

Polynomial Calculus for MaxSAT

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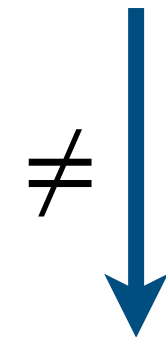


Alghero, July 7 2023 (SAT)

This presentation

SAT

Polynomial Calculus

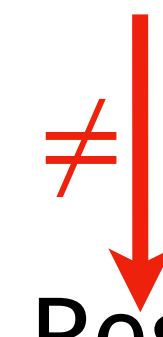


Resolution



MaxSAT

Polynomial Calculus for MaxSAT



MaxSAT Resolution
Weighted Resolution

Why?

Polynomials are more expressive than clauses

Natural/non-Boolean encodings of CSPs

Sometimes useful in practice (coloring, multiplier circuits)

$P \longrightarrow Q$: P is stronger than Q

weighted MaxSAT

input $F = \{(C_1, w_1), \dots, (C_m, w_m)\}$ a multiset

goal find $\min_{\alpha} \text{cost}(F, \alpha)$

$\text{cost}(\{(C_1, w_1), \dots, (C_m, w_m)\}, \alpha) := \text{sum of all } w_i \text{ s.t. } \alpha \text{ falsifies } C_i$

C_i : clauses

w_i : weights (in \mathbb{Z})

α : truth assignment

F_0 (the input, a multiset of weighted clauses)

F_1

F_2

\vdots

$F_{\ell-1}$

$F_\ell = \{(\perp, w), (D_1, w_1), \dots, (D_s, w_s)\}$

(★) for every i and every α ,
 $\text{cost}(F_i, \alpha) = \text{cost}(F_{i-1}, \alpha)$

Theorem

If $\{D_1, \dots, D_s\}$ is satisfiable and (★), then $\min_{\alpha} \text{cost}(F_0, \alpha) = w$

Theorem

If $w_1 \geq 0, \dots, w_s \geq 0$ and (★), then $\min_{\alpha} \text{cost}(F_0, \alpha) \geq w$

\perp : empty clause

D_i : clauses

F_i : multisets of
weighted clauses

α : truth assignment

\mathbb{Z} -weighted Resolution

Substitution rules to ensure that for every α , $\text{cost}(F_i, \alpha) = \text{cost}(F_{i-1}, \alpha)$

$$\frac{F_{i-1} \quad (C \vee x, w) \quad (C \vee \bar{x}, w)}{F_i \quad (C, w)}$$

or

$$\frac{F_{i-1} \quad (C, w)}{F_i \quad (C \vee x, w) \quad (C \vee \bar{x}, w)}$$

