Assignment - 3
Numerical solution of the two-dimensional Poisson's equation with the Finite-Difference Method

Numerical Analysis For PDE's (WI4014TU)
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Consider the following boundary value problem for the Poisson equation

$$\Delta u = f, (x, y) \in \Omega$$
$$\text{u(x)} = 0,\text{(x,y)} \in \partial\Omega$$
$$\text{f(x,y)}= 20 \sin(\pi y) sin(\pi x + \pi), (x, y) \in \overline{\Omega}$$
$$\text{Domain} \quad \Omega : (x, y) \in \Omega, if \ (x^2 + y^2 - 1)^3 - x^2y^3 < 0,$$
$$\text{Boundary} \quad \partial\Omega : (x, y) \in \partial\Omega, if \ (x^2 + y^2 - 1)^3 - x^2y^3 = 0.$$

1. ! Finite Difference discretization. First consider Ω to be a square domain (-1.5,1.5) X (-1.5,1.5) . Divide this domain in 4 equal steps along both axes.

   (a) What are the parameters N_x and N_y and the grid steps h_x and h_y?
   The parameters N_x and N_y is 4, whereas h_x and h_y is 0.75

   $$h_x = \frac{3}{N_x} = 0.75$$

   $$h_y = \frac{3}{N_y} = 0.75$$

   (b) How many internal and boundary points are there?
   The total internal boundary points are $(N_x - 1) \times (N_y - 1)$ which is 9, whereas the number of boundary points is $2 \times (N_x + N_y) = 16$

   (c) Give the $\mathcal{O}(h^2)$ approximation of the negative Laplacian operator in two-dimensions

   $$\mathcal{O}(h_x^2) + \mathcal{O}(h_y^2) = -\Delta u_{i,j} - \left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} \right) \tag{1}$$

   (d) Using correct point labels, write down one internal point FD equation, 4 near-boundary-point equations (one for each boundary segment), and 4 near-corner-point equation
   Internal points are expressed as :

   $$\frac{-u_{i+1,j} + 2u_{i,j} - u_{i-1,j}}{h_x^2} + \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{h_y^2} = f_{i,j}$$

   where $i = 1, 2, .., N_x - 1$; $j = 1, 2, .., N_y - 1$
   Near-Boundary point equations are as follows:

   $$\frac{-u_{i+1,j} + 2u_{i,j} - u_{i-1,j}}{h_x^2} + \frac{-u_{i,j-1} + 2u_{i,j}}{h_y^2} = f_{i,j}$$

   where, $i = 2, 3, ..., N_x - 2$; $j = N_y - 1$ North boundary side

   $$\frac{-u_{i+1,j} + 2u_{i,j} - u_{i-1,j}}{h_x^2} + \frac{2u_{i,j} - u_{i,j+1}}{h_y^2} = f_{i,j}$$

where, $i = 2, 3, ..., N_x - 2$; $j = 1$ South boundary side

$$\frac{2u_{i,j} - u_{i-1,j}}{h_x^2} + \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{h_y^2} = f_{i,j}$$

where $i = N_x - 1$; $j = 2, ..., N_y - 2$ East boundary side

$$\frac{-u_{i+1,j} + 2u_{i,j}}{h_x^2} + \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{h_y^2} = f_{i,j}$$

where $i = 1$; $j = 2, ..., N_y - 2$ West boundary side

Near-Corner point equations are as follows:

$$\frac{-u_{i+1,j} + 2u_{i,j}}{h_x^2} + \frac{2u_{i,j} - u_{i,j+1}}{h_y^2} = f_{i,j}$$

where $i = 1$; $j = 1$ Near South-West Corner point

$$\frac{2u_{i,j} - u_{i-1,j}}{h_x^2} + \frac{2u_{i,j} - u_{i,j+1}}{h_y^2} = f_{i,j}$$

where $i = N_x - 1$; $j = 1$ Near South-East Corner point

$$\frac{-u_{i+1,j} + 2u_{i,j}}{h_x^2} + \frac{-u_{i,j-1} + 2u_{i,j}}{h_y^2} = f_{i,j}$$

where $i = 1$; $j = N_y - 1$ Near North-West Corner point

$$\frac{2u_{i,j} - u_{i-1,j}}{h_x^2} + \frac{-u_{i,j-1} + 2u_{i,j}}{h_y^2} = f_{i,j}$$

where $i = N_x - 1$; $j = N_y - 1$ Near North-East Corner point

(e) How many nonzero elements will the system matrix have in total?
The total non-zero elements in the system matrix will be 33.

