \sim \mathcal{O}(1), \quad S_{BH} \sim \mathcal{O}(N^2)$$

$\Rightarrow$ Index cannot reproduce the black hole entropy!

AdS_5: Recent Progress

- Entropy of BPS AdS_5 black hole:
  ('17 Hosseini, Hristov, Zaffaroni)
  Legendre transformation of $\log Z$
- Different approaches (allow complex chemical potentials):
  ('18)
  — Localization of $\mathcal{N} = 4$ SYM in complex backgrounds
  (Cabo-Bizet, Cassani, Martelli, Murthy)
  — Free $Z_{\mathcal{N}=4\ SYM}$ with complex fugacities
  (Choi, Kim, Kim and Nahmgoong)
  — $\mathcal{I}_{\mathcal{N}=4\ SYM}$ with complex fugacities
  (Benini, Milan)
- Later, generalized to other dimensions

AdS_5 BPS BH

Compute $\log Z$ from field theory

$$\log Z \simeq \mathcal{F} = \frac{N^2-1}{2}\frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2} \quad \text{with} \ \sum_I \Delta_I - \sum_i \omega_i = 2\pi i n$$

Define its entropy via Legendre transform

$$\begin{aligned} S(\Delta_I, \omega_i; Q_I, J_i) =& \frac{N^2}{2}\frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2} + \sum_{I=1}^{3} Q_I\Delta_I + \sum_{i=1}^{2} J_i\omega_i \\ & - \Lambda\left(\sum_{I=1}^{3}\Delta_I + \sum_{i=1}^{2}\omega_i - 2\pi i\right) \end{aligned}$$

Extremization:

$$\frac{\partial S}{\partial \Lambda} = 0, \quad \frac{\partial S}{\partial \Delta_I} = 0, \quad \frac{\partial S}{\partial \omega_i} = 0$$

Results from the gravity side ('06 Kim, Lee)

Generalization of Casimir Energy

- Casimir energy in curved space $S^{d-1} \times \mathbb{R}$: ('15 Assel, Cassani, Pietro, Komargodski, Lorenzen, Martelli)

$$E_0 = \int_{S^{d-1}} \mathrm{d}^{d-1}x\sqrt{g}\langle T_{\tau\tau}\rangle$$

$\Rightarrow d = 2 \qquad E_0 = -\frac{c}{12r_1}$

$\Rightarrow d = 4 \qquad E_0 = \frac{3}{4r_3}(a - \frac{b}{2})$

c: central charge; b: scheme related parameter;

- SUSY Casimir energy independent of coupling constants ('13 Closset, Dumitrescu, Festuccia, Komargodski)

$$E_{susy} = -\lim_{\beta\to\infty} \frac{\mathrm{d}}{\mathrm{d}\beta}\log Z^{susy}_{M_3\times S^1_\beta}$$

$$Z^{susy}_{M_3\times S^1_\beta} \equiv \mathrm{Tr}[(-1)^F e^{-\beta H_{susy}}]$$

Casimir Energy and Gravity

The Casimir energy of $\mathcal{N} = 1$ 4d field theory with R-symmetry:

$$E_{susy} = \frac{4}{27 r_3}(a + 3c)$$

$\Rightarrow$ Anomaly Polynomial Interpretation of $a + 3c$
('15 Bobev, Bullimore, Kim)
$\Rightarrow$ Prefactor $\mathcal{F}$

$$\mathcal{F}(\omega_1, \omega_2, \varphi) = -(3c - 2a)\frac{16}{27}\frac{\varphi^3}{\omega_1\omega_2} + (a - c)16\pi i \Psi_2^{(0)}$$
$$Z(\omega_1, \omega_2, \varphi) = e^{-\mathcal{F}(\omega_1,\omega_2,\varphi)}\mathcal{I}(\omega_1, \omega_2, \varphi)$$

The index scales at large N limit:

$$-\log \mathcal{I}_{\mathcal{N}=4} \xrightarrow{N\to\infty} \frac{N^2}{2}\frac{\varphi_1\varphi_2\varphi_3}{\omega_1\omega_2} = -\mathcal{F}_{\mathcal{N}=4}$$

Which matches precisely the gravitational on-shell action.

