Chapter 1

ELEMENTARY SET THEORY

1.1 Introduction

Set theory is a natural choice of a field where students can first become acquainted with an axiomatic development of a mathematical discipline.
The central concept in this chapter revolves around a set, which is simply a collection, group, conglomerate, aggregate of objects.
All fundamental tools of elementary set theory as needed in mathematics and elsewhere in the sciences and social sciences are included in detailed exposition in this chapter.

Objectives
At the end of this lecture, you should be able to:
• Define a set.
• Carry out some set operations.
• Apply set theory in solving some practical problems.

1.2 Rudiments of Set Theory

Definition 1.1 *A **set** is any well-defined `collection, group, aggregate, class` or `conglomerate` of objects.*

These objects (which may be cities, years, numbers, letters, or anything else) are called

elements of the set, and are often said to be **members** of the set.
A set is often specified by

- ⊙ listing its elements inside a pair of braces or curly brackets or parentheses
- ⊙ means of a property of its elements.

Example 1.1

The set whose elements are the first six letters of the alphabet is written

$$\{a, b, c, d, e, f\}$$

Example 1.2

The set whose elements are the even integers between 1 and 11 is written

$$\{2, 4, 6, 8, 10\}$$

We can also specify a set by giving a description of its elements (without actually listing the elements).

Example 1.3

The set $\{a, b, c, d, e, f\}$ can also be written

$$\{The\ first\ six\ letters\ of\ the\ alphabet\}$$

Example 1.4

The set $\{2, 4, 6, 8, 10\}$ can also be written

$$\{all\ even\ integers\ between\ 1\ and\ 11\}$$

1.2.1 Notation and Terminology

For convenience, we usually denote sets by capital letters of the alphabet A, B, C, and so on. We use lowercase letters of the alphabet to represent elements of a set.
For a set A, we write $x \in A$ if x is a member of A or belongs to A. We write $x \notin A$ to mean that x is not a member of A or does not belong to A.

Example 1.5

If $\mathbb{E}$ denotes the set of even integers, then $4 \in \mathbb{E}$ but $7 \notin \mathbb{E}$.

Definition 1.2 *An **empty set** is a set with no elements.*

An empty set is usually denoted by $\emptyset$. It is a set that arises in a variety of guises.

Example 1.6

Let $A = \{x : x \text{ is a real number and } x^2 < 0\}$. Clearly A has no elements since the square of any real number is non-negative.

Example 1.7

Let $B = \{People\ taller\ than\ the\ Eiffel\ Tower\ inFrance\}$. It is clear that B is empty.

Definition 1.3 *Let A and B be two sets. If every element of A is an element of B, we say that A is a **subset** of B, and we write $A \subseteq B$. We also say that A is **contained** in B.*

Definition 1.4 *If $A \subseteq B$ and $B \subseteq A$, then we say that A and B are **equal**, and write $A = B$. Two sets A and B are said to be **equal** if and only if they contain exactly the same elements. This is called the **Principle of Extensionality**.*

Theorem 1.1 *Let A and B be sets. If every element of A is an element of B and every element of B is an element of A, then $A = B$.*

Note that neither order nor repetition is of importance or relevance for a general set. Consequently, we find, for example that $\{1, 2, 3\} = \{2, 2, 1, 3\} = \{1, 2, 1, 3, 1\}$.

Example 1.8

The following sets, although described differently, are equal.

1. The set of all real numbers such that $x^2 + 3x + 2 = 0$.
2. The set of all integers such that $1 \leq x < 3$.
3. The set containing exactly the natural numbers 1 and 2.

If S and T are two sets that do not contain exactly the same elements, then we say that the sets are **unequal** and we write $S \neq T$.

Definition 1.5 *If $A \subseteq B$ and $A \neq B$, we say that A is a* ***proper subset*** *of B, or A is* ***properly contained*** *in B, and write $A \subset B$.*

We also write $B \supseteq A$ instead of $A \subseteq B$ and $B \supset A$ instead of $A \subset B$.

Remark 1.1

Note that since the empty set $\emptyset$ has no elements, every element in $\emptyset$ is also in any given set A. Hence $\emptyset \subseteq A$. By the definition of subset, every set is a subset of itself. That is, for any set A we have $A \subseteq A$.

Lemma 1.2 *(**Uniqueness of the Empty Set**). There exists only one set with no elements.*

Proof. Assume A and B are sets with no elements. Then every element of A is an element of B (since A has no elements). Similarly, every element of B is an element of A (since B has no elements). Therefore, $A = B$, by the Principle of Extensionality. $\square$

Definition 1.6 *(**Cardinality of a Set**). The number of elements in a set A is called the* ***cardinality*** *of A, and is denoted $n(A)$ or $|A|$.*

Note that cardinality of a set is always a non-negative integer or infinity. A set with one element is called a **singleton set**. A set A is said to be **finite** if $n(A) < \infty$. A set A is said to be **infinite** if $n(A) = \infty$.
Note that $n(\emptyset) = 0$.

Definition 1.7 *(**Universal Set**) A* ***Universal set*** *$\mathcal{U}$ is a set which contains all elements under consideration. It is also called the* ***universe of discourse*** *or simply* ***universe****.*

Example 1.9

(a). If one considers the set of men and women, then a universal set is probably the set of human beings.
(b). If one considers sets such as pigs, cows, chickens, or horses, the universal set is probably the set of animals.

(c). If $A = \{1, 2, 5\}$ and $B = \{4, 7, 9\}$, then a universal set is probably $\mathcal{U} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Note that a universal set is not unique, unless specified.

1.2.2 Fundamental Operations on Sets

We introduce simple set-theoretic operations on sets and prove some of their properties. Given two or more sets, we can form a new set using these operations.

1. **Complement of a Set**

Let $\mathcal{U}$ be the universal set and let A be any set. The **complement** of A, written $A^{\complement}$ or sometimes $\overline{A}$ is defined as

$$A^{\complement} = \Big\{x \in \mathcal{U} : x \notin A\Big\}$$

Example 1.10

Let the universal set be $\mathcal{U} = \{0, 1, 2, 3, 5, 6\}$ and $A = \{3, 5\}$. Clearly, $A^{\complement} = \{0, 1, 2, 6\}$.

2. **Union of Sets**

Let A and B be sets. The **union** of A and B, denoted by $A \cup B$ is

$$A \cup B = \Big\{x : x \in A \textit{ or } x \in B \textit{ or both}\Big\}$$

More generally, if $A_1, A_2, ..., A_n$ are sets, then their union is the set of all objects which belong to at least one of them, and is denoted by

$$A_1 \cup A_2 \cup \cdots \cup A_n$$

or by

$$\bigcup_{i=1}^{n} A_i$$

This is the set of elements which belong to at least one A_i, $i = 1, 2, ..., n$.

Example 1.11

(a). If $A = \{2, 5, 7\}$ and $B = \{Tom, Bush, Mary\}$, then $A \cup B = \{2, 5, 7, Tom, Bush, Mary\}$.
(b). If $A_1 = \{x, y, t, s\}, A_2 = \{q, r, f\}, A_3 = \{0, 1, 3, 4, 5, 6, 7, 8, 20\}$, then

$$A_1 \cup A_2 \cup A_3 = \{x, y, t, s, q, r, f, 0, 1, 3, 4, 5, 6, 7, 8, 20\}$$

3. **Intersection of Sets**

Let A and B be sets. The **intersection** of A and B, denoted by $A \cap B$ is defined as

$$A \cap B = \Big\{x : x \in A \ and \ x \in B\Big\}$$

In general, if $A_1, A_2, ..., A_n$ are sets, then their intersection denoted by

$$A_1 \cap A_2 \cap A_n$$

or

$$\bigcap_{i=1}^{n} A_i$$

is given by

$$\bigcap_{i=1}^{n} A_i = \Big\{x : x \in A_i \ for \ each \ A_i, \quad n = 1, 2, ..., n\Big\}$$

Clearly, this is the set of elements which belong to all A_i, $i = 1, 2, ..., n$.

Example 1.12

(a). If $A = \{2, 5, 8\}$ and $B = \{0, 2, 6, 9\}$, then $A \cap B = \{2\}$.
(b). If $A_1 = \{x, y, t, s\}, A_2 = \{q, r, f, s, x\}, A_3 = \{0, 1, 3, 4, 5, 6, 7, 8, 20, x, f, s\}, A_4 = \{x, s, z, w\}$, then

$$A_1 \cup A_2 \cup A_3 \cap A_4 = \{x, s\}$$

(c). If $A = \{3, 1, 2, 6\}$ and $B = \{4, 7\}$, then $A \cap B = \emptyset$.

Definition 1.8 *Two sets A and B are said to be **disjoint** if they do not have a member in common. That is, $A \cap B = \emptyset$. If this is the case, we say that A and B do not intersect. If $A \cap B \neq \emptyset$, we say that A and B intersect.*

4. **Set Difference**

Let A and B be sets. The **set difference** or relative complement of A with respect to B, denoted by $A - B$ is defined as

$$A - B = \Big\{x : x \in A \ and \ x \notin B\Big\}$$

Example 1.13

(a). If $A = \{1, 2, 3, 5, 6, 7\}$ and $B = \{3, 5, 9\}$, then $A - B = \{1, 2, 6, 7\}$ and $B - A = \{9\}$.
(b). If

$$A = \{NewYork, Cairo, Mumbai, Seoul, Beijing, Moscow, London\}$$

and

$$B = \{Nairobi, Kigali, Pretoria, Beijing, Harare, Paris, London\},$$

then

$$A - B = \{NewYork, Cairo, Mumbai, Seoul, Moscow\}$$

and

$$B - A = \{Nairobi, Kigali, Pretoria, Harare, Paris\}$$

Clearly, if $A - B = \emptyset$ and $B - A = \emptyset$, then $A = B$.
It is easy to verify that $A - B = A \cap B^{\complement}$.
Note that $A^{\complement} = \mathcal{U} - A$.

5. **Symmetric Difference of Two Sets**

Let A and B be sets. The symmetric difference of A and B, denoted by $A \triangle B$ is defined as

$$A \triangle B = \Big\{x : x \in A \ or \ x \in B, \ but \ not \ both\Big\}$$

Clearly,

$$\begin{aligned} A \triangle B &= \Big\{x : x \in A \ or \ x \in B, \ but \ not \ both\Big\} \\ &= \Big\{x : x \in A \ or \ x \in B, \ and \ x \notin A \cap B\Big\} \\ &= \Big\{x : x \in A \cup B, \ and \ x \notin A \cap B\Big\} \\ &= \Big\{x : x \in (A \cup B) - (A \cap B)\Big\} \\ &= (A - B) \cup (B - A). \\ &= (A \cap B^{\complement}) \cup (B \cap A^{\complement}) \end{aligned}$$

The symmetric difference of two sets is also called the **Boolean sum** of the two sets.

Example 1.14

If $A = \{2, 1, 3, 5\}$ and $B = \{x, t, 7, 1\}$, then $A \cup B = \{1, 2, 3, 5, x, t, 7\}$ and $A \cap B = \{1\}$. Therefore,

$$A \triangle B = \Big\{2, 3, 5, x, t, 7\Big\}$$

6. **Cartesian Product of Sets**

Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$ is defined as

$$A \times B = \Big\{(a, b) : a \in A \ and \ b \in B\Big\}$$

More generally, the Cartesian product of n sets $A_1, A_2, ..., A_n$ is defined as

$$A_1 \times A_2 \times \cdots \times A_n = \Big\{(a_1, a_2, ..., a_n) : a_i \in A_i, \ i = 1, 2, 3, ..., n\Big\}$$

The expression $(a_1, a_2, ..., a_n)$ is called an `ordered n-tuple`.

Example 1.15

If $A = \{0, 1, 2\}$ and $B = \{a, b\}$, then

$$A \times B = \Big\{(0, a), (0, b), (1, a), (1, b), (2, 1), (2, b)\Big\}$$

$$B \times A = \Big\{(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2)\Big\}$$

$$A \times A = \Big\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\Big\}$$

Example 1.16

Let $\mathbb{R}$ be the set of real numbers. Then the Cartesian product

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &= \Big\{(x, y) : x, y \in \mathbb{R}\Big\} \\ &= \mathbb{R}^2 \end{aligned}$$

or the 2-dimensional Cartesian plane or the xy-plane.
The Cartesian product

$$\begin{aligned} \mathbb{R} \times \mathbb{R} \times \mathbb{R} &= \Big\{(x, y, z) : x, y, z \in \mathbb{R}\Big\} \\ &= \mathbb{R}^3 \end{aligned}$$

is the ordinary space of elementary geometry or the 3-dimensional Euclidean space.
In general, the product $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, x_2, ..., x_n) : x_i \in \mathbb{R}\}$ which is the set of n-tuples of real numbers, is called the n-dimensional Euclidean space.

Remark 1.2

Clearly, the elements of a set may themselves be sets. A special class of such sets is the `power set`.

Definition 1.9 *Let A be a given set. The* ***power set*** *of A denoted by $\mathcal{P}(A)$, is a family of sets such that if $X \subseteq A$, then $X \in \mathcal{P}(A)$. Symbolically, $\mathcal{P}(A) = \Big\{X : X \subseteq A\Big\}$. In other words, the power set of A is the collection of all subsets of A.*

Example 1.17

If $A = \{1, 2\}$, then $\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{1, 2\}\Big\}$.
For any set A with cardinality $n(A) = n \geq 0$, the power set has 2^n elements. For any $0 \geq k \geq n$, there are $\binom{n}{k}$ subsets of size k. Suppose A has n elements. Counting the subsets of A according to the number k of elements in a set, we have the combinatorial identity

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad for\ n \geq 0$$

1.3 Laws of Set Theory

For any sets A, B and C taken from a universal set $\mathcal{U}$:

1. $\left(A^{\complement}\right)^{\complement} = A$ `Law of Double Complement or Involution`

2. $(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$ `De Morgan's Laws`
$(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$

3. $A \cup B = B \cup A$ `Commutative Laws`
$A \cap B = B \cap A$

4. $A \cup (B \cup C) = (A \cup B) \cup C$ `Associative Laws`
$A \cap (B \cap c) = (A \cap B) \cap C$

5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ `Distributive Laws`
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6. $A \cup A = A$ `Idempotence Laws`
$A \cap A = A$

7. $A \cup \emptyset = A$ `Identity Laws`
$A \cap \mathcal{U} = A$

8. $A \cup A^{\complement} = \mathcal{U}$ `Inverse Laws`
$A \cap A^{\complement} = \emptyset$

9. $A \cup \mathcal{U} = \mathcal{U}$ `Domination Laws`
$A \cap \emptyset = \emptyset$

10. $A \cup (A \cap B) = A$ `Absorption Laws`
$A \cap (A \cup B) = A$

1.3.1 Venn Diagrams

It is often useful a diagram called a **Venn diagram**(named after John Venn, a British Mathematician and philosopher (1834-1923))or sometimes **Euler diagram** (after Leonard Euler, who first introduced them) to visualize and prove some of the various properties of set operations. Venn diagrams are useful in many fields, including set theory, probability, logic, statistics and computer science.

In a Venn diagram, the universal set $\mathcal{U}$ is represented/depicted by the interior of a large rectangular area/region. Subsets within this universe are represented by interiors of circular areas/regions and wanted regions are to be shaded. For a set A, the region/area outside the cire for A represents $A^{\complement}$.

