Data Science 07-09-15: Game Theory

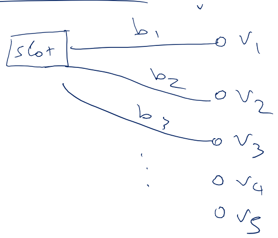
Prisoner's Dilemma

	Confess	Silent
Confess	4 \ 4 ~~1~~	2 \ 5
Silent	5 \	2 \ 2

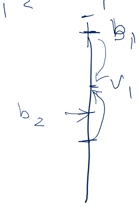
Dominant Strategy:

- Strategy s.t no matter what the other player does you are better off playing it.

Second Price Auction



$u_i b_2 \longrightarrow v_i \cdot x_i - p_i$



Tragedy of the Commons

- n players
- $x_i \in [0,1]$
- $u_i = x_i \left(1 - \sum_j x_j\right)$

$t = \sum_{j \neq i} x_j$

$u_i = x_i (1 - t - x_i)$

$u_i'(x_i) = 1 - t - 2x_i$

$x_i = \frac{1-t}{2}$

$x_i = \frac{1 - \sum_{j \neq i} x_j}{n+2}$ $\Rightarrow$ $x_i = \frac{1 - \frac{n-1}{n+1}}{2} = \frac{\frac{n+1-(n-1)}{n+1}}{2}$

$x_i = \frac{1}{}$

$= \frac{2}{2(n+1)} = \frac{1}{n+1}$

Nash Equilibrium

A profile of strategies $s_1, s_2, \dots, s_n$

$$u_i(s_1, \dots, s_n) \geq u_i(\underbrace{s_i', s_{-i}}_{(s_1, \dots, s_{i-1}, s_i', s_{i+1}, \dots, s_n)})$$

Social Inefficiency

$\sum_i u_i = \frac{1}{n+1}\left(1 - \frac{n}{n+1}\right) = \frac{1}{(n+1)^2}$

$u_i = \frac{n}{(n+1)^2} \approx \frac{1}{n}$

$x_i' = \frac{1}{2n}$ $\Rightarrow$ $SW(x') = n \, \frac{1}{2n}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$

Battle of the Sexes

		W: Shop	W: Foot
M	Shop	3, 2	1, 1
	Foot	0, 0	2, 3

Two Nash-Equilibria

Non-unique prediction

strategies $\vec{x}_1, \dots, \vec{x}_n$

~~Mixed Nash Equilibrium~~

$$E_{s \sim x}[u_i(s)] \geq E_{s_{-i} \sim x_{-i}}[u_i(s_i', s_{-i})]$$

A profile of randomized

Matching Pennies

		Mismatch: H	Mismatch: T
Match	H	1, -1	-1, 1
	T	-1, 1	1, -1

$s \sim x$

$$\begin{array}{c} \ \ C \ m \ j \\ A = \\ R \ \ n \ i \end{array} \begin{pmatrix} & & \\ & \boxed{u_{ij}} & \\ & & \end{pmatrix}$$

$$B = -A$$

$x = (x_1, \dots, x_n)$ $\qquad u_R = x^T A y$

$y = (y_1, \dots, y_n)$

$$\vec{u}_R = Ay = \begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$(x_1, \dots, x_n) \begin{pmatrix} E u_1 \\ E u_2 \\ \vdots \\ E u_n \end{pmatrix}$$

(x^*, y^*) : $\qquad x^T A y \geq (A y)_i$ $\qquad x^T A y \leq (x^T A)_j$

$$v^* = \max_x \min_y x^T A y$$

$x^* = \max_x \min_y$

y $\|$

$y^* = \min_y \max_x x^T A y$

Fictitious Play

i_t | $\hat{x}^i_t = \frac{1}{t} \sum_{\tau=1}^{t-1} \mathbb{1}\{i_\tau = i\}$: Belief about Row player

j_t

$\hat{y}^j_t = \frac{1}{t} \sum_{\tau=1}^{t-1} \mathbb{1}\{j_\tau = j\}$: Belief about Column Player.

Row Player:

$i_{t+1} = \arg\max_i (A \hat{y}_t)_i$

$j_{t+1} = \arg\min_j (\hat{x}_t^T A)_j$

$\hat{y}_t \to y_*$ is Nash Equilibrium.

$\hat{x}_t \to x_*$

No-Regret Learning

$R: \quad \frac{1}{T} \sum_{t=1}^{T} x_t^T A y_t \geq \frac{1}{T} \sum_{t=1}^{T} (A \hat{y}_t)_i$

$C:$ —— $''$ -

i

$T A \hat{\ }$ $\mathbb{1}R(x)$

$$x^{i}_{t+1} = \arg\max_{x} \; x^T A \hat{y}_t - \frac{1}{\eta} R(x)$$

$$R(x) = \sum_i x_i \log(x_i)$$

$$x_{t+1} = \frac{e^{\eta \cdot (A\hat{y}_t)_i}}{\sum_{i'} e^{\eta (A\hat{y}_t)_{i'}}}$$

Multiplicative Weight Updates Algorithm.

$$x_{t+1} = \arg\max \; \frac{1}{T}\sum_t x^T A y_t - \frac{1}{\eta} R(x)$$

$$= \arg\max \; \sum_t x^T A y_t - \frac{T}{\eta} R(x)$$

$$\frac{T}{\eta} \approx \sqrt{T} \Rightarrow \eta = \frac{1}{\sqrt{T}}$$

$$\text{Regret} \leq \frac{1}{\sqrt{T}}$$