7. RC circuits

Electronic filters can generally be classified into passive and active types. Active filters use op-amps and we will look at this later in the course. Passive filters are built using resistors (R), capacitors (C) and inductors (L).

Learn it in lab
You will build and characterise a simple RC low-pass filter in the first lab session and use a high-pass filter in the second session.

7.1. The RC circuit

To see how to apply AC circuit analysis we will study the series RC circuit in some detail. This also provides a simple illustration of several important concepts for the course. Because the capacitor has an impedance which is frequency-dependent, the RC circuit is an example of the class of systems known as *filters*, which will be very important for our studies. It is also a *Linear Time-Invariant (LTI)* system (cf. Section 14).

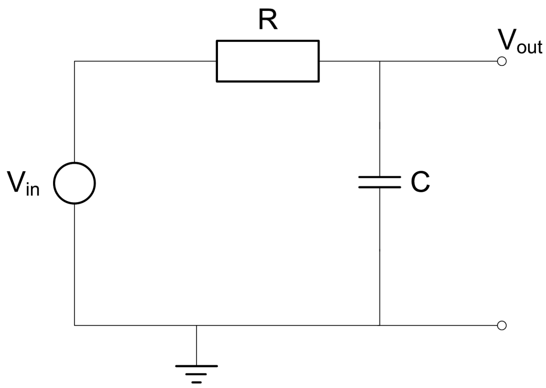

Figure 7.1: Simple RC circuit– The low-pass filter.

As shown in Figure 7.1, if we apply an input voltage across resistor and capacitor we can choose to take the output across either the resistor or the capacitor. Here we take the output across the capacitor. Consider the limiting cases of frequency: as $\omega \to \infty$, then the impedance of the capacitor tends to zero (effectively a short-circuit), so the entire input voltage appears across the resistor – by Kirchhoff's voltage law (KVL) – and $v_C \to 0$. Conversely, at DC (i.e. if we apply a constant voltage) then the impedance of the capacitor tends to ∞, the current flowing tends to zero, hence $v_R \to 0$, and the entire applied voltage appears as v_C. We will study two intermediate behaviours in more detail: first, the behaviour under steady-state AC conditions, and secondly the transient response.

7.1.1. Steady-state behaviour

In the following discussion, voltages and impedances are complex quantities. In the steady-state condition, we are working with periodic signals which have been present at the input for sufficiently long time that there is no remaining "transient behaviour" (which we will cover in the next section). The voltage $\mathbf{v_{in}}$ is an AC sinusoidal waveform and the voltage across the capacitor we label as $\mathbf{v_{out}}$. Complex Ohm's law gives

$$\begin{aligned} \mathbf{i} &= \frac{\mathbf{v_{in}}}{\mathbf{Z}} = \frac{\mathbf{v_{in}}}{R - j/\omega C} \\ \mathbf{v_{out}} &= -\mathbf{i}\frac{j}{\omega C} . \end{aligned}$$

We want to find the output in terms of the input, which we define as the *complex gain* (also denoted in bold-face to stress that this too is a complex quantity):

$$\mathbf{G} = \frac{\mathbf{v_{out}}}{\mathbf{v_{in}}} = \frac{-j}{\omega RC - j} .$$

Multiplying by the complex conjugate of the denominator and simplifying leads to

$$\mathbf{G} = \frac{1 - j\omega RC}{\omega^2 R^2 C^2 + 1} . \tag{7.1}$$

As anticipated, the gain is both frequency-dependent and complex (implying a phase change at the output). To visualise this we need to plot the magnitude and the phase of the gain:

$$|\mathbf{G}| = \sqrt{\mathbf{G} \times \mathbf{G}^*} = \frac{1}{\sqrt{\omega^2 R^2 C^2 + 1}} , \tag{7.2}$$

$$\begin{aligned} \phi &= \tan^{-1}\left(\frac{\mathcal{I}\{\mathbf{G}\}}{\mathcal{R}\{\mathbf{G}\}}\right) \\ &= \tan^{-1}(-\omega RC) \\ &= -\tan^{-1}(\omega RC) . \end{aligned} \tag{7.3}$$

7.1.2. The Bode plot

The gain and phase plots together constitute the *Bode plot* for the RC filter system. In Figure 7.2 the frequency axis has been normalised to units of $\omega_c = 1/RC = 1$ rad/s. For the Bode plot the gain magnitude is traditionally plotted log/log in dB, i.e. $20\log_{10}|\mathbf{G}|$. The phase is usually given in degrees.

