6. Fourier theory of signals

Generally, a signal is any time- or spatially-varying quantity. We are used to time-varying quantities such as voltage, current, light intensity and sound pressure. A spatially-varying signal might be, for example, an image. Typically, a signal represents something physical.

Continuous and discrete signals. Both the sound pressure and its electrical representation are signals in their own right, and both these types of signals are continuous functions of time. Generally, any physical quantity we may wish to deal with is represented by a continuous parameter such as mass, length, voltage, current, etc. However, the very act of measurement creates a discrete signal. Each time we make a measurement of a quantity we are generating a discrete signal. For example, the height of the sea measured every hour, or the temperature in a reaction chamber measured every second. These are examples of sampled signals. In physics and engineering we encounter mainly signals which have been both sampled and digitised. In this section we will be looking at *continuous signals*; however, it is important to always bear in mind that a discrete equivalent exists.

Learn it in lab
In the third lab session you will look at the representation of signals in the frequency domain, and characterise the components of a system in frequency space by measuring and analysing their *frequency response functions.* The Fourier treatment of signals introduced here (and extended in later sections to the more generic Laplace transform treatment) is at the heart of this topic.

6.1. Periodic continuous signals

Figure 6.1 shows six common waveforms. The simplest is the last (f), which is the sinusoid. This might represent, for example, the position of a mass oscillating with simple harmonic motion. This figure shows both the time and frequency domain representations. For the latter, the relative amplitudes of the first six Fourier coefficients are given. By convention, f is the fundamental frequency, and $2f$, $3f$ are the 2^{nd} and 3^{rd} harmonics, and so on. Our sinusoid of course has only the fundamental, but the rectified version (e) has both a "DC" (constant) component at 0 Hz but also lots of higher harmonics which are due to the sharp corners in the waveform. Saw-tooth (d), triangular (c) and square (b) waveforms are frequently seen in electronics. The triangular and square waves have the characteristic that all the even harmonics are zero. However, all three have the characteristic that their Fourier series is infinite, i.e. an infinite number of harmonics is required to accurately represent these waveform.

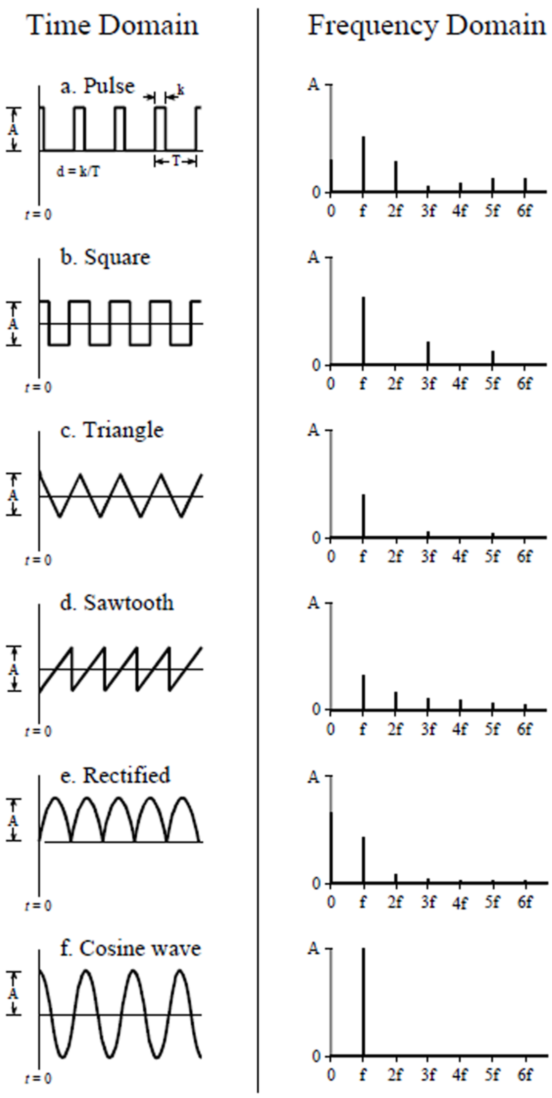

Figure 6.1: Common waveforms in physics or engineering.

6.1.1. Fourier series of periodic waveforms

Wherever there exists periodicity, we should seek a Fourier representation, and a Fourier understanding of signals is important in instrumentation as it allows us to predict a *linear system's* output in response to a periodic input. In practice, all physically realistic *periodic* signals obey the Dirichlet conditions[7] and are therefore transformable into a Fourier series. Hence, any signal $f(t)$ with period T can be expressed as

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{+\infty} \alpha_{\mathbf{n}}\, e^{jn\omega_0 t} \\ &= \sum_{n=-\infty}^{+\infty} |\alpha_{\mathbf{n}}|\, e^{j(n\omega_0 t+\phi_n)}\,, \end{aligned} \tag{6.1}$$

[7]See Poularikas section 3.2.

