MGFs & Continuous R.V.s Cheat Sheet

Sections 3.9-3.10, 4.2-4.7, 4.9-4.10

PART 1: MGFs (Sec 3.9, 4.9)

Definition & Core Concept

$m(t) = E[e^{tY}]$ **packages all moments about origin**

Discrete: $m(t) = \sum e^{ty} p(y)$ **Continuous:** $m(t) = \int e^{ty} f(y) dy$

▷ **Property 1: Generate Moments**

$\mu'_k = m^{(k)}(0) = \frac{d^k}{dt^k} m(t)\big|_{t=0}$

Mean: $\mu = m'(0)$ $\mathbf{E(Y^2)}$: $E(Y^2) = m''(0)$

Variance: $\sigma^2 = m''(0) - [m'(0)]^2$

▷ **Property 2: Uniqueness Theorem**

If $m_X(t) = m_Y(t)$**, then** X **and** Y **have same distribution. Use to identify distributions.**

Advanced Properties

Linear Transform: $m_{aY+b}(t) = e^{tb} m_Y(at)$

Sum of Independent: $m_{X+Y}(t) = m_X(t) \cdot m_Y(t)$

Example: Poisson MGF $\rightarrow \mu, \sigma^2$

Given $m(t) = e^{\lambda(e^t - 1)}$

$m'(t) = \lambda e^t e^{\lambda(e^t-1)} \implies m'(0) = \lambda$

$m''(t) = (\lambda e^t)^2 e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)}$

$m''(0) = \lambda^2 + \lambda \implies \sigma^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$

Example: Sum of Independent Poissons

$X \sim$ **Pois**(λ_1), $Y \sim$ **Pois**(λ_2) **independent. Find dist. of** $W = X + Y$.

$m_W(t) = m_X(t) m_Y(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}$

By uniqueness: $W \sim$ **Pois**$(\lambda_1 + \lambda_2)$

Example: Linear Transform

Y **has** $m_Y(t) = \frac{0.16}{0.16-t}$. **Find** $M_W(t)$ **for** $W = 0.7Y$.

$m_W(t) = m_Y(0.7t) = \frac{0.16}{0.16-0.7t}$

Example: Sum $J + K + L$ **(Gamma)**

$M_J(t) = (1-2t)^{-3}$, $M_K(t) = (1-2t)^{-2.5}$, $M_L(t) = (1-2t)^{-4.5}$

$m_X(t) = (1-2t)^{-3}(1-2t)^{-2.5}(1-2t)^{-4.5} = (1-2t)^{-10}$

Therefore $X \sim$ **Gamma**$(10, 2)$

▷ **Distribution Identification by MGF**

Discrete:

$(pe^t + q)^n \rightarrow$ **Binomial**(n, p) $e^{\lambda(e^t-1)} \rightarrow$ **Poisson**(λ)

Continuous:

$(1-\beta t)^{-\alpha} \rightarrow$ **Gamma**(α, β) $(1-\beta t)^{-1} \rightarrow$ **Exp**(β)

$(1-2t)^{-v/2} \rightarrow \chi^2(v)$ $e^{\mu t + \frac{1}{2}\sigma^2 t^2} \rightarrow$ **Normal**(μ, σ^2)

PART 2: PGFs (Sec 3.10)

Definition (Non-negative integers only)

$P(t) = E[t^Y] = \sum p(y) t^y$ $(y = 0, 1, 2, \ldots)$

▷ **Generate Probabilities**

$p(k) =$ **coeff of** t^k **or** $p(k) = \frac{1}{k!} P^{(k)}(0)$

▷ **Generate Factorial Moments**

$\mu_{[k]} = E[Y(Y-1)\cdots(Y-k+1)] = P^{(k)}(1)$

Mean: $\mu = P'(1)$ **Variance:** $\sigma^2 = P''(1) + P'(1) - [P'(1)]^2$

PGF Table

Distribution	PGF
Binomial(n, p)	$(pt + q)^n$
Geometric(p)	$\frac{pt}{1-qt}$
Poisson(λ)	$e^{\lambda(t-1)}$

▷ **MGF vs PGF**

$m(t) = P(e^t)$ $P(t) = m(\ln t)$

MGF: Evaluate at $t = 0$ **PGF: Evaluate at** $t = 1$

PART 3: CONTINUOUS R.V.s (Sec 4.2)

