Polynomial Calculus for MaxSAT

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This presentation

SAT

Polynomial Calculus

$\neq \downarrow$

Resolution

MaxSAT

MaxSAT Resolution

Weighted Resolution

Why?

Polynomials are more expressive than clauses

Natural/non-Boolean encodings of CSPs

Sometimes useful in practice (coloring, multiplier circuits)

P $\longrightarrow$ Q : P is stronger than Q

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weighted MaxSAT

input $F = \{(C_1, w_1), \ldots, (C_m, w_m)\}$ a multiset

goal find $\min_{\alpha} \text{cost}(F, \alpha)$

$$\text{cost}(\{(C_1, w_1), \ldots, (C_m, w_m)\}, \alpha) := \text{sum of all } w_i \text{ s.t. } \alpha \text{ falsifies } C_i$$

C_i : clauses

w_i : weights (in $\mathbb{Z}$)

α : truth assignment

weighted MaxSAT

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polynomials

C_i : ~~clauses~~

w_i : weights (in $\mathbb{Z}$)

α : truth ? assignment

F_0 (the input, a multiset of weighted clauses)

$\perp$: empty clause

D_i : clauses

F_i : multisets of weighted clauses

α : truth assignment

F_0	(the input, a multiset of weighted clauses)
F_1	

$\perp$: empty clause

D_i : clauses

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F_2	

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$\vdots$

$F_{\ell-1}$

$F_\ell = \{(\perp, w), \quad (D_1, w_1), \ldots, (D_s, w_s)\}$

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($\star$) for every i and every α,

$\text{cost}(F_i, \alpha) = \text{cost}(F_{i-1}, \alpha)$

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Theorem

If $\{D_1, \ldots, D_s\}$ is satisfiable and ($\star$), *then* $\min_\alpha \text{cost}(F_0, \alpha) = w$

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If $w_1 \geq 0, \ldots, w_s \geq 0$ and ($\star$), *then* $\min_\alpha \mathrm{cost}(F_0, \alpha) \geq w$

$\perp$: empty clause

D_i : clauses

F_i : multisets of weighted clauses

α : truth assignment

$\mathbb{Z}$-weighted Resolution

Substitution rules to ensure that for every α, cost$(F_i, \alpha) =$ cost(F_{i-1}, α)

$$\frac{(C \vee x, w) \quad (C \vee \bar{x}, w)}{(C, w)}$$

or

$$\frac{(C, w)}{(C \vee x, w) \quad (C \vee \bar{x}, w)}$$

or

$$\frac{}{(C, w) \quad (C, -w)}$$

+ natural “structural” rules

$w \in \mathbb{Z}$

C : clause

x : variable

$\frac{premises}{conclusions}$: substitution rules

α : truth assignment

F_i : multisets of weighted clauses

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$$\begin{array}{ll} F_{i-1} & \dfrac{(C \vee x, w) \quad (C \vee \bar{x}, w)}{(C, w)} \\ F_i & \end{array}$$

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$\mathbb{Z}$-weighted Resolution

aka Sherali-Adams **[BL20]**

aka Circular Resolution **[AL19]**

$\neq$

$\mathbb{N}$-weighted Resolution

$\neq$?

MaxSAT Resolution

P $\longrightarrow$ Q : P is stronger than Q

(Unary versions of the systems omitted)

weighted MaxSAT on polynomials

input $F = \{(p_1, w_1), \ldots, (p_m, w_m)\}$ a multiset

goal find $\min_{\alpha} \text{cost}(F, \alpha)$

$$\text{cost}(\{(p_1, w_1), \ldots, (p_m, w_m)\}, \alpha) := \text{sum of all } w_i \text{ s.t. } p_i(\alpha) \neq 0$$

p_i : polynomials in $\mathbb{F}[x_1, \ldots, x_n]$

w_i : weights (in $\mathbb{Z}$)