Set Operation	*Symbol*
1. *Set B is contained in A*	$B \subseteq A$

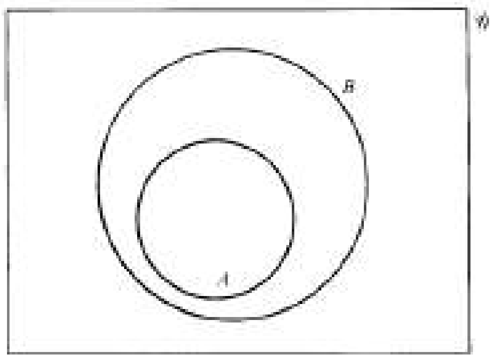


Figure 1.1: Venn diagram for $A \subset B$

$\underline{Set\ \ Operation}$	$\underline{Symbol}$
2. $Complement\ of\ set\ A$	A^{C}

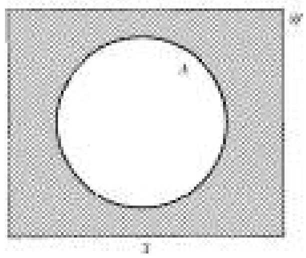


Figure 1.2: Venn diagram for A^{C}

$\underline{Set\ \ Operation}$	$\underline{Symbol}$
3. $Difference\ of\ A\ and\ B$	$A - B$

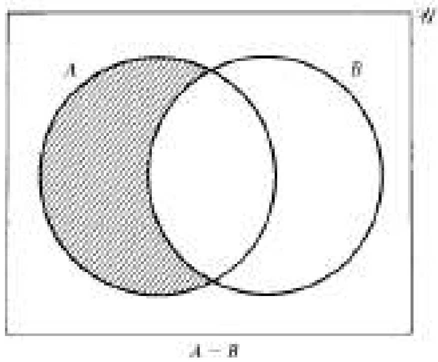


Figure 1.3: Venn diagram for $A - B$

Set Operation	*Symbol*
4. *Union of A and B*	$A \cup B$

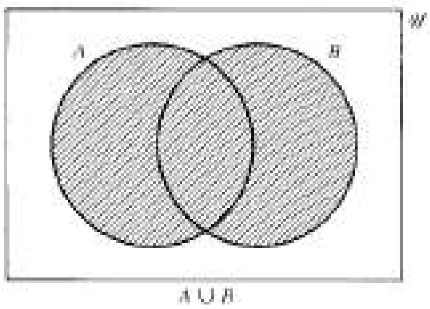


Figure 1.4: Venn diagram for $A \cup B$

Set Operation	*Symbol*
5. *Intersection of A and B*	$A \cap B$

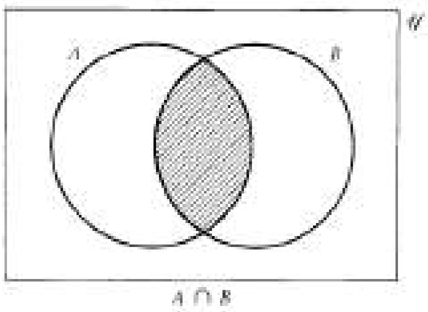


Figure 1.5: Venn diagram for $A \cap B$

Set Operation	*Symbol*
6. *Symmetric difference of A and B*	$A \triangle B$

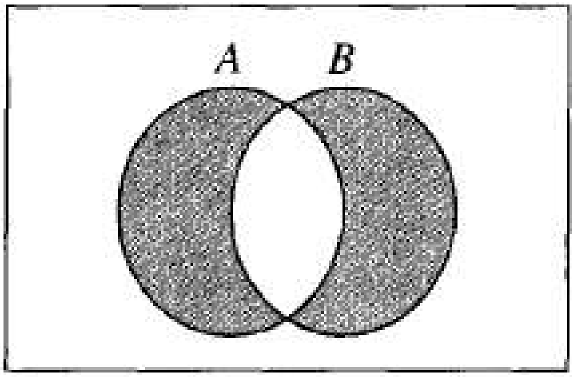


Figure 1.6: Venn diagram for $A \triangle B$

Set Operation	*Symbol*
7. *Intersection of two disjoint sets A and B*	

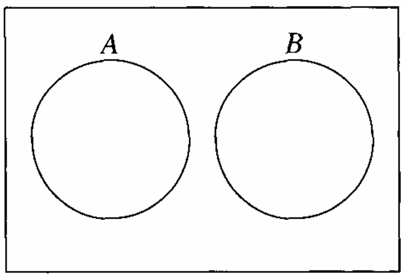


Figure 1.7: Venn diagram for $A \triangle B$

Set Operation	*Symbol*
8. *Union of two disjoint sets A and B*

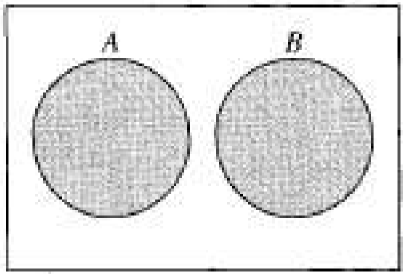


Figure 1.8: Venn diagram for $A \triangle B$

Example 1.18

De Morgan's laws can be established from Venn diagrams:

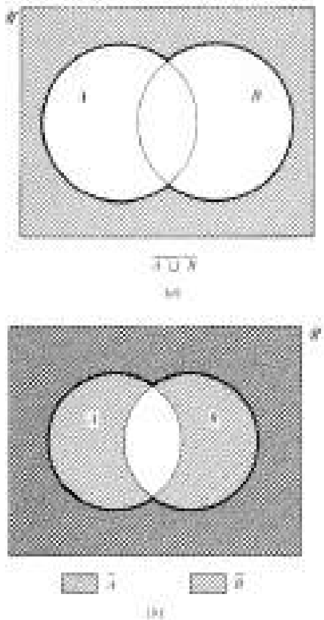


Figure 1.9: Venn diagram to prove that $(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$

Remark 1.3

The use of Venn diagrams may be appealing, especially to the reader who presently does not appreciate writing proofs. Venn diagrams may help us to understand certain mathematical situations, but when the number of sets involved exceeds three, the diagram could be difficult to draw.
The the `Elements Argument method`, which gives a detailed explanation is more rigorous than Venn diagrams techniques and is the more preferred method for proving results in set theory.

1.3.2 Elements Argument Method

Another way of to show that two sets are equal is to show that one of the sets is a subset of the other and vice versa. This method is known as the **Elements Argument** or **set membership method** or **formal proof**.

Example 1.19

Prove that $(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$ by the Elements Argument method.
Solution.
First, suppose $x \in (A \cap B)^{\complement}$. It follows that $x \notin (A \cap B)$. This implies that $x \notin A$ or $x \notin B$. Hence $x \in A^{\complement}$ or $x \in B^{\complement}$. Thus $x \in A^{\complement} \cup B^{\complement}$. This shows that

$$(A \cap B)^{\complement} \subseteq A^{\complement} \cup B^{\complement} \qquad (*)$$

Now suppose that $x \in A^{\complement} \cup B^{\complement}$. Then $x \in x \in A^{\complement}$ or $x \in x \in B^{\complement}$. It follows that $x \notin A$ or $x \notin B$. Hence, $x \notin A \cap B$. Therefore, $x \in (A \cap B)^{\complement}$. This demonstrates that

$$A^{\complement} \cup B^{\complement} \subseteq (A \cap B)^{\complement} \qquad (**)$$

From (*) and (**), we have equality. $\square$

Example 1.20 *Prove by the Elements Argument method that*
(a). $A \cup (B - A) = A \cup B$
(b). $\overline{A - (A \cap B)} = A \cap \overline{B}$
(c). $\overline{\Big((A \cap B) \cup C\Big)} = \overline{(A \cup C)} \cup \overline{(B \cup C)}$

Solution

(a). Let $x \in A \cup (B - A)$. Then $x \in A$ or $x \in B - A$. This implies that $x \in A$ or $x \in B$ and $x \notin A$. This means that $x \in A$ or $x \in B$ or $x \in A$ and $x \notin A$. Therefore $x \in A$ or $x \in B$. That is, $x \in A \cup B$. This proves that

$$A \cup (B - A) \subseteq A \cup B \qquad (*)$$

Conversely, suppose $x \in A \cup B$. Then $x \in A$ or $x \in B$. Equivalently, $x \in A$ or $x \in B - \emptyset$. This means that $x \in A$ or $x \in B \cup \emptyset$. Thus, $x \in A$ or $x \in B \cup (A - A)$. That is $x \in A$ or $x \in B - A$. This means that $x \in A \cup (B - A)$. Thus

$$A \cup B \subseteq A \cup (B - A) \qquad (**)$$

From (*) and (**), equality follows. $\square$

(b) and (c) are left as exercises.

Example 1.21 *Let A, B and C be sets. Prove that*
(a). $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(b). $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof.

(a). Suppose $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$. If $y \in B$, then $(x, y) \in A \times B$. If $y \in C$, then $(x, y) \in A \times C$. In either case, $(x, y) \in (A \times B) \cup (A \times C)$. Thus

$$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C) \qquad (*)$$

Conversely, suppose $(x, y) \in (A \times B) \cup (A \times C)$. If $(x, y) \in A \times B$, then $x \in A$ and $y \in B$, so $y \in B \cup C$ and hence $(x, y) \in A \times (B \cup C)$. If $(x, y) \in A \times C$, then $x \in A$ and $y \in B \cup C$, so again $(x, y) \in A \times (B \cup C)$. Thus

$$(A \times B) \cup (A \times C) \subseteq A \times (B \cup C) \qquad (**)$$

From (*) and (**), equality follows. $\square$

(b). $(x,y) \in A \times (B \cap C)$ if and only if $x \in A$ and $y \in B \cap C$
if and only if $x \in A$ and $y \in B$ and $y \in C$
if and only if $x \in A$ and $y \in B$ and $x \in A$ and $y \in C$
if and only if $(x,y) \in A \times B$ and $(x,y) \in A \times C$
if and only if $(x,y) \in (A \times B) \cap (A \times C)$. $\square$

1.4 Fundamental Counting Principle

Some quite complex mathematical results rely for their proofs on counting arguments: counting the numbers of elements of various sets, the number of ways in which a certain outcome can be achieved, etc. Although counting may appear to be a rather elementary exercise, in practice it can be extremely complex and rather subtle.

A counting problem is one that requires us to determine the number of elements in a set.

Lemma 1.3 ***(Counting Principle 1)*** *If A and B are disjoint sets, then*

$$|A \cup B| = |A| + |B|$$

In many applications, of course, more than two sets are involved. The above principle easily generalizes to the following, which can be proved formally using mathematical induction (see chapter 2).

Lemma 1.4 ***(Counting Principle 2)*** *If $A_1, A_2, ..., A_n$ are pairwise disjoint sets, then*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

Sometimes the sets whose elements are to be counted will not satisfy the rather stringent condition in Lemma 1.3 and Lemma 1.4.

Theorem 1.5 ***(Inclusion-Exclusion Principle)*** *If A and B are finite sets, then*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof We can divide or partition $A \cap B$ into subsets $A - B$, $A \cap B$ and $B - A$, which are disjoint and hence satisfy Counting Principle 2 in Lemma 1.4

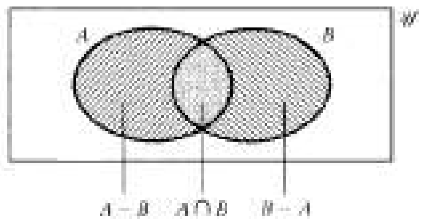


Figure 1.10: Venn diagram for partition of $(A \cup B)$

Therefore, by the Counting Principle 2 in Lemma 1.4,

$$|A \cup B| = |A - B| + |A \cap B| + |B - A| \tag{1.1}$$

The sets A and B can themselves be split into disjoint subsets $A - B, A \cap B$ and $B - A, A \cap B$, respectively. Thus

$$|A| = |A - B| + |A \cap B| \tag{1.2}$$

and

$$|B| = |B - A| + |A \cap B| \tag{1.3}$$

It is a simple exercise to combine (1.1), (1.2) and (1.3) to produce the required result. □

Remark 1.4

The Inclusion-Exclusion Principle is so called because to count the elements of $A \cup B$ we could have added the number of elements of A and the number of elements of B, in which case we have included the elements of $A \cap B$ twice: once as elements of A and once as elements of B. To obtain the correct number of elements in $A \cup B$, we would then need to exclude those in $A \cap B$ once, so that overall they are just counted once.

The Inclusion-Exclusion Principle can be extended to more than two sets.

Theorem 1.6 *If A, B and C are finite sets, then*

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

Theorem 1.7 *If A, B, C and D are finite sets, then*
$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D|$
$- |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D|$
$+ |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|.$

Remark 1.5

The general pattern should be evident:
Add the cardinalities of each of the sets, subtract the cardinalities of the intersections of all pairs of sets, ass the cardinalities of all intersections of the sets taken three at a time, subtract cardinalities of all intersections of the sets taken four at a time, and so on.
In general we have the generalized Inclusion-Exclusion Principle:

Theorem 1.8 ***(Generalized Inclusion-Exclusion Principle)*** *Given a finite number of finite sets $A_1, A_2, \cdots, A_n$, the number of elements in the union $A_1 \cup A_2 \cup \cdots \cup A_n$ is*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

where the first sum is over all i, the second sum is over all pairs i, j, with $i < j$, the third sum is over all triples i, j, k with $i < j < k$, and so forth.

1.4.1 Counting and Venn Diagrams

We can use Venn diagrams to solve counting problems.

Example 1.22 *In a class of* 50 *college freshmen,* 30 *are studying C^{++},* 25 *are studying Java, and* 10 *are studying both languages.*
(a). How many freshmen are studying either computer language?
(b). Determine $|A^{\complement} \cap B^{\complement}|$.

Solution
(a). We let the universal set be
$\mathcal{U} = \{class\ of\ 50\ freshmen\}$,
$A = \{students\ studying\ C^{++}\}$,
$B = \{students\ studying\ Java\}$
To answer the question we need $|A \cup B|$.
$|A| = 30, |B| = 25$ and $|A \cap B| = 10$ From the Inclusion-Exclusion Principle, we have that

$$|A \cup B| = |A| + |B| - |A \cap B| = 30 + 25 - 10 = 45$$

(b). By De Morgan's Law $A^{\mathsf{C}} \cap B^{\mathsf{C}} = (A \cup B)^{\mathsf{C}}$. Hence

$$|A^{\mathsf{C}} \cap B^{\mathsf{C}}| = |(A \cup B)^{\mathsf{C}}| = |\mathcal{U}| - |A \cup B| = |\mathcal{U}| - |A| - |B| + |A \cap B| = 50 - 30 - 25 + 10 = 5$$

Note that $|A^{\mathsf{C}} \cap B^{\mathsf{C}}|$ is the number of students who did not study any of the two languages. This problem can be solved using a venn diagram.

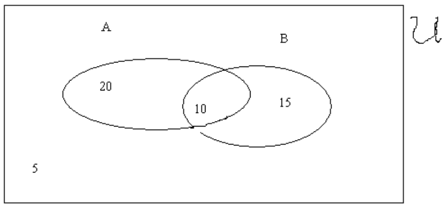


Figure 1.11: Venn diagram Example 1.20

Example 1.23 *In the year 2011, Fortune Magazine surveyed the presidents of the 500 largest corporations in the United States. Of these 500 people, 310 had degrees (of any sort) in business, 238 had undergraduate degrees in business, and 184 had postgraduate degrees in business.*
(a). How many presidents had both undergraduate and postgraduate degrees in business?
(b). How many presidents had no undergraduate and no postgraduate degree in business?
(c). How many presidents had undergraduate degree in business and no postgraduate degree in business?
(d). How many presidents had at most one degree?

Solution
Let the universal set be $\mathcal{U} = \{500 \ presidents\ of\ largest\ corporations\ in\ the\ US\}$
$S = \{presidents\ with\ an\ undergraduate\ degree\ in\ business\}$
$T = \{presidents\ with\ a\ postgraduate\ degree\ in\ business\}$
Then
$S \cup T = \{presidents\ with\ at\ least\ one\ degree\ in\ business\}$
$S \cap T = \{presidents\ with\ both\ undergraduate\ and\ postgraduate\ degrees\ in\ business\}$
We are given that

$$|S| = 238, \quad |T| = 184, \quad |S \cup T| = 310$$

(a). The problem asks for $|S \cap T|$. By the Inclusion-Exclusion Principle,

$$|S \cup T| = |S| + |T| - |S \cap T|$$

Thus $|S \cap T| = |S| + |T| - |S \cup T| = 238 + 184 - 310 = 112$.
That is,exactly 112 of the presidents had both undergraduate and postgraduate degrees in business.

(b). $|S^{\complement} \cap T^{\complement}| = |(S \cup T)^{\complement}| = |\mathcal{U}| - |S \cup T| = 500 - 310 = 190$.
That is, exactly 190 of the presidents had no undergraduate degree and no postgraduate degree in business.
The problem can be represented in A Venn diagram as below:

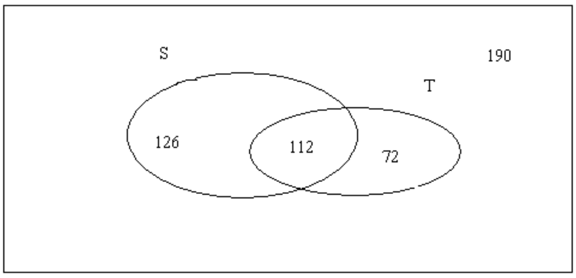


Figure 1.12: Venn diagram Example 1.21

(c). We need to find $|S \cap T^{\complement}|$. From the Venn diagram $|S \cap T^{\complement}| = 126$.

(d). At most one degree means either undergraduate or postgraduate degrees but not both. We need to find $|S \triangle T|$. Clearly,

$$|S \triangle T| = |(S-T) \cup (T-S)| = |(S \cup T) - (S \cap T)| = |(S \cup T)| - |(S \cap T)| = 310 - 112 = 98$$

This result can also be obtained from the Venn diagram by adding the two numbers 126 and 72.

Example 1.24 *Safaricom (Kenya Ltd) surveyed 400 of its customers to determine the way they learned about the new* `Jibambie` *tariff. The survey shows that 180 learned* *about the tariff from radio, 190 from television, 190 from newspapers, 80 from radio and television, 90 from radio and newspapers, 50 from television and newspapers, and 30 from all three forms of media.*

(a). Draw a Venn diagram to represent this information

Using your Venn diagram (together with the Inclusion-Exclusion Principle where need be), determine

(b). the number of customers who learned of the tariff from at least two of the three media.

(c). the number of customers who learned of the tariff from exactly one of the three media.

(d). the number of customers who did not learn of the tariff any of the three media.