▷ **Key Concept**

$P(Y = y) = 0$ **for all** y. **Only intervals have probability.**

$P(a \le Y \le b) = P(a < Y < b)$ **(inequalities equivalent)**

CDF: $F(y) = P(Y \le y)$

Properties: (1) $F(-\infty) = 0$, $F(\infty) = 1$ **(2) Non-decreasing (3) Continuous**

Find Probability: $P(a < Y \le b) = F(b) - F(a)$

PDF: $f(y) = F'(y)$

Valid if: (1) $f(y) \ge 0$ **(2)** $\int_{-\infty}^{\infty} f(y) dy = 1$

Probability: $P(a \le Y \le b) = \int_a^b f(y) dy$

▷ **CDF** $\leftrightarrow$ **PDF**

$F(y) = \int_{-\infty}^{y} f(t) dt$ $f(y) = \frac{d}{dy} F(y)$

Example: $f(y) = 3y^2$ **for** $0 \le y \le 1$**, find** $F(y)$

$$F(y) = \begin{cases} 0 & y < 0 \\ \int_0^y 3t^2 dt = y^3 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

Check: $P(0.5 \le Y \le 0.8) = F(0.8) - F(0.5) = 0.512 - 0.125 = \mathbf{0.387}$

Quantiles (Percentiles)

p**-th quantile** ϕ_p**: Solve** $F(\phi_p) = p$

Median: $\phi_{0.5}$ **where** $F(\phi_{0.5}) = 0.5$

PART 4: EXPECTED VALUES (Sec 4.3)

▷ **Main Formula**

$E[g(Y)] = \int_{-\infty}^{\infty} g(y) f(y) dy$

Key Expectations

Mean: $\mu = E(Y) = \int y f(y) dy$

$\mathbf{E(Y^2)}$: $E(Y^2) = \int y^2 f(y) dy$

Variance: $\sigma^2 = E(Y^2) - [E(Y)]^2$

Example: $f(y) = 3y^2$ **for** $0 \le y \le 1$

$E(Y) = \int_0^1 y(3y^2) dy = \int_0^1 3y^3 dy = [\frac{3y^4}{4}]_0^1 = \mathbf{3/4}$

$E(Y^2) = \int_0^1 y^2(3y^2) dy = [\frac{3y^5}{5}]_0^1 = \mathbf{3/5}$

$\sigma^2 = 3/5 - (3/4)^2 = 3/5 - 9/16 = \mathbf{3/80}$

Properties (Linearity)

$E(c) = c$ $E(aY + b) = aE(Y) + b$ $V(aY + b) = a^2 V(Y)$

Example: Cost Function (Beta)

$Y \sim$ **Beta**$(1, 2)$, **Cost** $C = 10 + 20Y + 4Y^2$. **Find** $E(C)$.

$E(C) = 10 + 20E(Y) + 4E(Y^2)$

Beta: $E(Y) = \frac{1}{3}$, $V(Y) = \frac{1}{18} \implies E(Y^2) = \frac{1}{18} + (\frac{1}{3})^2 = \frac{1}{6}$

$E(C) = 10 + 20(\frac{1}{3}) + 4(\frac{1}{6}) = 10 + \frac{22}{3} \approx \mathbf{17.33}$

Alt. Formula (Non-negative Y)

$E(Y) = \int_0^{\infty} [1 - F(y)] dy$

PART 5: UNIFORM (Sec 4.4)

$Y \sim U(a, b)$

PDF: $f(y) = \frac{1}{b-a}$ **for** $a \le y \le b$

CDF: $F(y) = \begin{cases} 0 & y < a \\ \frac{y-a}{b-a} & a \le y \le b \\ 1 & y > b \end{cases}$

Parameters

$\mu = \frac{a+b}{2}$ $\sigma^2 = \frac{(b-a)^2}{12}$ $m(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$

Probability (Rectangle Area)

$P(c \le Y \le d) = \frac{d-c}{b-a}$ **(for** $c, d \in [a, b]$**)**

PART 6: NORMAL (Sec 4.5)

$Y \sim N(\mu, \sigma^2)$

PDF: $f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/(2\sigma^2)}$ **(never integrate by hand!)**