where $\omega_0 = 2\pi/T$ (note: this is a constant) and the $\alpha_\mathbf{n}$ are complex constants given by

$$\begin{aligned} \alpha_\mathbf{n} &= \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_0 t}\,\mathrm{d}t \qquad (6.2)\\ &= |\alpha_\mathbf{n}|\, e^{j\phi_n} \\ &= |\alpha_\mathbf{n}|\cos\phi_n + j|\alpha_\mathbf{n}|\sin\phi_n\,. \qquad (6.3)\end{aligned}$$

Since the $\alpha_\mathbf{n}$ are determined from $f(t)$, and vice versa, they are known as a *Fourier transform pair.* At any time t_0 the Fourier expansion of the function converges to $f(t_0)$ as long as the function is continuous at t_0. If the function is discontinuous, then the Fourier expansion converges to a point mid-way between the discontinuity. If $f(t)$ is real, then

$$\begin{aligned} \alpha_{-\mathbf{n}} &= \alpha_\mathbf{n}^* \\ f(t) &= \alpha_0 + \sum_{n=1}^{+\infty}[(\alpha_\mathbf{n}+\alpha_\mathbf{n}^*)\cos n\omega_0 t \\ &\quad + j(\alpha_\mathbf{n}-\alpha_\mathbf{n}^*)\sin n\omega_0 t\,]\,, \qquad (6.4)\end{aligned}$$

which can be written in trigonometric form as[8]

$$\begin{aligned} f(t) &= \frac{A_0}{2} + \sum_{n=1}^{+\infty}(A_n\cos n\omega_0 t + B_n \sin n\omega_0 t) \\ A_0 &= \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\,\mathrm{d}t \\ A_n &= \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t\,\mathrm{d}t \\ B_n &= \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t\,\mathrm{d}t\,, \qquad (6.5)\end{aligned}$$

or

$$\begin{aligned} f(t) &= \frac{A_0}{2} + \sum_{n=1}^{+\infty} C_n\cos(n\omega_0 t + \phi_n) \\ C_n &= \sqrt{A_n^2 + B_n^2} \\ \phi_n &= \tan^{-1}\left(\frac{B_n}{A_n}\right)\,. \qquad (6.6)\end{aligned}$$

The square wave. The Fourier expansion of a square wave of unity amplitude and unity period is

$$f(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{\sin 7t}{7} + \ldots\right)\,, \qquad (6.7)$$

i.e. all the A's are zero, as are all the even B_n. Figure 6.2 illustrates the first nine coefficients and Figure 6.3 shows the square wave reproduced from the first 2, 3 and 9 *non-zero* coefficients. As n increases so does the fidelity, though we are always left with an overshoot of about 10% at the edges (Gibbs' phenomenon). Fidelity is worst at the edges, and this improves rapidly with n. From this we know that sharp edges are represented by high-frequencies in the expansion. We see this if we run a square wave through a low-pass filter: the edges are "rounded off" (see also the transient response of the RC filter in the next section).

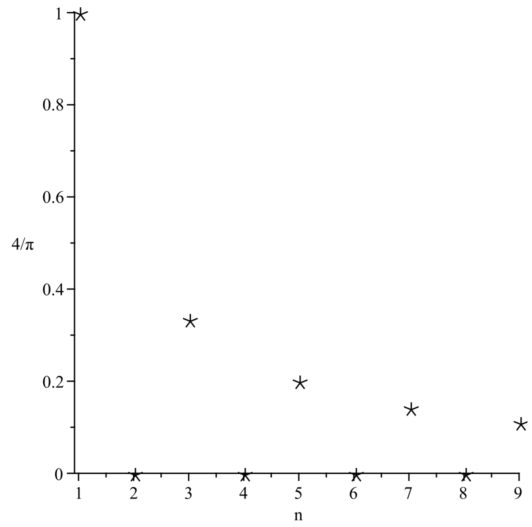

Figure 6.2: Square wave amplitude coefficients.

Related is the fact that the higher harmonics in the expansion contribute only to the detail of the waveform. We can see in Figure 6.3 where the centre plot shows a passable square wave from just the fundamental plus the 3$^{\text{rd}}$ and 5$^{\text{th}}$ harmonics (the first 3 non-zero coefficients). Figure 6.4 plots the power spectrum (amplitude coefficients squared, normalised to the power of the fundamental). The higher harmonics contribute a tiny fraction of the overall signal power.

The triangular wave. The Fourier expansion of a triangular wave of unity amplitude and unity period is

$$f(t) = \frac{8}{\pi^2}\left(\sin t - \frac{\sin 3t}{9} + \frac{\sin 5t}{25} - \frac{\sin 7t}{49} + \ldots\right)\,. \qquad (6.8)$$

Figure 6.5 shows the first 9 coefficients and subsequent expansion. Note that:

1 There are negative coefficients;
2 The coefficients are similar to those of the square wave but scaled by $1/n$. This makes sense when we consider the action of the low-pass filter on the square wave: the low-pass filter has a response $\propto 1/f$ for frequencies higher than the cut-off frequency.

6.1.2. Symmetry, spectra and energy

An even function has symmetry about the $t = 0$ axis (i.e. it is the same under reflection about the vertical axis, e.g. $\cos(t)$), while an odd function has rotational symmetry

[8]Note that you would not be expected to reproduce these formulae for an exam!

