# Numerical Analysis For PDE's (WI4014TU)
# Assignment - 1

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Consider an object occupying the simply-connected domain $\Omega \subset R^3$. An exothermic chemical reaction inside the object produces heat at the steady in time and uniform in space rates, $s \geq 0$ [W/m$^3$]. The thermal conductivity of the object $k > 0$ [W/(m K)], is also uniform in space. The surface of the object $\partial\Omega$ is kept at a constant temperature $T_0$ [K].

1. Considering the object to be in thermal equilibrium and in steady state, make a sketch and write down the global and local conservation laws describing the flow of heat. Does any heat flow through the boundary $\partial\Omega$ ?

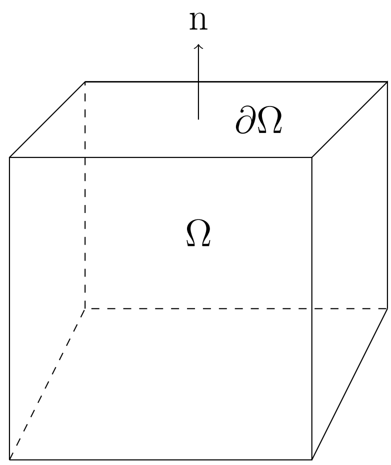


Figure.1

Let the heat flow inside the domain $\Omega$ shall be Q(t), $\Phi$ be the heat flux across the boundary $\partial\Omega$ with a normal vector n and $s$ be the heat generation rate inside the domain $\Omega$ as a function of space and time s(x,y,z,t). Let x denote (x,y,z). By the law of conservation of energy, the change in heat energy inside a domain can only be due to generation of heat energy or due to outward heat flux across the boundary. Therefore the global form of conservation law can be written as below.

$$\frac{dQ}{dt} = -\Phi + s \tag{1}$$

For steady state, $\frac{dQ}{dt} = 0$

$$s = \Phi \tag{2}$$

The heat flow rate in the domain $\Omega$ based on the temperature change for material density $\rho$ with heat capacity $C$ can be written as

$$\frac{\partial}{\partial t}\int_V \rho C T \, dV = -\int_{\partial\Omega} \mathbf{q}\cdot\mathbf{n} \; d(\partial\Omega) + \int_V s dV \tag{3}$$

According to Fourier's Law,

$$q = -k\nabla T \tag{4}$$

Applying Gauss' theorem and mean-value theorem, the local conservation law of heat flow for constant density and constant thermal conductivity k is given by

$$\rho C \frac{\partial T}{\partial t} = k \Delta T + s \tag{5}$$

For steady state, $\frac{\partial T}{\partial t} = 0$

$$-\Delta T = \frac{\mathbf{s}}{k} \tag{6}$$

In this case, the laplacian of temperature field depends on the heat generation rate and the temperature $T_0$ is constant along $\partial\Omega$, therefore in order for the system to be in equilibrium and steady state, the heat should flow across the boundary.

2. Formulate the boundary-value problem for the temperature distribution in the object

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = -\frac{\mathbf{s}}{k} \tag{7}$$

With $k > 0$, $\quad \forall\, \mathbf{x} \in \partial\Omega \cup \Omega$
$T(\mathbf{x}) = T_0$, $\quad \mathbf{x} \in \partial\Omega$

The temperature distribution of the object in steady state with source term $\mathbf{s}$ is defined

3. Using the definitions and theorems from the Chapters 1, 2 of the book and the slides of Lectures 1, 2, analyze the boundary value problem,e.g.:

   (a) What are the class/type of PDE and the type of boundary condition?

   $$a_{11} \frac{\partial^2 T}{\partial x^2} + a_{22} \frac{\partial^2 T}{\partial y^2} + 2a_{12} \frac{\partial^2 T}{\partial x \partial y} + F\left(x, y, \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right) = 0 \tag{8}$$

   Since the equation behaves the same way in both 2D and 3D domains, the 2D form of equation (7) is compared with equation (8)

   $$a_{12}^2 - (a_{11} a_{22}) = 0 - (1) = -1 \tag{9}$$

   Since, the equation (9) returns lesser than zero, the type of PDE is elliptic
   The type of boundary condition is Dirichlet boundary condition for the given, as it has a constant value for all x along the boundary $\partial\Omega$

   (b) Is the solution unique?
   Applying uniqueness theorem for Dirichlet boundary condition, Let $\Omega \subset R^3$ be a bounded region with boundary $\partial\Omega$, and $T \in C^2(\Omega) \cap C^1(\partial\Omega)$ satisfies

   $$-\Delta T = \frac{\mathbf{s}}{k}, \mathbf{x} \in \Omega \tag{10}$$

   $T(\mathbf{x}) = T_0$, $\quad \mathbf{x} \in \partial\Omega$

   The above condition is analogous to the theorem conditions, therefore the solution is unique

   (c) Is the solution positive/negative, where does it have a maximum/minimum?
   From equation (6), for $k > 0$, $\mathbf{s} > 0$

   $$-\Delta T > 0, \quad \mathbf{x} \in \Omega \tag{11}$$

   The function is superharmonic. Applying the minimum principle theorem for superharmonic function, the maximum exists inside the domain $\Omega$ and the minimum exists at the boundary $\partial\Omega$. For $T(\mathbf{x}) \geq 0$ where $\mathbf{x} \in \partial\Omega$, the solution in domain $\Omega$ is positive

(d) By how much would the temperature of the object change if the boundary temperature changed from $T_0$ to $T_1$?

Applying the stability theorem for Dirichlet boundary condition. Let $\Omega \subset R^3$ be a bounded region with boundary $\partial\Omega$, and suppose that $T_{x0}$, $T_{x1} \in C^2(\Omega\ ) \cap C^1(\partial\Omega)$ satisfy

$$\begin{array}{c} -\Delta T_{x0} = \frac{\mathbf{s}}{k},\ \mathbf{x} \in \Omega \\ T(\mathbf{x}){=}\mathrm{T}_0, \quad \mathbf{x} \in \partial\Omega \\ -\Delta T_{x1} = \frac{\mathbf{s}}{k},\ \mathbf{x} \in \Omega \\ T(\mathbf{x}){=}\mathrm{T}_1, \quad \mathbf{x} \in \partial\Omega \end{array}$$

Then, $|T_{x0} - T_{x1}| \leq max_{x\in\partial\Omega}|T_0 - T_1|$, for all x $\in \Omega$.

(e) Reformulate the problem by specifying the density of heat flux at the boundary, rather than the temperature. What happens (physically and mathematically) if you set the flow of heat to zero at the boundary?

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = -\frac{\mathbf{s}}{k} \tag{12}$$

$$\begin{array}{c} \text{With k} > 0, \quad \forall\, \mathbf{x} \in \partial\Omega \cup \Omega \\ \mathrm{k}\ \frac{\partial T(\mathbf{x})}{\partial \mathbf{n}}{=}\mathrm{q}, \quad \mathbf{x} \in \partial\Omega \end{array}$$

Now, it has become Neumann boundary condition. When the flow of heat becomes zero at the boundary, physically the system will become unsteady as it has heat generation inside the domain, while there will be no heat transferred out to maintain a steady state, equilibrium temperature in domain $\Omega$. Mathematically, the boundary condition changes as follows

$$\frac{\partial T(\mathbf{x})}{\partial \mathbf{n}} = 0, \quad \mathbf{x} \in \partial\Omega$$