$\alpha : \{x_1, \ldots, x_n\} \to \mathbb{F}$ assignment

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Example. Maxcut

$G = (V, E)$ a graph

Boolean variables x_v for each $v \in V$

$F = \{(x_v + x_w + 1,\ 1) : (v, w) \in E\}$

p_i : polynomials in $\mathbb{F}[x_1, \ldots, x_n]$

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$F = \{(x_v + x_w + 1,\ 1) : (v, w) \in E\}$

$$x_v + x_w + 1 \in \mathbb{F}_2[x_v, x_w]$$

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F_0 (the input, a multiset of weighted polynomials)
F_1
F_2
$\vdots$
$F_{\ell-1}$
$F_\ell = \{(1, w), \quad (p_1, w_1), \ldots, (p_s, w_s)\}$

($\star$) for every i and every α,

cost$(F_i, \alpha) =$ cost(F_{i-1}, α)

Theorem

If $\{p_1, \ldots, p_s\}$ have a common zero and ($\star$), *then* $\min_\alpha \text{cost}(F_0, \alpha) = w$

Theorem

If if $w_1 \geq 0, \ldots, w_s \geq 0$ and ($\star$), *then* $\min_\alpha \text{cost}(F_0, \alpha) \geq w$

1 : the polynomial 1

p_i : polynomials

F_i : multisets of weighted polynomials

α : truth assignment

weighted Polynomial Calculus over $\mathbb{F}_2$

Substitution rules to ensure that for every α, $\text{cost}(F_i, \alpha) = \text{cost}(F_{i-1}, \alpha)$

$$\frac{(p, w)}{(pq, w) \quad (p(q+1), w)}$$

or

$$\frac{(p, w) \quad (q, w)}{(p+q, w) \quad (pq, 2w)}$$

or

$$\frac{}{(p, w) \quad (p, -w)}$$

+ natural "structural" rules

e.g. identify the polynomials p^2 and p

$w \in \mathbb{Z}$

$p, q \in \mathbb{F}_2[x_1, \ldots, x_n]$

$\frac{\textit{premises}}{\textit{conclusions}}$: substitution rules

α : assignment

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$$\frac{\dfrac{(xz+y,1)}{((xz+y)z,\ 1)\quad((xz+y)(z+1),\ 1)}}{(xz+yz,\ 1)\quad(yz+y,\ 1)}$$

$$\frac{(p,w)}{(pq,w)\quad(p(q+1),w)}\ \text{SPLIT}$$

$$\frac{\dfrac{(x+y,\ 1)\quad(x+z,\ 1)}{(y+z,\ 1)\quad((x+z)(x+y),\ 2)}}{(y+z,\ 1)\quad(x+xy+xz+yz,\ 2)}$$

$$\frac{(p,w)\quad(q,w)}{(p+q,w)\quad(pq,\ 2w)}\ \text{SUM}$$

Weights

coefficients of the polynomials

	$\mathbb{N}$	$\mathbb{Z}$
$\mathbb{F}_2$	$\mathrm{PC}_{\mathbb{F}_2,\mathbb{N}}$	$\mathrm{PC}_{\mathbb{F}_2,\mathbb{Z}}$
$\mathbb{F}_q$	**Not in this talk**	

Also **not** in this talk:

- weights in *unary* or *binary*
- encodings (e.g. twin variables)
- how to deal with hard constraints

Theorem (soundness)

If there is a $\mathrm{PC}_{\mathbb{F}_2,\ \mathbb{Z}}$ derivation of $(1, w)$ from F, *then* $\min_{\alpha} \operatorname{cost}(F, \alpha) \geq w$

Theorem (completeness)

If $\min_{\alpha} \operatorname{cost}(F, \alpha) \geq w$, *then* there is a $\mathrm{PC}_{\mathbb{F}_2,\ \mathbb{N}}$ derivation of $(1, w)$ from F

via an algorithmic procedure

Remark.

F is a generic set of polynomials, it might be encoding a CNF formula or not

$w \in \mathbb{N}$

F : multiset of weighted polynomials over $\mathbb{F}_2$

α : assignment

Theorem (soundness)

If there is a $\mathrm{PC}_{\mathbb{F}_2,\,\mathbb{Z}}$ derivation of $(1,w)$ from F, *then* $\min_{\alpha} \mathrm{cost}(F, \alpha) \geq w$

Why the substitution rules preserve the cost?

$$\frac{(p, w)}{(pq, w) \quad (p(q+1), w)}\ \text{SPLIT}$$

$$\frac{(p, w) \quad (q, w)}{(p+q, w) \quad (pq, 2w)}\ \text{SUM}$$

Remark.