7.1.3. Phase interpretation

The interpretation of the gain magnitude is clear: it is the ratio of the amplitudes of the output voltage (that across the capacitor) to that of the input voltage (applied to both the capacitor and the resistor in series). The interpretation of the phase is more difficult. We understand that there is a phase difference, but between what and what? And does it lead or lag? This can be interpreted in two main ways.

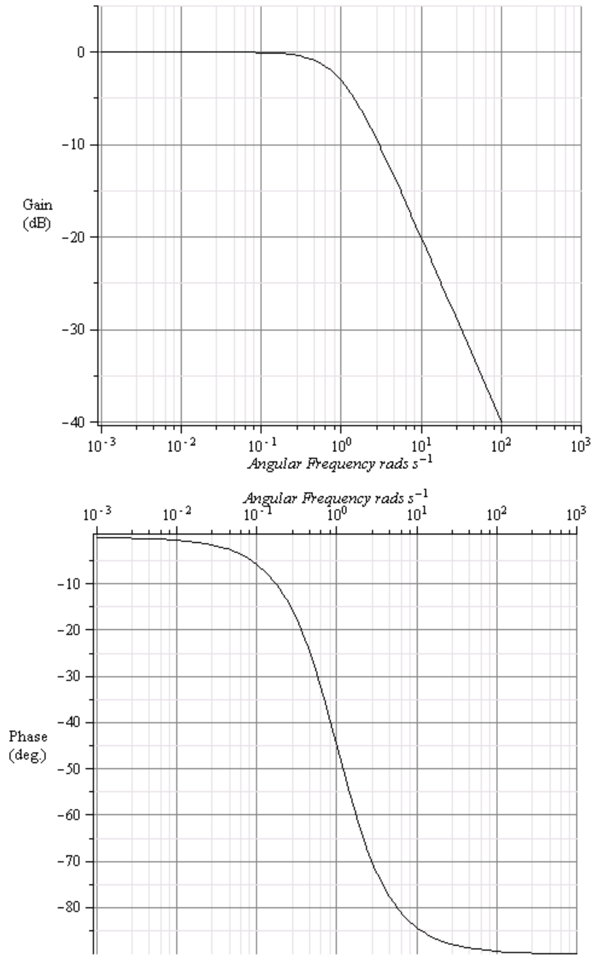

Figure 7.2: Bode plot of the low-pass (RC) filter.

Argand diagram. We can use the Argand diagram (Figure 7.3). At some instant in time we know that the voltage v_R across the resistor (and the current through it) is along the real axis, i.e. it has zero phase angle. The voltage v_C across the capacitor is always out of phase with the current, and we can see that it has a phase angle $-\pi/2$. The voltage on the input is along $\mathbf{Z}$, which is the equivalent (combined) impedance. The phase of the gain is defined as the angle by which the output voltage *leads* the input voltage, and here this is negative, i.e. $\phi = -\tan^{-1}(\omega RC)$.

Phasor representation. A simpler but perhaps less intuitive method is to consider the phasor representations of the quantities involved, and use

$$\mathbf{v_{out}} = \mathbf{G} \times \mathbf{v_{in}}\,.$$

We know that $\mathbf{v_{in}}$ lies along $\mathbf{Z}$ at $t = 0$, so we can write that the phase angle is (clockwise from the real axis) $\theta = 3\pi/2 + \phi$ and, consequently,

$$\begin{aligned}\mathbf{v_{out}} &= |\mathbf{G}|\, e^{j\phi}\, |\mathbf{v_{in}}|\, e^{j\theta} \\ &= |\mathbf{G}|\, |\mathbf{v_{in}}|\, e^{j(\theta+\phi)}\,.\end{aligned}$$

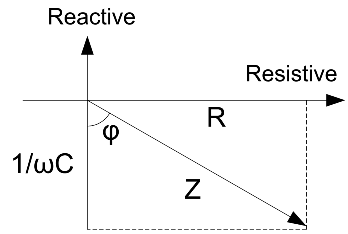

Figure 7.3: Argand diagram for the low-pass (RC) filter.