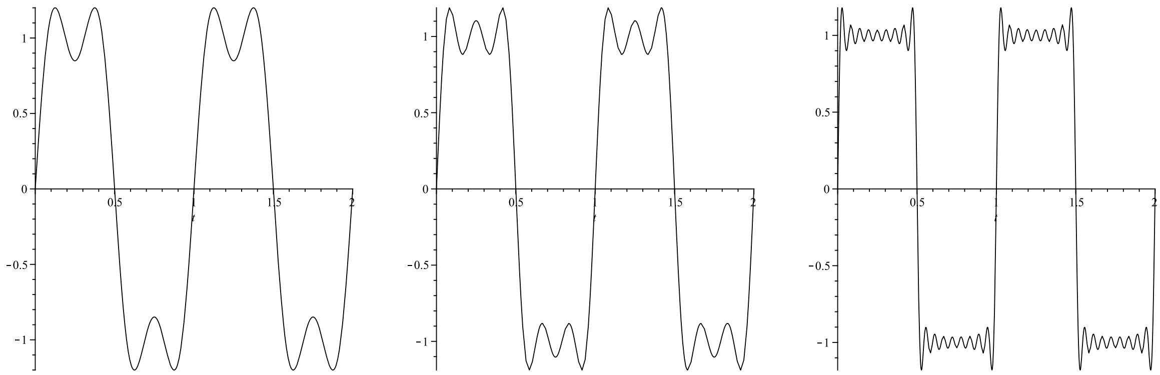

Figure 6.3: Square wave reproduced from the first 2, 3 and 9 non-zero Fourier coefficients.

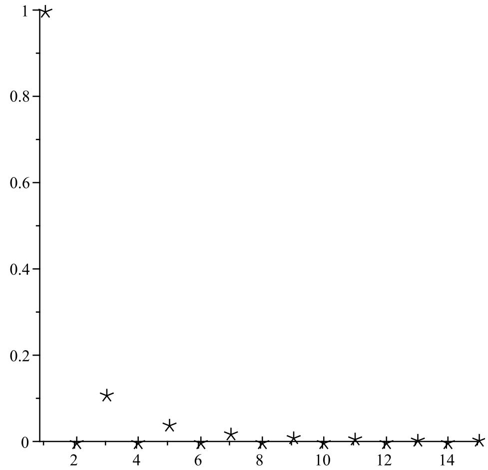

Figure 6.4: Power spectrum for the square wave.

about the origin (unchanged after rotation 180° around the origin, e.g. $\sin(t)$). Table 6.1 summarises these symmetries which we can exploit (noting that many functions have no overall symmetry).

Table 6.1: Symmetries and Fourier coefficients.

symmetry	functional form	A_0	A_n	B_n
even	$f(t) = f(-t)$	exist	exist	0
odd	$f(t) = -f(-t)$	0	0	exist

We should point out that symmetry can be a matter of phase. For example, in our expansion of the square and triangular waves above we see only sine terms (i.e. the B_n), suggesting that we started with an odd function. This is not what we represented in Figure 6.1, where these functions were even – but if we can shift periodic waveforms along in time (in the same way that we can turn a sine into a cosine) this can simplify things.

Typically, we might specify a spectrum of a signal by its coefficients C_n (known as the *amplitude spectrum*) and their corresponding phases ϕ_n (known as the *phase spectrum*). The spectra studied in the previous sections are special cases with zero phase spectra. As pointed out above, choosing an appropriate origin for the representation of a signal can significantly simplify its expansion.[9]

The energy relation. Parseval's theorem applied to a signal $f(t)$ expanded as a Fourier series gives

$$\frac{1}{T}\int_{t_0}^{t_0+T} f(t)^2 \,\mathrm{d}t = \frac{A_0^2}{4} + \sum_{n=1}^{+\infty} \frac{C_n^2}{2}\,. \tag{6.9}$$

If we consider $f(t)^2$ to be the power in the signal, then the LHS of equation (6.9) is the average power in the signal over the period of 1 cycle. The RHS is the power of the signal in the frequency domain, expressed as a sum of the power of the individual frequency components. Equation (6.9) is known as the *energy relation* since it tells us that energy measured in the time domain is the same as the energy measured in the frequency domain, a consequence of conservation of energy.

6.2. Realistic signals

The Fourier series is excellent for describing the periodic, repetitive signals frequently encountered in physics and engineering, such as the square, triangular and sawtooth waveforms and indeed any arbitrary waveform – *so long as it has a repeat period* T. We see in the Fourier series of the square and triangular waveforms a fundamental signal at the repeat period T which we can recognise in both time and frequency representations, and we can see that the fundamental carries most of the signal power. The Fourier series is very useful where we have a "steady-state" periodic signal input into a system. By *steady state* we mean to imply that the signal has existed long enough that it can be reasonably described as a Fourier series. Mathematically, this would mean that the signal should have existed for all time, which is of course not realistic; however, if the signal has existed long enough for any transient effects to have died away, then this is a reasonable

[9] See Poularikas section 3.3.