Euclidean Method

- The density matrix of a canonical ensemble

$$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}} \qquad Z = \mathrm{Tr} e^{-\beta \hat{H}} = \sum_i \langle i| e^{-\beta \hat{H}} |i\rangle$$

- The matrix elements of a time revolution operator $e^{-i\hat{H}t}$

$$\langle i| e^{-i\hat{H}t} |j\rangle$$

$\Rightarrow$

$$\beta = \int i\mathrm{d}t \equiv \int \mathrm{d}\tau$$

- Test on a static spherically symmetric metric (near-horizon)*

$$\mathrm{d}s^2 = f'_+ R \mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{f'_+ R} + r_+^2 \mathrm{d}\Omega^2 \equiv \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2 + r_+^2 \mathrm{d}\Omega^2$$

$\Rightarrow$ On-shell condition: $\beta = 4\pi / f'_+ = 1/T_H$

Euclidean Method

$$F = \beta^{-1} W, \quad S = (\beta \partial_\beta - 1) W$$

1. Partition function of a black hole system

$$Z(\beta) = \int [D\phi] e^{-I[\phi]} \equiv e^{-W(\beta)}$$

   $I[\phi]$: Euclidean Einstein-Hilbert action;
   $W(\beta)$: effective action

2. Consider fluctuation around a classical solution ϕ_0

$$I[\phi_0 + \tilde{\phi}] = I[\phi_0] + I_2[\tilde{\phi}] + \cdots$$

   $I_2[\tilde{\phi}]$ denotes for a second order correction

3. Correction for effective action

$$W_1(\beta) = -\log Z_1(\beta) = \frac{1}{2} \sum_j \mathrm{logdet}[-\mu^2 D_j(\phi_0)]$$

Calculation

('95 Frolov, Fursaev, Zelnikov)

$$W_1(\beta) = \frac{1}{2}\mathrm{logdet}(-\Delta)$$

Heat Kernel Expansion

$$W_1 = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}s}{s}\mathrm{Tr}e^{s\Delta}$$

Expand the factor $\mathrm{Tr}e^{s\Delta}$

$$\mathrm{Tr}e^{s\Delta} = \frac{1}{(4\pi s)^{d/2}}\sum_{n\in\mathbb{Z}_{\geq 0}} a_n^{(d)} s^n$$

For $d{=}2$, only $a_1^{(d)}$ contributes to W_1

$$W_1 = W_1^{bare} - W_1^{div} = \frac{1}{4\pi}\lim_{d\to 2}\frac{1}{d-2}[a_1^{(d)}(\tilde{\gamma}) - a_1^{(d)}(\gamma)]$$

Calculation

('95 Frolov, Fursaev, Zelnikov)

$$W_1(\beta) = \frac{1}{2}\mathrm{logdet}(-\Delta)$$

ζ-Function Regularization

$$W_1 = \frac{1}{2}\mathrm{logdet}\mathcal{O} = -\left.\frac{1}{2}\frac{\mathrm{d}\zeta_{\mathcal{O}(s)}}{\mathrm{d}s}\right|_{s=0}$$

where

$$\zeta_{\mathcal{O}}(s) \equiv \sum_n \frac{1}{\lambda_n^s}$$

$\mathcal{O}$: an operator with positive definite eigenvalues λ_n, here the operator is taken as $-\mu^2\Delta$

$$W_1 = -\frac{1}{2}\zeta'_{-\mu^2\Delta}(0) = \frac{1}{2}\zeta_\Delta(0)\log\mu^2 - \frac{1}{2}\zeta'_\Delta(0)$$

$\Rightarrow \lambda_n$ can be obtained from specific models.