As we know from above, ϕ is negative, so it is clear that $\mathbf{v_{out}}$ lags $\mathbf{v_{in}}$ by ϕ.

7.1.4. Low-pass filter

Frequencies $\omega < \omega_c$ pass from input to output with little attenuation (or phase change); ω_c is the *cut-off frequency*, i.e. the frequency at which the gain has fallen by 3 dB (i.e. to $1/\sqrt{2} \approx 70\%$ of the maximum). If we imagine that the filter is driving a load resistance R_{load} connected across its terminals, then the cut-off frequency is where the power delivered into R_{load} has fallen by one-half compared to the flat or "pass-band" range of frequencies. For $\omega \gg \omega_c$ we can see that

$$|\mathrm{G}| \propto \frac{1}{\omega}\,. \tag{7.4}$$

On the Bode plot we see this as a slope of -20 dB per decade of frequency (a decade is a factor of 10). This is equivalent to -6 dB per octave (an octave is a factor of two in frequency). Both these figures are often quoted as the characteristic "roll-off" frequency behaviour for a first-order low-pass filter.

High frequencies are severely attenuated, hence the understanding of this circuit as a *low-pass filter*. As an example, were we to build a filter with $R = 1\,\Omega$, $C = 1\,\mathrm{F}$ and apply a 1 V signal at $\omega = 1$ rad/s, then the waveforms would be as per Figure 7.4.

Note that, by KVL, the voltage across the resistor has the opposite behaviour, so if we swap resistor and capacitor we have a *high-pass filter* (this is often called the "CR circuit", as we take the output over the resistor instead of the capacitor).

7.1.5. Transient response of the high-pass filter

For situations where we do not have a "steady state", we need to consider the transient response of the circuit. This is common in instrumentation applications: something happens at time $t = 0$ and we need to observe how the system evolves for times $t > 0$. Consider the modified circuit in Figure 7.5. Note that in this figure we have swapped the positions of the resistor and capacitor. This is just because we want to observe the voltage across the resistor, but in terms of frequency-response this is a high-pass filter. If we set the applied voltage to be a constant V and then close the switch at time $t = 0$ then $v(t) = u(t)V$.

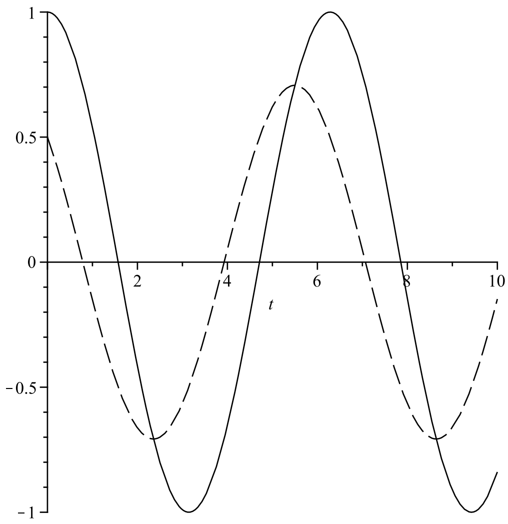

Figure 7.4: Low-pass filter waveforms (solid line is input, dashed line is output).

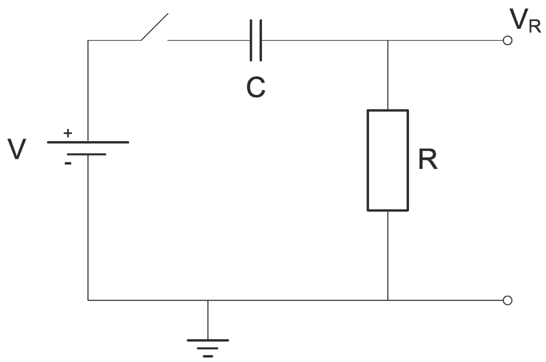

Figure 7.5: Switched CR circuit.

