Gamma & Beta Functions

Gamma Function

- $\Gamma(n) = (n-1)!$ **for** $n \in \mathbb{Z}^+$
- $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$
- $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$
- $\int_0^\infty x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$
- $(y+\alpha-1)! = \binom{y+\alpha-1}{y}(y!)(\alpha-1)!$

Beta Function

- $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
- $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt$

Expectation & Variance Laws

Law of Total Expectation

$E[X] = E[E[X|\theta]]$

Law of Total Variance

$\mathbf{Var}(X) = E[\mathbf{Var}(X|\theta)] + \mathbf{Var}(E[X|\theta])$

Variance Properties

- $\mathbf{Var}(X) = E[X^2] - (E[X])^2$
- $\mathbf{Var}(aX+b) = a^2\mathbf{Var}(X)$
- $\mathbf{Var}(aX+bY) = a^2\mathbf{Var}(X) + b^2\mathbf{Var}(Y) + 2ab\mathbf{Cov}(X,Y)$

Variance of Sum

$\mathbf{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathbf{Var}(X_i) + 2\sum_{1\le i<j\le n}\mathbf{Cov}(X_i,X_j)$

Covariance & Correlation

Covariance

- $\mathbf{Cov}(X,Y) = E[XY] - E[X]E[Y]$
- $\mathbf{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$
- $\mathbf{Cov}(X,X) = \mathbf{Var}(X)$
- $\mathbf{Cov}(aX+b, cY+d) = ac\cdot\mathbf{Cov}(X,Y)$
- $\mathbf{Cov}(X,Y+Z) = \mathbf{Cov}(X,Y) + \mathbf{Cov}(X,Z)$
- $\mathbf{Cov}(X,X+Y) = \mathbf{Var}(X) + \mathbf{Cov}(X,Y)$
- $\mathbf{Cov}(X,X-Y) = \mathbf{Var}(X) - \mathbf{Cov}(X,Y)$
- **If** X,Y **independent:** $\mathbf{Cov}(X,Y) = 0$

Correlation

- $\rho(X,Y) = \frac{\mathbf{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{\mathbf{Cov}(X,Y)}{\sqrt{\mathbf{Var}(X)\mathbf{Var}(Y)}}$
- $-1 \le \rho(X,Y) \le 1$
- **If** X,Y **independent:** $\rho(X,Y) = 0$

Conditional Distributions

Conditional Density/PMF

$f_{X|Y=b}(x) = \frac{f_{X,Y}(x,b)}{f_Y(b)}$

Conditional Expectation

$E[X|Y=b] = \int_{-\infty}^\infty x\cdot f_{X|Y=b}(x)dx$

$E[X|Y=b] = \sum_x x\cdot P(X=x|Y=b)$

Marginal Distributions

From Joint to Marginal

$f_X(x) = \int_{-\infty}^\infty f_{X,Y}(x,y)dy$

$P(X=x) = \sum_y P(X=x, Y=y)$

Hierarchical Models

$f_X(x) = \int_{-\infty}^\infty f_{X|\theta}(x|\theta)f_\theta(\theta)d\theta$

$P(X=x) = \sum_\theta P(X=x|\theta)P(\theta)$

Transformation Methods

CDF Method

- **Find** $F_Y(y) = P(Y\le y) = P(g(X)\le y)$
- **Solve for** X: $P(X\le h(y))$ **or** $P(X\ge h(y))$
- **Differentiate:** $f_Y(y) = \frac{d}{dy}F_Y(y)$

For $Y = g(X)$ (diff, monotonic)

$f_Y(y) = f_X(g^{-1}(y))\cdot\left|\frac{d}{dy}g^{-1}(y)\right|$

Common Distributions

Uniform: $X\sim\mathbf{Unif}[a,b]$

- **PDF:** $f(x) = \frac{1}{b-a}$ **for** $a\le x\le b$
- $E[X] = \frac{a+b}{2}$
- $\mathbf{Var}(X) = \frac{(b-a)^2}{12}$

Discrete Uniform on $\{a,\dots,b\}$

- **PMF:** $P(X=k) = \frac{1}{b-a+1}$
- $E[X] = \frac{a+b}{2}$
- $\mathbf{Var}(X) = \frac{(b-a+1)^2-1}{12}$

Exponential: $X\sim\mathbf{Exp}(\lambda)$

- **PDF:** $f(x) = \lambda e^{-\lambda x}$, $x\ge 0$
- $E[X] = \frac{1}{\lambda}$
- $\mathbf{Var}(X) = \frac{1}{\lambda^2}$
- **Memoryless:** $P(X>s+t|X>s) = P(X>t)$