Solution

(a). Let the universal set be

$\mathcal{U} = \{400 \ Safaricom \ customers\}$

$R = \{customers \ who \ learned \ about \ the \ tariff \ from \ Radio\}$

$T = \{customers \ who \ learned \ about \ the \ tariff \ from \ Television\}$

$N = \{customers \ who \ learned \ about \ the \ tariff \ from \ Newspapers\}$

We are given that

$$|\mathcal{U}| = 400, \ |R| = 180, \ |T| = 190, \ |N| = 190$$

and

$$|R \cap T| = 80, \ |R \cap N| = 90, \ |T \cap N| = 50, \ |R \cap T \cap N| = 30$$

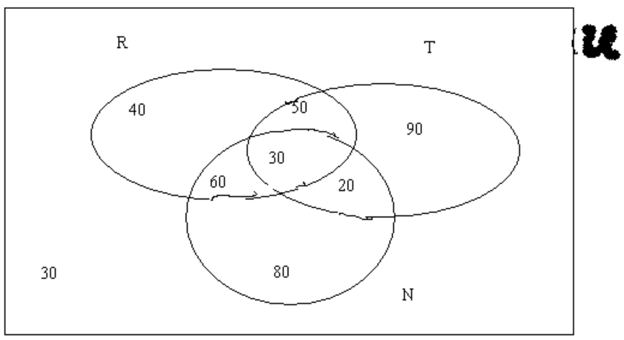


Figure 1.13: Venn diagram Example 1.22

b). At least two of the media means we add : 50,60, 20 and 30 to get 160. Thus exactly 160 customers learned of the tariff from at least two of the three media.

(c). Exactly one of the media means Radio only or Television only or Newspapers only. From the Venn diagram we have: $40 + 90 + 80 = 210$.

(d). We need $|(R \cup T \cup N)^{\complement}|$. But

$$|(R \cup T \cup N)^{\complement}| = |\mathcal{U} - (R \cup T \cup N)| = |\mathcal{U}| - |R \cup T \cup N| = 400 - 370 = 30.$$

Example 1.25 *Each of the 100 students in the first year of Open University's Computer Science Department studies at least one of the subsidiary subjects: mathematics, electronics and accounting. Given that 65 study mathematics, 45 study electronics, 42 study accounting, 20 study mathematics and electronics, 25 study mathematics and accounting, and 15 study electronics and accounting, find the number who study:*

(a) all three subsidiary subjects;

(b) mathematics and electronics but not accounting;

(c) only electronics as a subsidiary subject.

Solution

Let $\mathcal{U} = \{students\ in\ the\ first\ year\ of\ Utopias\ Computer\ Science\ Department\}$
$M = \{students\ studying\ mathematics\}$
$E = \{students\ studying\ electronics\}$
$A = \{students\ studying\ accounting\}$.
We are given the following information:

$$|\mathcal{U}| = 100, |M| = 65, |E| = 45, |A| = 42, |M \cap E| = 20, |M \cap A| = 25, |E \cap A| = 15.$$

Also, since every student takes at least one of three subjects as a subsidiary,
$\mathcal{U} = \mathrm{M} \cup E \cup A$.

Let $|M \cap E \cap A| = x$. Figure 1.14 shows the cardinalities of the various disjoint subsets of $\mathcal{U}$. These are calculated as follows, beginning with the innermost region representing $M \cap E \cap A$ and working outwards in stages.

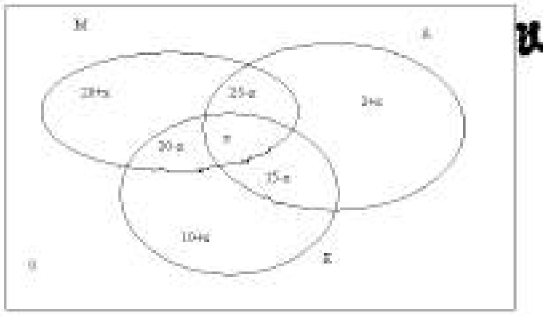


Figure 1.14: Venn diagram Example 1.22

By Counting Principle 1,

$$|M| = |M \cap A \cap E| + |(M \cap A) - E|$$

so

$$|(M \cap A) - E| = |M \cap A| - |M \cap A \cap E| = 25 - x.$$

Similarly

$$|(M \cap E) - A| = |M \cap E| - |M \cap E \cap A| = 20 - x$$

and

$$|(A \cap E) - M| = |A \cap E| - |M \cap E \cap A| = 15 - x.$$

Now consider set M. By Counting Principle 2,

$$|M| = |M - (A \cup E)| + |(M \cap A) - E| + |(M \cap E) - A| + |M \cap E \cap A|$$

so

$$\begin{aligned} |M - (A \cup E)| &= |M| - |(M \cap A) - E| - |(M \cap E) - A| - |M \cap E \cap A| \\ &= 65 - (25 - x) - (20 - x) - x \\ &= 20 + x. \end{aligned}$$

Similarly

$$\begin{aligned}|A-(M\cup E)| &= |A|-|(A\cap M)-E|-|(A\cap E)-M|-|M\cap E\cap A|\\ &= 42-(25-x)-(15-x)+x\\ &= 2+x\end{aligned}$$

and

$$\begin{aligned}|E-(M\cup A)| &= |E|-|(E\cap M)-A|-|(E\cap A)-M|-|M\cap E\cap A|\\ &= 45-(20-x)-(15-x)+x\\ &= 10+x.\end{aligned}$$

Now, using Counting Principle 2 again, $|M\cup A\cup E| = 100$ is the sum of the cardinalities of its seven disjoint subsets, so:

$$\begin{aligned}100 &= (20+x)+(2+x)+(10+x)+(25-x)+(20-x)+(15-x)+x\\ \implies 100 &= 92+x\\ \implies x &= 8\end{aligned}$$

This answers part (a).
We could now re-draw figure 1.14 showing the cardinality of each disjoint subset of $M\cup A\cup E$. However, this is not necessary to answer the three parts of the question.

(b). The number of students who study mathematics and electronics but not accounting is $|(M\cap E)-A| = 20-x = 20-8 = 12$.
(c). The number of students who study only electronics as a subsidiary subject is $|E-(M\cup A)| = 10+x = 10+8 = 18$.

1.5 Real Number Systems

There are certain sets of numbers that appear frequently in set theory and in many branches of mathematics. We begin this section by studying the decomposition of the real line into the following subsets:

1.1 **The Natural Numbers,** $\mathbb{N}$

$$\mathbb{N} = \{1,2,3,...\}$$

This set is also called the set of counting numbers.

Definition 1.10 *A non-empty set X is said to be closed with respect to a binary operation $*$ if for all $a, b \in X$, we have $a * b \in X$.*

Note that $\mathbb{N}$ is closed with respect to the usual addition and usual multiplication but not with respect to usual subtraction.
The set of natural numbers consists of the dark points on Fig. 1.15.

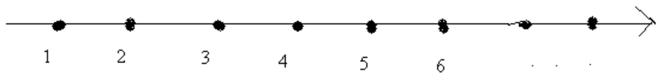


Figure 1.15: Set of Natural numbers

1.2 **The Whole Numbers, $\mathbb{W}$**

$$\mathbb{W} = \{0, 1, 2, 3, ...\}$$

Note that $\mathbb{W} = \{0\} \cup \mathbb{N}$.
Note that $\mathbb{W}$ is closed with respect to usual addition and multiplication but not under subtraction.
The set of whole numbers consist of the numbers represented by the dark dots in Fig 1.16

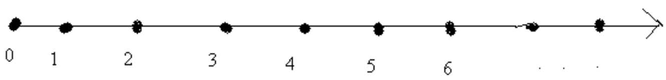


Figure 1.16: Set of Whole numbers

1.3 **The Integers, $\mathbb{Z}$**

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

Note that $\mathbb{Z} = -\mathbb{N} \cup \mathbb{W}$.
This system guarantees solutions to every equation $x + n = m$ with $n, m \in \mathbb{W}$. Clearly,

$\mathbb{Z}$ consists of numbers x such that $x \in \mathbb{N}$ or $x = 0$ or $-x \in \mathbb{N}$.
$\mathbb{Z}$ is closed with respect to usual addition and usual multiplication.
Note also that $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z}$.

Figure 1.17: Set of Integers

1.4 **The Rational Numbers**, $\mathbb{Q}$

A rational number r is one that can be expressed in the form $r = \frac{a}{b}$, for $a,\ b \in \mathbb{Z},\ b \neq 0$ and $gcd(a, b) = 1$, where $gcd(a, b)$ denotes the greatest common divisor of a and b.

Definition 1.11 *The set of rationals, denoted by* $\mathbb{Q}$*, is given by*

$$\mathbb{Q} = \{\tfrac{a}{b} : a, b \in \mathbb{Z}, b \neq 0, gcd(a, b) = 1\}.$$

With this system, solutions to all equations $nx + m = 0$ with $m, n \in \mathbb{Z}$, and $n \neq 0$ can be uniquely found: i.e. $x = -n^{-}1m = -\frac{m}{n}$.

Examples: 2, 0, $\frac{1}{2}$, $-\frac{5}{900}$.

Note that

$$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q}$$

1.5 **The Irrational Numbers**, $\mathbb{Q}^{\complement}$

An irrational number s is one that is not rational, i.e. s cannot be expressed as $s = \frac{a}{b},\ a, b \in \mathbb{Z}, b \neq 0$ and $(a, b) = 1$.
Note that the sets of rationals and irrationals are disjoint. That is, $\mathbb{Q} \cap \mathbb{Q}^{\complement} = \emptyset$.

Examples: $\sqrt{2}$, $\sqrt{3}$, π.

1.6 **The Real Numbers,** $\mathbb{R}$

The set of real numbers is the (disjoint) union of the set of rational numbers with the set of irrational numbers. That is, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^{\complement}$. Graphically, $\mathbb{R}$ is represented by the real number line and called the *real number system.* This means that a rational number is either rational or irrational but not both.

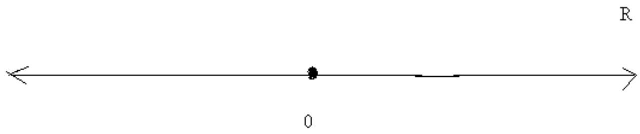


Figure 1.18: Set of Real numbers

Every point on the real line represents a real number.

1.5.1 Some Useful subsets of Real Numbers

The following subsets are frequently encountered in applications.

⊙ The set of positive integers

$$\mathbb{Z}^{+} = \{1, 2, 3, \cdots\}$$

That is $\mathbb{Z}^{+} = \{x \in \mathbb{Z} : x > 0\}$.
Clearly $\mathbb{Z}^{+} = \mathbb{N}$.

⊙ The set of negative integers

$$\mathbb{Z}^{-} = \{-1, -2, -3, \cdots\}$$

⊙ The set of non-negative integers

$$\mathbb{Z}^{*} = \{0, 1, 2, 3, ... \cdots\}$$

That is $\mathbb{Z}^{*} = \{x \in \mathbb{Z} : x \geq 0\}$.
Note also that $\mathbb{Z}^{*} = \mathbb{W}$.

⊙ The set of non-positive integers

$$\mathbb{Z}^{\dagger} = \{0, -1, -2, -3, \cdots\}$$

Note that $\mathbb{Z}^{\dagger} = \{x \in \mathbb{Z} : x \leq 0\}$.

⊙ The set of positive rational numbers

$$\mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$$

⊙ The set of negative rational numbers

$$\mathbb{Q}^- = \{x \in \mathbb{Q} : x < 0\}$$

⊙ The set of non-negative rational numbers

$$\mathbb{Q}^* = \{x \in \mathbb{Q} : x \geq 0\}$$

⊙ The set of non-positive rational numbers

$$\mathbb{Q}^\dagger = \{x \in \mathbb{Q} : x \leq 0\}$$

⊙ The set of positive real numbers

$$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$$

⊙ The set of negative real numbers

$$\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$$

1.5.2 Intervals

Bounded intervals

Let $a, b \in \mathbb{R}$ and suppose that $a < b$. We define the following special sets called **intervals**.

1. $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, called the `closed interval` between a and b.

2. $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, called the `open interval` between a and b.

3. $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ and $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ called the `half-open intervals` between a and b.

These intervals are bounded since both a and b are real numbers.

Unbounded intervals

We define unbounded intervals:

4. $[a,\infty)=\{x\in\mathbb{R}:x\geq a\}$

5. $(a,\infty)=\{x\in\mathbb{R}:x>a\}$

6. $(-\infty,b]=\{x\in\mathbb{R}:x\leq b\}$

7. $(-\infty,b)=\{x\in\mathbb{R}:x<b\}$

Remark 1.6

We define $(-\infty,\infty)=\mathbb{R}$. We also caution that $-\infty$ and ∞ are not real numbers but abstract symbols we use to symbolize "smallest" and "largest" real numbers, respectively. The `Extended Real Number System` is obtained by adjoining a largest element ∞ and a smallest element $-\infty$, to the real line $\mathbb{R}$ to get:

$$\mathbb{R}^{\sharp}=\mathbb{R}\cup\{-\infty,\infty\}$$

Definition 1.12 *Let I be a nonempty set and $\mathcal{U}$ be a universal set. For each $i\in I$ let $A_i\subseteq\mathcal{U}$. Then I is called an **indexing set** (or **set of indices**), and each $i\in I$ is called an **index**.*

Under these conditions

$$\bigcup_{i\in I}A_i=\{x:x\in A_i\ \textit{for at least one}\ i\in I\}$$

$$\bigcap_{i\in I}A_i=\{x:x\in A_i\ \textit{for every}\ i\in I\}$$

If the indexing set I is the set $\mathbb{Z}^+$ or $\mathbb{N}$, we can write

$$\bigcup_{i\in\mathbb{Z}^+}A_i=A_1\cup A_2\cup\cdots=\bigcup_{i=1}^{\infty}A_i$$

and

$$\bigcap_{i\in\mathbb{Z}^+} A_i = A_1 \cap A_2 \cap \cdots = \bigcap_{i=1}^{\infty} A_i$$

Example 1.26 *Let $I = \{3, 4, 5, 6, 7\}$, and for each $i \in I$ let $A_i = \{, 2, 3, ..., i\} \subseteq \mathcal{U} = \mathbb{Z}^+$.*
Find

$$(a) \qquad \bigcup_{i\in\mathbb{Z}^+} A_i$$

$$(b) \qquad \bigcap_{i\in\mathbb{Z}^+} A_i$$

Solution (a). Note that

$$\bigcup_{i\in I} A_i = A_1 \cup A_2 \cup \cdots \cup A_7 = \bigcup_{i=3}^{7} A_i = \{1, 2, 3, 4, 5, 6, 7\} = A_7$$

(b). Clearly

$$\bigcap_{i\in I} A_i = A_1 \cap A_2 \cap \cdots \cap A_7 = \bigcap_{i=3}^{7} A_i = \{1, 2, 3\} = A_3$$

Example 1.27 *Let $\mathcal{U} = \mathbb{R}$ and $I = \mathbb{R}^+$. For each $r \in \mathbb{R}^+$, define $A_r = [-r, r]$. Determine*

$$(a) \qquad \bigcup_{r\in I} A_r$$

$$(b) \qquad \bigcap_{iI} A_r$$

Solution (a). It is easy to show that

$$\bigcup_{r\in I} A_r = \mathbb{R}$$

(b).

$$\bigcap_{r\in I} A_r = \{0\}$$

Example 1.28 *Let $\mathcal{U} = \mathbb{R}$ and let $I = \mathbb{N}$. For each $n \in \mathbb{N}$, let $A_n = [-2n, 3n]$. Determine*
(a). $A_3 - A_4$

(b). $A_3 \triangle A_4$

(c).

$$\bigcup_{n=1}^{7} A_n$$

(d).

$$\bigcap_{n=1}^{7} A_n$$

(e).

$$\bigcup_{n\in I} A_n$$

(f).

$$\bigcap_{n\in I} A_n$$

Solution

(a). $A_3 = [-6, 9]$ and $A_4 = [-8, 12]$. Clearly $[-6, 9] - [-8, 12] = \emptyset$

(b). $A_3 \triangle A_4 = (A_3 \cup A_4) - (A_3 \cap A_4) = [-8, 12] - [-6, 9] = [-8, 6) \cup (9, 12]$

(c).

$$\begin{aligned} \bigcup_{n=1}^{7} A_n &= \bigcup_{n=1}^{7} A_n = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup \cup A_6 \cup A \\ &= [-2, 3] \cup [-4, 6] \cup [-6, 9] \cup \cdots \cup [-14, 21] \\ &= [-14, 21] \\ &= A_7 \end{aligned}$$

(d), (e) and (f) can be solved similarly and are therefore left as exercises.

1.6 Application of Laws of Set Theory

One of the applications of laws of set theory is in the simplification of complex set operations. For convenience, we use $\overline{A}$ to denote the complement of a set A.

Example 1.29 *Simplify the expression*

$$\overline{\overline{\Big((A \cup B) \cap C\Big)} \cup \overline{B}}$$

quoting all the laws you use.

