RadiiPolynomial

IlPreteRosso

November 20, 2025

Abstract

This blueprint presents a formalization in Lean 4 of the radii polynomial approach for proving existence of zeros of nonlinear functions on Banach spaces. The method combines the contraction mapping theorem with Newton-like operators, enabling computer-assisted proofs of existence and uniqueness of solutions.

The formalization covers both finite-dimensional and infinite-dimensional settings, with particular emphasis on the Banach Fixed Point Theorem (Contraction Mapping Theorem), Newton-like operators for zero-finding problems, radii polynomial approach in finite dimensions (Theorem 2.4.2), general radii polynomial approach on Banach spaces (Theorem 7.6.2), and Neumann series for operator invertibility.

The formalization extends classical results to maps between potentially different Banach spaces E and F, requiring careful treatment of approximate inverses and approximate derivatives.

Contents

Notations

General Notations

E, F, X, Y — Banach spaces over $\mathbb{R}$
$\|\cdot\|$ — norm on a normed space
$\|\cdot\|_{\mathcal{B}(X,Y)}$ — operator norm for continuous linear maps
$B_r(x)$ — open ball of radius r centered at x
$\overline{B}_r(x)$ — closed ball of radius r centered at x
I_E — identity operator on space E
$\mathrm{Df}(x)$ — Fréchet derivative of f at x

Radii Polynomial Notations

$\bar{x}$ — approximate zero or initial guess
$\tilde{x}$ — exact zero of function f
A — approximate inverse operator
$A^\dagger$ — approximate derivative operator
Y_0 — bound on $\|A(f(\bar{x}))\|$ (initial defect)
Z_0 — bound on $\|I_E - AA^\dagger\|$ (composition error)
Z_1 — bound on $\|A[\mathrm{Df}(\bar{x}) - A^\dagger]\|$ (derivative approximation error)
$Z_2(r)$ — bound on $\|A[\mathrm{Df}(c) - \mathrm{Df}(\bar{x})]\|$ for $c \in \overline{B}_r(\bar{x})$
$Z(r)$ — combined bound $Z_0 + Z_1 + Z_2(r) \cdot r$
$p(r)$ — radii polynomial

Special Cases

When $E = F$ and $A^\dagger = \mathrm{Df}(\bar{x})$, we have $Z_1 = 0$.
Simple radii polynomial: $p(r) = Z_2(r)r^2 - (1 - Z_0)r + Y_0$
General radii polynomial: $p(r) = Z_2(r)r^2 - (1 - Z_0 - Z_1)r + Y_0$

Chapter 1

Foundations

1.1 Contraction Mapping Theorem

The contraction mapping theorem is a fundamental tool for proving existence and uniqueness of fixed points. This section establishes the basic definitions and the classical theorem.

Definition 1.1.1 (Complete metric space)**.** *A metric space (X, d) is complete if every Cauchy sequence converges in X, i.e., if given any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that $n, m > N(\varepsilon)$ implies $d(x_n, x_m) < \varepsilon$, then there is some $y \in X$ with $\lim_{n\to\infty} x_n = y$.*

Definition 1.1.2 (Contraction mapping)**.** *Let (X, d) be a metric space. A function $T : X \to X$ is a contraction if there is a number $\kappa \in [0, 1)$, called a contraction constant, such that*

$$d(T(x), T(y)) \leq \kappa \cdot d(x, y)$$

for all $x, y \in X$.

Theorem 1.1.3 (Contraction Mapping Theorem)**.** *Let (X, d) be a complete metric space. If $T : X \to X$ is a contraction with contraction constant κ, then there exists a unique fixed point $\tilde{x} \in X$ of T. Furthermore, $\tilde{x}$ is globally attracting, and for any $x \in X$,*

$$d(T^n(x), \tilde{x}) \leq \frac{\kappa^n}{1-\kappa} d(T(x), x).$$

Remark 1.1.4. *Given a contraction mapping $T : X \to X$, the rate at which points in X converge to the globally attracting fixed point $\tilde{x}$ is determined by κ. In particular, the smaller κ is, the faster iterates under T converge to $\tilde{x}$.*

1.2 Mean Value Theorem

The mean value theorem and its corollary provide the analytical tools needed to verify contraction properties of Newton-like operators.

