Probability & Statistics Quick Reference

Gamma & Beta Functions

Gamma Function:

- $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}^+$
- $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$
- $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$
- $\int_0^\infty x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$
- $(y+\alpha-1)! = \binom{y+\alpha-1}{y}(y!)(\alpha-1)!$ for integer α

Beta Function:

- $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
- $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt$

Expectation & Variance Laws

Law of Total Expectation:

$$E[X] = E[E[X|\theta]]$$

Law of Total Variance:

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|\theta)] + \mathrm{Var}(E[X|\theta])$$

Variance Properties:

- $\mathrm{Var}(X) = E[X^2] - (E[X])^2$
- $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$
- $\mathrm{Var}(aX+bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\mathrm{Cov}(X,Y)$

Variance of Sum:

$$\mathrm{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathrm{Var}(X_i) + 2\sum_{1\le i<j\le n} \mathrm{Cov}(X_i, X_j)$$

Covariance & Correlation

Covariance:

- $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$
- $\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$
- $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$
- $\mathrm{Cov}(aX+b, cY+d) = ac \cdot \mathrm{Cov}(X,Y)$
- $\mathrm{Cov}(X, Y+Z) = \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Z)$
- $\mathrm{Cov}(X, X+Y) = \mathrm{Var}(X) + \mathrm{Cov}(X,Y)$
- $\mathrm{Cov}(X, X-Y) = \mathrm{Var}(X) - \mathrm{Cov}(X,Y)$
- If X, Y independent: $\mathrm{Cov}(X,Y) = 0$

Correlation:

- $\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$
- $-1 \le \rho(X,Y) \le 1$
- If X, Y independent: $\rho(X,Y) = 0$ (converse not true)

Conditional Distributions

Conditional Density/PMF:

$$f_{X|Y=b}(x) = \frac{f_{X,Y}(x,b)}{f_Y(b)}$$

Conditional Expectation:

$$E[X|Y=b] = \int_{-\infty}^{\infty} x \cdot f_{X|Y=b}(x) dx$$

$$E[X|Y=b] = \sum_x x \cdot P(X=x|Y=b)$$

Marginal Distributions

From Joint to Marginal:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$$

$$P(X=x) = \sum_y P(X=x, Y=y)$$

Hierarchical Models:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|\theta}(x|\theta) f_\theta(\theta) d\theta$$

$$P(X=x) = \sum_\theta P(X=x|\theta)P(\theta)$$

Transformation Methods

CDF Method:

- Find $F_Y(y) = P(Y \le y) = P(g(X) \le y)$
- Solve for X: $P(X \le h(y))$ or $P(X \ge h(y))$
- Differentiate: $f_Y(y) = \frac{d}{dy} F_Y(y)$

For $Y = g(X)$ (differentiable, monotonic):

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right|$$

Common Distributions

Uniform: $X \sim \textbf{Unif}[a,b]$

- PDF: $f(x) = \frac{1}{b-a}$ for $a \le x \le b$
- $E[X] = \frac{a+b}{2}$
- $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$

Discrete Uniform on $\{a, a+1, \ldots, b\}$:

- PMF: $P(X=k) = \frac{1}{b-a+1}$
- $E[X] = \frac{a+b}{2}$
- $\mathrm{Var}(X) = \frac{(b-a+1)^2-1}{12}$

Exponential: $X \sim \textbf{Exp}(\lambda)$

- PDF: $f(x) = \lambda e^{-\lambda x}$, $x \ge 0$
- $E[X] = \frac{1}{\lambda}$
- $\mathrm{Var}(X) = \frac{1}{\lambda^2}$
- Memoryless: $P(X > s+t | X > s) = P(X > t)$

Binomial: $X \sim \textbf{Bin}(n,p)$

- PMF: $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$
- $E[X] = np$
- $\mathrm{Var}(X) = np(1-p)$

Beta: $X \sim \textbf{Beta}(\alpha,\beta)$

- PDF: $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$
- $E[X] = \frac{\alpha}{\alpha+\beta}$
- $\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Poisson: $X \sim \textbf{Pois}(\lambda)$

- PMF: $P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$
- $E[X] = \lambda$
- $\mathrm{Var}(X) = \lambda$

Gamma: $X \sim \textbf{Gamma}(r,\beta)$

Shape-rate parameterization

- PDF: $f(x) = \frac{\beta^r}{\Gamma(r)} x^{r-1} e^{-\beta x}$, $x > 0$
- $E[X] = \frac{r}{\beta}$
- $\mathrm{Var}(X) = \frac{r}{\beta^2}$

Normal: $X \sim N(\mu, \sigma^2)$

- PDF: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$
- $E[X] = \mu$
- $\mathrm{Var}(X) = \sigma^2$
- If $Z \sim N(0,1)$, then $Z^2 \sim \chi^2_1 \sim \mathrm{Gamma}(1/2, 1/2)$

Chi-Square: $X \sim \chi^2_k$

- PDF: $f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$, $x > 0$
- $E[X] = k$
- $\mathrm{Var}(X) = 2k$
- Equivalent to $\mathrm{Gamma}(k/2, 1/2)$