Solution

$$\begin{aligned}
\overline{\overline{\Big((A \cup B) \cap C\Big)} \cup \overline{B}} &= \overline{\overline{\Big((A \cup B) \cap C\Big)}} \cap \overline{\overline{B}} && \textit{De Morgan's Law} \\
&= \Big((A \cup B) \cap C\Big) \cap B && \textit{Law of Double Complement} \\
&= (A \cup B) \cap (C \cap B) && \textit{Associative Law of Intersection} \\
&= (A \cup B) \cap (B \cap C) && \textit{Commutative Law of Intersection} \\
&= \Big[(A \cup B) \cap B\Big] \cap C && \textit{Associative Law of Intersection} \\
&= B \cap C && \textit{AbsorptionLaw}
\end{aligned}$$

Example 1.30 *Express $\overline{A - B}$ in terms of $\cup$ and $^-$.*

Solution From the definition of set difference

$$A - B = \{x : x \in A \;\; and \;\; x \notin B\} = A \cap \overline{B}$$

Therefore

$$\begin{aligned}
\overline{A - B} &= \overline{A \cap \overline{B}} && \textit{Definition} \\
&= \overline{A} \cup \overline{\overline{B}} && \textit{De Morgan's Law} \\
&= \overline{A} \cup B && \textit{Law of Double Complement}
\end{aligned}$$

Example 1.31 *Show that $\overline{A \triangle B} = \overline{A} \triangle B = A \triangle \overline{B}$*

Proof

First recall that

$$\begin{aligned}
A \triangle B &= \Big\{x : x \in A \cup B \;\; and \;\; x \notin A \cap B\Big\} \\
&= (A \cup B) - (A \cap B) \\
&= (A \cup B) \cap \overline{(A \cap B)}
\end{aligned}$$

Therefore

$$
\begin{array}{rcll}
\overline{A \triangle B} & = & \overline{(A \cup B) \cap \overline{(A \cap B)}} & \textit{Definition} \\
 & = & \overline{(A \cup B)} \cup \overline{\overline{(A \cap B)}} & \textit{De Morgan's Law} \\
 & = & \overline{(A \cup B)} \cup (A \cap B) & \textit{Law of Double Complement} \\
 & = & (A \cap B) \cup \overline{(A \cup B)} & \textit{Commutative Law} \\
 & = & (A \cap B) \cup \overline{A} \cap \overline{B} & \textit{De Morgan's Law} \\
 & = & \Big[(A \cap B) \cup \overline{A}\Big] \cap \Big[(A \cap B) \cup \overline{B}\Big] & \textit{Distributive Law} \\
 & = & \Big[(A \cup \overline{A}) \cap (B \cup \overline{A})\Big] \cap \Big[(A \cup \overline{B}) \cap (B \cup \overline{B})\Big] & \textit{Distributive Law} \\
 & = & \Big[\mathcal{U} \cap (B \cup \overline{A})\Big] \cap \Big[(A \cup \overline{B}) \cap \mathcal{U}\Big] & \textit{Inverse Law} \\
 & = & (B \cup \overline{A}) \cap (A \cup \overline{B}) & \textit{Identity Law} \\
 & = & (\overline{A} \cup B) \cap (A \cup \overline{B}) & \textit{Commutative Law} \\
 & = & (\overline{A} \cup B) \cap \overline{(\overline{A} \cap B)} & \textit{De Morgan's Law} \\
 & = & \overline{A} \triangle B & \textit{Definition} \\
 & = & (A \cup \overline{B}) \cap \overline{(\overline{A} \cup B)} & \textit{Commutative Law} \\
 & = & (A \cup \overline{B}) \cap \overline{(A \cap \overline{B})} & \textit{De Morgan's Law} \\
 & = & A \triangle \overline{B} &
\end{array}
$$

Example 1.32 *Using Laws of set theory, simplify each of the following*

(a). $A \cap (B - A)$

(b). $\Big[(A \cap B) \cup (A \cap B \cap \overline{C} \cap D)\Big] \cup (\overline{A} \cap B)$

(c). $(A - B) \cup (A \cap B)$

Solution

(a).

$$
\begin{array}{rcll}
A \cap (B - A) & = & A \cap (B \cap \overline{A}) & \textit{Definition} \\
 & = & B \cap (A \cap \overline{A}) & \textit{Commutativity and Associativity} \\
 & = & B \cap \emptyset & \textit{Inverse Law} \\
 & = & \emptyset & \textit{Domination Law}
\end{array}
$$

(b).

$$\begin{aligned}
\Big[(A\cap B)\cup(A\cap B\cap\overline{C}\cap D)\Big]\cup(\overline{A}\cap B) &= (A\cap B)\cup(\overline{A}\cap B) && \textit{Absorption Law}\\
&= (A\cup\overline{A})\cap B && \textit{Distributive Law}\\
&= \mathcal{U}\cap B && \textit{Inverse Law}\\
&= B && \textit{Identity Law}
\end{aligned}$$

1.7 Exercises

1. Let $S=\{1,2,3\}$. Find all the subsets of S.

2. Let A and B be sets. Prove that $A-B=A\cap B^{\complement}$.

3. Let A,B and C be sets. Show that $(A\bigtriangleup B)\bigtriangleup C=A\bigtriangleup(B\bigtriangleup C)$.

(**4**). Let $\mathcal{U}=\mathbb{R}$ and let the indexing set $I=\mathbb{Q}^+$. For each $q\in I$, let $A_q=[0,2q]$ and $B_q=(0,3q]$. Determine

(a). $A_3\bigtriangleup B_4$

(b).

$$\bigcup_{q\in I} A_q$$

(c).

$$\bigcap_{q\in I} A_q$$

5. Let the universal set be $\mathcal{U}=\{people\ at\ the\ University\ of\ Nairobi\}$ and let

$A=\{students\ at\ the\ University\ of\ Nairobi\}$

$B=\{lecturers\ at\ the\ University\ of\ Nairobi\}$

$C=\{females\ at\ the\ University\ of\ Nairobi\}$

$D=\{males\ at\ the\ University\ of\ Nairobi\}$

Describe verbally the following sets:

(a). $A\cap D$

(b). $B\cap C$

(c). $B\cup C$

(d). $A \cup \overline{C}$

(e). $(A \cap D)^{\complement}$

5. A large corporation classifies its many divisions by their performance in the preceding year. Let

$P = \{divisions\ that\ made\ a\ profit\}$

$L = \{divisions\ that\ had\ an\ increase\ in\ labour\}$

$T = \{divisions\ whose\ total\ revenue\ increased\}$

Describe symbolically the following sets:

(a). $\{divisions\ that\ had\ increases\ in\ labour\ costs\ or\ total\ revenue\}$

(b). $\{divisions\ that\ did\ not\ make\ a\ profit\}$

(c). $\{divisions\ that\ made\ a\ profit\ despite\ an\ increase\ in\ labour\ costs\}$

(d). $\{divisions\ that\ had\ an\ increase\ in\ labour\ costs\ and\ either\ were\ unprofitable$ $or\ did\ not\ increase\ their\ total\ revenue\}$

(e). $\{profitable\ divisions\ with\ increases\ in\ labour\ costs\ and\ total\ revenue\}$

(f). $\{profitable\ divisions\ that\ were\ unprofitable\ or\ did\ not\ have\ increases$ $in\ either\ labour\ costs\ or\ total\ revenue\}$

6. Prove by the Elements Argument method that

(a). $A - (A \cap B) = A \cap \overline{B}$

(b). $\overline{\Big((A \cap B) \cup C\Big)} = \overline{(A \cup C)} \cup \overline{(B \cup C)}$

(c). $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

7. Let $A = \{0, 1\}, B = \{1, 2\}$ and $C = \{0, 1, 2\}$. Find

(a). $A \times B$

(b). $A \times B \times C$

(c). $|A \times B \times C|$

8. Let A and B be non-empty sets. Prove that $A \times B = B \times A$ if and only if $A = B$.

9. Draw Venn diagrams for each of the following sets. Shade the region corresponding to each set.

(a). $A \cap (B \cup C)$
(b). $\overline{A} \cap \overline{B} \cap \overline{C}$
(c). $A - (A - B)$

10. In a survey of 1000 households, 275 owned a home computer, 455 a video, 405 two cars, and 265 households owned neither a home computer nor a video nor two cars. Given that 145 households owned both a home computer and a video, 195 both a video and two cars, and 110 both two cars and a home computer, find the number of households surveyed which owned
(a). a home computer, a video and two cars.
(b). a video only.
(c). two cars, a video but not a home computer.
(d). a video, a home computer but not two cars.

10. In a survey conducted on campus, it was found that students like watching the Barclays Premier League teams: ManU, Chelsea and Arsenal. It was also found that every student who is a fan of Arsenal is also a fan of ManU or Chlesea (or both), and 42 students were fans of ManU, 45 were fans of Chelsea, 7 where fans of both ManU and Chelsea, 11 were fans of of both ManU and Arsenal, 28 were fans of both Chelsea and Arsenal, and twice as many students were fans only of ManU as those who were fans only of Chelsea.
Find the number of students in the survey who were fans of
(a). all three football clubs
(b). Arsenal
(c). only ManU.

11. Let A, B and C be sets.
(a). Prove that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$.
(b). Determine $|A \cup B \cup C|$ when $|A| = 100$, $|B| = 200$ and $|C| = 3000$ if
(i). $A \subseteq B \subseteq C$.
(ii). $A \cap B = A \cap C = B \cap C = \emptyset$.

12. Given that $|A| = 55,\ |B| = 40,\ |C| = 80,\ |A \cap B| = 20,\ |A \cap B \cap C| = 17,\ |B \cap C| = 24$ and $|A \cup C| = 100$, find

(a). $|A \cap C|$

(b). $|C - B|$

(c). $|(B \cap C) - (A \cap B \cap C)|$

13. Find the following power sets

(a). $\mathcal{P}(\emptyset)$

(b). $\mathcal{P}(\mathcal{P}(\emptyset))$

(c). $\mathcal{P}(\{\emptyset\}, \{1, 2\})$

1.8 Complex Numbers

The last great extension of the real number system is to the set of complex numbers. The need for complex numbers must have been felt from the time that the formula for solving quadratic equations was discovered, especially due to the existence of square roots of negative numbers. The use of complex numbers simplifies many problems from the convergence of series to the evaluation of definite integrals. The set of real numbers might seem to be a large enough set of numbers to answer all our mathematical questions adequately. However, there are some natural mathematical questions that have no solution if answers are restricted to be real numbers. In particular, many simple equations have no solution in the realm of real numbers. For example, $x^2 + 1 = 0$. A solution would require a number whose square is 1. For many centuries, mathematicians were content with the answer, "there is no such number." Eventually, it became acceptable to allow existence of a number i, called the **imaginary unit**, such that $i^2 = -1$.

Definition 1.13 *The set of complex numbers, denoted by $\mathbb{C}$ is defined as*

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R} \text{ and } i^2 = -1\}$$

Clearly $\mathbb{R}$ is a subset of $\mathbb{C}$, since $x + 0i = x$ is a real number for every x.

$$\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

1.8.1 Arithmetic Operations of Complex Numbers

Addition and multiplication of complex numbers are defined by

$$(a \pm bi) + (c \pm di) = (a) + (b \pm d)i$$

$$(a + bi).(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Addition, subtraction and multiplication of complex numbers is generally quite straightforward, requiring little more than the application of elementary algebra. However, division of complex numbers is not straightforward. We need to develop the theory to enable us carry out division of complex numbers.

Definition 1.14 *Let $z = x + iy$ be a complex number. x is called the **real part** of z, and denoted by $Re(z)$, while y is the **imaginary part** of z, denoted by $Im(z)$.*

Two complex numbers are **equal** if and only if they have the same real part and the same imaginary part. Let $z = x + iy$ be a complex number. If $x = 0$, then z is said to be **purely imaginary**. If $y = 0$, then z is **real**. Note that 0 is the only number which is at once real and purely imaginary.

Example 1.33 *Let $z_1 = 2 + 3i$ and $z_2 = 4 + i$. Find*
(a). $z_1 + z_2$
(b). $z_1.z_2$

Solution
(a). $z_1 + z_2 = (2 + 3i) + (4 + i) = (2 + 4) + (3 + 4)i = 6 + 7i$
(b). $z_1.z_2 = (2 + 3i).(4 + i) = 8 + 14i + 3i^2 = 8 + 14i - 3 = 5 + 14i$

Example 1.34 *Simplify, leaving your answer in the form $a + bi$, $a, b \in \mathbb{R}$*
(a). $5i^4$
$(b).(8i)^3$
Solution
(a). $5i^4 = 5i^2.i^2 = 5(-1)(-1) = 5$
(b). $(8i)^3 = (8i).(8i).(8i) = 512i^3 = 512i^2.i = 512(-1)i = -512i$

1.8.2 The Complex Plane or The Argand Diagram or The Gauss Plane

Every point on the real line corresponds to a real number, and conversely. A similar relation exists between the set of points in the plane and the set of complex numbers. To the point with coordinates (a, b) corresponds the complex number $a + bi$. When a plane is used in this way to picture complex numbers, it is called the `complex number plane`. It is also called the `Argand Diagram`(after J.R. Argand who first used it in 1806) or `Gauss Plane`. The horizontal axis of the Argand Diagram is called the real axis and the **vertical axis** is called the **imaginary axis**.

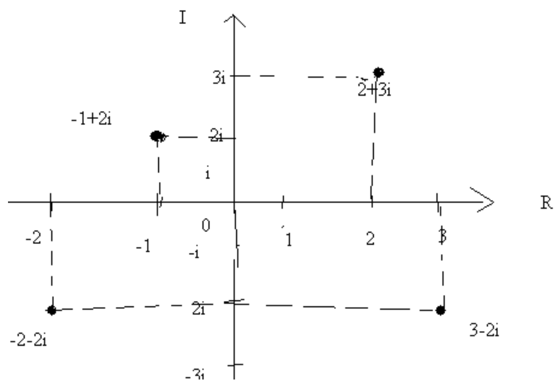


Figure 1.19: The Complex Plane

Exercise Plot the following numbers on the Argand diagram:

(a). $2 + 5i$

(b). $-2 - i$

(c). $2i$

(d). 8

1.8.3 Conjugates, Absolute Values and Arguments of Complex Numbers

Recall that every complex number z can be written as $z = Re(z) + i\ Im(z)$.

Definition 1.15 *The* ***complex conjugate*** *of a complex number z, denoted by $\overline{z}$ is the number defined by $z = Re(z) - iIm(z)$.*

If $z = a + bi$, then $\overline{z} = a - bi$. The numbers $a + bi$ and $a - bi$ are said to be complex conjugates and we write

$$\overline{a + bi} = a - bi$$

Theorem 1.9 *For any complex conjugate numbers w and z.*
(a). $\overline{w+z} = \overline{w} + \overline{z}$
(b). $\overline{w-z} = \overline{w} - \overline{z}$
(c). $\overline{wz} = \overline{w}.\overline{z}$
(c). $\overline{\frac{w}{z}} = \frac{\overline{w}}{\overline{z}}$
(d). $\overline{\overline{z}} = z$
(f). z *is real if and only if* $z = \overline{z}$
(d). z *is purely imaginary if and only if* $z = -\overline{z}$
(e). $Re(z) = \frac{1}{2}(z + \overline{z})$
(f). $Im(z) = \frac{1}{2i}(z - \overline{z})$

Remark 1.7

So far we are able to add and multiply complex numbers. When evaluating the quotient of complex numbers $\frac{a+bi}{c+di}$, we multiply both the numerator and denominator by the complex conjugate of the denominator and simplify . This process is called **complex rationalization**.

Example 1.35 *Simplify the following, leaving your answer in the form* $a+bi, \quad a, b \in \mathbb{R}$.
(a). $\frac{5+2i}{3+4i}$
(b). $\frac{5+2i}{i}$

Solution
(a). $\frac{5+2i}{3+4i} = \frac{(5+2i)(3-4i)}{(3+4i)(3-4i)} = \frac{23}{25} - \frac{14}{25}$

(b). $\frac{5+2i}{i} = \frac{(5+2i)(-i)}{(i)(-i)} = 2 - 5i$

Definition 1.16 *If* $z = Re(z) + i\ Im(z)$ *is a complex number, we define the absolute value of* z*, denoted by* $|z|$ *by*

$$|z| = \sqrt{\Big(Re(z)\Big)^2 + \Big(Im(z)\Big)^2},$$

where the square radical $\surd$ *denotes the unique nonnegative square root.*

This real number is also called the **modulus** of z. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$ and we write $|a + bi| = \sqrt{a^2 + b^2}$.