Definition 1.2.1 (Operator norm)**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces and let $A : X \to Y$ be a linear map. The operator norm on A is given by*

$$\|A\| := \sup_{x \in X\setminus\{0\}} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Ax\|_Y.$$

Proposition 1.2.2 (Properties of operator norm)**.** *Let $(X, \|\cdot\|)$ be a normed linear space and let $A, B : X \to X$ be linear maps. Then:*

(1) $\|Ax\| \le \|A\|\|x\|$ for all $x \in X$ (2) $\|AB\| \le \|A\|\|B\|$ (submultiplicativity)

Theorem 1.2.3 (Mean Value Theorem)**.** *Suppose that $U \subset \mathbb{R}^n$ is open and that $f : U \to \mathbb{R}^n$ is C^1. Consider $x, y \in U$ such that the line segment*

$$(1-t)x + ty, \quad t \in [0,1]$$

is contained in U. Then,

$$f(y) - f(x) = \left(\int_0^1 Df((1-t)x + ty)\, dt \right) (y - x),$$

where Df denotes the Jacobian of f.

Corollary 1.2.4 (Mean Value Inequality)**.** *Consider an open set $U \subset \mathbb{R}^n$. Let $f : U \to \mathbb{R}^n$ be a C^1 function. Fix a point $x_0 \in U$ and assume $\overline{B}_\rho(x_0) \subset U$ for some $\rho > 0$. Then, for all $x, y \in \overline{B}_\rho(x_0)$,*

$$\|f(y) - f(x)\|_\infty \le \left(\sup_{z \in \overline{B}_\rho(x_0)} \|Df(z)\|_\infty \right) \|y - x\|_\infty,$$

where $\|Df(\cdot)\|_\infty$ denotes the operator norm.

1.3 Newton's Method

Newton's method transforms the problem of finding zeros into the problem of finding fixed points. This section establishes the fundamental equivalence and introduces Newton-like operators.

Definition 1.3.1 (Newton-like map)**.** *Let E, F be Banach spaces, $f : E \to F$ a function, and $A : F \to E$ a continuous linear map. The Newton-like map is defined by*

$$T(x) = x - A(f(x)).$$

Proposition 1.3.2 (Fixed points $\iff$ Zeros)**.** *Let $f : E \to F$ and $A : F \to E$ be an injective linear map. Let $T(x) = x - A(f(x))$ be the Newton-like operator. Then:*

$$T(x) = x \quad \iff \quad f(x) = 0.$$

Proof. First direction ($T(x) = x \Rightarrow f(x) = 0$):

If $T(x) = x$, then $x - A(f(x)) = x$, which gives $A(f(x)) = 0$. Since A is linear, $A(0) = 0$. By injectivity of A, we have $A(f(x)) = A(0)$ implies $f(x) = 0$.

Second direction ($f(x) = 0 \Rightarrow T(x) = x$):

If $f(x) = 0$, then $T(x) = x - A(f(x)) = x - A(0) = x - 0 = x$. □

Definition 1.3.3 (Nondegenerate zero)**.** *Let $f \in C^1(U, \mathbb{R}^n)$ where $U \subset \mathbb{R}^n$ is an open set. A point $\tilde{x} \in U$ is a nondegenerate zero of f if $f(\tilde{x}) = 0$ and $Df(\tilde{x})$ is invertible.*

Definition 1.3.4 (Classical Newton operator)**.** *If $f \in C^2(U, \mathbb{R}^n)$ where $U \subset \mathbb{R}^n$ is an open set, the classical Newton operator is given by*

$$T(x) := x - (Df(x))^{-1} f(x).$$

Remark 1.3.5. *If $\tilde{x}$ is a nondegenerate zero of $f \in C^2$, then the derivative of the classical Newton operator satisfies*

$$DT(\tilde{x}) = I - (Df(\tilde{x}))^{-1} Df(\tilde{x}) = 0,$$

making T a strong contraction near $\tilde{x}$.

1.4 Radii Polynomials in Finite Dimensions

This section develops the radii polynomial method for proving existence of zeros in finite dimensions. The method provides explicit domains of existence and uniqueness.