Properties: Symmetric, Mean=Median=Mode=μ

▷ **Z-Score (Standardization)**

$$Z = \frac{Y - \mu}{\sigma} \sim N(0, 1)$$

Interpretation: # of std. deviations from mean

Find Probability ("Forward")

1. Standardize: $Z = \frac{y-\mu}{\sigma}$

2. Use given $P(Z < z)$ **value or symmetry**

Example: $Y \sim N(10, 4)$**, find** $P(Y > 13)$

$\mu = 10, \sigma^2 = 4 \implies \sigma = 2$

$Z = \frac{13-10}{2} = 1.5$

$P(Y > 13) = P(Z > 1.5)$. **Given** $P(Z > 1.5) = 0.0668$**, answer is 0.0668.**

Find Value ("Backward")

1. Find z **from given** $P(Z \le z)$

2. Un-standardize: $y = \mu + z\sigma$

Example: 90th percentile of $Y \sim N(100, 25)$

$\mu = 100, \sigma^2 = 25 \implies \sigma = 5$

Given $P(Z \le 1.28) = 0.90$**, so** $z = 1.28$

$y = 100 + (1.28)(5) = 100 + 6.4 = \mathbf{106.4}$

Example: Unit Conversion

Temp (C): $Y \sim N(10, 100)$. **Find** $P(\textbf{Temp} \le 59°F)$.

Convert: $C = (59 - 32) \times 5/9 = 15°C$

$Z = \frac{15-10}{10} = 0.5 \implies P(Y \le 15) = P(Z \le 0.5)$

Empirical Rule

$P(\mu \pm 1\sigma) \approx 68\%$ $P(\mu \pm 2\sigma) \approx 95\%$ $P(\mu \pm 3\sigma) \approx 99.7\%$

MGF

$m(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ **Standard Normal:** $m(t) = e^{t^2/2}$

Example: Identify $m(t) = e^{10t + 2t^2}$

Match: $\mu t = 10t \implies \mu = 10$

$\frac{1}{2}\sigma^2 t^2 = 2t^2 \implies \sigma^2 = 4$

Therefore: $Y \sim \mathbf{N(10, 4)}$

PART 7: GAMMA FAMILY (Sec 4.6)

Gamma Function

$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy$

▷ $\Gamma(n) = (n-1)!$ $\Gamma(1/2) = \sqrt{\pi}$ $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

Gamma Distribution: $Y \sim$ **Gamma**(α, β)

PDF: $f(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}$ $(y \ge 0)$

$\mu = \alpha\beta$ $\sigma^2 = \alpha\beta^2$ $m(t) = (1-\beta t)^{-\alpha}$

Exponential: $Y \sim$ **Exp**(β) **[Gamma with** $\alpha = 1$**]**

PDF: $f(y) = \frac{1}{\beta} e^{-y/\beta}$ $(y \ge 0)$

$\mu = \beta$ $\sigma^2 = \beta^2$ $m(t) = (1-\beta t)^{-1}$

▷ **Memoryless:** $P(Y > a + b | Y > a) = P(Y > b)$

Example: Earthquake Wait Time

$X \sim$ **Exp**$(\beta = 2.4)$. **Find** $P(X \ge 3 | X \ge 2)$.

Memoryless: $P(X \ge 2 + 1 | X \ge 2) = P(X \ge 1)$

$P(X \ge 1) = e^{-1/2.4} \approx \mathbf{0.659}$

Chi-Square: $Y \sim \chi^2(v)$ **[Gamma with** $\alpha = v/2, \beta = 2$**]**

PDF: $f(y) = \frac{1}{\Gamma(v/2) 2^{v/2}} y^{v/2-1} e^{-y/2}$ $(y \ge 0)$

$\mu = v$ $\sigma^2 = 2v$ $m(t) = (1-2t)^{-v/2}$

Gamma Family Summary

Dist.	μ	σ^2	MGF
Gamma	$\alpha\beta$	$\alpha\beta^2$	$(1-\beta t)^{-\alpha}$
Exp	β	β^2	$(1-\beta t)^{-1}$
χ^2	v	$2v$	$(1-2t)^{-v/2}$

Example: Identify from MGF

Given $M_J(t) = (1-2t)^{-3}$

Match Gamma form: $\beta = 2, \alpha = 3$

$\mu = (3)(2) = \mathbf{6}$ $\sigma^2 = (3)(2^2) = \mathbf{12}$

Example: Find Parameters (Backward)

$Y \sim$ **Gamma**(α, β) **with** $E(Y) = 20$, $V(Y) = 100$. **Find** α, β.

