Delft University of Technology

Numerical Analysis For PDE's
WI4014TU

Assignment: 5

Modeling atmospheric pollution with the Finite-Element Method

Ilambharathi Govindasamy
January 12, 2021

The spread of a polluting gas, such as NO_2 is modeled using Finite-Element method.The gas is emitted by several localized sources at a constant rate in time.The emission rate density is described by the function,

$$f(x,y) = \sum_{i=1}^{3} Ae^{-\alpha(x-x_i)^2-\alpha(y-y_i)^2} \tag{1}$$

The transport of pollution is defined by the Convection-Diffusion equation:

$$\frac{\partial u}{\partial t} - D\Delta u + \mathbf{w} \cdot \nabla u = f \tag{2}$$

D is the effective diffusivity describing the (approximately) diffusive transport of gas caused by the atmospheric turbulence and $\mathbf{w}$(x,y,t) = $\langle w_x, w_y \rangle$ is the vector eld of the mean wind velocity with its components given by,

$$w_x(x,y,t) = Bcos(\omega t), \qquad w_y(x,y,t) = Bsin(\omega t) \tag{3}$$

1 Formulation and weak form

(a) Considering the initial concentration of pollutant to be zero as well as assuming that the concentration of pollutant at the boundary of Ω is zero at all times, write down the boundary value problem in terms of u,f,$\mathbf{w}$,Ω,$\partial\Omega$,etc

$$\frac{\partial u}{\partial t} - D\Delta u + \mathbf{w} \cdot \nabla u = f, \qquad (x,y) \in \Omega \subset R^2, \qquad 0 < t \leq T \tag{4}$$

Boundary condition:

$$u(x,y,t) = 0, \qquad (x,y) \in \partial\Omega, \qquad 0 < t \leq T \tag{5}$$

Initial condition:

$$u(x,y,t) = 0, \qquad (x,y) \in \Omega, \qquad t = 0 \tag{6}$$

(b) Formulate the Backward-Euler time integration method for your PDE.

Backward-Euler Time integration method

$$u(x,y,t_{k+1}) = u(x,y,t_k) + h(f^{k+1}(x,y) + D\Delta u^{k+1}(x,y) - \mathbf{w} \cdot \nabla u^{k+1}(x,y)) \tag{7}$$

Rewritten as,

$$u(x,y,t_{k+1}) - h(D\Delta u^{k+1}(x,y)) + h(\mathbf{w} \cdot \nabla u^{k+1}(x,y)) = u(x,y,t_k) + h(f^{k+1}(x,y)) \tag{8}$$

(c) Derive the weak form of the Galerkin Finite-Element Method for your time-discretized problem and explain (in general terms) how the solution will be obtained at each time step.

Weak form of the Galerkin Finite-Element Method
Multiplying with test space v, applying u"v = (u'v)'-u'v' (Integrating throughout the domain, with Dirichlet boundary values, u'v =0, at $\partial\Omega$),

$$\int_\Omega \Big(u^{k+1}v + h(\mathbf{w} \cdot \nabla u^{k+1})v + h(D\nabla u^{k+1} \cdot \nabla v) \Big) d\Omega = \int_\Omega \Big(h(f^{k+1}v) + u^k \Big) d\Omega \tag{9}$$

$$a(u^{k+1}, v) = L_k(v) \tag{10}$$

Solve for u^{k+1} in each time step k.

(d) Discuss possible stability issues of the numerical solution, if aN_y, and how you intend to deal with them

The Backward-Euler method (Implicit) is stable provided that λ >0 and h>0.

Implementation and numerical solution in FEniCS (2,3)

The Numerical parameters are as given in Table.1

Variable	Value
$L_x = L_y$	10
T	10
A	5
α	50
x_1, x_2, x_3	1,1.5,4
y_1, y_2, y_3	2,2.5,3
B	0.5
ω	2 $\pi/(4T)$
D	0.005

Table.1

For N_x=N_y=40 and a time step of h = T/20, the weak form of Galerkin method shown in (9) is implemented in Python with FEniCS. The solution results are obtained as the following for t=5 and 10 s.

