Assignment - 2
Numerical solution of the one-dimensional Poisson's equation with the Finite-Difference Method

Numerical Analysis For PDE's (WI4014TU)
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Considering the following boundary-value problem:

$$-\frac{\partial^2 u_i}{\partial x^2} = f_i, \qquad x \in (-1,1) \tag{1}$$

$$u_i(-1) = 1, \qquad u_i(1) = 2, \qquad i = 1,2$$

With source functions

$$f_1(x) = 1, \qquad f_2(x) = e^x, \qquad x \in [-1,1] \tag{2}$$

1. Find the exact solutions u$_1^{ex}$,u$_2^{ex}$ of the problem(1) corresponding to f_1(x) and f_2(x) given in (2)

$$-u_1^{ex}(x) = \frac{x^2}{2} - \frac{x}{2} - 2 \tag{3}$$

$$-u_2^{ex}(x) = \; -e^x + 1.675x + 3.043 \tag{4}$$

2. Discretize the problem (1) using the Finite-Difference Method (FDM) on a uniform grid obtained by dividing the [-1,1] interval into n=5 sub-intervals of equal length

   (a) What is the step size h ?
   The step size h is given by,

$$h = \frac{D}{n} = \frac{1-(-1)}{5} = 0.4 \tag{5}$$

   (b) How many internal and boundary points do you get?
   The number of boundary points is found to be 2 and the number of internal points is 4, which is n-1.

   (c) How many unknowns does your numerical problem have?
   This numerical problem has 4 unknowns

   (d) What is Finite-difference $\mathcal{O}(h^2)$ (FD) approximation of the (negative) second derivative operator?

$$-\mathcal{O}(h^2) = u_i\,'' - \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{6}$$

   (e) Write down all discrete FD equations for your problem
   u_o and u_5 are the boundary conditions here

$$-\left(\frac{-2u_1 + u_2}{h^2}\right) = f_1 + \frac{u_o}{h^2} = f_1 + \frac{1}{0.4^2} \tag{7}$$

$$-\left(\frac{u_1 - 2u_2 + u_3}{h^2}\right) = f_2 \tag{8}$$

$$-\left(\frac{u_2 - 2u_3 + u_4}{h^2}\right) = f_3 \tag{9}$$

$$-\left(\frac{u_3 - 2u_4}{h^2}\right) = f_4 + \frac{u_5}{h^2} = f_4 + \frac{2}{0.4^2} \tag{10}$$

(f) Write your FD equations as a linear algebraic problem $\mathbf{A}u = \mathbf{f}$, show the system matrix $\mathbf{A}$ and the right-hand-side vectors $\mathbf{f}_1$ and $\mathbf{f}_2$, for both RHS functions (2)

$$\mathbf{A}u_1 = \mathbf{f}_1$$

$$-\frac{1}{0.4^2}\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 7.25 \\ 1 \\ 1 \\ 13.5 \end{bmatrix}$$

$$\mathbf{A}u_2 = \mathbf{f}_2$$

$$-\frac{1}{0.4^2}\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 6.798 \\ 0.818 \\ 1.221 \\ 14.322 \end{bmatrix}$$

(g) Compute the eigenvalues of A and present them as a column of a LaTeX table in your report
The eigen values are calculated by

$$\lambda_i = \frac{4}{h^2} sin^2\left(\frac{\pi i}{2n}\right), \qquad i = 1, 2..n-1 \tag{11}$$

The values are presented in the 1st column of Table.1

(h) Compute the first 4 (smallest) eigenvalues of the negative second derivative operator and present them as the second column in your table
The smallest eigen values of negative second derivative operator is calculated by

$$\widetilde{\lambda} = \left(\frac{\pi i}{D^2}\right) \tag{12}$$

The values are presented in the 2nd column of Table.1

2(g)(h), 5(d) are presented in the Table.1 as column 1,2 and 3 respectively

Eigen value of A	Smallest Eigen value	Numerical Eigen value
2.3872	2.4649	2.3872
8.6372	9.8596	8.6372
16.3627	22.1841	16.3627
22.6127	39.4384	22.6127

Table.1

4. Construct a uniform grid and display the source functions and the exact solutions.

A uniform grid is constructed in Python and the results are presented as Figure.1 for RHS functions and Figure.2 for exact solutions

