From Calderón Problem to Harry Potter's cloack

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The purpose of this discussion is to illustrate the idea behind the so-called *cloaking*, that is, how to make an object invisible to detections made by electromagnetic waves. The goal is to prevent observers not only from seeing the object, but also from being aware that something is being hidden.

We will show how to make an object invisible to Electrical Impedance Tomography (EIT), which is the analogue of "seeing" restricted to static fields only. Despite this restriction, we will see that the mathematical ideas covered apply to much more general contexts, such as non-zero frequency fields, that is to the whole electrodynamics.

We take a space $\Omega \subset \mathbb{R}^n$ and a tensor valued function on it $\sigma(x)$, the conductivity. In absence of electric charge it is well known that the potential $u(x)$ solves the equation

$$\nabla \cdot \sigma \nabla u = 0 \qquad\qquad u|_{\partial\Omega} = f$$

where f is the assigned boundary potential. We can therefore define the Dirichlet-To-Neumann map

$$\Lambda_\sigma : f \mapsto (\sigma \nabla u) \cdot \mathbf{n}|_{\partial\Omega}$$

The Calderón's Inverse Problem asks whether or not it is possible from the knowledge of Λ_σ to state uniquely what σ is.
It is shown that, given a diffeomorphism F which restricted to $\partial\Omega$ is the identity, considering the push-forward of σ through F we obtain

$$\Lambda_{F_*\sigma} \equiv \Lambda_\sigma$$

This result can be used to build an approximate first cloaking, which makes objects less visible, not invisible. Passing through the Fourier transform we will give an estimate in L^2 norm of the effectiveness of this method.

After that we will deal formally with the limit for the apparent size of the object going to zero, i.e. invisibility, and highlight the problems that are encountered. In fact, under that limit the conductivity σ no longer satisfies the hypotheses assumed for the previous theorems, in particular there are no longer two constants $c_0, c_1 > 0$ such that $c_0 x \leq \sigma(x) \leq c_1 x$ for each $x \in \Omega$. We will therefore take a different approach to treat this case rigorously too.

We will conclude with a theorem that extends the results found, calculated for convenience of treatment in a context with spherical symmetry, to sufficiently regular arbitrary forms. Therefore it is "possible" to build an Harry Potter's cloak only if it is bi-Lipschitz equivalent to a spherical shell.