With weights in $\mathbb{Z}$ the rules are <u>redundant</u>

$w \in \mathbb{Z}$
$p, q \in \mathbb{F}_2[x_1, \ldots, x_n]$

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Proof idea (similar to **[BLM07]**)

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~**Definition** A set S of polynomials is ***saturated*** w.r.t a variable x, if no SPLIT or SUM can be applied to get new *non-trivial* polynomials without the x

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Proposition

If a set S is saturated w.r.t. x, *then* setting $x = 0$ or $x = 1$ satisfies all the polynomials in S containing x

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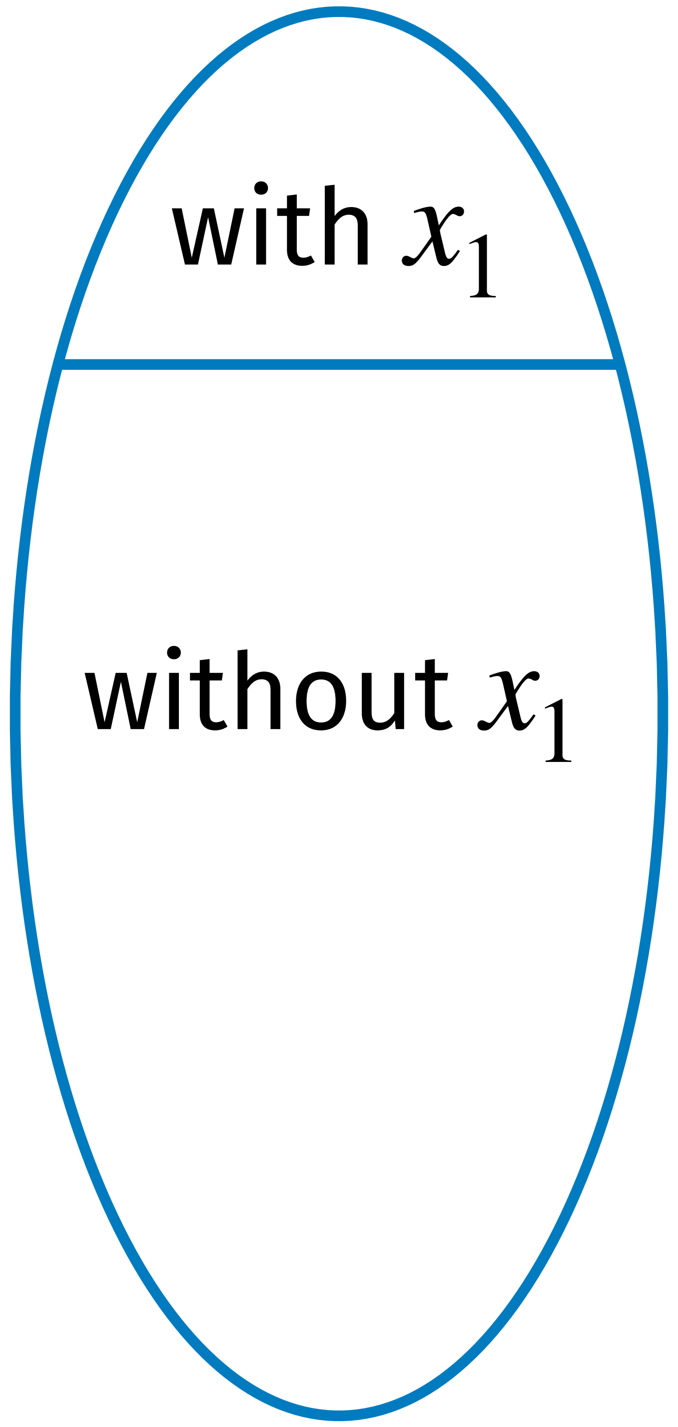

