Consider Equation (7.40) with $\lambda_0 = 1/3$ truncated at $N = 2$. This leads to the system quadratic of equations

$$
\begin{aligned}
a_0^2 - 1/3 &= 0 \\
2a_0 a_1 - 1 &= 0 \\
a_2 a_0 + a_1 a_1 + a_0 a_2 &= 0.
\end{aligned}
$$

An approximate solution is give by

$$
\bar{a} = \begin{pmatrix} \bar{a}_0 \\ \bar{a}_1 \\ \bar{a}_2 \end{pmatrix} = \begin{pmatrix} 0.57735026918962 \\ 0.86602540378443 \\ -0.64951905283832 \end{pmatrix}.
$$

Moreover the numerical matrix

$$
A^{(N)} = \begin{pmatrix} 0.86602540378443 & -0.00000000000000 & 0.00000000000000 \\ -1.29903810567665 & 0.86602540378443 & -0.00000000000000 \\ 2.92283573777248 & -1.29903810567665 & 0.86602540378443 \end{pmatrix}
$$

approximately inverts the matrix $DF^{(N)}(\bar{a})$.

We choose $\nu = 1/4$ and check that

$$
Y_0 \leq 0.016650268688483,
$$

$$
\|I - A^{(N)} DF^{(N)}(\bar{a})\| \leq 1.504019729180741 \times 10^{-15} =: Z_0,
$$

$$
Z_1 \leq 0.44531250,
$$

and

$$
Z_2 \leq 2.746924327628767
$$

Similarly we check that if

$$
0.03668029648410 = r_- \leq r \leq r_+ = 0.16525009323246,
$$

the

$$
p(r) \leq 0.
$$

It follows that the approximate solution

$$
x^{(N)}(\lambda) = \bar{a}_2(\lambda - 1/3)^2 + \bar{a}_1(\lambda - 1/3) + \bar{a}_0,
$$

satisfies

$$
\sup_{|\lambda| \leq 1/4} |x^{(N)}(\lambda) - \tilde{x}(\lambda)| \leq 0.03668029648410,
$$

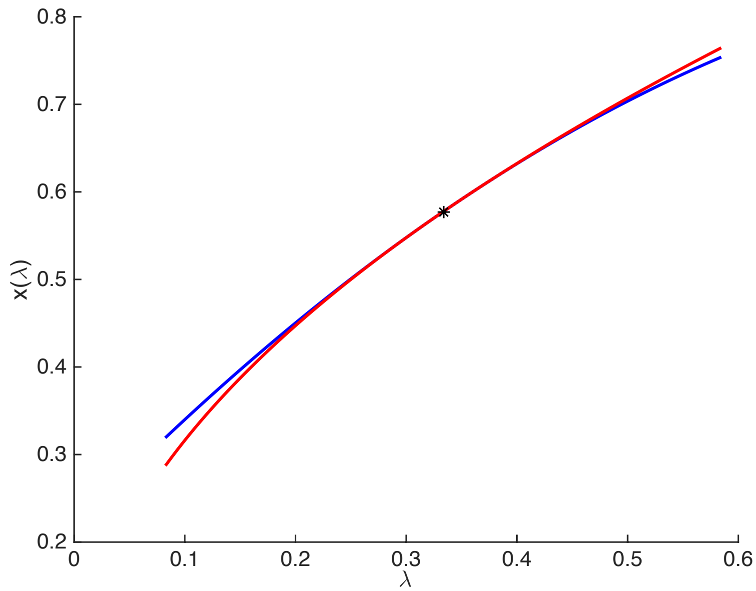

Figure 7.1: Plot of $\bar{x}^{(N)}(\lambda)$ for $N = 2$ show as a blue curve versus and $\tilde{x}(\lambda) = \sqrt{\lambda}$ shown in red. The point $(\lambda_0, x_0) = (1/3, \sqrt{1/3})$ is the black star. The approximation error is rigorously proven to be less than 0.037 in the indicated domain.

where $\tilde{x}$ is the unique function satisfying

$$\tilde{x}(\lambda)^2 - \lambda = 0, \qquad x(1/3)^2 = 1/3.$$

The example just discussed could be worked more or less by hand. Of course these results can be improved by increasing N and adjusting ν. Results of a number of additional computations are illustrated in Table 7.2.

Remark 7.7.2. In this case we can compare with the true solution. Figure 8.1 illustrates the graph of $x^{(N)}(\lambda)$ (with $N = 2$ and $\lambda_0 = 1/3$) versus the known true solution $\tilde{x}(\lambda) = \sqrt{\lambda}$. Indeed the explicit formula for the Taylor coefficients is given by

$$\tilde{a}_n = \frac{\sqrt{\lambda_0}}{\lambda_0^n} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 4^n}$$

The formula above provides valuable information about the precise decay rate of the power series coefficients. However for computational purposes it is difficult to argue that such complicated exact formulas are preferable to the validated Newton-like argument discussed above. Especially when we stop to consider that the "exact formula" for $\tilde{a}_n$ still involves $\sqrt{\lambda_0}$, a quantity which must itself be computed via a rigorously validated iterative argument.

λ_0	N	ν	r	$\lvert\bar{a}_N\rvert$
1/3	10	0.25	5.91×10^{-4}	3.1×10^{2}
1/3	18	0.25	2.9×10^{-5}	8.5×10^{5}
1/3	5	0.0025	2.22×10^{-15}	3.9
1/3	12	0.05	8.2×10^{-14}	2.15×10^{3}
0.1	10	0.01	3.0×10^{-14}	3×10^{7}
2	30	1	2.1×10^{-12}	2.3×10^{-12}
2	40	1	1.8×10^{-15}	1.5×10^{-15}
7	30	2.5	5×10^{-16}	3×10^{-28}

Figure 7.2: The table records the results of a number of additional computer-assisted proofs. Note that when $\lambda_0 < 1$ the coefficients of the power series grow. This has to be balanced by taking the domain ν smaller. It also makes high order computations numerically unstable. We achieve the best results when N is not too large. On the other hand when $\lambda_0 > 1$ the coefficients of the series decay. This stabilizes high order computations and allows us to take both N and ν larger.

7.8 Exercises

Exercise 7.8.1. Show that $C^1([a,b],\mathbb{R})$ with the norm

$$\|f\|_{C^1([a,b])} = \|f\|_{C^0([a,b])} + \|f'\|_{C^0([a,b])}.$$

is a Banach space.

Exercise 7.8.2. Let X and Y are vector spaces, and let $A : Y \to X$ and $B : X \to Y$ be a linear operators. Assume that AB is invertible. Give an example where A and B are not invertible.

Exercise 7.8.3. Let X and Y be Banach spaces and $T\colon X \to Y$ be a bijective linear map. Prove that the inverse of T is also linear.

Exercise 7.8.4. Let X and Y be normed linear spaces and $T\colon X \to Y$ be a linear operator. Prove that the sets

$$\text{image}(T) = \{y \in Y : y = T(x) \text{ for some } x \in X\},$$

and

$$\ker(T) = \{x \in X : T(x) = 0\},$$

are normed linear (sub)spaces. Prove that T is one-to-one if and only if $\ker(T) = 0$.

Prove that if X and Y are Banach spaces and $T \in B(X,Y)$, then $\ker(T)$ is a Banach space.

Exercise 7.8.5. Let ω be a one or two sided sequence of weights (depending on the context).