Assignment - 4
Numerical solution of the two-dimensional Poisson's equation with a varying co-efficient function using the Finite-Volume Method

Numerical Analysis For PDE's (WI4014TU)
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Consider the following boundary value problem for the Poisson equation

$$
\begin{aligned}
-\nabla \cdot (k\nabla u) &= f, \quad (x,y) \in \Omega = (0,8) \times (0,8), \\
u(x,y) &= 0, \quad (x,y) \in \partial\Omega \\
f(x,y) &= e^{\alpha(x-3)^2+\alpha(y-5.5)^2} + e^{\alpha(x-5)^2+\alpha(y-5.5)^2} \\
&\quad + e^{\alpha(x-1)^2+\alpha(y-1.7)^2} + e^{\alpha(x-7)^2+\alpha(y-1.7)^2} \\
&\quad + e^{\alpha(x-2)^2+\alpha(y-2.2)^2} + e^{\alpha(x-6)^2+\alpha(y-2.2)^2} \\
&\quad + e^{\alpha(x-3)^2+\alpha(y-2.5)^2} + e^{\alpha(x-5)^2+\alpha(y-2.5)^2} \\
&\quad + e^{\alpha(x-4)^2+\alpha(y-2.6)^2}, \\
&\quad \text{with } \alpha = -10, \quad (x,y) \in \bar{\Omega}
\end{aligned}
\tag{1}
$$

where $\bar{\Omega} = [0,8] \times [0,8]$ is a square with corners $(0,0), (8,0), (8,8)$, and $(0,8)$. The coefficient function $k(x,y)$ is given, but will be specified later.

1. Finite-Volume discretization.

   (a) If we want to have a doubly-uniform grid with the grid step $h_x = h_y = h$, what is the condition on the corresponding numbers of sub-intervals N_x and N_y along the x- and y-axis? Is it still possible to define a doubly uniform grid if, instead of a square, we had a rectangle with the sides L_x and L_y? If yes, how to choose N_x and N_y in that case?
   Answer: For a doubly uniform grid with $h_x = h_y = h$, N_x should be equal to N_y. Yes, it is possible to define a doubly uniform grid for rectangular domain. In such a case N_x and N_y are chosen by having a constant h, as shown in (2)

   $$N_x = \frac{L_x}{h}, N_y = \frac{L_y}{h} \tag{2}$$

   (b) For given N_x and N_y, what are the expected number of unknowns and the dimensions of the FVM system matrix?
   Answer: For, $N_x = 100$, $N_y = 100$,the expected number of unknowns is $(N_x - 1) \times (\mathrm{N}_y - 1) = 9801$ and the dimensions of FVM system matrix would be $((N_x - 1) \times (\mathrm{N}_y - 1)) \times ((\mathrm{N}_x - 1) \times (\mathrm{N}_y - 1))$ =9801×9801

   (c) Write down the discrete FVM equations for the inner point away from any boundary, 4 equations for near-boundary points, 4 equations for near-corner points.

   AU=f

   Where A is the system matrix, U is the unknown matrix and f is the source matrix.

The general inner point equation is as follows,

$$
\begin{aligned}
f_{i,j} = &-\frac{k_{i-\frac{1}{2},j}u_{i-1,j}+}{h_x^2} - \frac{k_{i+\frac{1}{2},j}u_{i+1,j}}{h_x^2} \\
&+(\frac{k_{i-\frac{1}{2},j}+k_{i+\frac{1}{2},j}}{h_x^2} + \frac{k_{i,j-\frac{1}{2}}+k_{i,j+\frac{1}{2}}}{h_y^2})u_{i,j} \\
&-\frac{k_{i,j-\frac{1}{2}}u_{i,j-1}}{h_y^2} - \frac{k_{i,j+\frac{1}{2}}u_{i,j+1}}{h_y^2}
\end{aligned} \tag{3}
$$

The inner point equation for $h_x = h_y = h$ can be written as follows

$$
\begin{aligned}
f_{i,j} = \frac{1}{h^2}\Big[&u_{i,j}(k_{i-\frac{1}{2},j}+k_{i+\frac{1}{2},j}+k_{i,j-\frac{1}{2}}+k_{i,j+\frac{1}{2}}) \\
&-k_{i-\frac{1}{2},j}(u_{i-1,j}) - k_{i+\frac{1}{2},j}(u_{i+1,j}) \\
&-k_{i,j-\frac{1}{2}}(u_{i,j-1}) - k_{i,j+\frac{1}{2}}(u_{i,j+1})\Big]
\end{aligned} \tag{4}
$$

Where, i=1...N_x-1,j=1...N_y-1

Near-Boundary point equations are as follows.