(f) Show for your grid that under lexicographic ordering, the FD matrix A of the negative 2D Laplacian operator can indeed be obtained as A=$I_y \otimes$L$_{xx}$ +$L_{yy} \otimes$I$_x$, where L_{xx} and L_{yy} are the FD matrices of negative 1D Laplacians in the x and y direction, respectively, and I_x,I_y are identity matrices of proper size. Also show that $L_{xx} = D_x^T D_y \text{and} L_{yy} = D_y^T D_y$, where D_x and D_y are matrices representing one-sided FD approximations of the first derivatives.

$$D_x = -\frac{1}{h_x}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad D_y = -\frac{1}{h_y}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \tag{2}$$

$$D_x^T = -\frac{1}{h_x}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad D_y^T = -\frac{1}{h_y}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{3}$$

$$L_{xx} = -\frac{1}{h_x^2}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \qquad L_{yy} = -\frac{1}{h_y^2}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \tag{4}$$

$$I_x = I_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$A = \begin{bmatrix} 7.11 & -1.77 & 0 & -1.77 & 0 & 0 & 0 & 0 & 0 \\ -1.77 & 7.11 & -1.77 & 0 & -1.77 & 0 & 0 & 0 & 0 \\ 0 & -1.77 & 7.11 & 0 & 0 & -1.77 & 0 & 0 & 0 \\ -1.77 & 0 & 0 & 7.11 & -1.77 & 0 & -1.77 & 0 & 0 \\ 0 & -1.77 & 0 & -1.77 & 7.11 & -1.77 & 0 & -1.77 & 0 \\ 0 & 0 & -1.77 & 0 & -1.77 & 7.11 & 0 & 0 & -1.77 \\ 0 & 0 & 0 & -1.77 & 0 & 0 & 7.11 & -1.77 & 0 \\ 0 & 0 & 0 & 0 & -1.77 & 0 & -1.77 & 7.11 & -1.77 \\ 0 & 0 & 0 & 0 & 0 & -1.77 & 0 & -1.77 & 7.11 \end{bmatrix} \tag{6}$$

(g) What happens to the vector of unknowns, the right-hand-side vector, and the system matrix, if the boundary of your domain is deformed by making one of the inner grid points next to the old boundary a new boundary point?

2. System matrix assembly for a rectangular domain
By using sparse negative 1D Laplacian matrices and kronecker product, the sparse negative 2D Laplacian matrix 'A' has been created in python environment and it is presented as spy plot for N_x , $N_y = 6$