Binomial: $X\sim\mathbf{Bin}(n,p)$

- **PMF:** $P(X=k) = \binom{n}{k}p^k(1-p)^{n-k}$
- $E[X] = np$
- $\mathbf{Var}(X) = np(1-p)$

Beta: $X\sim\mathbf{Beta}(\alpha,\beta)$

- **PDF:** $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $0<x<1$
- $E[X] = \frac{\alpha}{\alpha+\beta}$
- $\mathbf{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Poisson: $X\sim\mathbf{Pois}(\lambda)$

- **PMF:** $P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}$
- $E[X] = \lambda$
- $\mathbf{Var}(X) = \lambda$

Gamma: $X\sim\mathbf{Gamma}(r,\beta)$

Shape-rate parameterization

- **PDF:** $f(x) = \frac{\beta^r}{\Gamma(r)}x^{r-1}e^{-\beta x}$, $x>0$
- $E[X] = \frac{r}{\beta}$
- $\mathbf{Var}(X) = \frac{r}{\beta^2}$

Normal: $X\sim N(\mu,\sigma^2)$

- **PDF:** $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$
- $E[X] = \mu$
- $\mathbf{Var}(X) = \sigma^2$
- **If** $Z\sim N(0,1)$, **then** $Z^2\sim\chi_1^2\sim\mathbf{Gamma}(1/2,1/2)$

Chi-Square: $X\sim\chi_k^2$

- **PDF:** $f(x) = \frac{1}{2^{k/2}\Gamma(k/2)}x^{k/2-1}e^{-x/2}$, $x>0$
- $E[X] = k$
- $\mathbf{Var}(X) = 2k$
- **Equivalent to** $\mathbf{Gamma}(k/2,1/2)$

Negative Binomial: $X\sim\mathbf{NB}(r,p)$

- **PMF:** $P(X=k) = \binom{k+r-1}{k}p^r(1-p)^k$
- $E[X] = \frac{r(1-p)}{p}$
- $\mathbf{Var}(X) = \frac{r(1-p)}{p^2}$

Multinomial: $\mathbf{X}\sim\mathbf{Multi}(n,\mathbf{p})$

- **PMF:** $P(\mathbf{X}=\mathbf{x}) = \frac{n!}{x_1!\cdots x_k!}p_1^{x_1}\cdots p_k^{x_k}$
- $E[X_i] = np_i$
- $\mathbf{Var}(X_i) = np_i(1-p_i)$
- $\mathbf{Cov}(X_i,X_j) = -np_ip_j$ **for** $i\ne j$

Dirichlet: $\mathbf{p}\sim\mathbf{Dir}(\boldsymbol{\alpha})$

Let $\alpha_0 = \sum_{i=1}^k\alpha_i$

- **PDF:** $f(\mathbf{p}) = \frac{\Gamma(\alpha_0)}{\prod_{i=1}^k\Gamma(\alpha_i)}\prod_{i=1}^k p_i^{\alpha_i-1}$
- $E[p_i] = \frac{\alpha_i}{\alpha_0}$
- $\mathbf{Var}(p_i) = \frac{\alpha_i(\alpha_0-\alpha_i)}{\alpha_0^2(\alpha_0+1)}$
- $\mathbf{Cov}(p_i,p_j) = -\frac{\alpha_i\alpha_j}{\alpha_0^2(\alpha_0+1)}$ **for** $i\ne j$

Hierarchical Distributions

1. Beta-Binomial

$X|p\sim\mathbf{Bin}(n,p)$, $p\sim\mathbf{Beta}(\alpha,\beta)$

Marginal PMF: $P(X=k) = \binom{n}{k}\frac{\Gamma(\alpha+\beta)\Gamma(k+\alpha)\Gamma(n-k+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)}$

Moments: $E[X] = \frac{n\alpha}{\alpha+\beta}$

$\mathbf{Var}(X) = \frac{n\alpha\beta(n+\alpha+\beta)}{(\alpha+\beta)^2(\alpha+\beta+1)}$

2. Gamma-Poisson (Negative Binomial)

$X|\lambda\sim\mathbf{Pois}(\lambda)$, $\lambda\sim\mathbf{Gamma}(r,\beta)$

Marginal PMF: $P(X=k) = \binom{k+r-1}{k}\left(\frac{\beta}{\beta+1}\right)^r\left(\frac{1}{\beta+1}\right)^k$