Example 1.36 *Let $z = 4 + 3i$. Find $|z|$.*

Solution $|z| = |4 + 3i| = \sqrt{4^2 + 3^2} = 5$

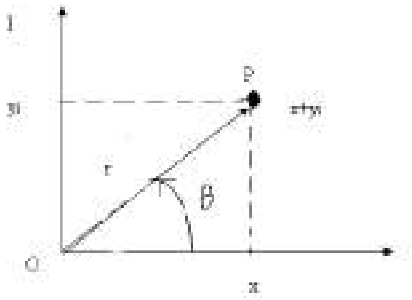


Figure 1.20: Argument of a Complex Number

Definition 1.17 *The angle between the radius vector OP positive x-axis (see Fig 1.20) is called the **argument** of the complex number $z = x + yi$, abbreviated $arg(z)$ and has infinitely many values. We usually take $180° < arg(z) < 180°$, or $-\pi \leq arg(z) < \pi$ to be the **principal value** of the argument.*

Example 1.37 *Find the principal values of the arguments of*

(a). $z = \cos 45° + i \sin 45°$

(b). $z = 1$

(c). $z = -i$

(d). $z = 1 - i$

(e). $z = \frac{2-i}{1+i}$

1.8.4 Polar Form of a Complex Number

A complex number can be completely specified by its modulus and its argument. This is because

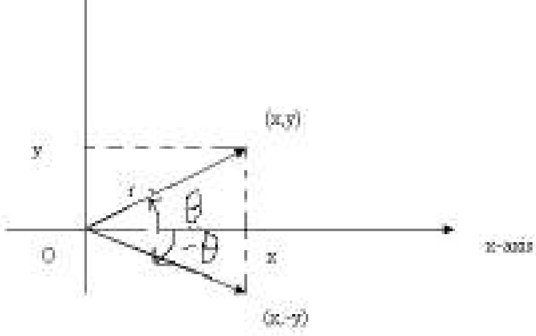


Figure 1.21: Polar Form of a Complex Number

From Fig. 1.21,

$$x = r\cos\theta, \quad y = r\sin\theta \tag{1.4}$$

If θ satisfies (1.4), then so does $\psi = \theta + 2n\pi$, for any integer $n = 0, \pm 1, \pm 2, ...$ The polar form of z is

$$\begin{aligned} z &= r\cos\theta + r\sin\theta \\ &= r(\cos\theta + \sin\theta) \\ &= re^{i\theta} \end{aligned}$$

From (1.4), we deduce that the argument and modulus of z are

$$\theta = \tan^{-1}(\frac{y}{x})$$

and

$$r = \sqrt{x^2 + y^2}$$

respectively.

Example 1.38 *Given* $|z| = 10$ *and* $arg(z) = 120°$, *find* z.

Solution

$z = 10(\cos 120° + i\sin 120°) = 10(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = -5 + i5\sqrt{3}$

Remark 1.8

It is easy to convert from rectangular form to polar form and vice versa by using (1.4) and by using scientific calculators.

1.8.5 De Movrie's Theorem

Theorem 1.10 ***(De Movrie's Theorem)*** *If* $z = x + iy = r(\cos\theta + i\sin\theta)$, *where* $\theta = \tan^{-1}(\frac{y}{x})$, *then*

$$z^n = r^n(\cos n\theta + i\sin n\theta),$$

where $n \in \mathbb{R}$.

Proof De Moivre's formula can be derived from Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and the exponential law

$$(e^{i\theta})^n = e^{in\theta}$$

Thus $(re^{i\theta})^n = r^n e^{in\theta} = r^n\cos n\theta + ir^n\sin n\theta = r^n(\cos n\theta + i\sin n\theta)$

Remark 1.9

This formula (named after Abrahame de Moivre, 1667-1754) is useful in finding explicit expressions for the n-th roots of complex numbers and more specifically n-th roots of unity, which are complex numbers z such that $z^n = 1$. The following are useful identities

- $e^{2n\pi i} = 1$
- $e^{\frac{\pi i}{2}} = i$
- $e^{\pi i} = -1$

If z is a complex number, written in polar form as

$$z = r(\cos\theta + i\sin\theta)$$

then

$$z^{\frac{1}{n}} = \Big[r(\cos\theta + i\sin\theta)\Big]^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$$

where k is an integer. To get the n different roots of z one only needs to consider values of k from 0 to $n-1$. That is, $k = 0, 1, 2, ..., n-1$.
If $k = 0$, then

$$r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right]$$

is called the `principal` n-th `root of` z.
The roots of unity are evenly distributed around the circle centered at the origin of radius 1, starting with the first (real) root at $1 + 0i$.

Example 1.39 *Let* $z = 1+i$. *Find* z^{100}, *leaving your answer in the form* $a+bi \;\; a, b \in \mathbb{R}$.

Example 1.40 *Find the square roots of the complex number* $z = 9 + 9i$. *Leave your answer in the form* $a + bi \quad a, b \in \mathbb{R}$.

Example 1.41 *Find all the cube roots of unity and plot them them on the Argand diagram.*

Solution

$$z^3 = 1 = e^{2n\pi i}$$

Therefore

$$z = e^{\frac{2n\pi i}{3}}, \quad n{=}0{,}1{,}2$$

$$\begin{aligned} z_0 &= e^0 &&= 1 \\ z_1 &= e^{\frac{2\pi i}{3}} &&= \cos(\tfrac{2\pi}{3}) + i\sin(\tfrac{2\pi}{3}) \\ z_2 &= e^{\frac{4\pi i}{3}} &&= \cos(\tfrac{4\pi}{3}) + i\sin(\tfrac{4\pi}{3}) \end{aligned}$$

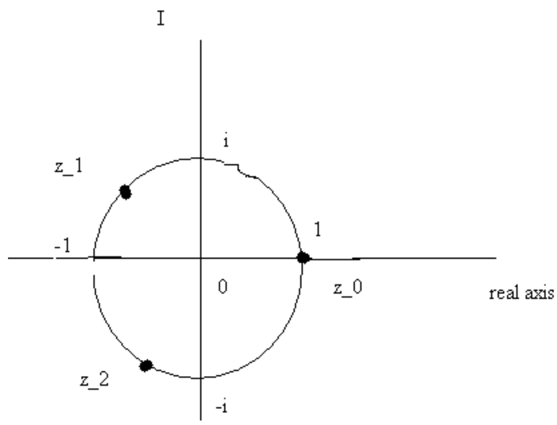


Figure 1.22: Cube roots of unity

Example 1.42 *Solve* $z^3 = i$.

Solution

Recall that $i = e^{\frac{\pi i}{2}} = e^{(\frac{\pi}{2}+2k\pi)i}, \quad k = 0, 1, 2, 3, ...$

Therefore $z^3 = e^{\frac{(4k+1)\pi i}{2}}, \quad k = 0, 1, 2, 3, ...$

Thus $z = e^{\frac{(4k+1)\pi i}{6}}, \quad k = 0, 1, 2$

$\odot$ When $k = 0$, $z_0 = e^{\frac{\pi i}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$\odot$ When $k = 1$, $z_1 = e^{\frac{5\pi i}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$\odot$ When $k = 2$, $z_1 = e^{\frac{5\pi i}{6}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + (-1)i = -i$

1.8.6 Application of Polar Form in Computing Products and Quotients of Complex Numbers

Polar representation is convenient for multiplication and division complex numbers.

Theorem 1.11 *Suppose* $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ *and* $z_2 = r_1(\cos\theta_2 + i\sin\theta_2)$. *Then*

$$z_1 z_2 = r_1 r_2 \Big[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\Big]$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\Big[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\Big]$$

Proof (**Hint**: apply the addition formulas of trigonometry) The proof is quite easy and is thus left as an exercise. □

Remark 1.10

Theorem 1.11 says that to multiply complex numbers, simply multiply their moduli and add their arguments and to divide complex numbers, simply divide their moduli and subtract the arguments. That is, z_1z_2 is the complex number with modulus r_1r_2 and argument $\theta_1+\theta_2$ and $\frac{z_1}{z_2}$ is the complex number with modulus $\frac{r_1}{r_2}$ and argument $\theta_1-\theta_2$.

Example 1.43 *Calculate* $\Big\{2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})\Big\}^5$

Solution Using De Moivre's Theorem

$$\Big\{2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})\Big\}^5 = 2^5\Big\{(\cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5})\Big\} = 32(\cos\pi + i\sin\pi) = -32$$

Example 1.44 *Simplify* $\frac{(1+i)^6}{(1-i\sqrt{3})^4}$

Solution

First we write our complex numbers in polar form:

$$1+i = \sqrt{2}\Big(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\Big)$$

$$1-i\sqrt{3} = 2\Big(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\Big)$$

Applying De Moivre's theorem, we have

$$(1+i)^6 = 8\Big(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\Big)$$
$$(1-i\sqrt{3})^4 = 16\Big(\cos\frac{-4\pi}{3} + i\sin\frac{-4\pi}{3}\Big)$$

Hence

$$\begin{aligned}\frac{(1+i)^6}{(1-i\sqrt{3})^4} &= \frac{8}{16}\Big(\cos(\tfrac{3\pi}{2}+\tfrac{4\pi}{3}) + i\sin(\tfrac{3\pi}{2}+\tfrac{4\pi}{3})\Big)\\ &= \frac{1}{2}\Big(\cos(\tfrac{17\pi}{6}) + i\sin(\tfrac{17\pi}{6})\Big)\end{aligned}$$

1.9 Exercises

1. Given that $z = 3+4i$ and $w = 12+5i$, write down the moduli and arguments of

(a). z

(b). $\frac{1}{z}$

(c). zw

(c). $\overline{zw}$

(d). w^2

2. Simplify, leaving your answer in the form $a+bi$, $\quad a, b \in \mathbb{R}$.

(a). $(1+i)^2$

(b). $(2-2i)^{100}$

(c). $\frac{6+6i}{3+4i}$

(d). $\frac{(3-3i)^4}{(\sqrt{3}+i)^3}$

3. Find the values of a and b such that $(a+ib)^2 = i$. Hence, or otherwise, solve the equation $z^2+2z+1-i=0$, leaving your answers in the form $a+bi$, $\quad a, b \in \mathbb{R}$.

4. Let $z = \cos\theta + i\sin\theta$, where θ is real.

(a). Show that

$$\frac{1}{1+z} = \frac{1}{2}\Big(1 - i\tan\frac{\theta}{2}\Big)$$

(b). Express the following in the form $a+ib$, where a and b are real functions of θ.

(i). $\frac{2z}{1+z^2}$

(ii). $\frac{1-z^2}{1+z^2}$

5. Mark on the Argand diagram the points representing the complex numbers

(a). $4+3i$

(b). $4-8i$

(c). $\frac{4+3i}{4-3i}$

6. Find

(a). the square roots of $z = 5+12i$

(b). the 8th roots of unity.

7. Prove that the modulus of $2+\cos\theta+i\sin\theta$ is $(5+4\cos\theta)^{\frac{1}{2}}$. Hence show that the modulus of

$$\frac{2+\cos\theta+i\sin\theta}{2+\cos\theta-i\sin\theta}$$

is 1.

Chapter 2

ELEMENTARY LOGIC

2.1 Introduction

Mathematical logic is the study of the processes used in mathematical deduction. The subject has origins in philosophy, and indeed it is also a legacy from philosophy that we can distinguish semantic reasoning (what is true) from syntactic reasoning (what can be shown). Logic is used to establish the validity of arguments. The rules of logic give precise meaning to mathematical statements. In addition to mathematical reasoning, logic has numerous applications in the design of computer circuits, construction of computer programs, verification of correctness of computer programs, and so on.

2.2 Mathematical Reasoning and Creativity

Human beings are expected to express themselves creatively in various fields. Mathematics is one of these fields, not just because of its nature but the manner of its presentation.

2.2.1 Inductive and Deductive Reasoning

Reasoning, or drawing conclusions can be classified in tow categories, namely `inductive reasoning` and `deductive reasoning`. When a person makes observations and on the basis of these observations reaches conclusions, he/she is said to reason `inductively`. For example, a young child touches a hot stove and concludes that stoves are hot. De-

ductive reasoning proceeds from assumptions rather than experience.
It is usually by inductive reasoning that mathematical results are discovered, and it is by deductive reasoning that they are proved.

Inductive Reasoning
Inductive reasoning is essential to mathematical activity. To engage in it, one makes observations, gets hunches, guesses, or makes conjectures.

Example 2.1 *Given the sequence*

$$2, 4, 6, 8, -$$

find the next number.

Solution It is *probably* 10, etc.

Deductive Reasoning
Arguments used in mathematical proofs most often proceed from some basic principles which are known or assumed. Such arguments are deductive.

Definition 2.1 *A* ***proof*** *is simply a convincing argument, a sequence of steps, an explanation and communication of ideas; a line of argument sufficient to convince a person of the validity of a certain result.*

As we shall find out, a disproof is also a proof.

Example 2.2 *If a figure is a triangle, then it is a polygon. Figure X is a triangle. What conclusion can be drawn?*

2.3 propositional Logic

In this section an elementary system of symbolic logic called propositional logic is presented. This system is built around propositions or statements.

2.3.1 Propositions and Truth Values

One of the basic ideas of logic is that of a proposition.

Definition 2.2 *A **proposition** is a declarative sentence, assertion which is capable of being classified as either true or false but not both.*

Propositions are sometimes called **statements**.

Example 2.3 *Examples of propositions are:*

1. It is raining

2. $3 + 2 = 4$

3. Nairobi is the capital city of Rwanda

4. $6 < 24$

5. Tomorrow is my birthday

Remark 2.1

Note that the same proposition may be sometimes be true and sometimes false depending on where and when it was stated and by whom. Whilst proposition 5 is true when stated by anyone whose birthday is tomorrow is true, it is false when stated by anyone else. Exclamations, questions and demands are not propositions since they can not be classified/declared as either true or false.

Example 2.4 *The following are not propositions*

6. Come here !

7. Keep off the grass

8. Long live the Queen !

9. Did you finish your homework?

10. Don't say that.

Definition 2.3 *(**Truth Values of a Proposition**) The truth or falsity of a proposition is called **truth value**. The truth value of a proposition is either true or false but not both. We denote the truth vales of propositions by T or F.*

Propositions are conventionally symbolized using letters $a, b, c, ...$ or $A, B, C, ...$

2.3.2 Logical Connectives and Truth Tables

Propositions 1-5 in Example 2.3 are simple propositions since they make only a single statement. In this subsection we look at how simple propositions can be combined to form more complicated propositions called **compound statements**. The devices we use are link pairs or more propositions are called logical connectives.

Definition 2.4 *In logic, a* ***logical connective*** *(also called a* ***logical operator****) is a word or symbol used to connect two or more propositions in a grammatically valid way, such that the compound statement produced has a truth value dependent on the respective truth values of the component simple propositions and the particular connective or connectives used to link them.*

Definition 2.5 *A table which summarizes truth values of a proposition is called a* ***truth table****.*

Definition 2.6 *A* ***negation*** *of a proposition P, usually denoted by $\overline{P}$ or $\sim P$ or $\neg P$ is denial of a proposition.*

The negation of a proposition P is read as "`not P`" . For example, let P denote the proposition: "`All dogs are black`". Then the following are some of its negations:

$\odot$ `It is not the case that all dogs are black.`

$\odot$ `Not all dogs are black.`

$\odot$ `Some dogs are not black.`

In accordance with ordinary language, the negation of a true proposition will be considered false, and the negation of a false proposition will be considered a true proposition. The truth table of a negation is given by

We consider four commonly used logical connectives: conjunction "`and`", inclusive disjunction "`inclusive or`", exclusive disjunction "`exclusive or`", conditional " `if ... then`" and biconditional "`if and only if`".

1. **Conjunction**

P	~P
T	F
F	T

Figure 2.1: Truth Table of a Negation

When two or more propositions are joined/combined by the logical connective "and", the resulting proposition is called a **conjunction** of its component simple propositions. It is read as "P and Q" and denoted by $P \wedge Q$. If P and Q are both true, then $P \wedge Q$ is true. If P and Q are both false or if just one of them is false, then $P \wedge Q$ is false.
The truth table of a conjunction $P \wedge Q$ is given by

P	Q	P∧Q
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**F**

Figure 2.2: Truth Table of a Conjunction

2. **Disjunction**

The word "`or`" can be used to link two simple propositions P and Q. The compound proposition so formed is called a **disjunction** of P and Q. It is read as "`P or` Q". P and Q are called `dijuncts`. In logic we distinguish between two different types of disjunctions: the inclusive and exclusive disjunctions. The word "`or`" in natural language is ambiguous in conveying which type of disjunction we mean. We use $P \vee Q$ to denote the inclusive disjunction (inclusive "`or`" -meaning one or the other or both) of P and Q. This compound statement is true when either or both of its components are true and is false otherwise.
The truth table of a disjunction is give by

P	Q	P∨Q
T	T	**T**
T	F	**T**
F	T	**T**
F	F	**F**

Figure 2.3: Truth Table of an Inclusive Disjunction

The exclusive disjunction of P and Q is symbolized $P \veebar Q$. This compound proposition is true when exactly one (i.e. one or other but not both) of its components is true. The truth table of of $P \veebar Q$ is given by

P	Q	$P \veebar Q$
T	T	F
T	F	T
F	T	T
F	F	F

Figure 2.4: Truth Table of an Exclusive Disjunction

The context of a disjunction will often provide the clue as to whether the inclusive or exclusive sense is intended. For example " `Tomorrow I will go swimming or play golf`" seems to suggest that I will not do both and therefore points to an exclusive interpretation. On the other hand, the proposition "`Applicants for this post must be over 25 years or have at least 3 years relevant experience`" suggests that applicants who satisfy both criteria will be considered. In this case "or" should be interpreted inclusively.

3. **Conditional Propositions**

A proposition such as "If P then Q" is called a **conditional** proposition, or simply a conditional. Such propositions are extremely important in mathematical proofs. In a deductive argument, something is assumed and something is concluded. If P represents what is assumed and Q represents what is concluded, then the main structure of the argument is evidently expressible by the conditional "`If P then Q`" and symbolized $P \Longrightarrow Q$. The proposition "`If P then Q`" is also read as "`P implies Q`", "`P only if Q`", "`P is a sufficient condition for Q`", "`Q is a necessary condition for P`".

In a conditional, $P \Longrightarrow Q$, the proposition P is sometimes called **antecedent** and Q the **consequent**.