$\alpha\beta = 20$ **and** $\alpha\beta^2 = 100$

Divide: $\frac{\alpha\beta^2}{\alpha\beta} = \frac{100}{20} \implies \beta = \mathbf{5}$

$\alpha(5) = 20 \implies \alpha = \mathbf{4}$

PART 8: BETA (Sec 4.7)

Beta Function
$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

Beta Distribution: $Y \sim$ **Beta**(α,β)

PDF: $f(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}$ $(0 \le y \le 1)$

$\mu = \frac{\alpha}{\alpha+\beta}$ $\quad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

▷ **Identification Trick**

Given $f(y) = ky^a(1-y)^b$ **for** $0 \le y \le 1$:

1. Match exponents: $\alpha = a+1$, $\beta = b+1$

2. $k = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$

3. Use shortcut formulas for μ, σ^2

Example: $f(y) = ky^3(1-y)^2$, **find** k, μ, σ^2

Match: $\alpha - 1 = 3 \implies \alpha = 4$, $\beta - 1 = 2 \implies \beta = 3$

$k = \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} = \frac{6!}{3!\cdot 2!} = \frac{720}{12} = \mathbf{60}$

$\mu = \frac{4}{4+3} = \mathbf{4/7}$ $\quad \sigma^2 = \frac{(4)(3)}{(7)^2(8)} = \frac{12}{392} = \mathbf{3/98}$

Special Case

Beta$(1,1) =$ **Uniform**$(0,1)$

PART 9: TCHEBYSHEFF (Sec 4.10)

▷ **The Theorem (ANY distribution)**

For $k > 1$:

$P(|Y-\mu| < k\sigma) \ge 1 - \frac{1}{k^2}$ $\quad$ **(within** $k\sigma$**)**

$P(|Y-\mu| \ge k\sigma) \le \frac{1}{k^2}$ $\quad$ **(outside** $k\sigma$**)**

Common Values

$k = 2$: **At least 75% within** $\mu \pm 2\sigma$

$k = 3$: **At least 88.9% within** $\mu \pm 3\sigma$

Example: $\mu = 50, \sigma = 5$, **find** $P(40 < Y < 60)$

Interval is $\mu \pm k\sigma$: $k\sigma = 60 - 50 = 10 \implies k = 2$

$P(40 < Y < 60) \ge 1 - \frac{1}{4} = \mathbf{0.75}$ **(at least 75%)**

Find Interval ("Backward")

Given probability p, **solve:** $1 - \frac{1}{k^2} = p \implies k = \sqrt{\frac{1}{1-p}}$

Interval: $(\mu - k\sigma, \mu + k\sigma)$

Example: Cover 8/9 of data, $\mu = 1000, \sigma = 100$

$1 - \frac{1}{k^2} = 8/9 \implies k^2 = 9 \implies k = 3$

Interval: $(1000 - 300, 1000 + 300) = (\mathbf{700}, \mathbf{1300})$

Tchebysheff vs Empirical Rule

Interval	Tcheb. (Any)	Emp. (Normal)
$\mu \pm 1\sigma$	$\ge 0\%$	$\approx 68\%$
$\mu \pm 2\sigma$	$\ge 75\%$	$\approx 95\%$
$\mu \pm 3\sigma$	$\ge 88.9\%$	$\approx 99.7\%$

PART 10: ADVANCED TOPICS

▷ **Mixed Distributions (Sec 4.11)**

Has discrete spikes (PMF) + continuous regions (PDF)

CDF: $F(y) = c_1 F_{disc}(y) + c_2 F_{cont}(y)$ **where** $c_1 + c_2 = 1$

E(Y): $E(Y) = c_1 E(Y_{disc}) + c_2 E(Y_{cont})$

MGF: $m(t) = c_1 m_{disc}(t) + c_2 m_{cont}(t)$

Example: $M(t) = \frac{1}{3(2-t)} + \frac{2}{3(3-t)}$

Rewrite: $\frac{1}{3} \cdot \frac{0.5}{1-0.5t} + \frac{2}{3} \cdot \frac{1/3}{1-(1/3)t}$

Mixture: $c_1 = 1/3$ **for Exp**(0.5), $c_2 = 2/3$ **for Exp**$(1/3)$

Example: Bank Wait Time (CDF)

$P(\mathbf{0}$ **people**$) = 0.5$, $P(\mathbf{1}$ **person**$) = 0.5$. **Service** $S \sim$ **Exp**(λ).