This is NB $\left(r,\frac{\beta}{\beta+1}\right)$

Moments: $E[X] = \frac{r}{\beta}$

$\mathbf{Var}(X) = \frac{r(\beta+1)}{\beta^2}$

3. Normal-Normal (Random Mean)

$X|\mu\sim N(\mu,\sigma^2)$, $\mu\sim N(\mu_0,\tau^2)$

Marginal: $X\sim N(\mu_0,\sigma^2+\tau^2)$

Moments: $E[X] = \mu_0$

$\mathbf{Var}(X) = \sigma^2+\tau^2$

4. Dirichlet-Multinomial

$\mathbf{X}|\mathbf{p}\sim\mathbf{Multi}(n,\mathbf{p})$, $\mathbf{p}\sim\mathbf{Dir}(\boldsymbol{\alpha})$

Marginal PMF: $P(\mathbf{X}=\mathbf{x}) = \binom{n}{x_1,\dots,x_k}\frac{\Gamma(\alpha_0)}{\Gamma(n+\alpha_0)}\prod_{i=1}^k\frac{\Gamma(x_i+\alpha_i)}{\Gamma(\alpha_i)}$

Moments: $E[X_i] = \frac{n\alpha_i}{\alpha_0}$

$\mathbf{Var}(X_i) = \frac{n\alpha_i(\alpha_0-\alpha_i)(n+\alpha_0)}{\alpha_0^2(\alpha_0+1)}$

$\mathbf{Cov}(X_i,X_j) = -\frac{n\alpha_i\alpha_j(n+\alpha_0)}{\alpha_0^2(\alpha_0+1)}$ **for** $i\ne j$

Joint Distributions

Independence

- $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$
- $P(X=x,Y=y) = P(X=x)\cdot P(Y=y)$
- **If independent:** $E[XY] = E[X]E[Y]$
- **If independent:** $\mathbf{Var}(X+Y) = \mathbf{Var}(X)+\mathbf{Var}(Y)$

Sum of Independent Variables

- **If** $X\sim N(\mu_1,\sigma_1^2)$, $Y\sim N(\mu_2,\sigma_2^2)$ **independent:** $X+Y\sim N(\mu_1+\mu_2,\sigma_1^2+\sigma_2^2)$
- **If** $X\sim\mathbf{Pois}(\lambda_1)$, $Y\sim\mathbf{Pois}(\lambda_2)$ **independent:** $X+Y\sim\mathbf{Pois}(\lambda_1+\lambda_2)$
- **If** $X\sim\mathbf{Gamma}(r_1,\beta)$, $Y\sim\mathbf{Gamma}(r_2,\beta)$ **independent:** $X+Y\sim\mathbf{Gamma}(r_1+r_2,\beta)$

Common Transformation Examples

If $X\sim\mathbf{Unif}[0,1]$

- $Y=\sqrt{X}$ **has PDF** $f_Y(y) = 2y$, $0\le y\le 1$
- $Y=e^X$ **has PDF** $f_Y(y) = \frac{1}{y}$, $1\le y\le e$
- $Y=-\ln X\sim\mathbf{Exp}(1)$

If $Z\sim N(0,1)$

- $Z^2\sim\chi_1^2$
- $Y=\mu+\sigma Z\sim N(\mu,\sigma^2)$

Example Problems & Solutions

Ex 1: Gamma-Poisson Derivation

Given $Y|\Lambda \sim$ Pois(Λ) and $\Lambda \sim$ Gamma(α, β), find marginal of Y.

Marginal: $f_Y(y) = \int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta} d\lambda$

$= \frac{1}{y!\Gamma(\alpha)\beta^\alpha} \int_0^\infty \lambda^{y+\alpha-1} \exp\left[-\lambda\left(1+\frac{1}{\beta}\right)\right] d\lambda$

Using $\int_0^\infty x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$**:**

$f_Y(y) = \frac{\Gamma(y+\alpha)}{y!\Gamma(\alpha)\beta^\alpha} \left(\frac{\beta}{\beta+1}\right)^{y+\alpha}$

For integer α**:** $\Gamma(y+\alpha) = \binom{y+\alpha-1}{y}(y!)(\alpha-1)!$

Result: $P(Y=y) = \binom{y+\alpha-1}{y}\left(\frac{\beta}{\beta+1}\right)^\alpha \left(\frac{1}{\beta+1}\right)^y$ **(NB dist)**

Ex 2: Beta-Binomial Derivation

Given $Y|p \sim$ Bin(n,p) and $p \sim$ Beta(α, β), find marginal of Y.