From now on we will just consider $t > 0$ so:

$$\begin{aligned} v(t) &= V \\ v_R(t) &= i(t)R \\ v_C(t) &= \frac{q(t)}{C}\,. \end{aligned}$$

Using KVL we find

$$v(t) = v_R(t) + v_C(t)\,,$$

and, taking the time derivative, we obtain

$$0 = RC\frac{\mathrm{d}i}{\mathrm{d}t} + i\,. \tag{7.5}$$

Note that we have set up a first-order ordinary differential equation. As a fundamental rule, there must be as many arbitrary constants in the solution to the equation as the order of the equation. This means we need additional information if we are to find a complete solution, and this information comes from a knowledge of the state of the system, usually taken at the initial time $t = 0$. You will have come across several methods for solving ODEs; the form of equation (7.5), where the function and its derivative must add to zero, suggests we attempt a trial solution of the form

$$i(t) = A\,e^{st}\,. \tag{7.6}$$

Substituting into (7.5),

$$RCsAe^{st} + Ae^{st} = 0\,. \tag{7.7}$$

yields

$$s = -\frac{1}{RC}\,.$$

To find the unknown A we apply knowledge of the initial condition of the circuit. This is assumed to be *relaxed*, that is to say the capacitor is uncharged, therefore $v_C(0) = 0$, and we can take $v_R(0) = V$. A physical understanding of this is that, the instant the switch is closed, there is no charge on the capacitor so an instantaneous current $i(0) = V/R$ will flow. Substituting into equation (7.5) gives

$$i(0) = A = \frac{V}{R}\,.$$

So, the full solution is

$$i(t) = \frac{V}{R}\,e^{-t/RC}\,,$$

from which

$$v_R(t) = i(t)R = V\,e^{-t/RC}\,. \tag{7.8}$$

The exponential is characteristic of first-order systems. Figure 7.6 gives the response of a high-pass filter with $R = 10\,\mathrm{k\Omega}$, $C = 1\,\mu\mathrm{F}$ to a step input of $2\,\mathrm{V}$. As $t \to \infty$ the capacitor achieves full-charge and hence $v_R \to 0$ as no current flows. Note that had we chosen a different initial condition then the solution would have been different.

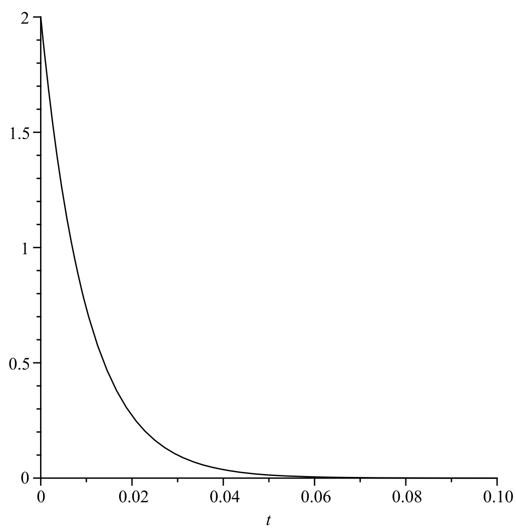

Figure 7.6: Time-domain response of high-pass filter to a step input.

The *time constant* of the system is RC. Note the relationship between the time constant and the cut-off frequency:

$$\omega_c = \frac{1}{\tau} = \frac{1}{RC}\,. \tag{7.9}$$

Initial conditions. There are three factors which determine how the voltage in our circuit evolves:

1. The differential equations which govern the circuit. These are entirely fixed and governed by the physics of the circuit;
2. The applied signal, in our case the voltage $v(t)$. This is frequently referred to as the *forcing function*; it is the externally applied stimulus which causes our system to respond;
3. The initial conditions of the system. Previously, we specified that the system was *initially relaxed*, that is to say the capacitor carried no initial charge. This is not always the case.