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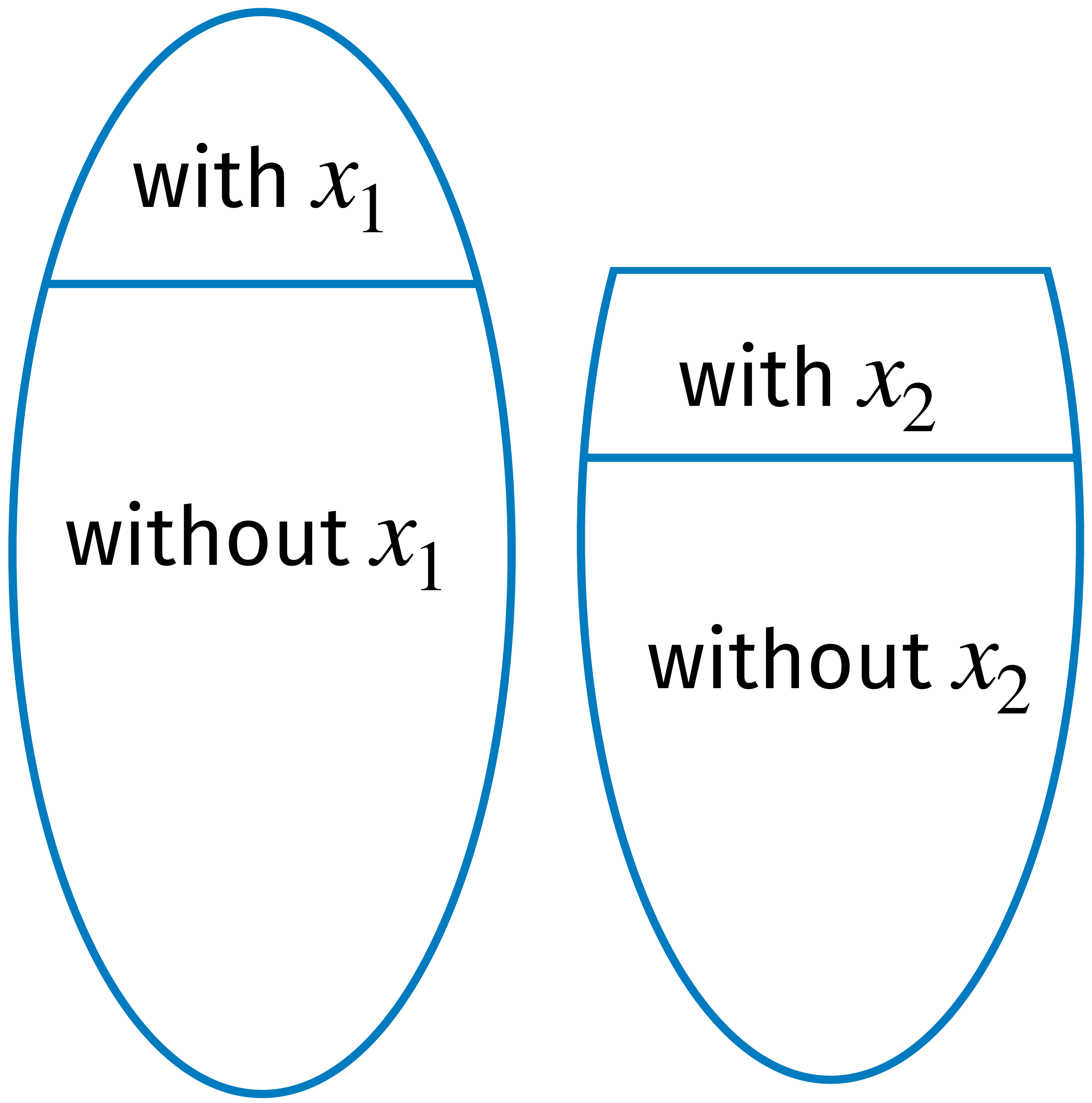