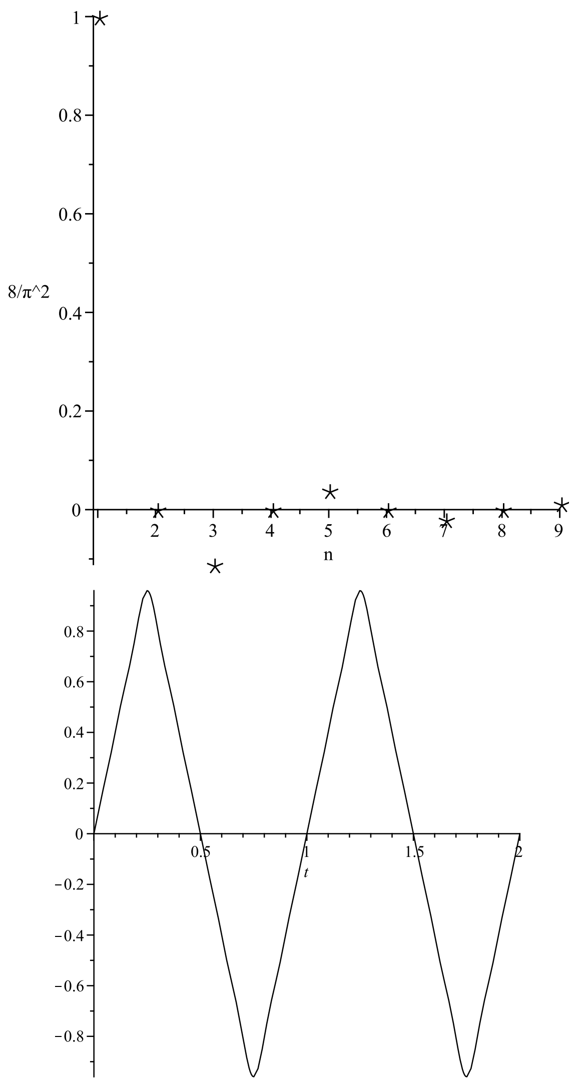

Figure 6.5: Triangular wave coefficients and waveform.

assumption. By *transient effects* we mean the initial response of the system to a sudden start-up of the signal, for example if we take a system and suddenly apply a square wave input then the initial response, say the first few cycles of the square wave, may be rather different from the behaviour after a thousand cycles. As a simple example imagine pushing someone on a playground swing: the periodic push is the input but it takes some number of cycles to build up to the full amplitude. We will come back to these ideas of transient effects later on and see how to deal with them; however, for now it is important to understand that the Fourier series is an excellent way of describing *periodic signals* that have existed for a time much longer than any *time constant* of the system being studied. Conversely, we must recognise that no signal persists infinitely in time, and therefore even a pure sinusoid truncated in time will exhibit some measure of spectral width.

6.3. Non-periodic signals

An example of a non-periodic function is the *unit pulse* function (Figure 6.6) which has width $2a$ and height 1:

$$P_a(t) = \begin{cases} 0, & |t| > a \\ 1, & |t| < a \end{cases} . \tag{6.10}$$

It can be scaled, e.g. $hP_a(t)$, which may represent a voltage h applied for time $2a$. To find the frequency "content" of a non-periodic function such as this rectangular pulse $P_a(t)$ we need the *Fourier integral*, which is in essence the *limiting case* of the Fourier series as the period $T \to \infty$.

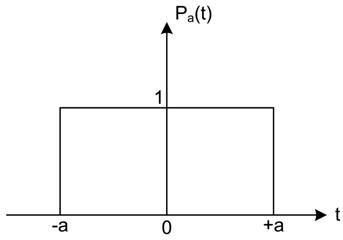

Figure 6.6: The unit pulse function.

6.3.1. Continuous functions

The Fourier transform of a *continuous* function $f(t)$ is written $\mathcal{F}\{f(t)\} = F(\omega)$ and the inverse is $f(t) = \mathcal{F}^{-1}\{F(\omega)\}$. So, we have the *forward transform*:

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, \mathrm{d}t\,, \tag{6.11}$$

and the *inverse transform*

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\, e^{j\omega t}\, \mathrm{d}\omega\,. \tag{6.12}$$

$F(\omega)$ is the *spectrum function* of $f(t)$. Since $F(\omega)$ is usually complex, we can represent it as a plot of the real and imaginary parts versus ω. It is usually more meaningful to plot the absolute value

$$|F(\omega)| = \sqrt{\mathcal{R}\{F(\omega)\}^2 + \mathcal{I}\{F(\omega)\}^2}$$

and the argument

$$\operatorname{Arg} F(\omega) = \tan^{-1}\frac{\mathcal{I}\{F(\omega)\}}{\mathcal{R}\{F(\omega)\}}\,,$$

in which case $|F(\omega)|$ is called the *amplitude spectrum* and $\operatorname{Arg} F(\omega)$ is the *phase spectrum*.

6.3.2. Real functions

In addition to continuous, physical signals are also *real* functions of time. Consider the function $f(t) = u(t)\, e^{-t}$ shown in Figure 6.7, which introduces another useful function: the *unit step* $u(t)$, which is zero for negative times and unity otherwise. The exponential pulse has the Fourier transform

$$F(\omega) = \frac{1}{1 + j\omega}\,.$$

The real, imaginary, magnitude and argument representations of the spectrum are given in Figure 6.8. There are some rules for real functions which can simplify their transforms.