or

$$\frac{F_{i-1}}{F_i \quad (C, w) \quad (C, -w)}$$

+ natural “structural” rules

$w \in \mathbb{Z}$

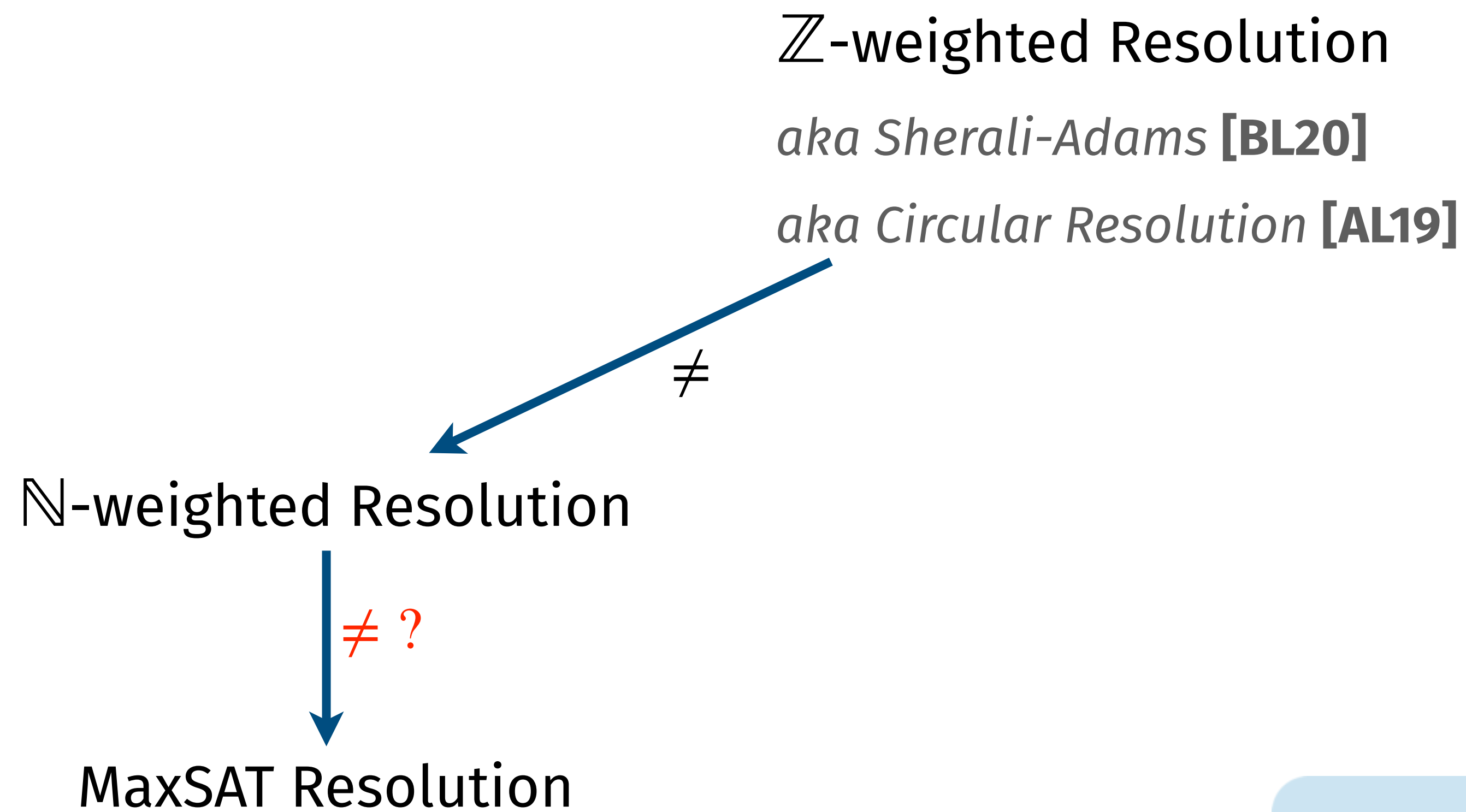
C : clause

x : variable

$\frac{\text{premises}}{\text{conclusions}}$: substitution rules

α : truth assignment

F_i : multisets of weighted clauses



$P \longrightarrow Q : P$ is stronger than Q

(Unary versions of the systems omitted)

weighted MaxSAT on polynomials

input $F = \{(p_1, w_1), \dots, (p_m, w_m)\}$ a multiset

goal find $\min_{\alpha} \text{cost}(F, \alpha)$

$\text{cost}(\{(p_1, w_1), \dots, (p_m, w_m)\}, \alpha) := \text{sum of all } w_i \text{ s.t. } p_i(\alpha) \neq 0$

Example. Maxcut

$G = (V, E)$ a graph

Boolean variables x_v for each $v \in V$

$F = \{(x_v + x_w + 1, 1) : (v, w) \in E\}$

$$x_v + x_w + 1 \in \mathbb{F}_2[x_v, x_w]$$

p_i : polynomials in $\mathbb{F}[x_1, \dots, x_n]$

w_i : weights (in \mathbb{Z})

$\alpha : \{x_1, \dots, x_n\} \rightarrow \mathbb{F}$ assignment

F_0 (the input, a multiset of weighted polynomials)

F_1

F_2

\vdots

$F_{\ell-1}$

$F_\ell = \{(\textcolor{red}{1}, \textcolor{blue}{w}), (p_1, w_1), \dots, (p_s, w_s)\}$

(★) for every i and every α ,
 $\text{cost}(F_i, \alpha) = \text{cost}(F_{i-1}, \alpha)$

Theorem

If $\{p_1, \dots, p_s\}$ have a common zero and (★), then $\min_{\alpha} \text{cost}(F_0, \alpha) = w$

Theorem

If $w_1 \geq 0, \dots, w_s \geq 0$ and (★), then $\min_{\alpha} \text{cost}(F_0, \alpha) \geq w$

