Problem Sheet 3

1. SEMINAR QUESTION

(a) You are asked by a company to design an instrument that produces short pulses with a perfect "top-hat" shape:

$$f(t) = P_a(t) = \begin{cases} 1, & |t| \leq a \\ 0, & |t| > a \end{cases}$$

They need to have a temporal full-width at half-max (FWHM) of $2a$, where a is a small constant. Use the integral form of the Fourier transform to show that the frequency content of the pulse is given by:

$$F(\omega) = \frac{2\sin\omega a}{\omega}.$$

(b) Should you take the contract to build this instrument? It's worth rather a lot of money, but the contract also includes penalties for non-completion of the work to the design specification. Explain your answer with reference to the frequency spectrum you have derived.

(c) State $\mathrm{F}(\omega \to 0)$ and ω for the first instance of $\mathrm{F}(\omega) = 0$.

(d) As a decreases describe qualitatively the change is the shape of $f(t)$ and $F(\omega)$. What is the physical interpretation of this behaviour in terms of pulse duration and signal bandwidth?

(e) As a increases how does $F(\omega)$ behave? What common function does this suggest and what is the physical interpretation?

2.

(a) A laser oscillator produces very short pulses of nearly Gaussian form where the electric field strength is approximated by

$$f(t) = Ae^{-b^2t^2} .$$

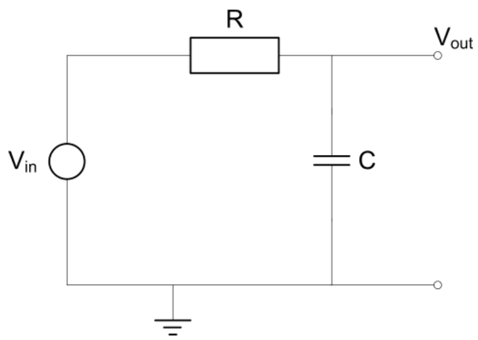
The intensity of the light is therefore proportional to $f(t)^2$. In laser physics, the pulse duration is taken to be the FWHM of the intensity profile, and the bandwidth is by convention the FWHM of the "power density" $F(\omega)^2$ in Hz. Show that the minimum time-bandwidth product is 0.44.

(b) For a pulse with FWHM=167 fs what is the FWHM bandwidth in Hz required to support this pulse width?

(c) If the pulse has a centre wavelength of 800 nm, what is the minimum bandwidth in nm?

3.

(a) The circuit in the figure shows a low-pass filter. For a step-input $V_{in} = V_0 u(t)$ find an expression for the output voltage V_{out} as a function of time $t > 0$.



(b) How should the circuit be modified to turn it into a high-pass filter?

(c) For the high-pass filter, find an expression for the complex gain of the filter as a function of angular frequency ω.

(d) What are the limiting values for the gain magnitude for the cases $\omega \to \infty$ and $\omega \to 0$?

(e) Find the cut-off frequency of the filter in terms of R and C.

(f) For frequencies below the cut-off, show that the gain increases 20 dB for every decade in frequency.

4.

(a) A 12-bit analogue-to-digital converter is used to measure a positive voltage signal with a resolution of 1 mV. What is the maximum digitsable input signal in volts?

(b) Give the dynamic range of the digitiser in dB;

(c) State the maximum quantisation error;

(d) Give the RMS quantisation noise added.

5.

(a) A signal analyser digitises at 10^6 samples per second (S/s). What is the Nyquist frequency of the digitised signal, in Hz?

(b) Which of the following frequencies can be properly sampled? 266 kHz, 498 kHz, 502 kHz, 1.6 MHz.

(c) For those improperly sampled, give the alias frequency.

Numerical Answers and Hints

1. (b) Consider the bandwidth the instrument would need to have; (c) $2a$, $\pm {}^{\pi}/_{a}$.

2. (b) 2.64×10^{12} Hz (c) Find the centre frequency and split the bandwidth in (a) equally either side of this to get $\Delta\lambda \sim 5.6$ nm

4. (a) 4.096V (b) 72.3 dB (c) ± 0.5 mV (d) 0.29 mV.

5. (a) 500 kHz (b) 266 and 498 kHz (c) 498 and 400 kHz.