Imagine now that at some time after the capacitor is fully charged we open the switch. No current can flow round the circuit, therefore the capacitor remains charged: $v_C = V$. If we now close the switch again, nothing happens – no current flows and v_R remains zero. Imagine that we now instantaneously change our forcing function to zero, i.e. $v(t) = 0$. Then, following the same methods as above, we will find that

$$v_R(t) = -Ve^{-t/RC}\,.$$

7.1.6. Integration and differentiation

If we now repeat this on/off sequence every T seconds where, $T \gg RC$, then our forcing function is a square wave between zero and V with period T. Plotted on a scale of several input cycles, the output looks like a sequence of spikes, positive and negative – see the upper sketch in Figure 7.7. The output is then *approximately* the time-derivative of the input signal. Mathematically, we can show this by

$$v_R(t) = iR = RC\frac{\mathrm{d}v_c}{\mathrm{d}t} \approx RC\frac{\mathrm{d}v_{in}}{\mathrm{d}t}\,.$$

For $\omega \gg \omega_c$ we find the impedance of the capacitor dominates over that of the resistor, therefore $v_{in} \approx v_C$.

Conversely, were we to go back to the low-pass configuration, but increase the values of R and C to make the time constant $T \ll RC$, we would find the circuit integrates the square wave to give a triangle output. We shall come back to this later on, but for now it is important to have a physical understanding of these RC circuits and how they behave in both the time and frequency domains. The principal characteristics are summarised in Table 7.1.

7.1.7. Physical analogues

The RC circuit is a good analogue for physical systems where the rate of change of a parameter is proportional to the magnitude of the parameter itself – see equation (7.5). For example, many thermal calculations apply the knowledge that heat-flow (and hence rate of change of temperature) is proportional to temperature difference. In mechanics, we might have a situation where a friction-force

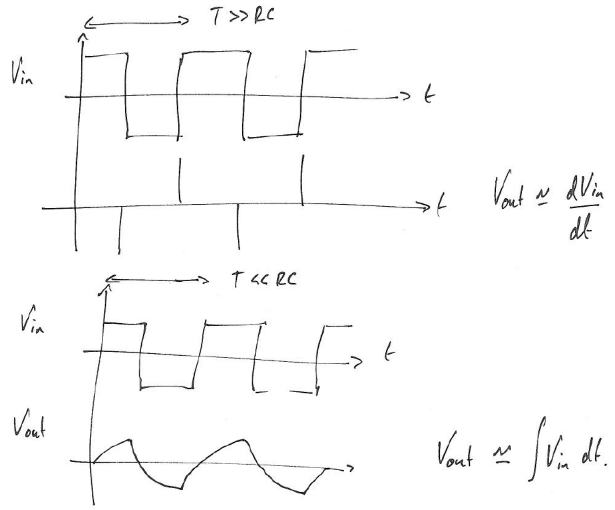

Figure 7.7: Differentiation of a square wave with a frequency above the cut-off of an RC high-pass filter (upper panel) and integration of the same signal for a frequency below the cut-off of an RC low-pass filter (lower panel).

(hence rate of change of velocity) is proportional to velocity. Such systems can be modelled by RC circuits, and indeed in the past before the use of numerical methods and computational models, it was common practice to model physical systems with such “analogue computers”.

7.1.8. Linearity of the RC filter

The reason we have spent this much effort studying the RC filter is that it is a *Linear Time-Invariant (LTI) system* (see Section 13). We will see how we can predict the behaviour of linear systems not just under sinusoidal inputs but with any arbitrary input. This will become a key theme in our studies, and in later parts of the course we will show how to use Fourier and Laplace transforms to gain easy solutions to some seemingly very complex problems in systems analysis, with the important proviso that all the component parts of the system are themselves linear.

To summarise what we know about the RC filter we start with the complex gain (here for the low-pass filter) given by:

$$\mathbf{G} = \frac{1 - j\omega RC}{\omega^2(RC)^2 + 1}\,. \qquad (7.10)$$

- The filter gain is a complex quantity and also a function of frequency. The plot of $\mathbf{G}(\omega)$ is the Bode plot;
- For any frequency ω the gain gives us the output sinusoid $\mathbf{v_{out}}$ amplitude and phase compared to the input $\mathbf{v_{in}}$;
- Output and input sinusoid have the same frequency ω but an adjusted amplitude and phase;
- The behaviour of the RC filter is governed by first order ordinary differential equations (for example equation (7.5)), hence it is a first-order linear system;

Table 7.1: Summary of RC circuit characteristics.