Theorem 1.4.1 (Fixed point theorem with radii polynomial). *Consider a map $T \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and let $\bar{x} \in \mathbb{R}^n$. Let $Y_0 \geq 0$ and $Z : (0, \infty) \to [0, \infty)$ be a non-negative function satisfying*

$$\|T(\bar{x}) - \bar{x}\| \leq Y_0$$

$$\|DT(c)\| \leq Z(r), \quad \text{for all } c \in \overline{B}_r(\bar{x}) \text{ and all } r > 0.$$

Define

$$p(r) := (Z(r) - 1)r + Y_0.$$

If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ such that $T(\tilde{x}) = \tilde{x}$.

Definition 1.4.2 (Radii polynomial). *Given constants $Y_0, Z_0 \geq 0$ and a function $Z_2 : (0, \infty) \to [0, \infty)$, the radii polynomial is defined by*

$$p(r) := Z_2(r)r^2 - (1 - Z_0)r + Y_0.$$

Definition 1.4.3 (Combined bound). *The combined bound is defined as*

$$Z(r) := Z_0 + Z_2(r) \cdot r.$$

Lemma 1.4.4 (Alternative form of radii polynomial). *The radii polynomial can be rewritten as*

$$p(r) = (Z(r) - 1)r + Y_0.$$

Lemma 1.4.5 (Polynomial negativity implies contraction). *If $Y_0 \geq 0$, $r_0 > 0$, and $p(r_0) < 0$, then $Z(r_0) < 1$.*

Theorem 1.4.6 (Radii polynomial theorem in finite dimensions). *Consider $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $\bar{x} \in \mathbb{R}^n$ and $A \in M_n(\mathbb{R})$. Let Y_0 and Z_0 be non-negative constants and $Z_2 : (0, \infty) \to [0, \infty)$ be a non-negative function satisfying*

$$\|Af(\bar{x})\| \leq Y_0$$

$$\|I - ADf(\bar{x})\| \leq Z_0$$

$$\|A[Df(c) - Df(\bar{x})]\| \leq Z_2(r) \cdot r, \quad \text{for all } c \in \overline{B}_r(\bar{x}) \text{ and all } r > 0.$$

Define

$$p(r) := Z_2(r)r^2 - (1 - Z_0)r + Y_0.$$

If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ satisfying $f(\tilde{x}) = 0$ and $Df(\tilde{x})$ is invertible (hence $\tilde{x}$ is a nondegenerate zero).

Proof. Define the Newton-like mapping $T(x) = x - Af(x)$.

Step 1: Verify bounds. We have $\|T(\bar{x}) - \bar{x}\| = \|Af(\bar{x})\| \leq Y_0$ by (**??**).

For $c \in \overline{B}_{r_0}(\bar{x})$,

$$\begin{aligned} \|\mathrm{DT}(c)\| &= \|I - A\mathrm{Df}(c)\| \\ &\leq \|I - A\mathrm{Df}(\bar{x})\| + \|A[\mathrm{Df}(\bar{x}) - \mathrm{Df}(c)]\| \\ &\leq Z_0 + Z_2(r_0) \cdot r_0 \quad \text{by } (\mathbf{??}) \text{ and } (\mathbf{??}) \end{aligned}$$

$$= Z(r_0).$$

Step 2: Apply fixed point theorem. Since $p(r_0) = (Z(r_0)-1)r_0 + Y_0 < 0$ by assumption, Theorem 1.4.1 gives a unique fixed point $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ with $T(\tilde{x}) = \tilde{x}$.

Step 3: Show A is invertible. From $p(r_0) < 0$ we get $Z(r_0) < 1$ by Lemma 1.4.5. In particular, $\|I - A\mathrm{Df}(\bar{x})\| \le Z_0 \le Z(r_0) < 1$, so $A\mathrm{Df}(\bar{x})$ is invertible by the Neumann series. Since $\mathrm{Df}(\bar{x})$ is square and finite-dimensional, this implies A is invertible.

Step 4: Convert fixed point to zero. By Proposition 1.3.2 and invertibility of A, $T(\tilde{x}) = \tilde{x}$ implies $f(\tilde{x}) = 0$.