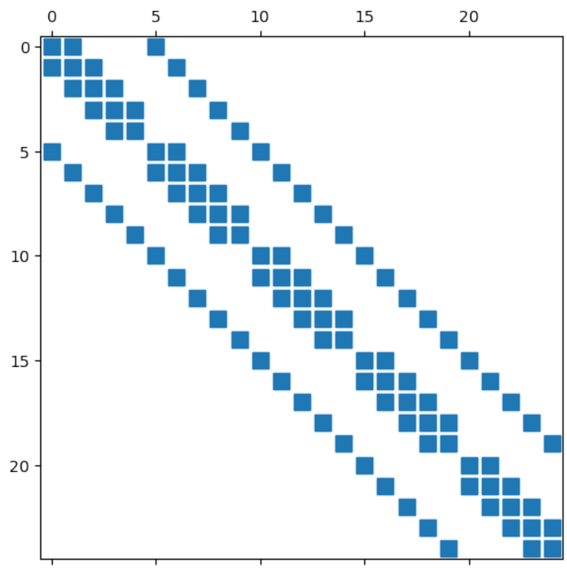

Figure.1

3. Construction of the two-dimensional grid and evaluation of the source function on a rectangular domain

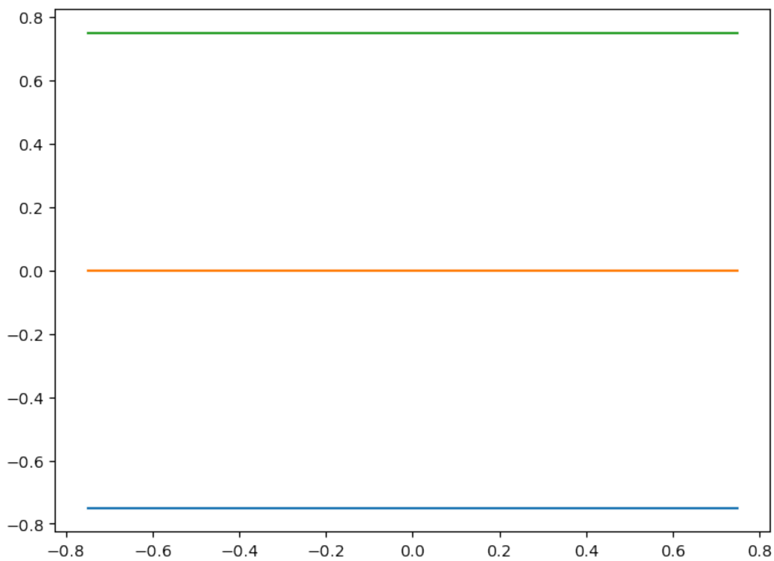

Figure.2

By plotting X and Y against each other, shows that Y values remain constant for varying X subsequently, the lexicographic ordering is shown in Figure.2

4. Rectangular and Deformed domain

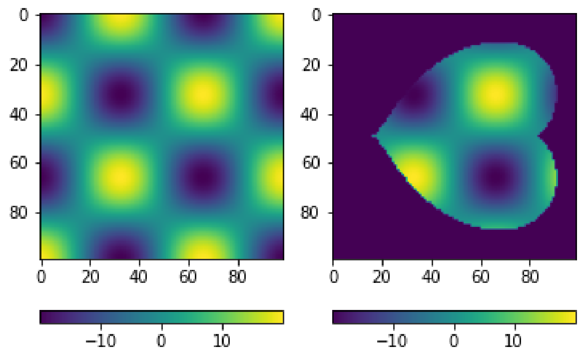

Figure.3: Source functions of rectangular and deformed domains

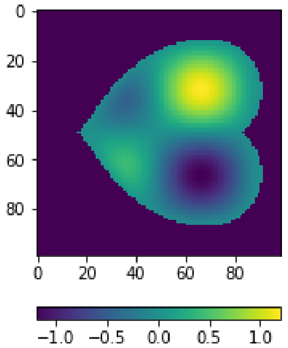

Figure.4: Solution vector plot

5. Is it possible to express the numerical solution of the following modified boundary problem in terms of the solution that you already have? If so, how?

$$-k\Delta u = f, \quad (x, y) \in \Omega,$$

$$u(x, y) = T_0, \quad (x, y) \in \partial\Omega,$$

$$f(x, y) = 20\sin(\pi y)\sin(\pi x + \pi), \quad (x, y) \in \bar{\Omega}$$

$$\text{Domain}\,\Omega : \quad (x, y) \in \Omega \quad \text{if} \quad (x^2 + y^2 - 1)^3 - x^2y^3 < 0,$$

$$\text{Boundary}\ \partial\Omega : \quad (x, y) \in \partial\Omega \ \text{if} \quad (x^2 + y^2 - 1)^3 - x^2y^3 = 0,$$

where both k and T_0 are some positive constants.

The RHS (Right Hand Side) matrix is multiplied by a factor of $\frac{1}{k}$ and the boundary values are incorporated in the RHS depending on the proximity to the boundary. The solution would be similar to our present solution except that amplitudes will be different for each point. Therefore it is possible to represent the modified solution using the current solution