Behaviour	Low-Pass (RC)	High-Pass (CR)
Measure output across	C	R
Time constant	$\tau = RC$	$\tau = RC$
Cut-off frequency	$\omega_c = 1/RC$	$\omega_c = 1/RC$
$\omega \ll \omega_c$	flat gain 0 dB	gain slope +20 dB/decade (differentiates)
$\omega \gg \omega_c$	gain slope −20 dB/decade (integrates)	flat gain 0 dB

- The time domain response to a step input is the familiar exponential form with a characteristic time constant.

Here is where our understanding of signals as complex quantities becomes useful. If we write the input as a sinusoid

$$\mathbf{v_{in}} = A\, e^{j\theta}\, e^{j\omega t}\,,$$

and the gain (evaluated at frequency ω) as

$$\mathbf{G} = G\, e^{j\phi}\,,$$

then

$$\mathbf{v_{out}} = \mathbf{G}\,\mathbf{v_{in}} = G\, e^{j\phi}\, A\, e^{j\theta}\, e^{j\omega t}\,.$$

As was stated previously, the time dependent part of the expression is frequently omitted since it is common to input and output. We can do this because the *linear system always preserves the frequency of the input.*

$$\mathbf{v_{out}} = G\, A\, e^{j(\theta+\phi)}\,.$$

That is to say, the output is the input scaled by a factor G (the magnitude of the complex gain **G**) and rotated by an angle ϕ (the phase angle of the complex gain). Now comes the most important point which will be essential for our future studies. For a linear system, the output is a linear superposition of the input. If our input is composed a several sinusoids, we can apply the above calculation for each input component and sum the results to get the output. Since Fourier tells us *any* input can be decomposed into sinusoids, *this means that the complex gain tells us what the output will be for any arbitrary input waveform.* This is important, as it allows use to use Fourier (and related) techniques to solve problems.

7.2. The RLC circuit

The *RLC circuit* shown in Figure 7.8 is the electrical analogue of the damped mechanical oscillator. In fact, we can use an analogue electronic system to simulate a complex mechanical system because of this equivalence. Whilst this is quite a powerful technique, it has been superseded by numerical computer simulations these days. In the RLC circuit we have the equivalence with the spring balance or damped harmonic oscillator.

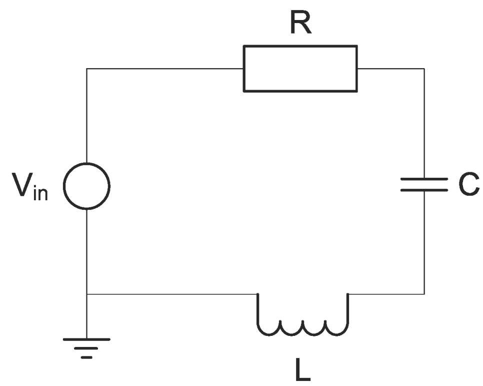

Figure 7.8: The RLC circuit.

As with the mechanical spring-balance we can determine the differential equations which govern the behaviour through use of some physical laws of conservation; in this case we use KVL:

$$v_R + v_C + v_L = v_{in}\,.$$

which we can re-write as

$$iR + L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int_{-\infty}^{t} i(t')\mathrm{d}t' = v_{in}\,.$$

If we take v_{in} to be a step input at $t = 0$ then, for all times $t > 0$, we can differentiate and write

$$L\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + R\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}i = 0\,.$$

Therefore, the series RLC circuit shows the same sort of transient response as the mechanical oscillator with undamped natural frequency ω_0 and damping ratio ξ:[15]

$$\begin{aligned} \omega_0^2 &= \frac{a_0}{a_2} = \frac{1}{LC} \\ \xi &= \frac{a_1}{2\sqrt{a_0 a_2}} = \frac{R}{2}\sqrt{\frac{C}{L}}\,; \end{aligned} \qquad (7.11)$$

We can understand the frequency-domain behaviour of the circuit best using the complex impedance approach given

[15] We define the a constants in Section 14, unimportant here.

previously. The total impedance of the circuit is given by

$$\mathbf{Z} = R + j\omega L - \frac{j}{\omega C}\,.$$

Since R is fixed, the circuit has minimum impedance when

$$j\omega L = \frac{j}{\omega C}\,,$$

which is the same result as equation (7.11). Further, since $\mathbf{Z}$ is a function of frequency, it is interesting to plot the response of the system as a function of frequency. Figure 7.9 shows the current flowing in the circuit as a function of frequency. For this figure, we take $L = R = C = 1$ and the applied voltage is 1 V in amplitude and swept in the range $\omega = 0.01...100$ rad/s. The *resonance* at $\omega_0 = 1$ rad/s is clearly visible.[16] At the resonance, the reactive part of the impedance is zero, so we have only resistive impedance, hence the current is 1 A.