Near-South boundary

$$
f_{i,j} = -\frac{k_{i-\frac{1}{2},j}u_{i-1,j}+}{h_x^2} - \frac{k_{i+\frac{1}{2},j}u_{i+1,j}}{h_x^2} + (\frac{k_{i-\frac{1}{2},j}+k_{i+\frac{1}{2},j}}{h_x^2} + \frac{k_{i,j-\frac{1}{2}}+k_{i,j+\frac{1}{2}}}{h_y^2})u_{i,j} - \frac{k_{i,j+\frac{1}{2}}u_{i,j+1}}{h_y^2} \tag{5}
$$

where, $i = 2, 3, ..., N_x - 2$; $j = 1$

Near-North boundary:

$$
f_{i,j} = -\frac{k_{i-\frac{1}{2},j}u_{i-1,j}+}{h_x^2} - \frac{k_{i+\frac{1}{2},j}u_{i+1,j}}{h_x^2} + (\frac{k_{i-\frac{1}{2},j}+k_{i+\frac{1}{2},j}}{h_x^2} + \frac{k_{i,j-\frac{1}{2}}+k_{i,j+\frac{1}{2}}}{h_y^2})u_{i,j} - \frac{k_{i,j-\frac{1}{2}}u_{i,j-1}}{h_y^2} \tag{6}
$$

where, $i = 2, 3, ..., N_x - 2$; $j = N_y - 1$

Near-West boundary:

$$
f_{i,j} = -\frac{k_{i+\frac{1}{2},j}u_{i+1,j}}{h_x^2} + (\frac{k_{i-\frac{1}{2},j}+k_{i+\frac{1}{2},j}}{h_x^2} + \frac{k_{i,j-\frac{1}{2}}+k_{i,j+\frac{1}{2}}}{h_y^2})u_{i,j} - \frac{k_{i,j-\frac{1}{2}}u_{i,j-1}}{h_y^2} - \frac{k_{i,j+\frac{1}{2}}u_{i,j+1}}{h_y^2} \tag{7}
$$

where $i = 1$; $j = 2, ..., N_y - 2$

Near-East boundary

$$
f_{i,j} = -\frac{k_{i-\frac{1}{2},j}u_{i-1,j}+}{h_x^2} + (\frac{k_{i-\frac{1}{2},j}+k_{i+\frac{1}{2},j}}{h_x^2} + \frac{k_{i,j-\frac{1}{2}}+k_{i,j+\frac{1}{2}}}{h_y^2})u_{i,j} - \frac{k_{i,j-\frac{1}{2}}u_{i,j-1}}{h_y^2} - \frac{k_{i,j+\frac{1}{2}}u_{i,j+1}}{h_y^2} \tag{8}
$$

where $i = N_x - 1$; $j = 2, ..., N_y - 2$

Near-Corner point Equations are as follows:

Near South-West Corner point:

$$
f_{i,j} = -\frac{k_{i+\frac{1}{2},j}u_{i+1,j}}{h_x^2} + (\frac{k_{i-\frac{1}{2},j}+k_{i+\frac{1}{2},j}}{h_x^2} + \frac{k_{i,j-\frac{1}{2}}+k_{i,j+\frac{1}{2}}}{h_y^2})u_{i,j} - \frac{k_{i,j+\frac{1}{2}}u_{i,j+1}}{h_y^2} \tag{9}
$$

where $i = 1$; $j = 1$

Near South-East Corner point:

$$
f_{i,j} = -\frac{k_{i-\frac{1}{2},j}u_{i-1,j}+}{h_x^2} + (\frac{k_{i-\frac{1}{2},j}+k_{i+\frac{1}{2},j}}{h_x^2} + \frac{k_{i,j-\frac{1}{2}}+k_{i,j+\frac{1}{2}}}{h_y^2})u_{i,j} - \frac{k_{i,j+\frac{1}{2}}u_{i,j+1}}{h_y^2} \tag{10}
$$

where $i = N_x - 1$; $j = 1$

Near North-West Corner point:

$$f_{i,j} = -\frac{k_{i+\frac{1}{2},j}u_{i+1,j}}{h_x^2} + (\frac{k_{i-\frac{1}{2},j} + k_{i+\frac{1}{2},j}}{h_x^2} + \frac{k_{i,j-\frac{1}{2}} + k_{i,j+\frac{1}{2}}}{h_y^2})u_{i,j} - \frac{k_{i,j-\frac{1}{2}}u_{i,j-1}}{h_y^2} \quad (11)$$

where $i = 1$; $j = N_y - 1$

Near North-East Corner point:

$$f_{i,j} = -\frac{k_{i-\frac{1}{2},j}u_{i-1,j}+}{h_x^2} + (\frac{k_{i-\frac{1}{2},j} + k_{i+\frac{1}{2},j}}{h_x^2} + \frac{k_{i,j-\frac{1}{2}} + k_{i,j+\frac{1}{2}}}{h_y^2})u_{i,j} - \frac{k_{i,j-\frac{1}{2}}u_{i,j-1}}{h_y^2} \quad (12)$$

where $i = N_x - 1$; $j = N_y - 1$

(d) What are the possible numbers of non-zero elements in the rows of the system matrix?
Answer: The possible numbers of non-zero elements in the rows of the system can be 3, 4 and 5.

