Gamma & Beta Functions

Gamma Function
- $\Gamma(n) = (n-1)!$ **for** $n \in \mathbb{Z}^+$
- $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$
- $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$
- $\int_0^\infty x^{\alpha-1} e^{-\beta x} dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$
- $(y+\alpha-1)! = \binom{y+\alpha-1}{y}(y!)(\alpha-1)!$

Beta Function
- $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
- $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt$

Expectation & Variance Laws

Law of Total Expectation
$E[X] = E[E[X|\theta]]$

Law of Total Variance
$\mathbf{Var}(X) = E[\mathbf{Var}(X|\theta)] + \mathbf{Var}(E[X|\theta])$

Variance Properties
- $\mathbf{Var}(X) = E[X^2] - (E[X])^2$
- $\mathbf{Var}(aX+b) = a^2\mathbf{Var}(X)$
- $\mathbf{Var}(aX+bY) = a^2\mathbf{Var}(X) + b^2\mathbf{Var}(Y) + 2ab\mathbf{Cov}(X,Y)$

Variance of Sum
$\mathbf{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathbf{Var}(X_i) + 2\sum_{1\le i<j\le n} \mathbf{Cov}(X_i, X_j)$

Covariance & Correlation

Covariance
- $\mathbf{Cov}(X,Y) = E[XY] - E[X]E[Y]$
- $\mathbf{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$
- $\mathbf{Cov}(X,X) = \mathbf{Var}(X)$
- $\mathbf{Cov}(aX+b, cY+d) = ac \cdot \mathbf{Cov}(X,Y)$
- $\mathbf{Cov}(X, Y+Z) = \mathbf{Cov}(X,Y) + \mathbf{Cov}(X,Z)$
- $\mathbf{Cov}(X, X+Y) = \mathbf{Var}(X) + \mathbf{Cov}(X,Y)$
- $\mathbf{Cov}(X, X-Y) = \mathbf{Var}(X) - \mathbf{Cov}(X,Y)$
- **If** X, Y **independent:** $\mathbf{Cov}(X,Y) = 0$

Correlation
- $\rho(X,Y) = \frac{\mathbf{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{\mathbf{Cov}(X,Y)}{\sqrt{\mathbf{Var}(X)\mathbf{Var}(Y)}}$
- $-1 \le \rho(X,Y) \le 1$
- **If** X, Y **independent:** $\rho(X,Y) = 0$

Conditional Distributions

Conditional Density/PMF
$f_{X|Y=b}(x) = \frac{f_{X,Y}(x,b)}{f_Y(b)}$

Conditional Expectation
$E[X|Y=b] = \int_{-\infty}^{\infty} x \cdot f_{X|Y=b}(x)dx$

$E[X|Y=b] = \sum_x x \cdot P(X=x|Y=b)$

Marginal Distributions

From Joint to Marginal
$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$

$P(X=x) = \sum_y P(X=x, Y=y)$

Hierarchical Models
$f_X(x) = \int_{-\infty}^{\infty} f_{X|\theta}(x|\theta) f_\theta(\theta) d\theta$

$P(X=x) = \sum_\theta P(X=x|\theta)P(\theta)$

Transformation Methods

CDF Method
- **Find** $F_Y(y) = P(Y \le y) = P(g(X) \le y)$
- **Solve for** X**:** $P(X \le h(y))$ **or** $P(X \ge h(y))$
- **Differentiate:** $f_Y(y) = \frac{d}{dy}F_Y(y)$

For $Y = g(X)$ (diff, monotonic)
$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy}g^{-1}(y)\right|$

Common Distributions

Uniform: $X \sim \mathbf{Unif}[a,b]$
- **PDF:** $f(x) = \frac{1}{b-a}$ **for** $a \le x \le b$
- $E[X] = \frac{a+b}{2}$
- $\mathbf{Var}(X) = \frac{(b-a)^2}{12}$

Discrete Uniform on $\{a, \ldots, b\}$
- **PMF:** $P(X=k) = \frac{1}{b-a+1}$
- $E[X] = \frac{a+b}{2}$
- $\mathbf{Var}(X) = \frac{(b-a+1)^2-1}{12}$