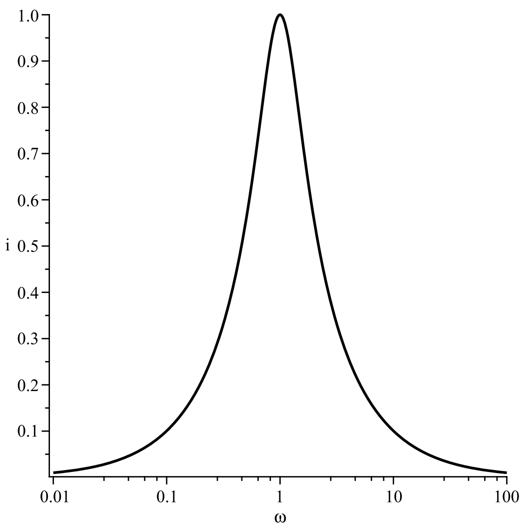

Figure 7.9: Current flowing in RLC circuit as a function of frequency.

At low frequencies the impedance $|\mathbf{Z}| \to \infty$ because of the capacitor, and at high frequencies $|\mathbf{Z}| \to \infty$ because of the inductor. Therefore, the overall shape of the curve is physically understandable. The mechanical oscillator would also show a similar behaviour in response to a swept-sinusoidal forcing function (force). Moreover, the RLC circuit will show similar time-domain response to a step input.

[16]Note that the *natural frequency* is the *un-driven* oscillation frequency, as we saw for the spring-balance where y = constant. For the RLC circuit we found the resonant frequency under driven conditions (swept sinusoid input). In general, for second-order systems the natural frequency and resonant frequency are close but not identical. But the RLC circuit is one where they are identical.

8. Filters

A filter is used to modify the frequency content of a signal: usually, either to reject some unwanted component, such as interference at a particular frequency (e.g. mains at 50 Hz, or DAB radio near 200 MHz), or more generally to remove broad-band noise over a range of frequencies which are above/below the signal we are interested in. Passive filters are made from the R, L and C components. They offer no amplification – the output signal can never be larger than the input – hence the maximum *gain* of a passive filter is 0 dB.

8.1. Filter types

Low-pass/high-pass. We have studied these in some depth, and we have seen how to implement simple versions of these with RC and CR circuits. The *bandwidth* of the low-pass filter is easy to define: from 0 Hz to the *cut-off frequency* (-3 dB point). The high-pass configuration (assuming the same cut-off) *blocks* this range of frequencies and *passes* frequencies above.

Band-pass. The band-pass filter has both a lower and a higher cut-off: it passes frequencies in-between. Therefore, it has a clearly defined bandwidth, and we want it to be flat in the *pass-band* region. In its simplest form the band-pass filter can be a RC-CR combination with appropriately chosen components.

Band-stop. Similarly, we can design a filter to reject a range of frequencies.

Notch filter. A variation on the previous two is the notch filter which is designed to pass (or reject) a particular frequency. In contrast to the band-pass/stop, the notch filter is designed to be as narrow as possible so that it only filters a particular frequency. These are used to select (or reject) a particular frequency component contained within a signal. Notch filters can be made using the RLC resonant circuit as seen in Section 7.

8.2. Filter characteristics

The ideal filter rejects all frequencies outside the pass-band and passes all other frequencies perfectly. The ideal low-pass filter would be 'square', as shown in Figure 8.1. The pass-band is flat, the *knee* is perfectly sharp, and the slope of the *roll-off* is infinite.

We have seen that the passive RC filter, when used in either low- or high-pass configuration, has a gain roll-off of 20 dB/decade (see Figure 5.5, e.g. between 10 rad/s and 100 rad/s). This is equivalent to 6 dB/octave, and it is a useful exercise to check that this is so. This behaviour arises because – for frequencies well above the cut-off – the gain has a $1/f$ characteristic. In the first lab exercise the RC filter is used as an anti-aliasing filter, which exposes its limitations. Frequencies above the Nyquist are attenuated, however they are still present, and consequently they alias.