(e) Explain why the system matrix is penta-diagonal (has 5 non-zero diagonals) if unknowns are stored in lexicographic order.
Answer: The system matrix is penta-diagonal when the unknowns are stored in Lexicographic order, because the inner point equation has the co-efficients of $u_{i,j}$, $u_{i,j+1}$,$u_{i,j-1}$,$u_{i+1,j}$ and $u_{i-1,j}$, which describe the inflow and outflow in two dimensions . In these the x-value is changed with constant y for lexicographic ordering and it reflects in the system matrix such that the co-efficients of the unknowns are multiplied accordingly to get the inner point equation. Hence, each point has an equation with 5 co-efficients.

(f) What will be the positions of these diagonals in terms of N_x and N_y? How long is each diagonal? Are there any zeros along any of the five diagonals? If yes, where exactly?
Answer: When the **position** of main diagonal (co-efficient of $u_{i,j}$) of dimension $(N_x - 1) \times (N_y - 1)$ is considered to be reference, the following applies.The position of the second lower diagonal for the co-efficient of $u_{i,j-1}$ will be -(N_x-1) from the main diagonal, whereas the second upper diagonal for the co-efficient of $u_{i,j+1}$ will be (N_x-1). For, both second diagonals, the dimension will be $(N_x-1)\times(N_y-1)$-(N_x-1) The first of lower(co-efficient of $u_{i-1,j}$) and upper diagonals (co-efficient of $u_{i+1,j}$) will be at -1 and +1 from the main diagonal. For both first diagonals, the dimension will be $(N_x-1)\times(N_y-1)$-1. Yes, there are zeroes along the first upper and lower diagonals. The near boundary points of x has zeroes in the first diagonal as they are the co-efficient of unknown at the boundaries which has to be made zero to make the boundary unknown zero.

2. System matrix and right hand side assembly
For the given lengths 8×8 with Nx=Ny=100, the source function,co-efficient functions K1 and K2 are created in a lexicographic order by nested loops. They are reshaped, transposed (because in lexicographic ordering y is kept constant for varying x, but we want to plot otherwise) and plotted as below.

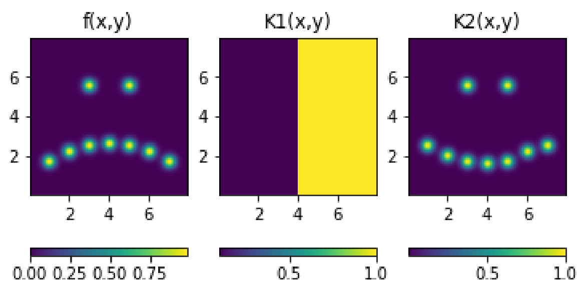

Figure.1 : Source function and co-efficient functions (K1 & K2) in the domain

The system matrix is constructed using lexicographic ordering with nested loops , conditional checks for boundary and corner checks. It is checked for bugs with k1=1 with Nx=Ny=4 (smaller grid). The Figure.2 explains the structure of matrix for a homogeneous coefficient function. It is as expected because, the co-efficient of $u_{i,j}$ is found along the main diagonal which is a negative summation of the the co-efficients of $u_{i,j+1}$,$u_{i,j-1}$,$u_{i+1,j}$ and $u_{i-1,j}$. Whereas the values of $u_{i+1,j}$ are in the immediate right of the main diagonal and left for

$u_{i-1,j}$. Similarly $u_{i,j+1}$,$u_{i,j-1}$ are on the corresponding extremes. The co-efficient values are zero elsewhere and even for the $u_{i,j+1}$,$u_{i,j-1}$,$u_{i+1,j}$ and $u_{i-1,j}$ depending on the closeness to boundary points or corner points.