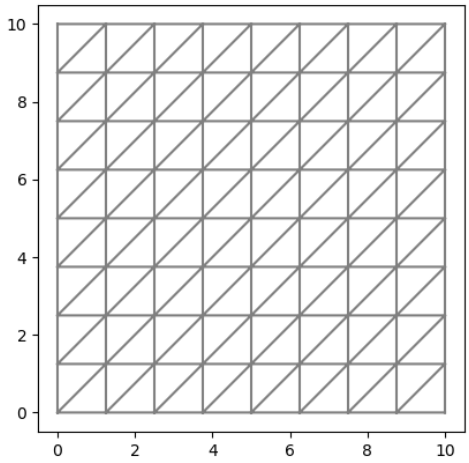

Figure.1 : Mesh for N_x=N_y=40

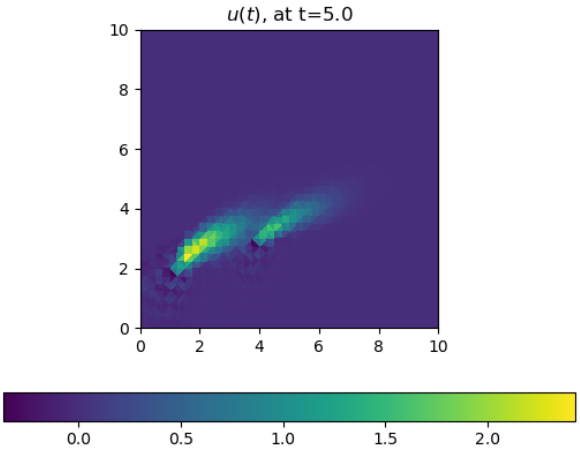

Figure.2 : N_x=N_y=40 (at t=5s)

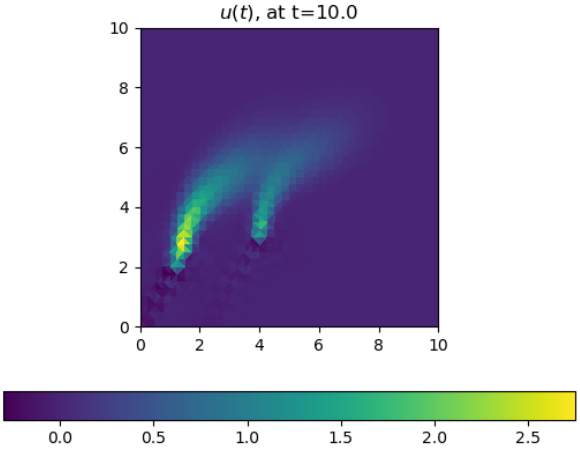

Figure.3 : N_x=N_y=40 (at t=10s)

The Figures 2 and 3 show the results for 5s and 10s respectively. It can be inferred that the gas diffuses and convects from source into the domain in North-East direction which looks to be same as that of wind direction, in the subsequent period of time, the wind flow changes its direction towards more Northern side. The solution is approximated in larger grid size, resulting in higher inaccuracies. Therefore N_x and N_y are **refined to 100** to produce more accurate results and are plotted as below.

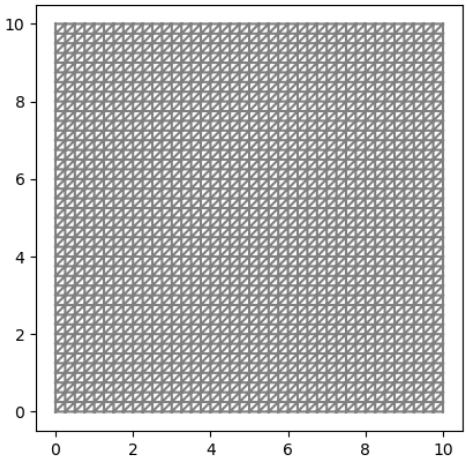

Figure.4 : Mesh for N_x=N_y=100

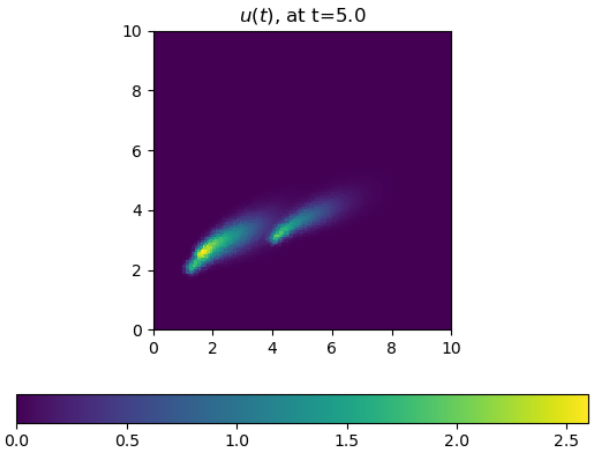

Figure.5 : N_x=N_y=100 (at t=5s)

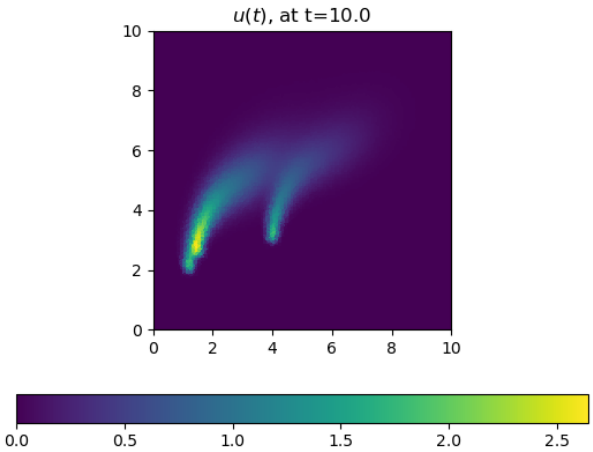

Figure.6 : N_x=N_y=100 (at t=10s)