where $h^{(N)} = (h_0, h_1, \dots, h_N) \in \mathbb{R}^{N+1}$ is the projection of h onto its first $(N+1)$ coordinates.

The fourth step towards applying Theorem 7.6.2 is to choose an injective linear operator A that is an approximate inverse of $A^\dagger$. Assuming $a_0 \neq 0$ define $A \colon \ell^1_\nu \to \ell^1_\nu$ by

$$(Ah)_n \stackrel{\text{def}}{=} \begin{cases} [A^{(N)} h^{(N)}]_n, & 0 \le n \le N \\ \\ \frac{1}{2\bar{a}_0} h_n, & n \ge N+1, \end{cases} \tag{7.48}$$

where $A^{(N)}$ is a numerical inverse of $DF^{(N)}(\bar{a})$.

Conceptually, it may be of use to think of $A^\dagger$ and A as infinite square matrices

$$A^\dagger = \begin{bmatrix} DF^{(N)}(\bar{a}) & 0 \\ 0 & \Lambda \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} A^{(N)} & 0 \\ 0 & \Lambda^{-1} \end{bmatrix}$$

where Λ is an infinite diagonal matrix with constant diagonal entries $2\bar{a}_0$. A direct application of Proposition 7.6.5 implies that if we identify a value r_0 at which the radii polynomial $p(r_0) < 0$, then A is injective.

Having defined the Banach spaces $X = Y = \ell^1_\nu$, the nonlinear function $F \colon \ell^1_\nu \to \ell^1_\nu$, the linear operators $A, A^\dagger \in B(X)$, and determining that A is invertible, we turn to fifth step towards applying Theorem 7.6.2 that of identifying the Radii Polynomial bounds Y_0, Z_0, Z_1, and Z_2. These are presented in the following theorem.

Theorem 7.7.1. *Fix $\nu > 0$ and define the constants*

$$\begin{aligned}
Y_0 &\stackrel{\text{def}}{=} \sum_{n=0}^{N} |[A^{(N)} F^{(N)}(\bar{a})]_n| \nu^n + \frac{1}{2|\bar{a}_0|} \sum_{n=N+1}^{2N} \sum_{j=0}^{2N-n} |\bar{a}_{N-j}||\bar{a}_{n-N+j}| \nu^n \\
Z_0 &\stackrel{\text{def}}{=} \left\| I - A^{(N)} DF^{(N)}(\bar{a}) \right\|_{1,\nu} \\
Z_1 &\stackrel{\text{def}}{=} \frac{1}{|\bar{a}_0|} \sum_{n=1}^{N} |\bar{a}_n| \nu^n \\
Z_2 &\stackrel{\text{def}}{=} 2 \max\left(\|A^{(N)}\|_{1,\nu}, \frac{1}{2|\bar{a}_0|} \right)
\end{aligned}$$

where, given $B \in M_{N+1}(\mathbb{R})$,

$$\|B\|_{1,\nu} \stackrel{\text{def}}{=} \max_{0 \le n \le N} \frac{1}{\nu^n} \sum_{m=0}^{N} |B_{m,n}| \, \nu^m.$$

Then, Y_0, Z_0, Z_1 and Z_2 as defined above satisfy the hypotheses of Theorem 7.6.2.

Proof. First note that

$$(\bar{a} * \bar{a})_n = \begin{cases} \displaystyle\sum_{j=0}^{n} \bar{a}_{n-j}\bar{a}_j, & 0 \leq n \leq N \\ \displaystyle\sum_{j=0}^{2N-n} \bar{a}_{N-j}\bar{a}_{n-N+j}, & N+1 \leq n \leq 2N \\ 0, & n \geq 2N+1 \end{cases}$$

as $\bar{a}_n = 0$ for $n \geq N+1$. Thus,

$$[AF(\bar{a})]_n = \begin{cases} [A^{(N)}F^{(N)}(\bar{a})]_n, & 0 \leq n \leq N \\ \dfrac{1}{2\bar{a}_0} \displaystyle\sum_{j=0}^{2N-n} \bar{a}_{N-j}\bar{a}_{n-N+j}, & N+1 \leq n \leq 2N \\ 0, & n \geq N+1 \end{cases}$$

and

$$\begin{aligned} \|AF(\bar{a})\|_{1,\nu} &= \sum_{n=0}^{\infty} |[AF(\bar{a})]_n|\nu^n \\ &= \sum_{n=0}^{N} |[A^{(N)}F^{(N)}(\bar{a})]_n|\nu^n + \sum_{n=N+1}^{2N} |[AF(\bar{a})]_n|\nu^n \\ &\leq \sum_{n=0}^{N} |[A^{(N)}F^{(N)}(\bar{a})]_n|\nu^n + \frac{1}{2|\bar{a}_0|} \sum_{n=N+1}^{2N} \sum_{j=0}^{2N-n} |\bar{a}_{N-j}||\bar{a}_{n-N+j}|\nu^n = Y_0. \end{aligned}$$