Step 5: Show Df$(\tilde{x})$ is invertible. Since $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$, we have $\|\mathrm{DT}(\tilde{x})\| = \|I - A\mathrm{Df}(\tilde{x})\| \le Z(r_0) < 1$. By the Neumann series, $A\mathrm{Df}(\tilde{x})$ is invertible, which implies $\mathrm{Df}(\tilde{x})$ is invertible (using that A is invertible). $\square$

Definition 1.4.7 (Existence interval)**.** *Consider a radii polynomial $p(r) = Z_2(r)r^2 - (1-Z_0)r + Y_0$ for a function $f : \mathbb{R}^n \to \mathbb{R}^n$ where $Z_2(r)$ is a polynomial with non-negative coefficients (or a non-decreasing function of r). The maximal interval $(r_-, r_+) \subset (0, \infty)$ over which $p(r) < 0$ is called the existence interval for the radii polynomial p, denoted by $EI(p)$.*

Remark 1.4.8. *If the existence interval for a radii polynomial is nonempty, then: (1) r_- provides tight bounds on the location of $\tilde{x}$ (2) r_+ provides information about the domain of isolation of $\tilde{x}$ (3) The unique zero $\tilde{x}$ satisfies $\tilde{x} \in \overline{B}_r(\bar{x})$ for all $r \in EI(p)$*

Chapter 2

Banach Space Theory

2.1 Radii Polynomial Approach on Banach Spaces

This section extends the radii polynomial method to infinite-dimensional Banach spaces, allowing for maps between potentially different spaces E and F.

2.1.1 Banach Space Setup

We work with two Banach spaces E and F over $\mathbb{R}$. For each space $X \in \{E, F\}$:

(1) `NormedAddCommGroup X`: X has a norm satisfying definiteness, symmetry, triangle inequality (2) `NormedSpace ℝ X`: The norm is compatible with scalar multiplication (3) `CompleteSpace X`: Every Cauchy sequence converges (crucial for fixed point theorems)

This framework supports: (4) Fréchet derivatives (via the norm structure) (5) Fixed point theorems (via completeness) (6) Mean Value Theorem (via the metric structure) (7) Linear operator theory (via the vector space structure)

2.1.2 Neumann Series and Operator Invertibility

The Neumann series provides a constructive way to show operators close to the identity are invertible.

Theorem 2.1.1 (Neumann series invertibility)**.** *Let E be a Banach space and $B : E \to E$ a continuous linear operator. If $\|I_E - B\| < 1$, then B is invertible (a unit in the multiplicative sense).*

Lemma 2.1.2 (Explicit two-sided inverse)**.** *If $\|I_E - B\| < 1$ for $B : E \to E$, then there exists $B^{-1} : E \to E$ such that*

$$B \circ B^{-1} = I_E \quad \textit{and} \quad B^{-1} \circ B = I_E.$$

Lemma 2.1.3 (Composition form)**.** *If $\|I_E - B\| < 1$, then there exists B^{-1} such that*

$$B.comp(B^{-1}) = I_E \quad \textit{and} \quad B^{-1}.comp(B) = I_E.$$

2.1.3 Newton-Like Operators for E to F Maps

Definition 2.1.4 (Newton-like map for E to F)**.** *For a function $f : E \to F$ and an approximate inverse $A : F \to E$, the Newton-like map is*

$$T(x) = x - A(f(x)).$$

Note that $T : E \to E$ even though f maps between different spaces.

Proposition 2.1.5 (Fixed points Zeros for E to F)**.** *Let $f : E \to F$ and $A : F \to E$ be injective. Then for the Newton-like operator $T(x) = x - A(f(x))$:*

$$T(x) = x \quad \Longleftrightarrow \quad f(x) = 0.$$

This holds even when $E \neq F$; injectivity of A is sufficient.

2.1.4 Radii Polynomial Definitions

Definition 2.1.6 (General radii polynomial)**.** *For constants $Y_0, Z_0, Z_1 \geq 0$ and function $Z_2 : (0, \infty) \to [0, \infty)$, the general radii polynomial is*

$$p(r) := Z_2(r)r^2 - (1 - Z_0 - Z_1)r + Y_0.$$

Definition 2.1.7 (Combined bound (general case))**.** *The combined bound is*

$$Z(r) := Z_0 + Z_1 + Z_2(r) \cdot r.$$

Lemma 2.1.8 (Alternative form (general))**.** *The general radii polynomial can be rewritten as*

$$p(r) = (Z(r) - 1)r + Y_0.$$

Definition 2.1.9 (Simple radii polynomial)**.** *For $Y_0 \geq 0$ and $Z : (0, \infty) \to [0, \infty)$, the simple radii polynomial is*