$\textcolor{red}{1}$: the polynomial 1

p_i : polynomials

F_i : multisets of
weighted polynomials

α : truth assignment

weighted Polynomial Calculus over \mathbb{F}_2

Substitution rules to ensure that for every α , $\text{cost}(F_i, \alpha) = \text{cost}(F_{i-1}, \alpha)$

$$\frac{F_{i-1} \quad (p, w)}{F_i \quad (pq, w) \quad (p(q+1), w)}$$

or

$$\frac{F_{i-1} \quad (p, w) \quad (q, w)}{F_i \quad (p+q, w) \quad (pq, 2w)}$$

or

$$\frac{F_{i-1}}{F_i \quad (p, w) \quad (p, -w)}$$

+ natural “structural” rules

e.g. identify the polynomials p^2 and p

$$w \in \mathbb{Z}$$

$$p, q \in \mathbb{F}_2[x_1, \dots, x_n]$$

$\frac{\text{premises}}{\text{conclusions}}$: substitution rules

α : assignment

F_i : multisets of weighted polynomials

$$\begin{array}{c}
 (xz + y, 1) \\
 \hline
 ((xz + y)z, 1) \quad ((xz + y)(z + 1), 1) \\
 \hline
 (xz + yz, 1) \quad (yz + y, 1)
 \end{array}$$

$$\begin{array}{c}
 (x + y, 1) \quad (x + z, 1) \\
 \hline
 (y + z, 1) \quad ((x + z)(x + y), 2) \\
 \hline
 (y + z, 1) \quad (x + xy + xz + yz, 2)
 \end{array}$$

$$\frac{(p, w)}{(pq, w) \quad (p(q + 1), w)} \text{ SPLIT}$$

$$\frac{(p, w) \quad (q, w)}{(p + q, w) \quad (pq, 2w)} \text{ SUM}$$

coefficients of the polynomials

		Weights	
		\mathbb{N}	\mathbb{Z}
\mathbb{F}_2		$\text{PC}_{\mathbb{F}_2, \mathbb{N}}$	$\text{PC}_{\mathbb{F}_2, \mathbb{Z}}$
\mathbb{F}_q		Not in this talk	

Also **not** in this talk:

- weights in *unary* or *binary*
- encodings (e.g. twin variables)
- how to deal with hard constraints

Theorem (soundness)

If there is a $\text{PC}_{\mathbb{F}_2, \mathbb{Z}}$ derivation of $(1, w)$ from F , then $\min_{\alpha} \text{cost}(F, \alpha) \geq w$

Theorem (completeness)

If $\min_{\alpha} \text{cost}(F, \alpha) = w$, then there is a $\text{PC}_{\mathbb{F}_2, \mathbb{N}}$ derivation of $(1, w)$ from F
via an algorithmic procedure

Remark.

F is a generic set of polynomials,
it might be encoding a CNF
formula or not

$w \in \mathbb{N}$

F : multiset of weighted polynomials over \mathbb{F}_2

α : assignment

Theorem (soundness)

If there is a $\text{PC}_{\mathbb{F}_2, \mathbb{Z}}$ derivation of $(1, w)$ from F , then $\min_{\alpha} \text{cost}(F, \alpha) \geq w$

Why the substitution rules preserve the cost?

$$\frac{(p, w)}{(pq, w) \quad (p(q+1), w)} \text{ SPLIT}$$

$$\frac{(p, w) \quad (q, w)}{(p+q, w) \quad (pq, 2w)} \text{ SUM}$$

Remark.

With weights in \mathbb{Z} the rules are redundant

$$w \in \mathbb{Z}$$
$$p, q \in \mathbb{F}_2[x_1, \dots, x_n]$$

Theorem (completeness)