Let P and Q are propositions. Then $P \Longrightarrow Q$ is true if both P and Q are true or both false or if P is false, and is false if P is true and Q is false.

The truth table of a conditional $P \Longrightarrow Q$ is given by

P	Q	P⇒Q
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Figure 2.5: Truth Table of an Implication

Example 2.5 *Let*

P: I eat breakfast

Q: I don't eat lunch. Then

$P \Longrightarrow Q$*:* `If I eat breakfast then I don't eat lunch.`

Alternative expressions for $P \Longrightarrow Q$ in this example are:

⊙ I eat breakfast only if I don't eat lunch.

⊙ Whenever I eat breakfast, I don't eat lunch.

⊙ That I eat breakfast implies that I don't eat lunch.

4. **Biconditional Propositions**

The biconditional connective is symbolized $\Longleftrightarrow$ and expressed by "`If and only if`" and shortened as "`Iff`". Let P and Q be propositions. We define a proposition "`P if and only if Q`" denoted by $P \Longleftrightarrow Q$, which is true if both P and Q are true or if both P and Q are false, and which is false if P is true while Q is false, and if P is false while Q is true.

The statement $P \Longleftrightarrow Q$ means that P implies Q and Q implies P. Two other ways of saying $P \Longleftrightarrow Q$ are:

⊙ P is equivalent to Q

⊙ P is a necessary and sufficient condition for Q.

Example 2.6 *If P and Q are as in Example 2.5, then*

P *:* `I eat breakfast if and only if I don't eat lunch.`

Equivalently `If and only if I don't eat breakfast, then I don't eat lunch.`

The truth table of a biconditional $P \iff Q$ is given by

P	Q	P⇔Q
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

Figure 2.6: Truth Table of an Implication

Example 2.7 *Consider the following propositions:*
P: Mathematician are generous.
Q: Spiders hate algebra.

Write the compound propositions symbolized by
(a). $P \vee \sim Q$
(b). $\sim (P \wedge Q)$
(c). $\sim P \Longrightarrow \sim Q$
(d). $\sim P \Longleftrightarrow \sim Q$

Solution
(a). Mathematicians are generous or spiders don't hate algebra.
(b). It is not the case that spiders hate algebra and mathematicians are generous.
(c). If Mathematicians are not generous then spiders hate algebra.
(d). Mathematicians are not generous if and only if spiders don't hate algebra.

Example 2.8 *Let P be the propositon "`Today is Monday`" and Q be "`I will go to London`". Write the following propositions symbolically:*
(a). If today is Monday, then I won't go to London.
(b). Today is Monday or I will go to London but not both
(c). I will go to London and today is not Monday.
(d). If and only if today is not Monday then I will go to London.

Solution
(a). $P \Longrightarrow \sim Q$

(b). $P \veebar Q$

(c). $Q \wedge \sim P$

(d). $\sim P \Longleftrightarrow Q$.

Example 2.9 *Construct truth tables for the following compound propositions.*

(a). $\sim P \vee Q$

(b). $\sim P \wedge \sim Q$

(c). $\sim Q \Longrightarrow P$

(d). $\sim P \Longleftrightarrow \sim Q$

Solution. Left as an exercise.

2.3.3 Tautologies and Contradictions

There are certain compound propositions which are always true no matter what the truth values of their simple components are and there are others which are always false regardless of the truth values of their components.

Definition 2.7 *A **tautology** is a compound proposition which is true under all circumstances regardless of the truth values of its simple components.*
*A **contradiction** is a compound proposition which is false no matter what the truth values of its simple components are.*

We shall denote a tautology by t and a contradiction by c.

Example 2.10 *The following are tautologies.*

(a). $P \vee \sim P$

(b). $(P \wedge Q) \vee \sim (\wedge Q)$

Solution `Hint`: Their truth tables return a column true values T.

Example 2.11 *The following are contradictions.*

(a). $(P \wedge Q) \wedge (\sim P \vee Q)$

(b). $P \wedge \sim P$

Solution `Hint`: Their truth tables return a column false values F.

2.3.4 Logical Equivalence and Logical Implication

Definition 2.8 *Two propositions P and Q are said to be* ***logically equivalent****, denoted by $P \equiv Q$ if they have the same or identical truth values for every set of truth values of their components.*

Example 2.12 *Show that $P \iff Q$ is equivalent to $(P \implies Q) \wedge (Q \implies P)$.*

Solution

The truth table for the two propositions is given below.

P	Q	P⇒Q	Q⇒P	P⇔Q	(P∧Q)⇔ (Q∧P)
T	T	T	T	**T**	**T**
T	F	F	T	**F**	**F**
F	T	T	F	**F**	**F**
F	F	T	T	**T**	**T**

Figure 2.7: Logical Equivalence

The last two columns are the same and hence the two propositions are logically equivalent.

Remark 2.2

Note that if $P \equiv Q$ then $P \iff Q$ is a tautology and that if $P \iff Q$ is a tautology, then $P \equiv Q$.

Definition 2.9 *A proposition P is said to* ***logically imply*** *a proposition Q, if whenever P is true, then Q is also true.*

2.3.5 The Algebra of Logical Equivalence of Propositions

Let P, Q, R be propositions and let t and c denote a tautology and contradiction, respectively. Then:

1. $\sim(\sim P) = P$ `Double Negation or Involution Law`

2. $(P \vee Q) \equiv (Q \vee P)$ `Commutative Laws`
   $(P \wedge Q) \equiv (Q \wedge P)$
   $(P \veebar Q) \equiv (Q \veebar P)$

$(P \iff Q) \equiv (Q \iff P)$

3. $[(P \vee Q) \vee R] \equiv [P \vee (Q \vee R)]$ `Associative Laws`

$[(P \wedge Q) \wedge R] \equiv [P \wedge (Q \wedge R)]$

$[(P \veebar Q) \veebar R] \equiv [P \veebar (Q \veebar R)]$

(P$\iff Q) \iff R \equiv P \iff (Q \iff R)$

4.(P$\vee Q) \wedge R \equiv (P \vee Q) \wedge (P \vee R)$ `Distributive Laws`

$(P \wedge Q) \vee R \equiv (P \wedge Q) \vee (P \wedge R)$

5. $P \vee P \equiv P$ `Idempotent Laws`

$P \wedge P \equiv P$

6. $P \vee c \equiv P$ `Identity Laws`

$P \vee t \equiv t$

$P \wedge c \equiv c$

$P \wedge t \equiv P$

7. $P \vee \sim P \equiv t$ `Complement Laws`

$P \wedge \sim P \equiv c$

$\sim c \equiv t$

$\sim t \equiv c$

8. $\sim (P \vee Q) \equiv (\sim P \wedge \sim Q)$ `De Morgan's Laws`

$\sim (P \wedge Q) \equiv (\sim P \vee \sim Q$

9. $(P \implies Q) \equiv (\sim Q \implies \sim P)$

10. $(P \implies Q) \equiv (\sim P \vee Q)$ `Implication`

$(P \implies Q) \equiv \ \sim (P \wedge \sim Q)$

11. $(P \iff Q) \equiv [(P \implies Q) \wedge (Q \implies P)]$ `Equivalence`

12. $(P \Longrightarrow Q) \equiv [(P\wedge \sim Q) \Longrightarrow c]$ `Reduction ad absurdum` or `Proof by Contradiction`

13. $P \wedge (P \vee Q) \equiv P$ `Absorption Laws`
$P \vee (P \wedge Q) \equiv P$

Note that the negation of a tautology is a contradiction and vice versa.

Example 2.13 *Prove that* $(\sim P \wedge Q) \vee \sim (P \vee Q) \equiv P$*, quoting all the laws you use.*

Solution

$$\begin{aligned}(\sim P \wedge Q) \vee \sim (P \vee Q) &= (\sim P \wedge Q) \vee (\sim P \wedge \sim Q) && \texttt{De Morgan's Laws}\\ &= \sim P \wedge (Q \vee \sim Q) && \texttt{Distributive Laws}\\ &= \sim P \wedge t && \texttt{Complement Laws}\\ &= \sim P && \texttt{Identity Laws}\end{aligned}$$

2.3.6 Relationship between Converse, Inverse and the Contrapositive of a Conditional Proposition

Given the conditional proposition $P \Longrightarrow Q$, we define the following
(a). the **converse** of $P \Longrightarrow Q$: $Q \Longrightarrow P$
(b). the **inverse** of $P \Longrightarrow Q$: $\sim P \Longrightarrow \sim Q$
(c). the **contrapositive** of $P \Longrightarrow Q$: $\sim Q \Longrightarrow \sim P$
The truth table below shows the relationship of these concepts.

P	Q	P⇒Q	Q⇒P	~P⇒ ~Q	~Q⇒~P
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Figure 2.8: Truth Table for a Conditional and its Converse, Inverse and Contrapositive

From the table, we note the following useful results:
⊙ a conditional proposition and its contrapositive are logically equivalent.
⊙ converse and inverse of a conditional proposition are logically equivalent.
⊙ a conditional proposition is not logically equivalent to either its converse or inverse.

Example 2.14 *State the converse, inverse and contrapositive of the proposition "*`If Jack plays his guitar then Sara will sing.`*"*

Solution

We define

P: Jack plays his guitar.

Q: Sara will sing

so that

$P \Longrightarrow Q$: If Jack plays his guitar then Sarah will sing.

`Converse`: $Q \Longrightarrow P$: If Sara will sing then Jack plays his guitar.

`Inverse`: $\sim P \Longrightarrow \ \sim Q$: If Jack doesn't paly his guitar then Sara won't sing.

`Contrapositive`: $\sim Q \Longrightarrow \ \sim P$: If Sara won't sing then Jack doesn't play his guitar.

2.4 Predicate Calculus

The sentence "The number x is even" is not a statement because we do not know to what x refers. For example if $x = 3$, then the statement is false. If $x = 6$, then the statement is true.

Definition 2.10 *An* ***open sentence*** $p(x)$ *is a declarative sentence that becomes a statement when* x *is given a particular value chosen from a universe of discourse of values* $\mathcal{U}$.

An open sentence is also called a **predicate**. A predicate is a statement $p(x_1, x_2, ..., x_n)$ involving variables $x_1, x_2, ..., x_n$ with the property that when $x_1, x_2, ..., x_n$ are given specific values, then resulting statement is either true or false(i.e., becomes a proposition). Thus a predicate is a statement that could be a proposition, except for ambiguity that exists because the scope or range of possibilities for the variables in the statement is not specified. The scope of a variable is usually specified by the use of quantifiers.

2.4.1 Universal Quantifier

The statement "For all $x \in \mathcal{U}, p(x)$" is symbolized $\forall x \in \mathcal{U}, p(x)$. We call the symbol $\forall$ the **universal quantifier** and we read it as "`for all`", "`for every`", "`for each`".

Example 2.15 *"For all $x \in \mathcal{U}$, if $x > 4$, then $x + 10 > 14$" is a trues statement for $\mathcal{U} = \{1, 2, 3, ...\}$, the universe.*

Example 2.16 *Consider the proposition "All rats are grey". One way in which we can paraphrase this proposition is "For every x, if x is a rat, then x is grey".*
$R(x)$: x is a rat
$G(x)$: x is grey
We can then write "All rats are grey" as

$$\forall x \Big[R(x) \Longrightarrow G(x)\Big]$$

Note that the statement $\forall x \in \mathcal{U}, p(x)$ is true if and only if $p(x)$ is true for every $x \in \mathcal{U}$.

Example 2.17 *Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6\}$. Determine the truth value of the statement*

$$\forall x[(x-4)(x-8) > 0]$$

Solution
Let $p(x)$ be the open sentence "`(x-4)(x-8)>0`". We consider the truth values of $p(1), p(2), ..., p(6)$. p(1) is true because $(1-4)(1-8) = (-3)(-7) = 21 > 0$. p(2) and p(3) are also true. However, p(4) is false because $(4-4)(4-8) = 0$. We need not check any other values from $\mathcal{U}$.
Therefore $[\forall x \in \mathcal{U} p(x)]$ is false.

2.4.2 Existential Quantifier

The statement "There exists an x in $\mathcal{U}$ such that $p(x)$" is symbolized $\exists x \in \mathcal{U} \;\; p(x)$. The symbol is true if and only if there is at least one element $x \in \mathcal{U}$ such that $p(x)$ is true. The symbol $\exists$ is called the **existential quantifier** and is read as "`there exists` x `such that` $p(x)$", "`for some` x, `p(x)`", or "`there is some` x `for which` $p(x)$".

Example 2.18 *Consider the proposition "some rats are grey". That is, there is at least one rat which is is grey. This can be paraphrased as "There exists at least one x such that x is a rat and x is grey". Thus if we define*
R(x): x is a rat

G(x): x is grey

and denote "there exists at least one x" by $\exists x$, then "some rats are grey" can be written

$$\exists x\Big[R(x) \wedge G(x)\Big]$$

Example 2.19 *Symbolize "some people think of no one but themselves"*

Solution Define

$p(x)$: x is a person

$n(x)$: x thinks of no one but himself

Then "some people think of no one but themselves" can be written

$$\exists x[p(x) \wedge n(x)].$$

2.4.3 Negation of a Quantified Statement

The statement $\forall x\ p(x)$ states that for all x in the universe of discourse, x has the property defined by the predicate p. The negation of this statement is $\sim \forall x\ p(x)$, and states that "It is not the case that all x has the property defined by p". That is, there is at least one x that does not have the property p. This is symbolized by

$$\exists x[\sim p(x)]$$

Therefore, for any predicate $p(x)$, the statements

$\sim \forall x\ p(x)$ and $\exists x\ [\sim p(x)]$

have the same truth tables and are therefore equivalent. That is

$$\sim \forall x\ p(x) \equiv \exists x\ [\sim p(x)]$$

Similarly, the negation of $\exists x \in \mathcal{U}\ \ p(x)$ symbolized by $\sim \exists x\ p(x)$, is equivalent to $\forall x[\sim p(x)]$.

Note also that

$$\sim \exists x[\sim p(x)] \equiv \forall x\ p(x),$$

since

$$\begin{aligned} \sim \exists x[\sim p(x)] &\equiv \forall x[\sim\sim p(x)] \\ &\equiv \forall x\ p(x) \end{aligned}$$

Similarly, we can show that

$$\sim \forall x[\sim p(x)] \equiv \exists x\ p(x)$$

Example 2.20 *We define the following on the universe of men.*
M(x): x is mortal
C(x): x lives in the city
Symbolize the negation of the following propositions changing the quantifier.
(a). All men are mortal.
(b). Some men live in the city.

Solution
(a). The proposition can be symbolized by $\forall x[\sim M(x)]$. The negation of this proposition is give by

$$\sim \forall x[\sim M(x)] \equiv \exists x\ M(x)$$

The resulting proposition is "some men are mortal."
(b). "some men live in the city" is symbolized by $\exists x\ C(x)$. Its negation is

$$\sim \exists x\ C(x) \equiv \forall x[\sim C(x)]$$

That is, "All men live out of the city".

Remark 2.3

Notice how moving the negation across a quantifier switches it from universal to existential and vice versa.

2.5 Application of Logic In Mathematical Proof

The discipline of Mathematics is characterized by the concept of proof. Many mathematical theorem are statements that a certain implication is true. We give some methods of proof.

2.5.1 Proof by a Counter-example

To show that a theorem or a step in a proof is false, it is enough to find a single case where the implication does not hold. This is called **proof by a counter-example**.

Example 2.21 *Consider the proposition P be*

$$x^2 < 1 \Longrightarrow 0 < x < 1$$

It is easy to show that P is false by producing an x such that $x^2 < 1$ but $x \notin (0, 1)$. For instance, $x = -\frac{1}{2}$ is a counter-example to the statement. This is a specific example which proves that an implication is false.

2.5.2 Direct Proof

Most theorem in mathematics are stated as implications $P \implies Q$. Sometimes, it is possible to prove such a statement *directly*; that is, by establishing the validity of a sequence of implications:

$$P \Longrightarrow P_1 \Longrightarrow P_2 \Longrightarrow ... \Longrightarrow Q$$

Example 2.22 *Prove that for all real numbers x, $x^2 - 4x + 17 \neq 0$.*

Proof
Observe that $x^2 - 4x + 17 = (x - 2)^2 + 13$ is the sum of 13 and a number $(x - 2)^2$, which is never negative. So $x^2 - 4x + 17 \geq 13$ for any x. In particular, $x^2 - 4x + 17 \neq 0$.

2.5.3 Proof by Cases

Sometimes a direct argument is made simpler by breaking it into a number of cases, one of which must hold and each of which leads to the desired conclusion.

Example 2.23 *Let n be an integer. Prove that $9n^2 + 3n - 2$ is even.*

Proof We can consider the cases: n is even and n is odd.
Case 1: n is even.
The product of an even integer and any integer is even. Since n is even, $9n^2$ and $3n$ are even too. Thus $9n^2 + 3n - 2$ is even since it is the sum of three even integers. **Case 2**: n is odd.
The product of odd integers is odd. In this case, since n is odd, $9n^2$ and $3n$ are odd and hence the sum $9n^2 + 3n$ is even. So $9n^2 + 3n - 2$ is even.