$P(Y=y) = \int_0^1 \binom{n}{y} p^y (1-p)^{n-y} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} dp$

$= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 p^{y+\alpha-1}(1-p)^{n+\beta-y-1} dp$

Using Beta function: $\int_0^1 p^{a-1}(1-p)^{b-1} dp = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Result: $P(Y=y) = \binom{n}{y} \frac{\Gamma(\alpha+\beta)\Gamma(y+\alpha)\Gamma(n+\beta-y)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)}$

Ex 3: Variance of Sum Proof

Prove: $\mathbf{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathbf{Var}(X_i) + 2\sum_{1 \le i < j \le n} \mathbf{Cov}(X_i, X_j)$

Let $S = \sum_{i=1}^n X_i$**. Then** $S^2 = \sum_{i=1}^n X_i^2 + 2\sum_{i<j} X_i X_j$

Taking expectations: $E[S^2] = \sum_{i=1}^n E[X_i^2] + 2\sum_{i<j} E[X_i X_j]$

Also: $(E[S])^2 = \left(\sum_{i=1}^n E[X_i]\right)^2 = \sum_{i=1}^n (E[X_i])^2 + 2\sum_{i<j} E[X_i]E[X_j]$

Since $\mathbf{Var}(S) = E[S^2] - (E[S])^2$**:**

$\mathbf{Var}(S) = \sum_{i=1}^n (E[X_i^2] - (E[X_i])^2) + 2\sum_{i<j}(E[X_i X_j] - E[X_i]E[X_j])$

$= \sum_{i=1}^n \mathbf{Var}(X_i) + 2\sum_{i<j} \mathbf{Cov}(X_i, X_j)$

Ex 4: Joint Distribution Problem

(X,Y) **has PDF** $f_{X,Y}(x,y) = 6(1-y)$ **for** $0 \le x \le y \le 1$.

(a) Find $E[X]$**:**

Marginal: $f_X(x) = \int_x^1 6(1-y)dy = [6y - 3y^2]_x^1 = 3(x-1)^2$ **for** $0 \le x \le 1$

$E[X] = \int_0^1 3x(x-1)^2 dx = 3\int_0^1 (x^3 - 2x^2 + x)dx = 3\left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{1}{4}$

(b) Find $E[X|Y=b]$**:**

Marginal of Y**:** $f_Y(y) = \int_0^y 6(1-y)dx = 6y(1-y)$ **for** $0 \le y \le 1$

Conditional: $f_{X|Y=b}(x) = \frac{6(1-b)}{6b(1-b)} = \frac{1}{b}$ **for** $0 \le x \le b$ **(uniform)**

$E[X|Y=b] = \int_0^b x \cdot \frac{1}{b} dx = \frac{b}{2}$

(c) Verify law of total expectation:

$E[X] = E[E[X|Y]] = E[Y/2] = \frac{1}{2}E[Y]$

$E[Y] = \int_0^1 6y^2(1-y)dy = 6\int_0^1 (y^2 - y^3)dy = 6\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{1}{2}$

Thus $E[X] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ ✓

Ex 5: Nested Discrete Uniforms

Roll fair die, get $X \in \{1,\ldots,6\}$**. Then roll** Y **uniform on** $\{1,\ldots,X\}$**.**

(a) Find $E[X]$ **and** $E[Y]$**:**

$E[X] = \frac{1+6}{2} = \frac{7}{2}$

For Y**:** $E[Y|X=k] = \frac{k+1}{2}$

$E[Y] = E[E[Y|X]] = \frac{1}{6}\sum_{i=1}^6 \frac{i+1}{2} = \frac{2+3+4+5+6+7}{12} = \frac{9}{4}$

(b) Find $\mathbf{Var}(X)$ **and** $\mathbf{Var}(Y)$**:**

$\mathbf{Var}(X) = \frac{6^2-1}{12} = \frac{35}{12}$

For Y**:** $\mathbf{Var}(Y|X=i) = \frac{(i-1)^2}{12}$

$\mathbf{Var}(E[Y|X]) = \mathbf{Var}\left(\frac{X+1}{2}\right) = \frac{1}{4}\mathbf{Var}(X) = \frac{35}{48}$

$E[\mathbf{Var}(Y|X)] = \frac{1}{6}\sum_{i=1}^6 \frac{(i-1)^2}{12} = \frac{0+1+4+9+16+25}{72} = \frac{55}{72}$