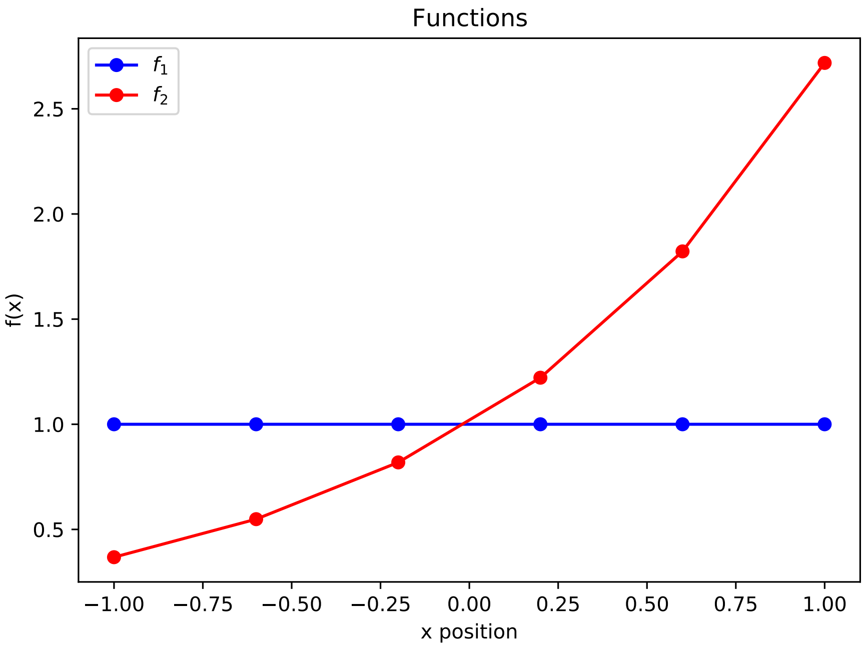

Figure.1

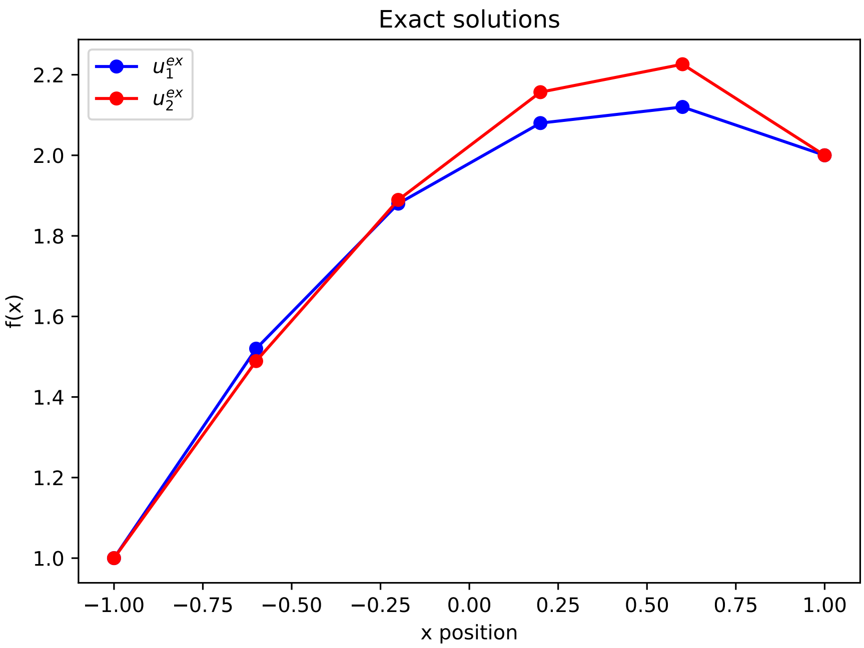

Figure.2

5. Assemble the Finite-Difference (negative) Laplacian matrix.

   The dimensions of A in terms of n is found to be $(n-1) \times (n-1)$. The structure of matrix $\mathbf{A}$ is shown in Figure.3 and the numerically calculated eigen values are shown in 3rd column of Table.1

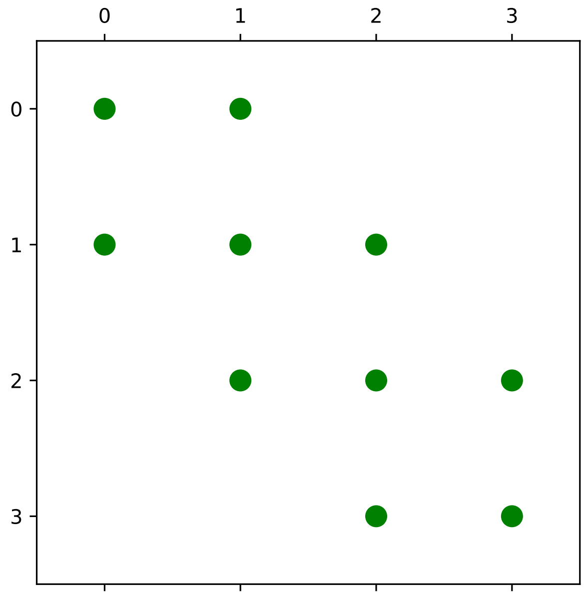

Figure.3

6. Solve the linear algebraic problem

   The problem is solved in python and compared with the exact solutions as shown in Figure.4