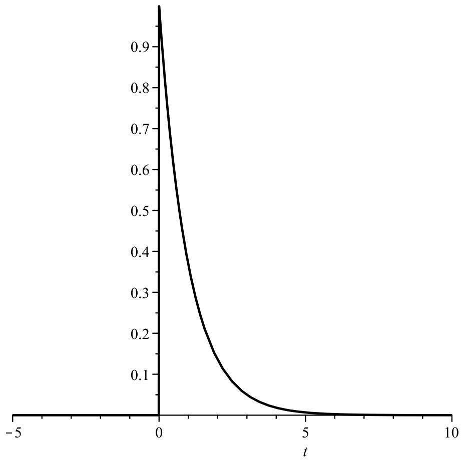

Figure 6.7: The exponential pulse, $f(t) = u(t)e^{-t}$.

Any real function. The reflected form of the spectrum (i.e. reflected about $\omega = 0$) is the complex conjugate of the spectrum:

$$F(-\omega) = F^*(\omega)\,.$$

Real and even functions. The imaginary part of the spectrum is zero:

$$\begin{aligned}\mathcal{R}\{F(\omega)\} &= 2\int_0^{+\infty} f(t)\cos\omega t\,\mathrm{d}t \\ \mathcal{I}\{F(\omega)\} &= 0\,.\end{aligned}$$

Real and odd functions. Conversely,

$$\begin{aligned}\mathcal{R}\{F(\omega)\} &= 0\,. \\ \mathcal{I}\{F(\omega)\} &= -j\int_{-\infty}^{+\infty} f(t)\sin\omega t\,\mathrm{d}t\end{aligned}$$

6.3.3. Properties

Fourier transforms have these important properties:

Linearity.

$$\mathcal{F}\{af_1(t) + bf_2(t)\} = aF_1(\omega) + bF_2(\omega)\,.$$

Scaling.

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)\,.$$

Derivative.

$$\mathcal{F}\left\{\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right\} = (j\omega)^n F(\omega)\,.$$

6.4. Some common signals

The *unit pulse* function and the *delta* function are both important signals for instrumentation applications. We may be trying to measure the height and/or width of a pulse generated by an experiment. In order to understand how our instrument will react to the pulse, we need an understanding of the pulse's frequency content. The delta function, because of its special frequency content, is frequently used as a "test" input into a system in order to determine a system's *impulse response.* We will deal with this later, for now it is important to appreciate the frequency content of these signals.

Pulse function. The Fourier transform of the pulse function (time-domain) is the "sinc" function[10] (frequency domain), and the Fourier transform of the sinc function (time domain) is the pulse function (frequency domain):

$$\begin{aligned}\mathcal{F}\{P_a(t)\} &= \mathrm{sinc}_a(\omega) \\ \mathcal{F}\{\mathrm{sinc}_a(t)\} &= P_a(\omega)\,.\end{aligned}$$

This result means that the pulse function, which is finite in time, has an infinite range of frequencies associated with it. Conversely, the sinc function in the time domain, which exists for all time $-\infty < t < +\infty$, has only a finite range of frequencies. Since all instrumentation has frequency-dependent behaviour, the implications of this are important. It is well worth sketching these for a few different values of a since this is an important topic.

Delta function. The Fourier transform of the delta function is a constant, and *vice-versa*:[11]

$$\begin{aligned}\mathcal{F}\{\delta(t)\} &= 1 \\ \mathcal{F}\{A\delta(t)\} &= A \\ \mathcal{F}\{A\} &= 2\pi A\delta(\omega)\,.\end{aligned}$$

Once again the implications are important. The delta function contains all frequencies in equal measure – see Figure 6.9. A physical example of a delta function input to a system is hitting a mass with a hammer. The hammer transfers a defined amount of energy to the mass in a very short period of time (ideally applying infinite force for an infinitely short period of time such that the total momentum transferred is 1). This means that the hammer blow excites all frequencies *simultaneously.* We can use this to test a system's frequency response. Mechanical engineers do this in practice: strike the object to be tested with a hammer and record the spectrum of frequencies which results. The peaks in the spectrum are where the object resonates.

[10] Recall $\mathrm{sinc}(x) = \sin(x)/x$, and we define $\mathrm{sinc_a}(x) = \sin(ax)/x$.

[11] A simple way to remember this is to consider the delta function as an infinitely narrow Gaussian: the Fourier transform of a narrow Gaussian in one domain is a wide Gaussian in the other domain, and an infinitely wide Gaussian is a constant.