Negative Binomial: $X \sim \textbf{NB}(r,p)$

- PMF: $P(X=k) = \binom{k+r-1}{k} p^r (1-p)^k$
- $E[X] = \frac{r(1-p)}{p}$
- $\mathrm{Var}(X) = \frac{r(1-p)}{p^2}$

Multinomial: $\mathbf{X} \sim \textbf{Multi}(n, \mathbf{p})$

- PMF: $P(\mathbf{X} = \mathbf{x}) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$
- $E[X_i] = np_i$
- $\text{Var}(X_i) = np_i(1 - p_i)$
- $\text{Cov}(X_i, X_j) = -np_i p_j$ for $i \neq j$

Dirichlet: $\mathbf{p} \sim \textbf{Dir}(\boldsymbol{\alpha})$

Let $\alpha_0 = \sum_{i=1}^k \alpha_i$

- PDF: $f(\mathbf{p}) = \frac{\Gamma(\alpha_0)}{\prod_{i=1}^k \Gamma(\alpha_i)} \prod_{i=1}^k p_i^{\alpha_i - 1}$
- $E[p_i] = \frac{\alpha_i}{\alpha_0}$
- $\text{Var}(p_i) = \frac{\alpha_i(\alpha_0 - \alpha_i)}{\alpha_0^2(\alpha_0 + 1)}$
- $\text{Cov}(p_i, p_j) = -\frac{\alpha_i \alpha_j}{\alpha_0^2(\alpha_0 + 1)}$ for $i \neq j$

Hierarchical Distributions

1. Beta-Binomial

$X|p \sim \text{Bin}(n, p)$, $p \sim \text{Beta}(\alpha, \beta)$

Marginal PMF:

$$P(X = k) = \binom{n}{k} \frac{\Gamma(\alpha + \beta)\Gamma(k + \alpha)\Gamma(n - k + \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n + \alpha + \beta)}$$

Moments:

$$E[X] = \frac{n\alpha}{\alpha + \beta}$$

$$\text{Var}(X) = \frac{n\alpha\beta(n + \alpha + \beta)}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

2. Gamma-Poisson (Negative Binomial)

$X|\lambda \sim \text{Pois}(\lambda)$, $\lambda \sim \text{Gamma}(r, \beta)$

Marginal PMF:

$$P(X = k) = \binom{k + r - 1}{k} \left(\frac{\beta}{\beta + 1}\right)^r \left(\frac{1}{\beta + 1}\right)^k$$

This is NB $\left(r, \frac{\beta}{\beta+1}\right)$

Moments:

$$E[X] = \frac{r}{\beta}$$

$$\text{Var}(X) = \frac{r(\beta + 1)}{\beta^2}$$

3. Normal-Normal (Random Mean)

$X|\mu \sim N(\mu, \sigma^2)$, $\mu \sim N(\mu_0, \tau^2)$

Marginal: $X \sim N(\mu_0, \sigma^2 + \tau^2)$

Moments:

$$E[X] = \mu_0$$
$$\text{Var}(X) = \sigma^2 + \tau^2$$

4. Dirichlet-Multinomial

$\mathbf{X}|\mathbf{p} \sim \text{Multi}(n, \mathbf{p})$, $\mathbf{p} \sim \text{Dir}(\boldsymbol{\alpha})$

Marginal PMF:

$$P(\mathbf{X} = \mathbf{x}) = \binom{n}{x_1, \ldots, x_k} \frac{\Gamma(\alpha_0)}{\Gamma(n + \alpha_0)} \prod_{i=1}^k \frac{\Gamma(x_i + \alpha_i)}{\Gamma(\alpha_i)}$$

Moments:

$$E[X_i] = \frac{n\alpha_i}{\alpha_0}$$

$$\text{Var}(X_i) = \frac{n\alpha_i(\alpha_0 - \alpha_i)(n + \alpha_0)}{\alpha_0^2(\alpha_0 + 1)}$$

$$\text{Cov}(X_i, X_j) = -\frac{n\alpha_i\alpha_j(n + \alpha_0)}{\alpha_0^2(\alpha_0 + 1)} \text{ for } i \neq j$$

Joint Distributions

Independence:

- $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$
- $P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$
- If independent: $E[XY] = E[X]E[Y]$
- If independent: $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

Sum of Independent Variables:

- If $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ independent:
$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$
- If $X \sim \text{Pois}(\lambda_1)$, $Y \sim \text{Pois}(\lambda_2)$ independent:
$$X + Y \sim \text{Pois}(\lambda_1 + \lambda_2)$$
- If $X \sim \text{Gamma}(r_1, \beta)$, $Y \sim \text{Gamma}(r_2, \beta)$ independent:
$$X + Y \sim \text{Gamma}(r_1 + r_2, \beta)$$

Common Transformation Examples

If $X \sim \textbf{Unif}[0, 1]$:

- $Y = \sqrt{X}$ has PDF $f_Y(y) = 2y$, $0 \leq y \leq 1$
- $Y = e^X$ has PDF $f_Y(y) = \frac{1}{y}$, $1 \leq y \leq e$
- $Y = -\ln X \sim \text{Exp}(1)$

If $Z \sim N(0, 1)$:

- $Z^2 \sim \chi_1^2$
- $Y = \mu + \sigma Z \sim N(\mu, \sigma^2)$