2.5.4 Proof by the Contrapositive

Recall that a conditional proposition and its contrapositive are equivalent. Thus the implication $P \implies Q$ is true if and only if the contrapositive $\sim Q \implies \sim P$ is true. If we can establish the truth of the contrapositive, we can deduce that the conditional is also true.

Example 2.24 *If the average of four different integers is 10, prove that one of the integers is greater than 11.*

Proof
Let P and Q be the statements
P: "The average of four integers, all different, is 10".
Q: "One of the fout=r integers is greater than 11".
We are asked to prove the truth of $P \implies Q$. Instead we prove the truth of the contraposition $\sim Q \implies \sim P$ from which the truth follows. Call given integers a, b, c, d. If Q is false, then each of these numbers is at most 11 and since they are different, the biggest value for $a + b + c + d$ is $11 + 10 + 9 + 8 = 38$. So the biggest possible average would be $\frac{38}{4}$, which is less than 10, so P is false.

2.5.5 Proof by Contradiction

Sometimes a direct proof of a statement P seems hopeless. We simply do not know how to begin. In this case, we sometimes make progress by assuming that the negation of P is true. If this assumption leads to a statement which is obviously false (an absurdity) or to a statement which contradicts something else, then we will have shown that $\sim P$ is false. So P must be true.

Example 2.25 *Show that there is no largest integer.*

Proof Let P be the statement "There is no largest integer". If P is false, then there is a largest integer N. This is absurd, however, because $N + 1$ is an integer larger than N. Thus $\sim P$ is false, so P is true.

2.5.6 Proof by Mathematical Induction

Mathematical Induction is appropriate for proving that a result holds for all positive integers. It consists of the following steps:
(a). Prove that the conjecture holds for $n = 1$
(b). Prove that, for all $k \geq 1$, if the result holds for $n = k$, then it must also hold for $n = k + 1$.

Theorem 2.1 ***(Principle of Finite Mathematical Induction)*** *Let*

$$p(m),\ p(m+1), ...,\ p(n)$$

be a finite sequence of propositions. If
(i). $p(m)$ is true, and
(ii). $p(k+1)$ is true whenever $p(k)$ is true and $m \leq k \leq n$, then all the propositions are true.

In many applications, m will be 0 or 1.

Theorem 2.2 ***(Principle of Infinite Mathematical Induction)*** *Let*

$$p(m),\ p(m+1), ...,$$

be a sequence of propositions. If
(i). $p(m)$ is true, and
(ii). $p(k+1)$ is true whenever $p(k)$ is true and $m \leq k$, then all the propositions are true.

Condition (i) is called the **basis**, and (ii) is called the **inductive step**. The basis is easy to check; the inductive step is sometimes quite a bit more complicated to verify. The principle tells that if we can show that (i) and (ii) holds, we are done.

Example 2.26 *Prove by Mathematical Induction that for every positive integer n,*

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

Proof
Let $p(n)$ be the proposition

$$"1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}"$$

then $p(1)$ is "$1 = \frac{1(1+1)}{2}$", which is true.
Assume inductively that $p(k)$ is true for some positive integer k. That is

$$1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}.$$

We want to show that this assumption implies that $p(k+1)$ is true. Now

$$\begin{aligned} 1 + 2 + 3 + ... + k + (k+1) &= \tfrac{k(k+1)}{2} + (k+1) \\ &= [\tfrac{k}{2} + 1](k+1) \\ &= \tfrac{k+2}{2}(k+1) \\ &= \tfrac{[(k+1)+1](k+1)}{2} \\ &= \tfrac{(k+1)(k+2)}{2} \end{aligned}$$

Example 2.27 *Use the Principle of Mathematical Induction to prove that for any natural number $n \geq 1$*

$$1^2 + 2^2 + 3^3 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof Clearly the statement holds or $n = 1$ since $1^2 = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$.
Now we assume the statement is true for some $k \in \mathbb{N}$. That is,

$$1^2 + 2^2 + 3^3 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$$

From this assumption we want to deduce that the statement holds for $n = k + 1$. Thus

$$\begin{aligned} 1^2 + 2^2 + 3^3 + ... + k^2 + (k+1)^2 &= \tfrac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1)\left[\tfrac{k(2k+1)}{6} + (k+1)\right] \\ &= (k+1)\left[\tfrac{2k^2+7k+6}{6}\right] \\ &= \tfrac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

and the general result follows by the Principle of Mathematical induction.

2.6 Applications of Logic

Logic is very applicable in the sciences and social sciences. Some of the applications include

▶ the compositions of logical expressions in computer programs (software engineering) and language design

▶ automatic reasoning systems (robotics/cybernetics-study of robotics)and artificial intelligence

▶ deductive proofs of program correctness

▶ the programming language Prolog, as a model of computation-the lambda calculus has a special role especially in designing systems and defining the semantics of programming languages

▶ the design of digital circuits, logic gates in digital electronics

▶ deductive science-Mathematics is a deductive science

▶ constraint logic programming- The marriage of logic programming with linear programming techniques has enabled rapid and efficient solutions to many difficult scheduling type problems in operations research

2.7 Exercises

1. In each of the following, construct the conjunction and disjunction of the set of simple propositions. Decide if you can, the truth value of each compound statement.

(a). July has 29 days.

Christmas is December 25th.

(b). $3 + 4 = 9$

$9 - 5 = 5 - 7$

2. Let P, Q and R be propositions. Construct a truth table for

(a). $P \wedge Q \wedge R$.

(b). $P \vee Q \vee R$.

3. Write the negation of each of the following propositions.

(a). "The smallest prime number is 2".

(b). "Mathematicians are smart people".

4. Determine whether each of the following is a tautology, a contradiction or neither.

(a). $P \Longrightarrow (P \vee Q)$

(b). $(P \Longrightarrow Q) \wedge (\sim P \vee Q)$

(c). $\Big[(P \Longrightarrow Q) \wedge (Q \Longrightarrow R)\Big] \Longrightarrow (P \Longrightarrow R)$

(d). $\Big[(P \vee Q) \Longrightarrow \quad \sim R\Big] \veebar (\sim P \vee \sim Q)$

5. Prove each of the following for all $n \geq 1$ by the Principle of Mathematical Induction.

(a). $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

(b). $1.3 + 2.4 + 3.5 + ... + n(n+2) = \frac{n(n+1)(2n+7)}{6}$

(c). $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

(d). $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$

(e). $\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$

(f). $\sum_{n=1}^{n} (i)(i!) = (n+1)! - 1$

(g). $1^3 + 2^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

6. Simplify each of the following propositions, quoting the laws you use.

(a). $(P \wedge Q) \vee \Big[\sim (\sim P \vee Q)\Big]$

(b). $(P \vee Q) \wedge \Big[\sim (\sim P \wedge Q)\Big]$

(c). $\sim \Big[\sim [(P \vee Q) \wedge R] \vee \sim Q\Big]$

(d). $\Big[(P \Longrightarrow Q) \vee (Q \Longrightarrow R)\Big] \wedge (R \Longrightarrow S)$

7. Write each of the following statements using the quantifiers $\forall$ and $\exists$.

(a). Not all countable sets are finite.

(b). 1 is the smallest positive integer.

8. Which of the following are propositions

(a). As the world turns.

(b). An apple a day keeps the doctor away

(c). If x is even then $x > 3$.

9. My parents promised to buy me a car if I get A's in Basic Mathematics and Calculus. I received an A in Basic Mathematics and a B in Calculus. Are my parents obligated to buy me a car? Give a reason for your answer.

10. Suppose you order a chicken sandwich at a Kenchic restaurant. The waitress tells you that the sandwich comes with soup or salad. Is the waitress most likely to be using an inclusive OR or an exclusive OR? Give a reason for your answer.

11. Consider the propositions

p: `Felix laughs`

q: `Jacinta cries`

r: `John shouts`

Write in words the following compound propositions

(a). $p \Longrightarrow (q \veebar r)$

(b). $(r \wedge q) \Longleftrightarrow p$

(c). $(p \vee r) \Longleftrightarrow \sim q$

12. Consider the propositions

p: `Bats are blind`

q: `Sheep eat grass`

r: `Ants have long teeth`

Express the following compound propositions symbolically

(a). If bats are blind the sheep don't eat grass.

(b). If and only if bats are blind or sheep eat grass then ants don't have long teeth.

(c). Ants don't have long teeth and, if bats are blind, then sheep don't eat grass.

(d). Bats are blind or sheep eat grass and, if sheep don't eat grass, then ants don't have long teeth.

13. Show that $(p \Longleftrightarrow q) \wedge q$ logically implies p.

14. Prove by Mathematical Induction that for every positive integer n, the expression $2^{n+2} + 3^{2n+1}$ is divisible by 7.

15. Give a direct proof that if n is odd then n^2 is odd.

16. Prove by the contrapositive[indirect proof] the theorem " `If` $3n+2$ `is odd, then` n `is even`".

17. Give a proof by contradiction of the theorem "`If` $3n+2$ `is odd, then` n `is odd`".

18. Find a counterexample to the proposition: "`For every prime number` n, $n+2$ `is prime`".

Chapter 3

PERMUTATIONS AND COMBINATIONS

3.1 Introduction

Permutations and combinations arise when a subset is to be chosen from a set. They are types of arrangements of elements of a set.

3.2 Basic Counting Principle

Counting problems arise throughout mathematics and computer science. Counting elements in a probability problem or occurrence problem individually may be extremely tedious (or even prohibitive). We shall spend some time developing efficient counting techniques. We begin by motivating the basic counting principle which is useful in solving a wide variety of problems.

Theorem 3.1 ***(Basic Counting Principle (BCP))*** *Suppose that a procedure involves a sequence of k stages. Let n_1 be the number of ways the first can occur and n_2 be the number of ways the second can occur after the first stage has occurred. Continuing in this way, let n_k be the number of ways the kth stage can occur after the first $k-1$ stages have occurred. Then the total number of different ways the procedure can occur is*

$$n_1.n_2.n_3 \cdots n_k$$

Example 3.1 *(**Travel Routes**) Two roads connect cities A and B, four roads connect B and C, and five roads connect C and D. To drive from A to B, to C, and then to city D, how many different routes are possible?*

Solution

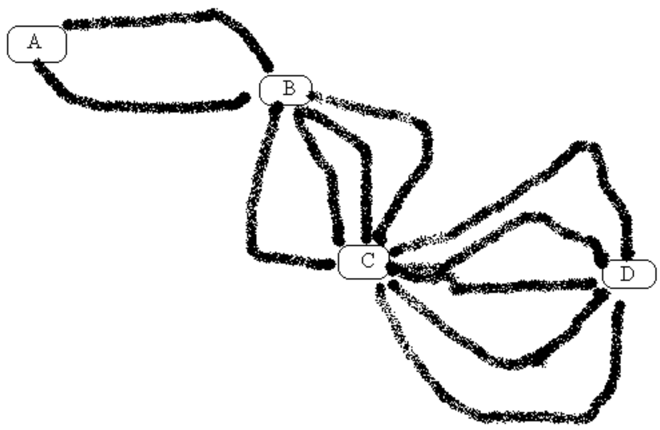


Figure 3.1: Routes from A to D

Here we have a three-stage procedure: The first (A $\rightarrow$ B) has two possibilities, the second (B$\rightarrow$ C) has four possibilities, and the third (C$\rightarrow$ D) has five.
By the Basic Counting Principle, the total number of routes is 2.4.5 = 40.

Example 3.2 *The chairs of a lecture room are to be labeled with a letter of the alphabet and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?*

Solution
The procedure of labeling a chair consists of two tasks, namely, assigning one of the letters and the assigning one of the 100 possible integers. The first task can be done in 26 different ways while the second task can be done in 100 different ways. By the BCP, there are $26 \times 100 = 2600$ different ways that a chair can be labeled. Therefore the largest number of chairs that can be labeled differently is 2600.

Example 3.3 *In how many different ways can a quiz be answered under each of the following conditions?*
(a). The quiz consists of three multiple-choice questions with four choices for each.

(b). The quiz consist of three multiple-choice questions(with four choices for each) and five true-false questions.

Solution

(a). Successively answering the three questions is a three-stage procedure. The first question can be answered in any of four ways. Likewise, each of the other two questions can be answered in four ways. By the Basic Counting Principle, the number of ways to answer the quiz is

$$4.4.4 = 64$$

(b). Answering the quiz can be considered a two-stage procedure. First we can answer the multiple-choice questions(first stage) and then we can answer the true-false questions(second stage). From part (a), the three multiple-choice questions can be answered in $4.4.4 = 64$ ways. Each of the true-false questions has two choices (`true` or `false`). So the total number of ways of answering all five of them is $2.2.2.2.2 = 32$. By the BCP, the number of ways the entire quiz can be answered is

$$64.32 = 2048.$$

Example 3.4 ***(Letter arrangements)*** *From the five letters A,B,C,D and E, how many three-letter horizontal arrangements (called "words") are possible if no letter may be repeated?*

Solution

To form a word we must successively fill the positions the first, second and third position/slots with letters. This is a three-stage procedure. For the first position, we can choose any of the five letters. After filling the first position with some letter, we can fill the second position with any of the remaining four letters. After that position is filled, the third position can be filled with any of the three letters that have not yet been used. By the Basic Counting Principle, the total number of three-letter words is

$$5.4.3 = 60.$$

3.3 Permutations

In Example 3.4, we selected three different letters from five letters and arranged them in an order. Each result is called a permutation of the five letters taken three at a time.

Definition 3.1 *An ordered arrangement of r objects, without repetition, selected from n distinct objects is called a* ***permutation*** *of the n objects taken r at a time.*

The number of such permutations is denoted $P(n, r)$ or ${}^{n}_{r}$P. For instance Example 3.4 can be solved as: $P(5, 3) = 5.4.3 = 60$.
Using the Basic Counting Principle, we can calculate the number of ways n different objects can be arranged in a specified order. The first object may be selected in n ways, the second in $(n - 1)$ ways, the third in $(n - 2)$, and so on until only one choice is remaining for the last object.So the total number of possible arrangements is

$$n(n - 1)(n - 2) \cdots 3.2.1$$

This expression is normally abbreviated to $n!$ (read as n `factorial` or `factorial` n). Thus the number of permutations of n different objects is $n!$

3.3.1 General Formula for $P(n, r)$

In making an ordered arrangement of r objects from n objects,for the first position we may choose any of the n objects. After the first position is filled, there remain $n - 1$ objects that may be chosen from for the second position. After that position is filled, there are $n - 2$ objects that may be chosen for the third position. Continuing in this way and using the Basic Counting Principle, we arrive at the following formula

$$P(n, r) = \underbrace{n(n - 1)(n - 2) \cdots (n - r + 1)}_{r \quad factors} \tag{3.1}$$

The formula $P(n, r)$ can be expressed in terms of factorials. Multiplying the right side of equation (3.1) by $\frac{(n-r)(n-r-1)\cdots 2\times 1}{(n-r)(n-r-1)\ldots 2\times 1}$ gives

$$P(n, r) = \frac{n(n - 1)(n - 2) \cdots (n - r + 1).(n - r)(n - r - 1) \cdots 2 \times 1}{(n - r)(n - r - 1) \cdots 2 \times 1}$$

The numerator is simply $n!$ and the denominator is $(n - r)!$ Thus we have the following result: The number of permutations of n objects taken r at a time is given by

$$P(n, r) = \frac{n!}{(n - r)!}$$

Example 3.5 *A club has 20 members. The offices of president, vice president, secretary and treasurer are to be filled, and no member may serve in more than one office. How many different slates of candidates are possible?*

Solution We shall consider a slate in the order of president, vice president, secretary and treasurer. Each ordering of four members constitutes a slate, so the number of possible slates is

$$P(20,4) = \frac{20!}{(20-4)!} = \frac{20!}{16!} = \frac{20 \times 19 \times 18 \times 17 \times 16!}{16!} = 20.19.18.17. = 116,280$$

Example 3.6 *In how many ways can 10 people sit at a round table?*

Solution On a round table we assume that the position of the people relative to the table is of no consequence, unless a special place of honour (head of table) is specified. This means that if all the occupants were to move one place to the left, or one place to the right, the new arrangement would still be regarded as the same arrangement, as each occupant would be finding the other occupants in the same places relative to him/her. We fix one person. The other nine can then be arranged in 9! ways or 362,880 ways.

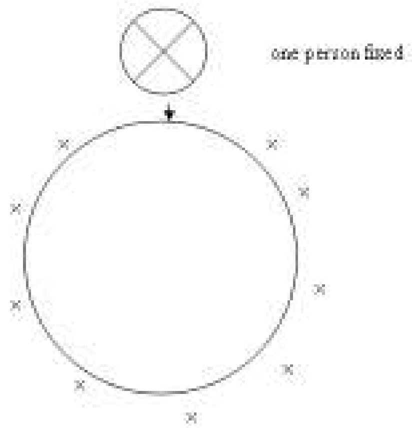


Figure 3.2: Arrangement on a round table

3.3.2 Permutations of Repeated Objects

So far we have discussed permutations of objects that were all different.

Example 3.7 *Determine the number of different permutations of the seven letters in the word **SUCCESS**.*

Solution The letters C nd S are repeated. If the two Cs were interchanged, the resulting permutation would be indistinguishable from SUCCESS. Thus, the number of distinguishable permutations is not 7! as it would be with 7 different objects.
We now give a formula to enable us solve similar problems.