On-shell Conclusion

Classical action of 2D dilaton gravity

$$W_{cl} = -\frac{1}{4}\int_{M^2}[r^2 R + 2(\nabla r)^2 + 2]\sqrt{\gamma}\mathrm{d}^2 x \\ -\frac{1}{2}\int_{\partial M^2} r^2(k-k_0)\mathrm{d}y + \frac{1}{2}\int \sqrt{\gamma}\phi_{,\mu}\phi^{,\mu}\mathrm{d}x$$

Ansatz of the corresponding EoMs gives a Gibbs-Hawking instanton:

$$\mathrm{d}s^2 = f\mathrm{d}\tau^2 + f^{-1}\mathrm{d}r^2, \quad f = 1 - r_+/r$$

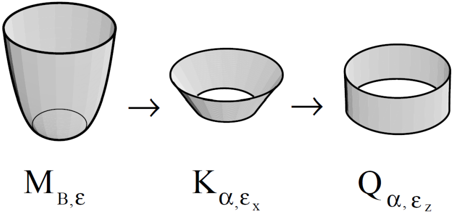
Which can be conformally transformed

$$\mathrm{d}s^2 = f\mathrm{d}\tau^2 + f^{-1}\mathrm{d}r^2 \equiv e^{2\sigma}\mathrm{d}\tilde{s}^2, \\ \mathrm{d}\tilde{s}^2 = \mu^2(x^2\mathrm{d}\tilde{\tau}^2 + \mathrm{d}x^2)$$

Then we can perform the calculation under a flat background.

Off-shell Models

- Brick-wall Model ('85 't Hooft):



$$\tilde{W}_1^{BW}(\beta, \alpha = 1, y, \epsilon) = \tilde{W}_1(\beta, y) + \frac{1}{6}\log\epsilon - \frac{1}{2}\log\frac{\pi}{\log(\beta/2\pi\epsilon)} + \mathcal{O}\log^{-1}\left(\frac{\beta}{\epsilon}\right)$$

$$S_1^{BW}(\beta, \alpha, y, \epsilon) = \frac{1}{12\alpha}\left(2\log\frac{\beta}{2\pi\alpha\epsilon} - \log y + \frac{1}{y} - 1\right) + \frac{1}{2}\log\frac{\pi\alpha}{\log(\beta/2\pi\alpha\epsilon)} + \mathcal{O}\log^{-1}\left(\frac{\beta}{\epsilon}\right)$$

Off-shell Models

Define a shared term of effective action:

$$U(\beta,\alpha,y) \sim (\alpha+\frac{1}{\alpha})\log\alpha$$

$$\tilde{W}_1^{BW}(\beta,\alpha,y,\epsilon) = U(\beta,\alpha,y) + \frac{1}{12}\left(\alpha+\frac{1}{\alpha}\right)\log\left(\frac{\epsilon}{\mu}\right) - \frac{1}{2}\log\frac{\pi\alpha}{\log(\beta/2\pi\alpha\epsilon)}$$

- Conical Singularity Method ('94 Susskind, Uglum):

$$\tilde{W}_1^{CS}(\beta,\alpha,y) = U(\beta,\alpha,y) + C(\alpha) \simeq \tilde{W}_1^{BC}$$

- Blunt Cone Method ('95 Solodukhin):
- Volume Cut-off Method('93 Frolov, Novikov):

$$\tilde{W}_1^{VC}(\beta,\alpha,y,\epsilon) = U(\beta,\alpha,y) + \frac{1}{12}\left(\alpha+\frac{1}{\alpha}\right)\log\left(\frac{\epsilon}{\mu}\right)$$

Statistical-Mechanical Entropy

We expect the statistical-mechanical entropy to take this form

$$S = -\mathrm{Tr}(\hat{\rho}\log\hat{\rho})$$

Q: the density matrix $\hat{\rho}$?

Partition function in Hamiltonian formalism

$$Z = \mathrm{Tr} e^{-\beta \hat{H}} = \sum_n e^{-\beta\omega_n} \equiv e^{-\beta\mathcal{F}}$$

Define

$$\hat{\rho} \equiv \frac{1}{Z}\sum_n e^{-\beta\omega_n} \left|\psi_n\right\rangle \left\langle\psi_n\right|$$

$$\Rightarrow \quad -\mathrm{Tr}(\hat{\rho}\log\hat{\rho}) = \beta\sum_n \omega_n \frac{e^{-\beta\omega_n}}{Z} + \log Z$$