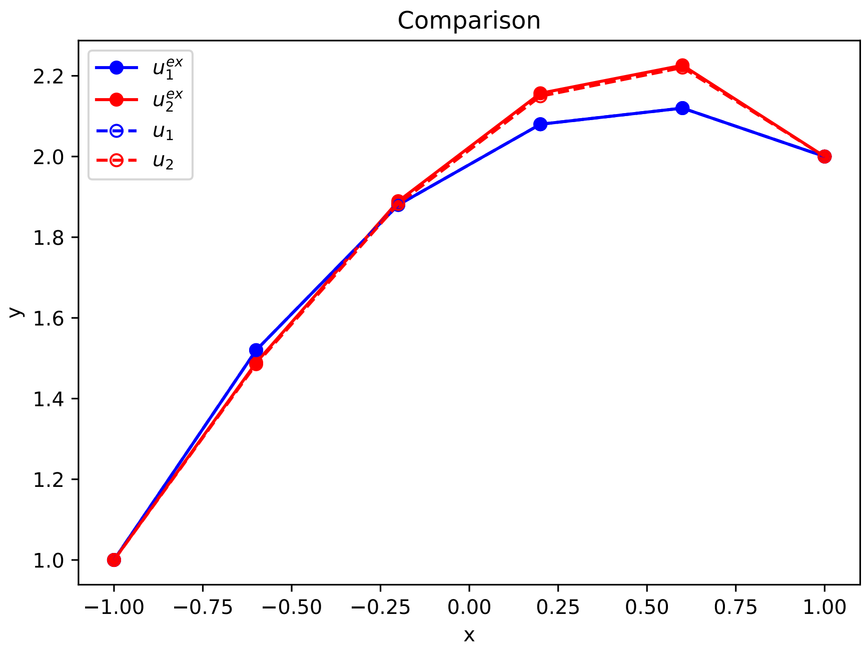

Figure.4

7. Analyze your results

(a) Compute the global error between the numerical and the exact solutions for $n = 5$. Explain your results

The global errors between exact solution and numerical solution for $f_1(x)$ and $f_2(x)$ are found to be $3.33e^{-16}$ and 0.005968 respectively. This very low error for $f_1(x)$ is because that $f_1(x)$ does not depend on x.

(b) Study the rate of convergence of the Finite-Difference Method by changing n. Display the plt.loglog() plot of the global error as a function of n. Explain your results.

It has been found that the rate of convergence increases with the n value, which implies that the error decreases for $f_2(x)$ when the function depends on x, but not for $f_1(x)$ as the source function does not have any dependence with x. The global error has been plotted for n = 5 to 100 and it is as shown in Figure.5 and Figure.6 for $f_1(x)$ and $f_2(x)$ respectively. Since the log-log plot represnets the relationship $Error = kn^C$, that is $Log(Error) = C(Log(n)) + Log(k)$. In this the slope C is found to be -2.02 for $f_2(x)$, which is the order in which the error decreases, since the convergence is inversely proportional that order of convergence could be 2.02

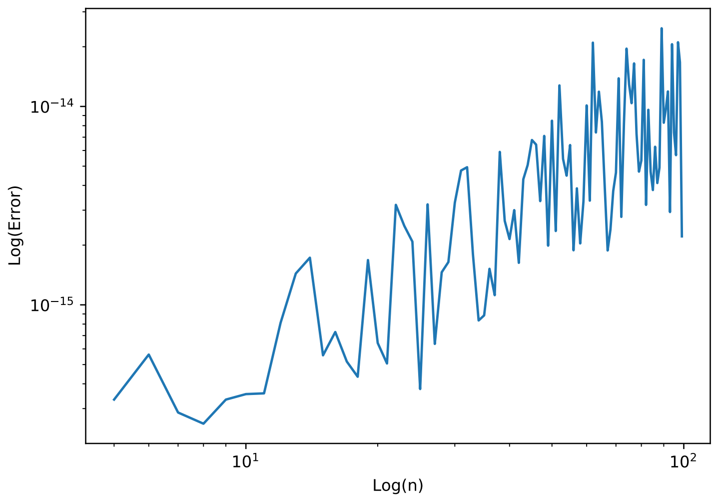

Figure.5

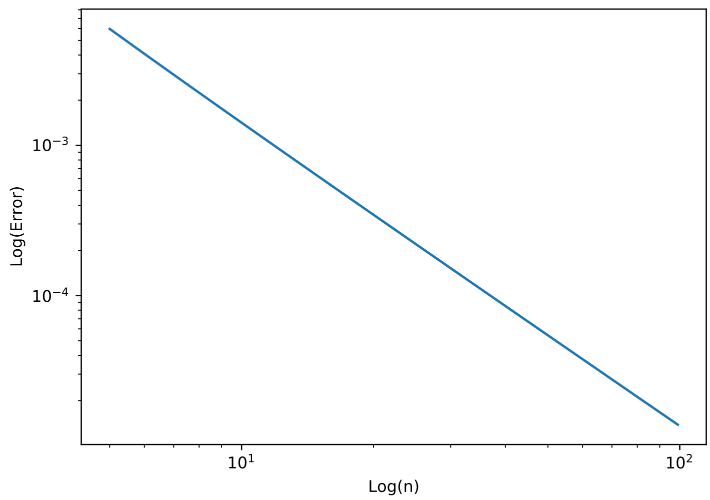

Figure.6