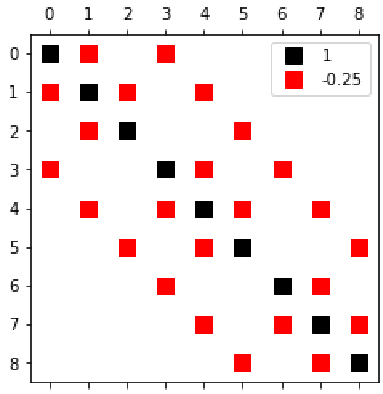

Figure.2: System matrix with homogeneous co-efficient k1=1

In order to check for the varying co-efficient, the k1 is changed back to the given function and the system matrix is found as shown in the Figure.3

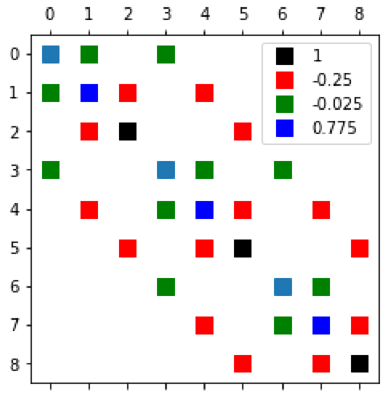

Figure.3: System matrix with varying co-efficient (given k1)

From the Figure.3, it can be verified that for varying co-efficient function, the system matrix also varies in a pattern throughout the domain

3. Solution of the linear algebraic problem.

   (a) The solution is found for the given coefficient function (k1) and source function (f). The solution is plotted in Figure.4

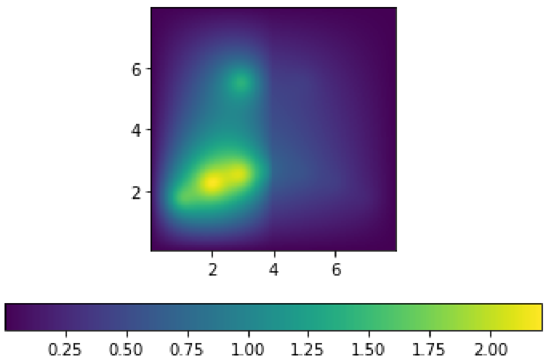

Figure.4: Solution with given varying co-efficient of k1

   (b) Explain the observed result from the physical point of view.
   The given equation, $-\nabla \cdot (k \nabla u) = f$ is analogous to steady state heat diffusion equation, in which u represents Temperature, k represents the thermal conductivity of the material

and f represents the heat source. From the Figure.4, it can be clearly understood that the temperature or u diffuses from the source to the surrounding regions. It can also be seen that temperature along source points are considerably higher as the heat diffusion is very low till it reaches x=4, which is due to lower thermal conductivity till x=4 which is given for the function of k1.

(c) The numerical FVM solution for $k_2(x, y)$ with $k_0 = 1$ and k_0 =0.001

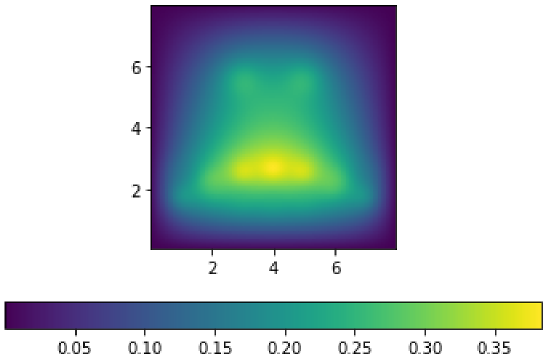

Figure.5: Solution for k_2 with $k_0 = 1$

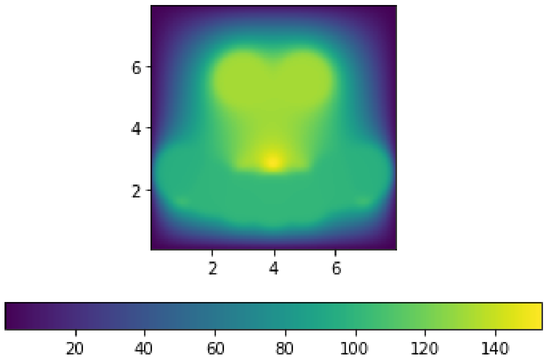

Figure.6: Solution for k_2 with $k_0 = 0.001$

(d) For Figure.5, comparing k_2 and f in Figure.1, it can be seen that as the source regions and high thermal conductivity regions are closer the rate of diffusion is higher from the source resulting in relatively lower temperature compared to the regions where they are away from each other (The regions around x=4 in the source spots). Comparing Figure.6 with Figure.5, the source function remains same. But the thermal conductivity is low, which means the rate of diffusion is also low. Therefore the area diffused has increased at the top two point regions, whereas in the lower curve although the thermal conductivity is low, it is higher than zero, therefore the temperature is higher at the source region where the distance is higher from conductive spots.

(e) Explain the potential mathematical problems with k_0=0.001
The potential problems could be,

- The mass diffusion in gases using Fick's second law with spatially varying Diffusion co-efficient (K_0 of D can be of order 0.001).
- Reduced Navier-Stokes equation for viscous flows with constant pressure and spatially varying viscosity (K_0 of μ can be of order 0.001)