$W = 0$ **if 0 people (discrete).** $W = S$ **if 1 person (continuous).**

$F(w) = 0.5(1) + 0.5(1 - e^{-\lambda w}) = 1 - 0.5e^{-\lambda w}$ $(w \ge 0)$

Note: $F(0) = 0.5$ **(jump at 0)**

▷ **Reading Discrete MGF**

$m(t) = \sum e^{ty}p(y)$ $\quad$ **Read probabilities from terms**

Example: $m(t) = \frac{1}{6}e^{-2t} + \frac{1}{3}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{4}e^{2t}$

$P(Y = -2) = 1/6$, $P(Y = -1) = 1/3$, $P(Y = 1) = 1/4$, $P(Y = 2) = 1/4$

$P(|Y| \le 1) = P(-1) + P(1) = 1/3 + 1/4 = \mathbf{7/12}$

▷ **Higher Moments:** $E(Y^k) = m^{(k)}(0)$

Gamma Shortcut: $E(Y^k) = \frac{\Gamma(\alpha+k)\beta^k}{\Gamma(\alpha)}$

Example: $E(X^3)$ **for Gamma**$(10,2)$

$E(X^3) = \frac{\Gamma(13)\cdot 8}{\Gamma(10)} = \frac{12!\cdot 8}{9!} = 1320 \cdot 8 = \mathbf{10560}$

▷ **Cumulant Generating Function**

$S(t) = \ln[m(t)]$ $\quad S'(0) = \mu$ $\quad S''(0) = \sigma^2$

Faster than standard MGF method for some distributions

▷ **When MGFs Don't Exist**

MGF exists only if $E[e^{tY}]$ **is finite for** t **near 0**

Cauchy: $f(x) = \frac{1}{\pi(1+x^2)}$ **has no MGF, no moments**

▷ **Full E(Y) Formula (any R.V.)**

$E(Y) = \int_0^\infty [1 - F(y)]dy - \int_0^\infty F(-y)dy$

▷ **PDF from Geometric Probability**

Example: Dart hits circle radius r **uniformly.** $X =$ **distance from center.**

$F(x) = P(X \le x) = \frac{\pi x^2}{\pi r^2} = \frac{x^2}{r^2}$ $(0 \le x \le r)$

$f(x) = F'(x) = \frac{2x}{r^2}$ $(0 \le x \le r)$

▷ **Symmetry Shortcut (Odd Functions)**

Example: $f(x) = (\lambda/2)e^{-\lambda|x|}$. **Find** $E(X)$.

$g(x) = xe^{-\lambda|x|}$ **is odd:** $g(-x) = -g(x)$

Integral of odd function from $-\infty$ **to** $\infty = \mathbf{0}$

Therefore $E(X) = \mathbf{0}$ **by symmetry (no calculation needed)**

QUICK REFERENCE

Key Formulas

Z-Score: $Z = \frac{Y-\mu}{\sigma}$ $\quad$ **Var Shortcut:** $\sigma^2 = E(Y^2) - [E(Y)]^2$

PDF↔CDF: $F(y) = \int_{-\infty}^{y} f(t)dt$ $\quad f(y) = F'(y)$

Tcheb: $P(|Y-\mu| < k\sigma) \ge 1 - \frac{1}{k^2}$

Distribution Identification Flowchart

MGF Form → **Distribution**

$(pe^t + q)^n \to$ **Binomial**(n,p)

$e^{\lambda(e^t-1)} \to$ **Poisson**(λ)

$(1-\beta t)^{-\alpha} \to$ **Gamma**(α,β)

$(1-\beta t)^{-1} \to$ **Exp**(β)

$(1-2t)^{-v/2} \to \chi^2(v)$

$e^{\mu t + \frac{1}{2}\sigma^2 t^2} \to$ **Normal**(μ,σ^2)