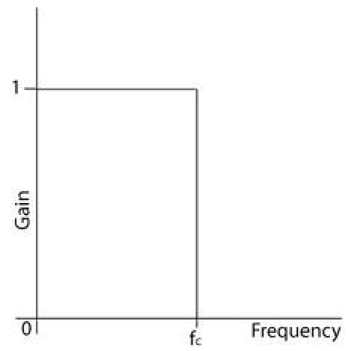

Figure 8.1: The ideal low-pass filter.

8.3. Higher order filters

We can chain together two RC filters to make a second-order filter which should have a roll-off of 40 dB/decade. Three or more would increase the effect. This requires some care if we are to avoid *loading*. The filter is a frequency dependent voltage divider. Ideally, the signal source will be low-impedance (e.g. a signal generator) and the filter will pass the signal to a high impedance load (e.g. an oscilloscope). The filter's impedance must not be too low otherwise it will load the source. The load (scope) must have a higher impedance otherwise it will load the filter.

Consider two chained low-pass filters. The RC of the second filter is actually in parallel with the C of the first. So, if this is not to load the first filter, we must make the impedance of the second filter somewhat higher than the first. A factor of ten is a good rule of thumb. Note that we can achieve this even for two filters with the same cut-off since it is the product of R and C which is important, not the individual values. In practice it is hard to make good sharp and steep filters with purely passive components.

8.4. Introducing active components

When loading is a problem, we can use a buffer, which is an active electronic device (typically an op-amp). We will look at the design later in the course. The buffer has a high input impedance, low output impedance, and a gain of 1. It can be used to isolate two circuit blocks from each other to prevent the second from loading the first. This means we could – in principle – chain together as many identical RC sections as we like without fear of loading. We would get a fast roll-off but the knee would still be quite blunt.

If we are going to use op-amps then there are better active filters where the op-amp is part of the filter design and with remarkably few components we can achieve filters which are much better than the passive types. Some examples are shown in Figure 8.2. The problem is that the analysis of these circuits is fiendishly complicated. For the purposes of this course it is sufficient to know that:

- Active filters use op-amps as an integral part of the filter design;

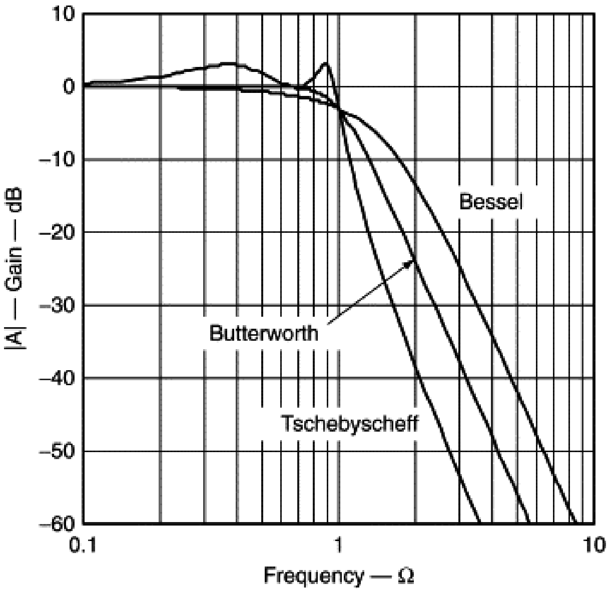

Figure 8.2: Some active low-pass filters (T. Kugelstadt, in *Op Amps for Everyone* (3$^{\text{rd}}$ edition), 2009). Typical features are the flat pass-band of the Butterworth filter and the sharper "knee" of the Chebyshev filter at the expense of some pass-band ripple.

- There are several standard designs to choose from; the user needs to look-up the correct component values to use in order to achieve the desired performance;
- Flatness of the pass-band is often compromised to achieve a 'sharp knee' or a 'fast roll-off;
- The op-amp provides buffering, so the active filter will not load the source;
- Signal amplification is possible.

We will come back to this topic again later in the course.