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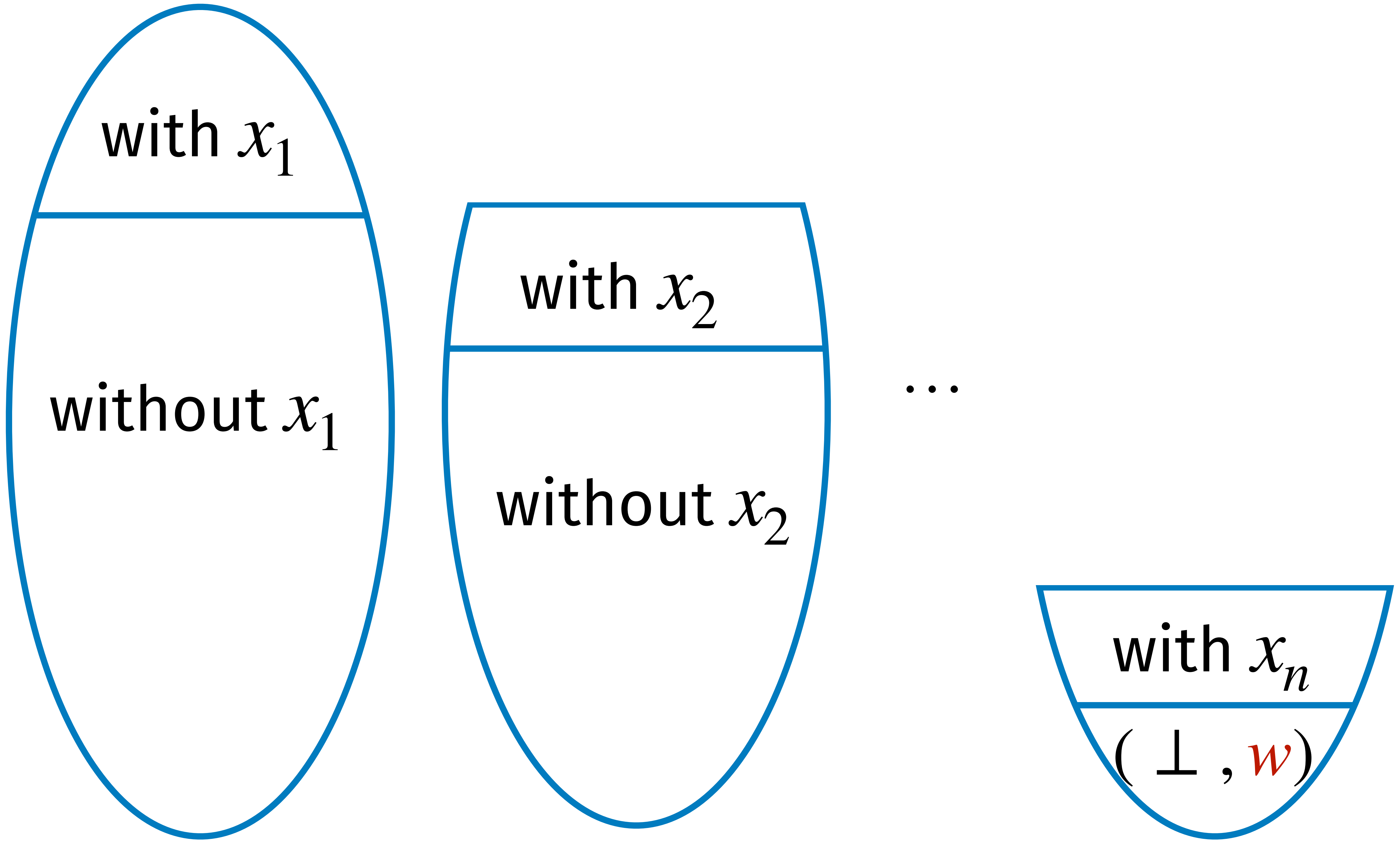

