Knapsack Problem with Placement Constraints

1 Formal Problem Definition

1.1 Inputs

- n: number of products
- x: height of the suitcase in millimeters
- y: width of the suitcase in millimeters
- c: limit to the total weight of the suitcase in grams
- p_i: price of the i-th product in euros
- w_i: weight of the i-th product in grams
- s_i: side length of the i-th product's (square) box in millimeters

1.2 Outputs

- Indices of the products chosen to maximize the accumulated price
- Arrangement of the products in the suitcase

2 Mathematical Formulation

2.1 Decision Variables

- Chosen_i: binary variable that is 1 if object i is chosen, and 0 otherwise.
- PointsX_i: the x-coordinate of the bottom-left corner of object i.
- PointsY_i: the y-coordinate of the bottom-left corner of object i.
- $\text{Overlap}_{i,j,d}$: binary variable indicating if objects i and j do not overlap in direction d, where $d \in \{1, 2, 3, 4\}$.

2.2 Objective Function

Maximize the total price of the chosen objects:

$$\text{maximize} \sum_{i=1}^{n} p_i \cdot \text{Chosen}_i$$

2.3 Constraints

2.3.1 Max Weight Constraint

Ensure the total weight of the chosen objects does not exceed the suitcase's capacity:

$$\sum_{i=1}^{n} w_i \cdot \text{Chosen}_i \leq c$$

2.3.2 Coordinate Bounds Constraints

Ensure each object lies entirely within the suitcase's boundaries:

$$\forall i \in \{1, \ldots, n\}, \quad \text{PointsX}_i \geq 1$$

$$\forall i \in \{1, \ldots, n\}, \quad \text{PointsY}_i \geq 1$$

$$\forall i \in \{1, \ldots, n\}, \quad \text{PointsX}_i + s_i - 1 \leq x$$

$$\forall i \in \{1, \ldots, n\}, \quad \text{PointsY}_i + s_i - 1 \leq y$$

2.3.3 Non-Overlapping Constraints

Ensure no two chosen objects overlap within the suitcase using the big-M method:

1. Horizontal non-overlapping to the right:

$$\forall i, j \in \{1, \ldots, n\}, i \neq j, \quad \text{PointsX}_i - \text{PointsX}_j + s_i \leq -M \cdot (\text{Chosen}_i + \text{Chosen}_j + \text{Overlap}_{i,j,1} - 3)$$

2. Horizontal non-overlapping to the left:

$$\forall i, j \in \{1, \ldots, n\}, i \neq j, \quad \text{PointsX}_j - \text{PointsX}_i + s_j \leq -M \cdot (\text{Chosen}_i + \text{Chosen}_j + \text{Overlap}_{i,j,2} - 3)$$

3. Vertical non-overlapping upwards:

$$\forall i, j \in \{1, \ldots, n\}, i \neq j, \quad \text{PointsY}_i - \text{PointsY}_j + s_i \leq -M \cdot (\text{Chosen}_i + \text{Chosen}_j + \text{Overlap}_{i,j,3} - 3)$$

4. Vertical non-overlapping downwards:

$$\forall i, j \in \{1, \ldots, n\}, i \neq j, \quad \text{PointsY}_j - \text{PointsY}_i + s_j \leq -M \cdot (\text{Chosen}_i + \text{Chosen}_j + \text{Overlap}_{i,j,4} - 3)$$

2.3.4 At Least One Not Overlapping Constraint

Ensure that for any two objects, at least one of the non-overlapping conditions is satisfied:

$$\forall i, j \in \{1, \ldots, n\}, i \neq j, \quad \sum_{d=1}^{4} \text{Overlap}_{i,j,d} \geq 1$$