Next, for $h \in \ell^1_\nu$,

$$\left[\left(I - AA^\dagger\right)h\right]_n = \begin{cases} \left[\left(I - A^{(N)}DF^{(N)}(\bar{a})\right)h^{(N)}\right]_n, & 0 \leq n \leq N \\ 0, & n \geq N+1. \end{cases}$$

Then

$$\|I - AA^\dagger\| = \sup_{\|h\|_{1,\nu}=1} \|[I - AA^\dagger]h\|_{1,\nu} \leq \|I - A^{(N)}DF^{(N)}(\bar{a})\|_{1,\nu} = Z_0.$$

By Remark 7.4.8,

$$DF(a)h = 2a * h,$$

for $h \in \ell^1_\nu$. Moreover,

$$\begin{aligned} [(DF(\bar{a}) - A^\dagger)h]_n &= \begin{cases} [DF^{(N)}(\bar{a})h^{(N)}]_n - [DF^{(N)}(\bar{a})h^{(N)}]_n, & 0 \leq n \leq N \\ 2(\bar{a} * h)_n - 2\bar{a}_0 h_n, & n \geq N+1 \end{cases} \\ &= \begin{cases} 0, & 0 \leq n \leq N \\ 2\displaystyle\sum_{j=1}^{N} h_{n-j}\bar{a}_j, & n \geq N+1, \end{cases} \end{aligned}$$

which exploits the fact that $\bar{a}_n = 0$ for $n \geq N+1$. Then

$$[A(DF(\bar{a}) - A^\dagger)h]_n = \begin{cases} 0, & 0 \leq n \leq N \\ \dfrac{1}{\bar{a}_0} \displaystyle\sum_{j=1}^{N} h_{n-j}\bar{a}_j, & n \geq N+1. \end{cases}$$

In order to better understand this term we define $\hat{a} \in \ell^1_\nu$ by $\hat{a}_n = 0$ if $n = 0$, or $n \geq N+1$ and $\hat{a}_n = \bar{a}^{(N)}_n$ for $1 \leq n \leq N$, i.e. $\hat{a} = (0, \bar{a}_1, \ldots, \bar{a}_N, 0, \ldots)$. Now for any $h \in \ell^1_\nu$ with $\|h\|_{1,\nu} = 1$,

$$\begin{aligned} \|A[DF(\bar{a}) - A^\dagger]h\|_{1,\nu} &= \sum_{n=N+1}^{\infty} \frac{1}{|\bar{a}_0|} \left| \sum_{j=1}^{N} h_{n-j}\bar{a}_j \right| \nu^n \\ &\leq \frac{1}{|\bar{a}_0|} \sum_{n=N+1}^{\infty} \left| \sum_{j=0}^{n} h_{n-j}\bar{a}_j \right| \nu^n \\ &\leq \frac{1}{|\bar{a}_0|} \sum_{n=0}^{\infty} \left| \sum_{j=0}^{n} h_{n-j}\bar{a}_j \right| \nu^n \\ &= \frac{1}{|\bar{a}_0|} \|h * \bar{a}\|_{1,\nu} \\ &\leq \frac{1}{|\bar{a}_0|} \|h\|_{1,\nu} \|\bar{a}\|_{1,\nu} \\ &\leq \frac{1}{|\bar{a}_0|} \sum_{n=1}^{N} |\bar{a}_n| \nu^n = Z_1. \end{aligned}$$

Hence, $\|A[DF(\bar{a}) - A^\dagger]\| \leq Z_1$.

Since $DF(a)h = 2a * h$, then

$$\|A[DF(c) - DF(\bar{x})]\| \leq 2\|A\| \|c - \bar{x}\|_{1,\nu} \leq 2\|A\| r. \tag{7.49}$$

Now, since A defined by (7.48) has the form

$$A = \begin{bmatrix} A^{(N)} & & 0 & \\ & \frac{1}{2\bar{a}_0} & & \\ & & \frac{1}{2\bar{a}_0} & \\ & 0 & & \ddots \end{bmatrix},$$

then by Proposition 7.3.14, $\|A\| \leq \max\left(\|A^{(N)}\|_{1,\nu}, \frac{1}{2|\bar{a}_0|}\right)$. From (7.49), we set

$$Z_2 = 2\max\left(\|A^{(N)}\|_{1,\nu}, \frac{1}{2|\bar{a}_0|}\right).$$

□