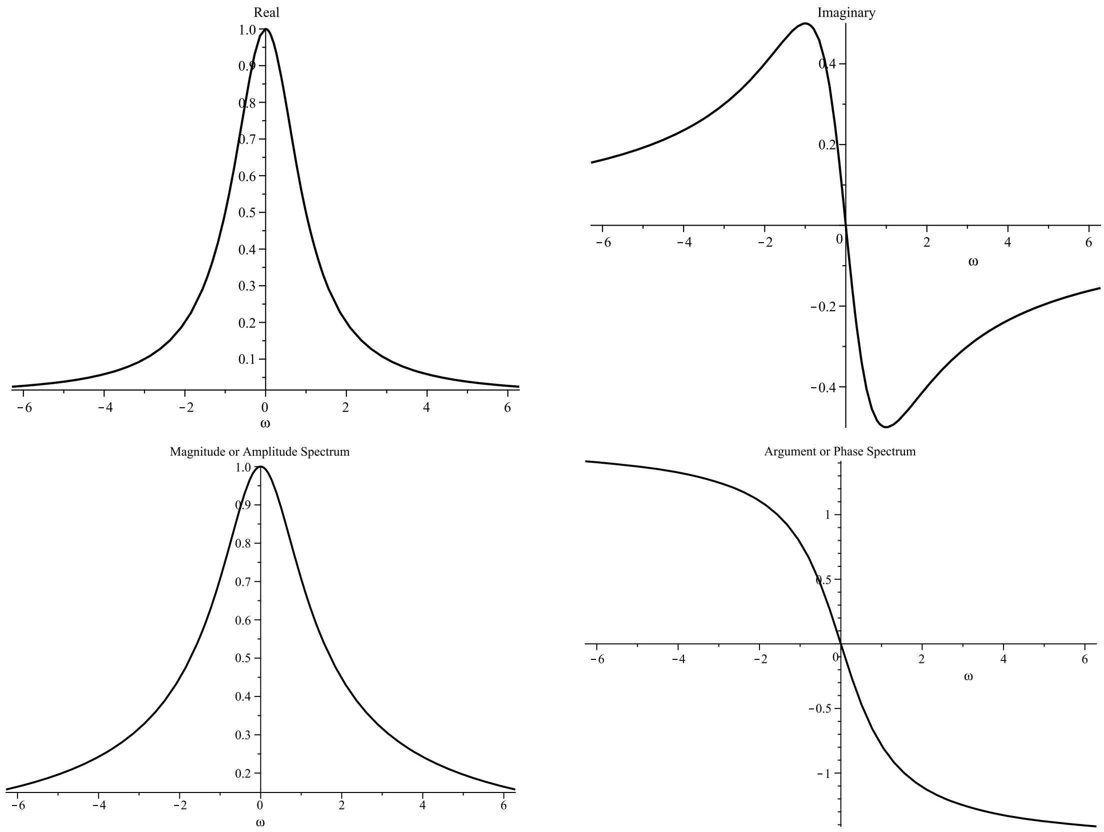

Figure 6.8: $F(\omega)$ representations in terms of $\mathcal{R}\{F(\omega)\}$ and $\mathcal{I}\{F(\omega)\}$ (top two panels) and magnitude and phase spectrum (bottom two panels). This refers to the transform of the exponentially-decaying pulse shown in Figure 6.7.

Sinusoid. The sinusoid contains a single frequency, and the frequency spectrum is represented by the delta function. Mathematically, it exists at both $\pm\omega_0$, though only positive frequencies exist in reality. It is useful to sketch the transform given here:[12]

$$\mathcal{F}\{A\cos(\omega_0 t)\} = \pi A[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]\,.$$

Naturally, all real signals are *finite* in duration, and so even a sinusoidal signal will have a finite (albeit it very small) spectral width. A real-world sine wave may be constructed by multiplying the ideal (infinite) one by a unit pulse, which kills the amplitude outside of some time $|t| > a$. What does the frequency spectrum look like for this real sinusoid? In this case we actually know the Fourier transforms of the individual signals (a delta function and a sinc function), and hence we can use the *Convolution theorem* that states that *the Fourier transform of the convolution of two functions is equal to the point-wise product of their Fourier transforms*: i.e. point-wise multiplication in the time domain equals convolution in the frequency domain and *vice-versa*. So, in this case we expect the delta functions at $\pm\omega_0$ in the frequency domain to be replaced by sinc functions centred at those frequencies. It is worth working through and sketching this example, as it illustrates an important point as well as a typical use of the convolution theorem.

[12]Figure 4.26 of Poularikas gives a very nice pictorial representation of the Fourier representation for many other common signals. It is well worth browsing this to get a feel for the behaviour of signals in the frequency domain.

6.5. Pulsed signals: limiting cases

Figure 6.10 shows the Fourier transform of the pulse function $P_a(t)$ for various values of a. The spectrum is the sinc function, as discussed above. Observe that the lowest frequency at which the function crosses the time axis is at $\omega = \pi/2a$. This means that as a increases (wider pulse) the frequency spectrum gets narrower. In the limit $a \to \infty$, as expected from the previous section, the frequency function becomes infinitely narrow and tends to a delta function.