Compared with

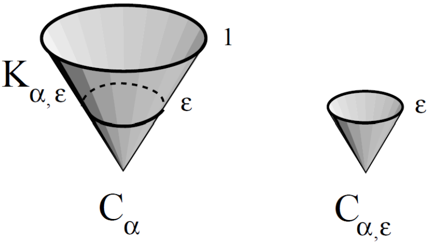
$$S = \frac{1}{T}E - \frac{F}{T}$$

Quantum Fluctuation on Boundary

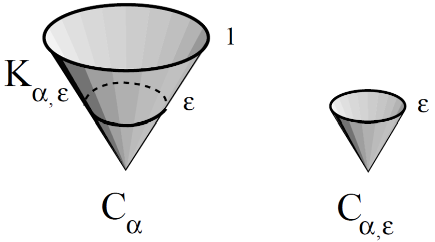
QFT path integral $\mathcal{Z}$ is given by

$$\mathcal{Z} = e^{-W_1}, \quad Z = e^{-\beta\mathcal{F}}$$

where $W_1 = \beta\mathcal{F} + C\beta$ (C: a constant).
The linear term of β does not change the entropy.



Quantum Fluctuation on Boundary



Brick-wall model:

$$W_1^{BW} = W_1[K_{\alpha,\epsilon}] = W_1[C_\alpha] - W_1[C_{\alpha,\epsilon}] + W_1(2\pi\alpha, \alpha, \epsilon)$$

Calculate with appropriate boundary conditions

$$W_1(2\pi\alpha, \alpha, \epsilon) = -\frac{1}{2}\log\frac{\pi\alpha}{\log(1/\epsilon)}$$

Zero Point Energy

- The Casimir force calculated using vacuum fluctuations of the electromagnetic field

$$\mathcal{F} = -\frac{\hbar c \pi^2}{240 d^4}$$

- Drude model (Landau, Lifshitz, *Electrodynamics of Continuous Media*)
  Casimir effect depends on fine structure constant α

$$\mathcal{F} \sim \frac{c}{d} \ll \omega_{pl} \quad \Longleftrightarrow \quad \alpha \gg \frac{mc}{4\pi\hbar n d^2}$$

  ω_{pl}: plasma frequency.

- Difference: $\mathcal{F} \xrightarrow{\alpha \to \infty}$ finite value;
  Common ground: $\mathcal{F}$ vanishes as $\alpha \to 0$.

The Casimir Effect Without the Vacuum ('05 Jaffe)

Casimir effect in modern language

$$\mathcal{E} = \frac{\hbar}{2\pi}\int \mathrm{d}\omega\omega\mathrm{Tr}\int \mathrm{d}^3x[\mathcal{G}(x,x,\omega+i\epsilon)-\mathcal{G}_0(x,x,\omega+i\epsilon)]$$

$\mathcal{G}$: full Greens function for the fluctuating field;
$\mathcal{G}_0$: free Greens.

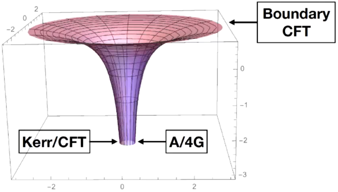
$$\frac{1}{\pi}\mathrm{Im}\int[\mathcal{G}(x,x,\omega+i\epsilon)-\mathcal{G}_0(x,x,\omega+i\epsilon)] = \frac{\mathrm{d}\Delta N}{\Delta\omega}$$

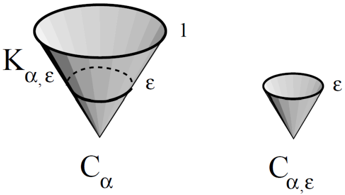
Consider the Casimir effect for a scalar field ϕ, the interaction

$$\mathcal{L}_{int} = \frac{1}{2}g\sigma(x)\phi^2(x)$$

Summing up all one loop Feynman diagram will give the Casimir energy.

Summary and Prospect





- Different dimensions SUSY Casimir energy;
- The origin of Casimir energy as fluctuation on black hole boundary;
- Cosmological constant problem;
- ...

Thank You!