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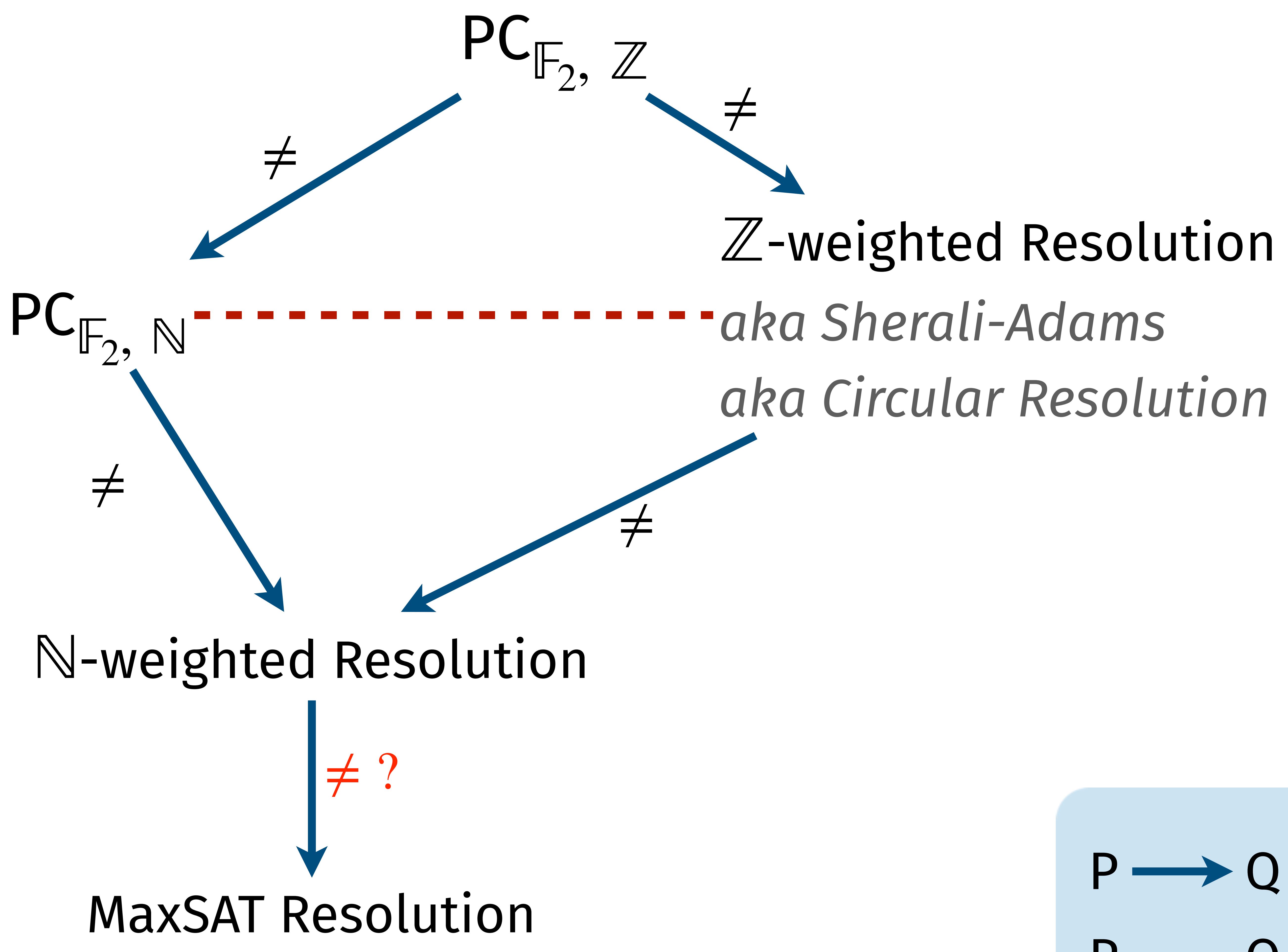
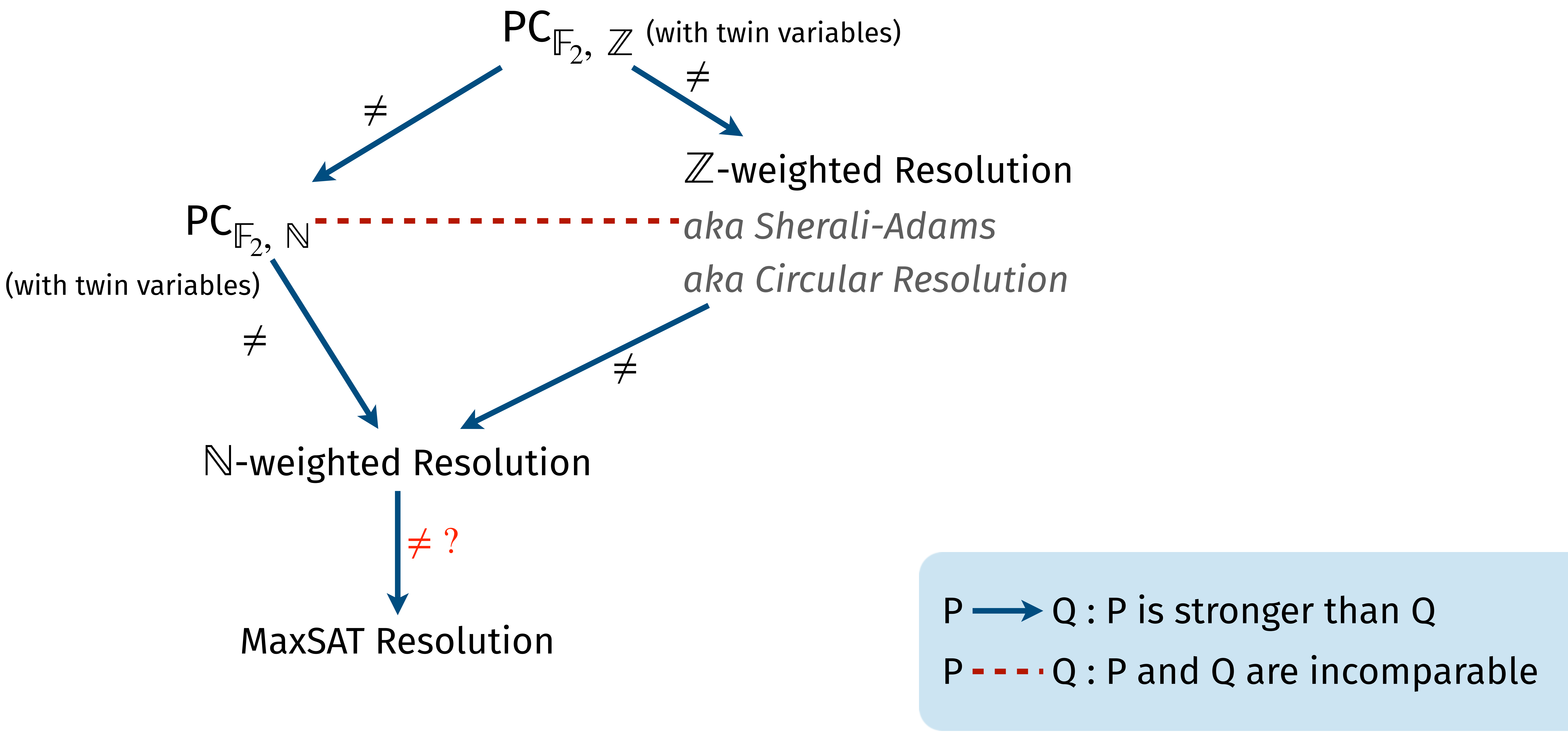