Exponential: $X \sim \mathbf{Exp}(\lambda)$
- **PDF:** $f(x) = \lambda e^{-\lambda x}$, $x \ge 0$
- $E[X] = \frac{1}{\lambda}$
- $\mathbf{Var}(X) = \frac{1}{\lambda^2}$
- **Memoryless:** $P(X > s+t | X > s) = P(X > t)$

Binomial: $X \sim \mathbf{Bin}(n,p)$
- **PMF:** $P(X=k) = \binom{n}{k}p^k(1-p)^{n-k}$
- $E[X] = np$
- $\mathbf{Var}(X) = np(1-p)$

Beta: $X \sim \mathbf{Beta}(\alpha,\beta)$
- **PDF:** $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$
- $E[X] = \frac{\alpha}{\alpha+\beta}$
- $\mathbf{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Poisson: $X \sim \mathbf{Pois}(\lambda)$
- **PMF:** $P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}$
- $E[X] = \lambda$
- $\mathbf{Var}(X) = \lambda$

Gamma: $X \sim \mathbf{Gamma}(r,\beta)$
Shape-rate parameterization
- **PDF:** $f(x) = \frac{\beta^r}{\Gamma(r)}x^{r-1}e^{-\beta x}$, $x > 0$
- $E[X] = \frac{r}{\beta}$
- $\mathbf{Var}(X) = \frac{r}{\beta^2}$

Normal: $X \sim N(\mu,\sigma^2)$
- **PDF:** $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$
- $E[X] = \mu$
- $\mathbf{Var}(X) = \sigma^2$
- **If** $Z \sim N(0,1)$**, then** $Z^2 \sim \chi_1^2 \sim \mathbf{Gamma}(1/2, 1/2)$

Chi-Square: $X \sim \chi_k^2$
- **PDF:** $f(x) = \frac{1}{2^{k/2}\Gamma(k/2)}x^{k/2-1}e^{-x/2}$, $x > 0$
- $E[X] = k$
- $\mathbf{Var}(X) = 2k$
- **Equivalent to** $\mathbf{Gamma}(k/2, 1/2)$

Negative Binomial: $X \sim \mathbf{NB}(r,p)$
- **PMF:** $P(X=k) = \binom{k+r-1}{k}p^r(1-p)^k$
- $E[X] = \frac{r(1-p)}{p}$
- $\mathbf{Var}(X) = \frac{r(1-p)}{p^2}$

Multinomial: $\mathbf{X} \sim \mathbf{Multi}(n,\mathbf{p})$
- **PMF:** $P(\mathbf{X}=\mathbf{x}) = \frac{n!}{x_1!\cdots x_k!}p_1^{x_1}\cdots p_k^{x_k}$
- $E[X_i] = np_i$
- $\mathbf{Var}(X_i) = np_i(1-p_i)$
- $\mathbf{Cov}(X_i, X_j) = -np_ip_j$ **for** $i \ne j$

Dirichlet: $\mathbf{p} \sim \mathbf{Dir}(\boldsymbol{\alpha})$
Let $\alpha_0 = \sum_{i=1}^k \alpha_i$
- **PDF:** $f(\mathbf{p}) = \frac{\Gamma(\alpha_0)}{\prod_{i=1}^k \Gamma(\alpha_i)}\prod_{i=1}^k p_i^{\alpha_i-1}$
- $E[p_i] = \frac{\alpha_i}{\alpha_0}$
- $\mathbf{Var}(p_i) = \frac{\alpha_i(\alpha_0-\alpha_i)}{\alpha_0^2(\alpha_0+1)}$
- $\mathbf{Cov}(p_i, p_j) = -\frac{\alpha_i\alpha_j}{\alpha_0^2(\alpha_0+1)}$ **for** $i \ne j$