We can state this as a general rule: *very short pulses (in the time domain) have very wide spectra, whilst very wide pulses have nicely compact spectra.* We can formalise this general rule using fundamental physical considerations deriving from the uncertainty principle. Ultimately, this leads to a dimensionless quantity called the *time-bandwidth product.* Note that the maximum height of the spectrum is at zero frequency and is given by

$$\begin{aligned} F(\omega) &= 2\frac{\sin a\omega}{\omega} \\ \lim_{\omega\to 0} F(\omega) &= 2a\,. \end{aligned}$$

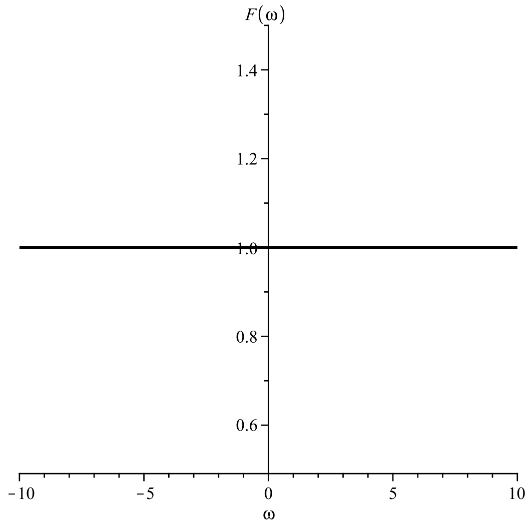

Figure 6.9: Fourier transform of the delta function, $\delta(t)$.

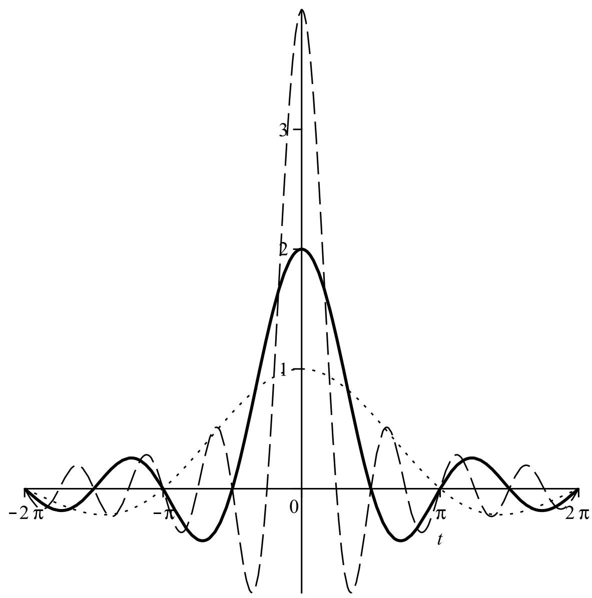

Figure 6.10: Fourier transform of the pulse function, $P_a(t)$ for $a = 1$ (solid line), $a = 2$ (dashed line) and $a = 1/2$ (dotted line) [Note: horizontal axis should be labeled ω].

So, as the pulse gets narrower, its spectrum becomes both wider and less tall. This is also physically intuitive: as the pulse contracts in time, its energy decreases, therefore in the frequency domain the total energy must also be conserved by the spectrum becoming wider but lower in amplitude (see also Parseval's theorem).

Finally, it is worth considering what would happen if, as the pulse gets narrower, we let it get taller such that the total *area* of the pulse remains 1. Then we would find that as $a \to 0$ the pulse height $\to \infty$, so we have a delta function in the time domain.[13] In the frequency domain we will see that this is a constant (see Figure 6.9) and from the above arguments and Figure 6.10 it is clear that this is the limiting case of the sinc function as $a \to 0$.[14]

6.6. Bandwidth

There is an intimate relationship between the duration of a pulse in the time domain and its range of frequencies in the Fourier transform. The latter is generally known as the *bandwidth* of the pulse. The practical importance of this comes from the fact that all instruments have a finite bandwidth, hence we need to be sure that this bandwidth is compatible with the spectral content of the signal. It is tempting to describe the width of a pulse in the time domain as its non-zero time range; however, many signals we can describe mathematically – such as the Gaussian pulse – would technically have infinite duration, even if this is not physical. An easier way to get a "measure" of the pulse is to take the Full-Width at Half Maximum (FWHM) – see Figure 6.11.

[13]One can also think of a delta function as an infinitely narrow rectangular pulse.

[14]There is a nice worked example of this in Poularikas (Example 4.19 in section 4.3). It is well worth looking this up.

6.6.1. Time-bandwidth product

A central theme in instrumentation, communications, optics and other branches of physics is that a signal that has a finite duration in time must therefore have associated with it some spread of frequencies. For any pulse shape we care to choose, with a given duration (FWHM), we can calculate the frequency spectrum by taking the Fourier transform and thus obtain its *spectral width* (FWHM). For any given pulse shape, the *time-bandwidth product* $\Delta\nu\Delta t$ is a constant that we can calculate. This is analogous to what you find in atomic, nuclear and particle physics eigenstates: the energy width of a particular state is inversely proportional to its lifetime and, conversely, only a truly stable state has zero energy width.