$$p(r) := (Z(r) - 1)r + Y_0.$$

2.1.5 Operator Bounds

Lemma 2.1.10 (Y bound for Newton operator)**.** *If $\|A(f(\bar{x}))\| \leq Y_0$, then for $T(x) = x - A(f(x))$:*

$$\|T(\bar{x}) - \bar{x}\| \leq Y_0.$$

Lemma 2.1.11 (Derivative of Newton operator)**.** *For $T(x) = x - A(f(x))$ where $f : E \to F$ is differentiable:*

$$DT(x) = I_E - A \circ Df(x).$$

Lemma 2.1.12 (General derivative bound)**.** *Suppose for all $c \in \overline{B}_r(\bar{x})$:*

$$\|I_E - A \circ A^\dagger\| \leq Z_0$$
$$\|A \circ (A^\dagger - Df(\bar{x}))\| \leq Z_1$$
$$\|A \circ (Df(c) - Df(\bar{x}))\| \leq Z_2(r) \cdot r$$

Then for $T(x) = x - A(f(x))$ and $c \in \overline{B}_r(\bar{x})$:

$$\|DT(c)\| \leq Z_0 + Z_1 + Z_2(r) \cdot r = Z(r).$$

Proof. For $c \in \overline{B}_r(\bar{x})$, decompose using $A^\dagger$:

$$\begin{aligned} \mathrm{DT}(c) &= I_E - A \circ \mathrm{Df}(c) \\ &= [I_E - A \circ A^\dagger] + A \circ [A^\dagger - \mathrm{Df}(\bar{x})] + A \circ [\mathrm{Df}(\bar{x}) - \mathrm{Df}(c)]. \end{aligned}$$

Apply triangle inequality and the three bounds (**??**), (**??**), (**??**):

$$\|\mathrm{DT}(c)\| \leq Z_0 + Z_1 + Z_2(r) \cdot r.$$

□

Lemma 2.1.13 (Simple derivative bound)**.** *When $A^\dagger = Df(\bar{x})$ (so $Z_1 = 0$), for all $c \in \overline{B}_r(\bar{x})$:*

$$\|I_E - A \circ Df(\bar{x})\| \leq Z_0$$

$$\|A \circ (Df(c) - Df(\bar{x}))\| \leq Z_2(r) \cdot r$$

imply $\|DT(c)\| \leq Z_0 + Z_2(r) \cdot r$.

2.1.6 Helper Lemmas

Lemma 2.1.14 (Closed balls are complete)**.** *In a complete space E, closed balls $\overline{B}_r(x)$ are complete.*

Lemma 2.1.15 (Extended distance is finite)**.** *In normed spaces, extended distance is always finite: $d_{ext}(x, y) \neq \top$.*

Lemma 2.1.16 (Constructing derivative inverse)**.** *If $A : F \to E$ is injective and $\|I_E - A \circ B\| < 1$ for $B : E \to F$, then B is invertible with inverse $B^{-1} = (A \circ B)^{-1} \circ A$.*

Lemma 2.1.17 (Ball mapping property)**.** *Given $T : E \to E$ differentiable with: (8) $\|T(\bar{x}) - \bar{x}\| \leq Y_0$ (9) $\|DT(c)\| \leq Z(r_0)$ for all $c \in \overline{B}_{r_0}(\bar{x})$ (10) $p(r_0) = (Z(r_0) - 1)r_0 + Y_0 < 0$*

Then $T : \overline{B}_{r_0}(\bar{x}) \to \overline{B}_{r_0}(\bar{x})$.

Proof. Let $x \in \overline{B}_{r_0}(\bar{x})$. From $p(r_0) < 0$:

$$Z(r_0) \cdot r_0 + Y_0 < r_0.$$

By Mean Value Theorem:

$$\|T(x) - T(\bar{x})\| \leq Z(r_0) \cdot \|x - \bar{x}\| \leq Z(r_0) \cdot r_0.$$

By triangle inequality:

$$\|T(x) - \bar{x}\| \leq \|T(x) - T(\bar{x})\| + \|T(\bar{x}) - \bar{x}\|$$

$$\leq Z(r_0) \cdot r_0 + Y_0 < r_0.$$

Therefore $T(x) \in \overline{B}_{r_0}(\bar{x})$. □

2.1.7 Main Theorems

Theorem 2.1.18 (General Fixed Point Theorem (Theorem 7.6.1))**.** *Let $T : E \to E$ be Fréchet differentiable and $\bar{x} \in E$. Suppose:*

$$\|T(\bar{x}) - \bar{x}\| \leq Y_0$$

$$\|DT(x)\| \leq Z(r) \quad \text{for all } x \in \overline{B}_r(\bar{x})$$

Define $p(r) := (Z(r) - 1)r + Y_0$.