Hierarchical Distributions

1. Beta-Binomial
$X|p \sim \mathbf{Bin}(n,p)$, $p \sim \mathbf{Beta}(\alpha,\beta)$

Marginal PMF: $P(X=k) = \binom{n}{k}\frac{\Gamma(\alpha+\beta)\Gamma(k+\alpha)\Gamma(n-k+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)}$

Moments: $E[X] = \frac{n\alpha}{\alpha+\beta}$

$\mathbf{Var}(X) = \frac{n\alpha\beta(n+\alpha+\beta)}{(\alpha+\beta)^2(\alpha+\beta+1)}$

2. Gamma-Poisson (Negative Binomial)
$X|\lambda \sim \mathbf{Pois}(\lambda)$, $\lambda \sim \mathbf{Gamma}(r,\beta)$

Marginal PMF: $P(X=k) = \binom{k+r-1}{k}\left(\frac{\beta}{\beta+1}\right)^r\left(\frac{1}{\beta+1}\right)^k$

This is NB $\left(r, \frac{\beta}{\beta+1}\right)$

Moments: $E[X] = \frac{r}{\beta}$

$\mathbf{Var}(X) = \frac{r(\beta+1)}{\beta^2}$

3. Normal-Normal (Random Mean)
$X|\mu \sim N(\mu,\sigma^2)$, $\mu \sim N(\mu_0,\tau^2)$

Marginal: $X \sim N(\mu_0, \sigma^2+\tau^2)$

Moments: $E[X] = \mu_0$

$\mathbf{Var}(X) = \sigma^2 + \tau^2$

4. Dirichlet-Multinomial
$\mathbf{X}|\mathbf{p} \sim \mathbf{Multi}(n,\mathbf{p})$, $\mathbf{p} \sim \mathbf{Dir}(\boldsymbol{\alpha})$

Marginal PMF: $P(\mathbf{X}=\mathbf{x}) = \binom{n}{x_1,\ldots,x_k}\frac{\Gamma(\alpha_0)}{\Gamma(n+\alpha_0)}\prod_{i=1}^k\frac{\Gamma(x_i+\alpha_i)}{\Gamma(\alpha_i)}$

Moments: $E[X_i] = \frac{n\alpha_i}{\alpha_0}$

$\mathbf{Var}(X_i) = \frac{n\alpha_i(\alpha_0-\alpha_i)(n+\alpha_0)}{\alpha_0^2(\alpha_0+1)}$

$\mathbf{Cov}(X_i, X_j) = -\frac{n\alpha_i\alpha_j(n+\alpha_0)}{\alpha_0^2(\alpha_0+1)}$ **for** $i \ne j$

Joint Distributions

Independence
- $f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$
- $P(X=x, Y=y) = P(X=x)\cdot P(Y=y)$
- **If independent:** $E[XY] = E[X]E[Y]$
- **If independent:** $\mathbf{Var}(X+Y) = \mathbf{Var}(X) + \mathbf{Var}(Y)$

Sum of Independent Variables
- **If** $X \sim N(\mu_1,\sigma_1^2)$, $Y \sim N(\mu_2,\sigma_2^2)$ **independent:** $X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$
- **If** $X \sim \mathbf{Pois}(\lambda_1)$, $Y \sim \mathbf{Pois}(\lambda_2)$ **independent:** $X+Y \sim \mathbf{Pois}(\lambda_1+\lambda_2)$
- **If** $X \sim \mathbf{Gamma}(r_1,\beta)$, $Y \sim \mathbf{Gamma}(r_2,\beta)$ **independent:** $X+Y \sim \mathbf{Gamma}(r_1+r_2,\beta)$

Common Transformation Examples
If $X \sim \mathbf{Unif}[0,1]$
- $Y = \sqrt{X}$ **has PDF** $f_Y(y) = 2y$, $0 \le y \le 1$
- $Y = e^X$ **has PDF** $f_Y(y) = \frac{1}{y}$, $1 \le y \le e$
- $Y = -\ln X \sim \mathbf{Exp}(1)$

If $Z \sim N(0,1)$
- $Z^2 \sim \chi_1^2$
- $Y = \mu + \sigma Z \sim N(\mu,\sigma^2)$