$\mathbf{Var}(Y) = \frac{35}{48} + \frac{55}{72} = \frac{175}{144}$

Ex 6: Conditional Probability on Triangle

$(X,Y) \sim$ **Uniform**$[0,1]^2$**. Let** $Z = X+Y$**. Find** $P(X<Y|Z \le 1)$**.**

Event $Z \le 1$**: region below line** $y = -x+1$ **(triangle with area** $\frac{1}{2}$**)**

Event $X < Y$**: region above line** $y = x$

By symmetry, exactly half of the triangle lies above $y = x$

Result: $P(X<Y|Z \le 1) = \frac{1}{2}$

Ex 7: Correlations with Sums/Differences

Let $X, Y \sim$ **Unif**$[0,1]$ **independent. Define** $Z = X+Y$**,** $W = X-Y$**.**

Basic facts: $E[X] = E[Y] = \frac{1}{2}$**,** $\mathbf{Var}(X) = \mathbf{Var}(Y) = \frac{1}{12}$

$E[Z] = 1$**,** $E[W] = 0$**,** $\mathbf{Var}(Z) = \mathbf{Var}(W) = \frac{1}{6}$

(a) $\rho(X,Y) = 0$ **(independent)**

(b) $\mathbf{Cov}(X,Z) = \mathbf{Cov}(X, X+Y) = \mathbf{Var}(X) = \frac{1}{12}$

$\rho(X,Z) = \frac{1/12}{\sqrt{(1/12)(1/6)}} = \frac{1}{\sqrt{2}}$

(c) $\mathbf{Cov}(Y,W) = \mathbf{Cov}(Y, X-Y) = -\mathbf{Var}(Y) = -\frac{1}{12}$

$\rho(Y,W) = \frac{-1/12}{\sqrt{(1/12)(1/6)}} = -\frac{1}{\sqrt{2}}$

(d) $E[ZW] = E[(X+Y)(X-Y)] = E[X^2 - Y^2] = \frac{1}{3} - \frac{1}{3} = 0$

$\mathbf{Cov}(Z,W) = 0 - 1 \cdot 0 = 0$**, so** $\rho(Z,W) = 0$

Ex 8: Chi-Square from Standard Normal

Let $Z \sim N(0,1)$**. Show** $Z^2 \sim \chi_1^2 \sim$ **Gamma**$(1/2, 1/2)$**.**

CDF: $P(Z^2 \le t) = P(-\sqrt{t} \le Z \le \sqrt{t}) = F_Z(\sqrt{t}) - F_Z(-\sqrt{t})$

PDF: $f_{Z^2}(t) = \frac{d}{dt}[F_Z(\sqrt{t}) - F_Z(-\sqrt{t})] = \frac{1}{2\sqrt{t}}[f_Z(\sqrt{t}) + f_Z(-\sqrt{t})]$

Since $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ **is symmetric:**

$f_{Z^2}(t) = \frac{1}{\sqrt{2\pi t}} \cdot 2 \cdot e^{-t/2} = \frac{1}{\sqrt{2\pi t}} e^{-t/2}$ **for** $t > 0$

This matches Gamma$(\alpha = 1/2, \beta = 1/2)$ **with** $\Gamma(1/2) = \sqrt{\pi}$

Ex 9: Transformations of Uniform

(a) If $X \sim$ **Unif**$[0,1]$**, find PDF of** $Y = \sqrt{X}$**.**

CDF: $F_Y(y) = P(\sqrt{X} \le y) = P(X \le y^2) = y^2$ **for** $0 \le y \le 1$

PDF: $f_Y(y) = 2y$ **for** $0 \le y \le 1$

(b) If $X \sim$ **Unif**$[0,1]$**, find PDF of** $Y = \ln X$**.**

CDF: $F_Y(y) = P(\ln X \le y) = P(X \le e^y) = e^y$ **for** $y \le 0$

PDF: $f_Y(y) = e^y$ **for** $y \le 0$ **(reflected exponential)**

Ex 10: General Exponential Transform

If X **has PDF** $f_X(x)$ **and** $Y = e^X$**, find PDF of** Y**.**

For $y > 0$**:** $P(Y \le y) = P(e^X \le y) = P(X \le \ln y) = F_X(\ln y)$

PDF: $f_Y(y) = f_X(\ln y) \cdot \frac{1}{y}$ **for** $y > 0$

Special case: If $X \sim$ **Unif**$[0,1]$**:**

$f_Y(y) = 1 \cdot \frac{1}{y} = \frac{1}{y}$ **for** $1 \le y \le e$