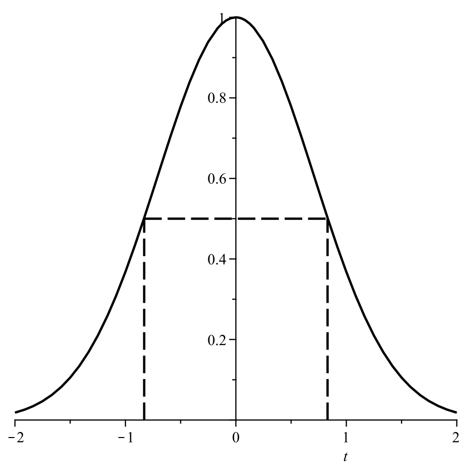

Figure 6.11: FWHM of a Gaussian pulse.

It is important to note that the definition of Δt and $\Delta\nu$ (or $\Delta\omega = 2\pi\Delta\nu$) which we are using here is the FWHM definition, and that often this is applied to power rather than amplitude; this determines the constant we actually get – so it is important to specify this. If we take a Gaussian-shaped pulse then, mathematically, it has infinite duration in the time domain and also also an infinite range of frequencies. However, if we use the FWHM definition then we can calculate the time-bandwidth product to be finite. Note that if Δt is the FWHM of the *power* in a pulse ($f(t)^2$), then the product must involve the width of the *power spectrum* ($|F(\omega)|^2$) – and in the Gaussian case this product is about 0.44, as in the example below.

The time-bandwidth product has interesting properties which make it a robust measure of the time and frequency spread of a signal: it is invariant under shifting the waveform in time and in frequency, and multiplication by a constant. It is essentially a property of the signal shape.

Example: Ultrafast laser pulse
The duration of a laser pulse is by convention taken to be the FWHM of the pulse's optical *intensity* (amplitude squared, $f(t)^2$), and its bandwidth is then the FWHM of the *power* spectrum (amplitude spectrum squared, $|F(\omega)|^2$). Using these definitions we can calculate the time-bandwidth product for any given pulse shape. For the Gaussian, this turns out to be about 0.44 (the proof of this is left as a problem sheet question). If we wish to produce a Gaussian-shaped pulse of duration 10 fs, we discover that we require a bandwidth of 44 THz!

Application to very short pulses. An alternative way of thinking about this is from the point-of-view of an instrument with limited bandwidth (as all instruments have, fundamentally). This means that the instrument can only reproduce signals over a finite range of frequencies, and ultimately this limits how short the pulse can be in the time domain. The time-bandwidth product is dependent on the shape of the pulse. Physically realisable pulses with values as low as 0.3 are possible, and some specialist lasers are able to generate pulses with durations close to this limit. This is not just of academic interest (see box below).

Real pulses are of course only approximations to the mathematical ideal. The "quality" of a very-short pulse is measured by how close we can get to the ideal time-bandwidth product.

6.6.2. Fourier transform limit

The *Fourier transform limit* gives us a useful tool that we can apply in the lab to test the validity of measurements such as the duration of an ultra-short laser pulse. We say that a pulse or signal is *transform-limited* if it contains (in the frequency domain) exactly the minimum range of frequencies required to support the pulse shape. Pulses with more frequencies than are required by the transform limit are physically possible, but those with less are not. One important result that follows from the transform limit is that that any signal that contains a sudden change (delta functions, step functions, square waves, and so forth) has associated with it a large spread of frequencies. Real instruments cannot deal with infinite frequency ranges: they always have some finite bandwidth. This means that while perfect pulses, steps, square-waves and so forth provide us with useful mathematical tools for analysing instruments, we never actually get to see them in real life.

Application: Optical fibre communications
To get the maximum data throughput on a optical fibre link it is necessary that the pulses (representing the data) are both short and very close together. Typically, a limiting factor is chromatic dispersion (frequency-dependent propagation speed). This results in the pulse shape getting distorted as it travels down the fibre, as different frequency components travel at different speeds. Ultimately, the problem is that the pulses merge together such that the receiver cannot decode the original train of pulses. For a given pulse duration, *transform-limited pulses* are those with the minimum possible spectral width. In optical communications, a transmitter emitting close to transform-limited pulses minimises the effect of chromatic dispersion, thus maximising the transmission distance.

Physics imposes a fundamental limit on how small the time-bandwidth product can be. We can show that this in both a purely classical formulation and (rather pleasingly) by using quantum mechanics too. If we apply a combination of a Fourier and an inverse Fourier transform to an arbitrary function of time $f(t)$, we find that there is a fixed relationship between the temporal width Δt and the bandwidth $\Delta\nu$ such that the time-bandwidth product $\Delta t\Delta\nu$ must obey the inequality

$$\Delta\nu\Delta t \geq \frac{1}{4\pi}$$

(or, equivalently, $\Delta\omega\Delta t \geq \frac{1}{2}$). If we look to quantum mechanics, the energy-time uncertainty principle gives us an equivalent result:

$$\Delta E\Delta t \geq \frac{\hbar}{2}\,.$$

Dividing the latter result by h we obtain the same transform limit which we can obtain classically (through the Fourier transform). The time-bandwidth product of a real pulse is always $> 1/4\pi$. By Fourier transforming real-world pulse shapes and calculating $\Delta\nu\Delta t$ we can obtain numerical values for these cases, e.g. the 0.441 product for the Gaussian pulse ($Ae^{-a^2t^2}$) or an even lower 0.315 for a $\mathrm{sech}^2(t)$ pulse.