P $\longrightarrow$ Q : P is stronger than Q

P - - - - Q : P and Q are incomparable

(Unary versions of the systems omitted)



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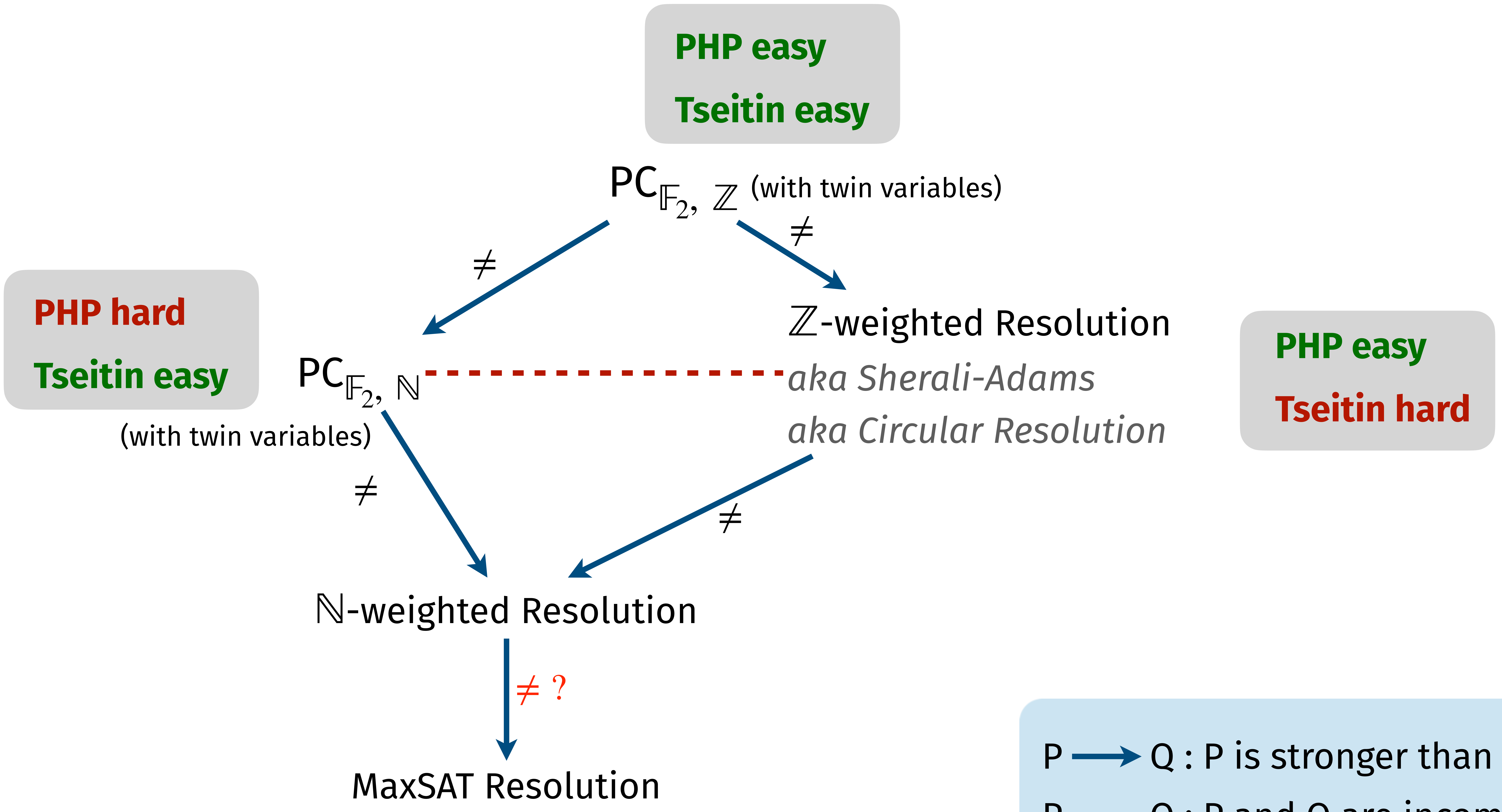

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Open problems

- Is $\mathrm{PC}_{\mathbb{F}_2,\mathbb{Z}}$ degree-automatable? I.e. can we find $\mathrm{PC}_{\mathbb{F}_2,\mathbb{Z}}$ proofs of degree d in time $n^{O(d)}$?
- Hardness results on $\mathrm{PC}_{\mathbb{F}_2,\mathbb{Z}}$?
- Is Polynomial Calculus on $\{\pm 1\}$-variables stronger than Resolution?
- Is $\mathbb{N}$-weighted Resolution equivalent to MaxSAT Resolution?

Thanks!

This presentation

Article relative to this talk

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