If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ such that $T(\tilde{x}) = \tilde{x}$.

Proof. **Step 1:** From $p(r_0) < 0$, we get $Z(r_0) < 1$ (contraction constant).

Step 2: By Lemma 2.1.17, $T : \overline{B}_{r_0}(\bar{x}) \to \overline{B}_{r_0}(\bar{x})$.

Step 3: T restricted to $\overline{B}_{r_0}(\bar{x})$ is a contraction with constant $Z(r_0) < 1$.

Step 4: The closed ball is complete by Lemma 2.1.14.

Step 5: Apply Banach Fixed Point Theorem to get unique fixed point. □

Theorem 2.1.19 (General Radii Polynomial Theorem (Theorem 7.6.2)). *Let E and F be Banach spaces and $f : E \to F$ be Fréchet differentiable. Suppose $\bar{x} \in E$, $A^\dagger : E \to F$, and $A : F \to E$ with A injective. Assume:*

$$\|A(f(\bar{x}))\| \leq Y_0$$

$$\|I_E - A \circ A^\dagger\| \leq Z_0$$

$$\|A \circ [Df(\bar{x}) - A^\dagger]\| \leq Z_1$$

$$\|A \circ [Df(c) - Df(\bar{x})]\| \leq Z_2(r) \cdot r \quad \textit{for } c \in \overline{B}_r(\bar{x})$$

Define

$$p(r) := Z_2(r)r^2 - (1 - Z_0 - Z_1)r + Y_0.$$

If there exists $r_0 > 0$ such that $p(r_0) < 0$, then there exists a unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ with $f(\tilde{x}) = 0$.

Proof. **Step 1: Define Newton operator.** Let $T(x) = x - A(f(x))$. Then $T : E \to E$ is differentiable.

Step 2: Verify conditions of Theorem 2.1.18. (11) By Lemma 2.1.10: $\|T(\bar{x}) - \bar{x}\| \leq Y_0$ (12) By Lemma 2.1.12: $\|\mathrm{DT}(c)\| \leq Z(r_0)$ for $c \in \overline{B}_{r_0}(\bar{x})$ (13) The polynomial condition: $p(r_0) = (Z(r_0) - 1)r_0 + Y_0 < 0$ by Lemma 2.1.8

Step 3: Apply Theorem 2.1.18. Get unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ with $T(\tilde{x}) = \tilde{x}$.

Step 4: Convert to zero. By Proposition 2.1.5 with injectivity of A: $f(\tilde{x}) = 0$. □

Theorem 2.1.20 (Simple Radii Polynomial (Same Space)). *Consider $f : E \to E$ Fréchet differentiable, $\bar{x} \in E$, and $A : E \to E$ injective. Assume:*

$$\|A(f(\bar{x}))\| \leq Y_0$$

$$\|I_E - A \circ Df(\bar{x})\| \leq Z_0$$

$$\|A \circ [Df(c) - Df(\bar{x})]\| \leq Z_2(r) \cdot r \quad \textit{for } c \in \overline{B}_r(\bar{x})$$

Define $p(r) := Z_2(r)r^2 - (1 - Z_0)r + Y_0$.

If $p(r_0) < 0$, then there exists unique $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ with $f(\tilde{x}) = 0$ and $Df(\tilde{x})$ invertible.

Proof. Apply Theorem 2.1.18 to get fixed point $\tilde{x}$. Convert to zero using Proposition 2.1.5.

For invertibility: Since $\tilde{x} \in \overline{B}_{r_0}(\bar{x})$ and $Z(r_0) < 1$, we have $\|I_E - A \circ \mathrm{Df}(\tilde{x})\| < 1$. Apply Lemma 2.1.16 to get $\mathrm{Df}(\tilde{x})$ invertible. □