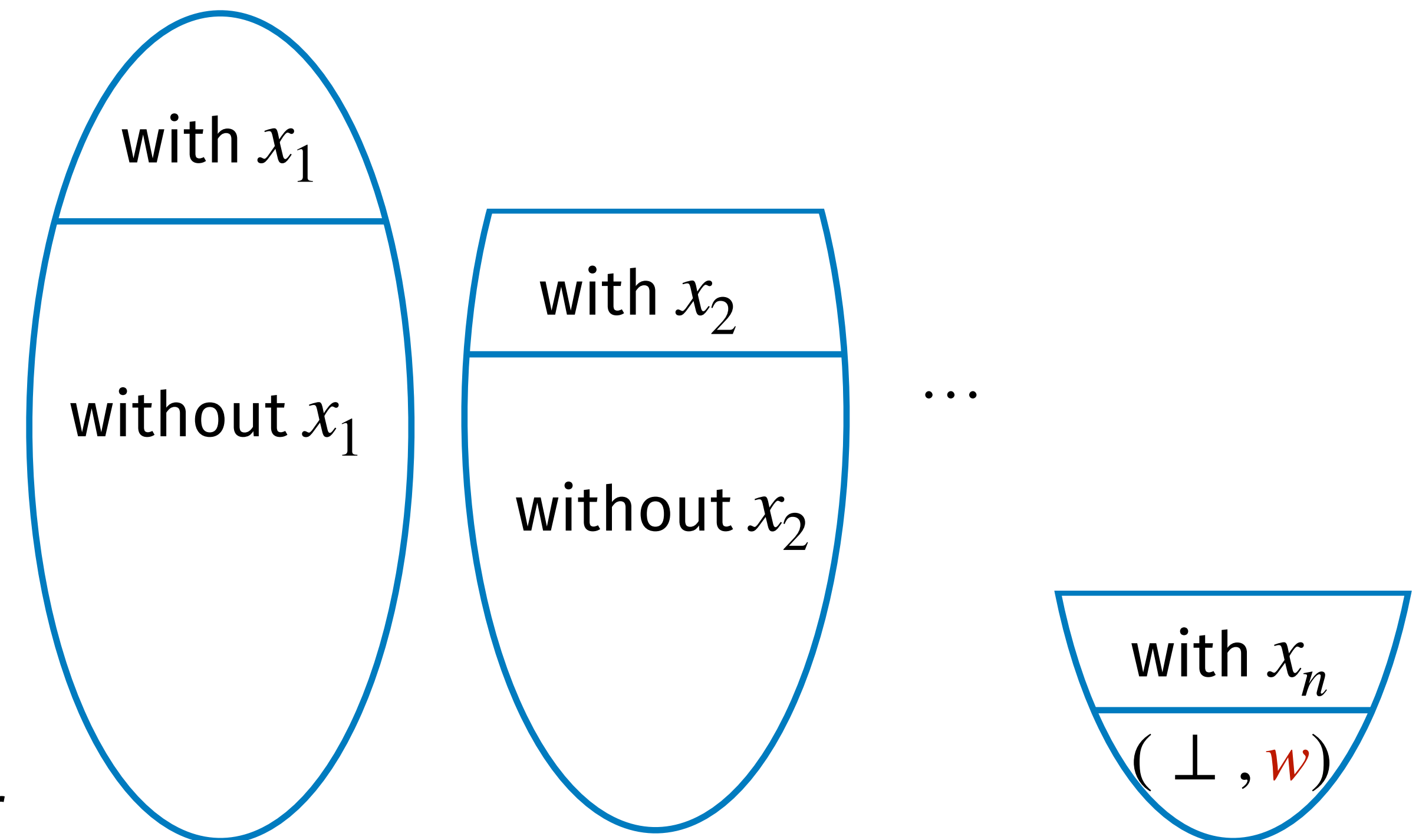
If $\min_{\alpha} \text{cost}(F, \alpha) = w$, then there is a $\text{PC}_{\mathbb{F}_2, \mathbb{N}}$ derivation of $(1, w)$ from F

Proof idea (similar to [BLM07])

~**Definition** A set S of polynomials is **saturated** w.r.t a variable x , if no SPLIT or SUM can be applied to get new *non-trivial* polynomials without the x

Proposition

If a set S is saturated w.r.t. x , then setting $x = 0$ or $x = 1$ satisfies all the polynomials in S containing x



PHP easy
Tseitin easy

$PC_{\mathbb{F}_2, \mathbb{Z}}$ (with twin variables)

\neq

\neq

\mathbb{Z} -weighted Resolution

aka Sherali-Adams

aka Circular Resolution

PHP hard
Tseitin easy

$PC_{\mathbb{F}_2, \mathbb{N}}$

(with twin variables)

\neq

\neq

\mathbb{N} -weighted Resolution

$\neq ?$

MaxSAT Resolution

PHP easy
Tseitin hard

$P \longrightarrow Q$: P is stronger than Q

$P \cdots Q$: P and Q are incomparable

(Unary versions of the systems omitted)

Open problems

- Is $\text{PC}_{\mathbb{F}_2, \mathbb{Z}}$ degree-automatable? I.e. can we find $\text{PC}_{\mathbb{F}_2, \mathbb{Z}}$ proofs of degree d in time $n^{O(d)}$?
- Hardness results on $\text{PC}_{\mathbb{F}_2, \mathbb{Z}}$?
- Is Polynomial Calculus on $\{\pm 1\}$ -variables stronger than Resolution?
- Is \mathbb{N} -weighted Resolution equivalent to MaxSAT Resolution?

Thanks!



This presentation



Article relative to this talk

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