Theorem 3.2 ***(Permutations with Repeated Objects)****The number of distinguishable permutations of n objects such that n_1 are of one type, n_2 are of a second type,..., and n_k are of a kth type , where $n_1 + n_2 + \cdots + n_k = n$ is*

$$\frac{n!}{n_1!\ n_2!\ \cdots\ n_k!}$$

Using this formula, the answer to Example 3.6 is easy to find. Note that the word SUCCESS has 7 characters: 3 Ss, 2 Cs, 1 U and 1 E. Therefore there are $\frac{7!}{3!2!1!1!} = 420$ distinguishable words.

Example 3.8 *(a).How many distinguishable arrangements are there of the word* ***MASSASAUGA****?*
(b). How many of these arrangements are the four A's together?

Solution(a). Massasauga is a white venomous snake indigenous in North America. Note that this word has 10 characters, some repeated: 4 As, 3 S, 1 M, 1 U, 1 G . Therefore there are

$$\frac{10!}{4!\ 3!\ 1!\ 1!\ 1!} = 25,200$$

possible arrangements.
(b). If we tie the four A's they form one object, denote it $\mathcal{A}$. Now we have 7 characters with some characters repeated: 1 $\mathcal{A}$, 3 Ss, 1 M, 1 U, 1 G . Therefore there are

$$\frac{7!}{3!\ 1!\ 1!\ 1!\ 1!} = 840$$

arrangements in which all four A's are together.

Remark 3.1

Cells or Compartments
Sometimes we want to find the number of ways in which objects can be placed into "compartments" or cells. For example, suppose that from a group of five people, three

are to be assigned to room A and two to room B. In how many ways can this be done? Obviously, order in which people are placed into the rooms is of no concern. The cells remind us of those permutations with repeated objects.
Hence the five people can be assigned in

$$\frac{5!}{3!\ 2!} = 10$$

ways.

Example 3.9 *A campaign manager must assign 15 campaigners to three matatus: 6 in the first matatu, 5 in the second, and 4 in the third. In how many ways can this be done.*

Solution Here people are placed into three cells (matatus): 6 in cell1, 5 in cell2, 4 in cell 3. Thus there are

$$\frac{15!}{6!\ 5!\ 4!} = 630,630$$

ways.

3.4 Combinations

Definition 3.2 *An arrangement of r objects, without regard to order and without repetition, selected from n distinct objects is called a* ***combination*** *of n objects taken r at a time.*

The number of such combinations is denoted by ${}^{n}{}_{r}\mathrm{C}$ or $n\ \mathrm{C}_r$ or $C(n,r)$ or $\binom{n}{r}$. Note that ABC, BAC is one combination but two permutations.

3.4.1 Comparing Combinations and Permutations

Example 3.10 *List all combinations and all permutations of the three letter A,B C when they are taken two at a time.*

Solution
The combinations are : AB, AC, BC
There are 3 combinations, so ${}^{3}{}_{2}\mathrm{C} = 3$.
The permutations are:

$AB \quad BC \quad AC$

$BA \quad CB \quad CA$

Thus there are 6 permutations.

With this observation, we can determine a formula for $\binom{n}{r}$. Suppose one such combination is

$$x_1 x_2 \cdots x_r$$

The number of permutations of these r objects is $r!$ Listing all such combinations and all permutations of these combinations, we obtain a list of permutations of the n objects taken r at a time. Thus

$$^{n}_{r}\mathrm{C}.\ r! = P(n, r) = \frac{n!}{(n-r)!}$$

Solving for $^{n}_{r}\mathrm{C}$ we get

$$^{n}_{r}\mathrm{C} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!\ r!}$$

Example 3.11 *If a club has 20 members, how many different four-member committees are possible?*

Solution

Order is not important because no matter how the members of a committee are arranged, we have the same committee. Thus, we simply have to compute the number of combinations of 20 objects taken four at a time,

$$^{20}_{4}\mathrm{C} = \frac{20!}{(20-4)!\ 4!} = 4845.$$

Example 3.12 *In how many ways can a class of 20 children be split into two groups of 8 and 12 respectively if there are two twins in the class who must not be separated?*

Solution Once the group of 8 has been selected then the remaining 12 children will automatically comprise the other group.

For the selection of those to join the group of 8 we have two cases:

(i). the twins are included,

(ii). the twins are not included.

$\odot$ If the twins are included we have to select 6 other children from 18, i.e. this can be done in $\binom{18}{6}$ ways.

$\odot$ If the twins are excluded we have to select 8 children, but still from 18 children, i.e.

this can be done in $\binom{18}{8}$ ways.
The total number of ways will be

$$\binom{18}{6} + \binom{18}{8} = \frac{18!}{12!\ 6!} + \frac{18!}{10!\ 8!} = 18,564 + 43,758 = 62,322$$

ways.

3.4.2 Combinations with Repetition

When we wish to select r objects out of n objects, with repetition allowed/permitted the number of combinations is

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!\ (n-1)!}$$

Example 3.13 *A donut shop offers 20 kinds kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop, we can select a dozen donuts in*

$$C(20+12-1, 12) = C(31, 12) = 141, 120.525$$

ways.

Example 3.14 *Four family members have just completed lunch and are ready to choose their afternoon fruit. There are bananas, apples, pears, kwi, apricots, and oranges in the house. In how many ways can a selection of four pieces of fruit be chosen?*

Solution
Note that only the selection of varieties (not which person eats what fruit) is of interest here. This is a combination with repetition problem. Thus the solution is

$$C(6, 4-1, 4) = C(9, 4) = \frac{9!}{4!\ 5!} = 126.$$

3.5 Problems involving Both Permutations and Combinations

Some problems require the concepts of both permutations(arrangements) and combinations.

Example 3.15 *Consider the word **TALLAHASSEE**.*
(a). How many distinguishable arrangements of the word are possible?
(b). How many of these arrangements have no adjacent A's?

Solution

(a). The number of distinguishable arrangements is

$$\frac{11!}{3!\ 2!\ 2!\ 2!\ 1!\ 1!} = 831,600.$$

(b). When we disregard the A's, there are

$$\frac{8!}{2!\ 2!\ 2!\ 1!\ 1!} = 5040$$

ways to arrange the remaining letters. One of these 5040 ways is shown in the following figure, where the arrows indicate nine possible locations for the three A's.

$$\begin{array}{ccccccccc} \mathbf{E} & \mathbf{E} & \mathbf{S} & \mathbf{T} & \mathbf{L} & \mathbf{L} & \mathbf{S} & \mathbf{H} & \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{array}$$

Figure 3.3: Possible locations for the A's

Three of these locations can be selected in $\binom{9}{3} = 84$ ways, and because this is also possible for all the other 5039 arrangements of E,E,E,T,L,L,S,H, by the Basic Counting Principle, there are $5040 \times 84 = 423,360$ arrangements of the letters in **TALLAHASSEE** with no consecutive A's.

Example 3.16 *How many committees of five people can be chosen from 20 men and 12 women*

(a). if exactly three men must be on each committee?

(b). if at least four women must be on each committee?

Solution

(a). We must choose three men from 20 and two women from 12. This can be done in

$$\binom{20}{3}\cdot\binom{12}{2} = 1140 \times 66 = 75,240$$

different ways.

(b). We calculate the cases of four women and five women separately and add the result. The answer is

$$\binom{12}{4}\cdot\binom{20}{1} + \binom{12}{5}\cdot\binom{20}{0} = 495(20) + 792 = 10,692$$

ways.

3.6 Applications of Combinations

3.6.1 Binomial Theorem

Theorem 3.3 ***(Binomial Theorem)*** *For any x and y any natural number n,*

$$\begin{aligned}(x+y)^n &= \textstyle\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \\ &= x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n-1}xy^{n-1}+y^n\end{aligned}$$

There are $\binom{n}{r}$ ways in which r brackets can be chosen from n brackets, so the term containing x^r is ${}^{n}_{r}Cx^ry^{n-r}$

Example 3.17 *(a). Use the binomial theorem to expand and simplify $(1+x)4$.*
(b). Use your result to approximate $(1.1)^4$.

Solution
(a).

$$\begin{aligned}(1+x)^4 &= 1+\binom{4}{1}x+\binom{4}{2}x^2+\binom{4}{3}x^3+\binom{4}{1}x^4 \\ &= 1+4x+6x^2+4x^3+x^4\end{aligned}$$

(b). Note that $1.1=1+0.1$. Therefore

$$(1.1)^4=(1+0.1)^4=1+0.4+0.06+0.004+0.0001=1.4641$$

Example 3.18 *Prove that*

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots\binom{n}{n}=2^n$$

for all natural numbers n.

Solution
Using Theorem 3.3 with $x=y=1$, we obtain

$$(1+1)^n=\sum_{k=0}^{n}\binom{n}{k}1^{n-k}1^k=\sum_{k=0}^{n}\binom{n}{k}$$

That is

$$2^n=\sum_{k=0}^{n}\binom{n}{k}$$

as desired.

3.7 Exercises

1. Determine the values of ${}^{6}_{4}\mathrm{P}$, ${}^{100}_{100}\mathrm{C}$, ${}^{99}_{2}\mathrm{C}$

2. Verify that $\binom{n}{r} = \binom{n}{n-r}$.

3. In a 10-question examination, each question is worth 10 points and is marked right or wrong. Considering the individual questions, in how many ways can a student score 80 or better?

4. In a certain African country, vehicle license plates consist of two letters of the alphabet, followed by four digits from 0 to 9, 0 and 9 included.

(a). How many different license plates are there if the second letter on the plate is either an '**O**' or a '**Q'** and the last digit is either a 3 or an 8?

(b). How many license plates are there if the first letter must be a '**K**' and the second letter must be an '**A**' and the last digit must be 5?

5. Because of overcrowding, five people out of nine people can enter an elevator. How many different groups can enter?

6. A carton contains 24 light bulbs, one of which is defective.

(a). In how many ways can three bulbs be selected?

(b). In how many ways can three bulbs be selected if one is defective.

7. How many distinguishable horizontal arrangements are there of the letters in the word **MISSISSIPPI**?

8. A group of tourists is composed of six from Nairobi, seven from London and eight from New York.

(a). In how many ways can a committee of six tourists be formed with two people from each city?

(b). In how many ways can a committee of seven tourists be formed with at least two tourists from each city?

9.(a). Find the Binomial expansion of $(2x + 3y)^n$.

(b). Find the coefficient of the fifth term.

(c). Use you result to find 4^5.

10. In how many ways can the letters in **WONDERING** be arranged with exactly two consecutive vowels?

11. Francesca has 20 different books but the shelf in her dormitory residence will hold only 12 of them.

(a). In how many ways can Francesca line up 12 of these books on her bookshelf?

(b). How many of the arrangements in part (a) include Francesca's three books on Calculus?

12. (a) Write down the first three terms in the expansion in ascending powers of x of

(i). $(1 - \frac{x}{2})^{10}$

(ii). $(3 - 2x)^8$

(b). Write down the Binomial expansion for $(1+x)^{20}$, and use it to approximate $(1.01)^{20}$, leaving your answer in 5 decimal places.

13. There are 8 persons, including a married couple Mr and Mrs Bush, from which a committee of 4 has to be chosen. In how many ways can the committee be chosen

(a). If both Mr and Mrs Bush are excluded,

(b). if Mrs Bush is included and Mr. Bush is excluded.

(c). if both Mr and Mrs Bush are included?

14. (a). Let n be a non-negative integer. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$$

(b). Let $n, r \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

(c). Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

[`Hint`: $\binom{n}{k}^2 = \binom{n}{k}.\binom{n}{k}$. Now replace the second $\binom{n}{k}$ with an equivalent expression]. (d). Prove that

$$r\binom{n+1}{r} = (n+1)\binom{n}{r-1}$$

15. How many code symbols can be formed using 5 out of the 6 letters G, H, I, J, K, L if the letters

(a). cannot be repeated?

(b). can be repeated?

(c). cannot be repeated but must begin with **K**.

(d). cannot be repeated but must end with **IGH**?

Chapter 4

RELATIONS AND FUNCTIONS

4.1 Introduction

In this chapter we extend the concepts of sets to include the concepts of relation and function. Relationships between elements of sets occur in many contexts. Such relationships are represented using the structure called a relation. Relations can be used to solve problems such as determining which pairs of cities are linked by airline flights in a network, or producing a useful way to store information in computer databases.

4.2 Cartesian Products and Relations

Recall that for sets A and B, the Cartesian product of A and B is

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

We say that the elements of $A \times B$ are ordered pairs.

Definition 4.1 *For sets A, B, any subset of $A \times B$ is called a* ***binary relation*** *from A to B. Any subset of $A \times A$ is called a binary relation on A.*

The most direct way to express a relationship between two sets is to use ordered pairs made up of two related elements. Thus a binary relation from A to B is a set $\mathcal{R}$ of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

Example 4.1 *With $A = \{2, 3, 4\}$ and $B = \{4, 5\}$, the following are some of the relations from A to B.*
(a). $\emptyset$
(b). $\{(2, 4)\}$
(c). $\{(2, 4), (2, 5)\}$
(d). $A \times B$

We use the notation $a\mathcal{R}b$ to denote that $(a, b) \in \mathcal{R}$ and $a \not\mathcal{R} b$. When $(a, b) \in \mathcal{R}$, a is said to be `related` to b by $\mathcal{R}$.

Example 4.2 *Let A be the set of cities, and B be the set of East African countries. Define the relation $\mathcal{R}$ by specifying that (a, b) belongs to $\mathcal{R}$ if city a is in country b. For instance, (Nairobi, Kenya), (Mombasa, Kenya), (Kigali, Rwanda), (Dar es salaam, Tanzania), (Kampala, Uganda), etc are to the relation $\mathcal{R}$.*

Example 4.3 *Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\mathcal{R} = \{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means for instance that $0\mathcal{R}a$, $1\mathcal{R}a$, etc.*

Relations can be represented graphically, using arrows to represent ordered pairs.

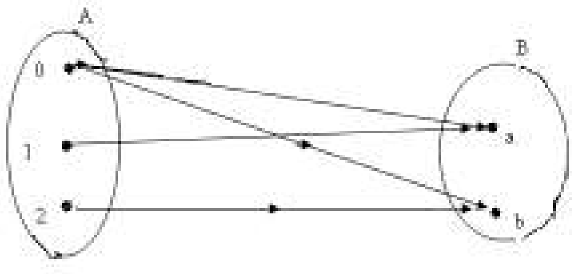


Figure 4.1: Graph of the relation in Example 4.3

4.2.1 Relations on a Set

Relations from a set A to itself are of special interest.

Definition 4.2 *A relation on a set A is a relation from A to A.*

Example 4.4 *Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation*
$\mathcal{R} = \{(a, b) : a \text{ divides } b\}$*?*

Solution
Since (a, b) is in $\mathcal{R}$ if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$\mathcal{R} = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$$

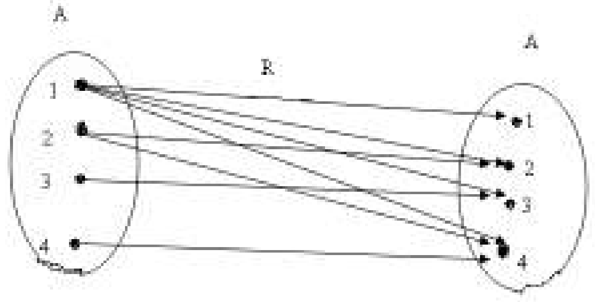


Figure 4.2: Graph of the relation in Example 4.4

Example 4.5 *How many relations are there on a set A with n elements?*

Solution
A relation on A is a subset of $A \times A$. Since $A \times A$ has n^2 elements, when A has n elements and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements.

4.3 Properties of Relations

There are several properties that are used to classify relations on a set.

Definition 4.3 *A relation $\mathcal{R}$ on a set A is said to be **reflexive** if for all $x \in A$, $(x, x) \in \mathcal{R}$.*

Example 4.6 *For $A = \{1, 2, 3, 4\}$, a relation $\mathcal{R} \subseteq A \times A$ will be reflexive if and only if $\mathcal{R} \supseteq \{(1,1),(2,2),(3,3),(4,4)\}$.*

Consequently, $\mathcal{R}_1 = \{(1,1),(2,2),(3,3)\}$ is not a reflexive relation on $A = \{1,2,3,4\}$ since $4 \in A$ but $(4,4) \notin \mathcal{R}_1$.
$\mathcal{R}_2 = \{(x, y) : x, y \in A, x \leq y)\}$ is reflexive on $A = \{1, 2, 3, 4\}$.

Example 4.7 *Consider the following relations on $A = \{1, 2, 3, 4\}$.*

$$\mathcal{R}_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$$

$$\mathcal{R}_2 = \{(1,1),(1,2),(2,1))\}$$

$$\mathcal{R}_3 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,1),(4,4)\}$$

Which of these relations is/are reflexive?

Solution

Only $\mathcal{R}_3$ is reflexive.

Definition 4.4 *A relation $\mathcal{R}$ on a set A is said to be **symmetric** if $(x, y) \in \mathcal{R}$ implies that $(y, x) \in \mathcal{R}$ for $x, y \in A$.*

Definition 4.5 *A relation $\mathcal{R}$ on a set A is said to be **antisymmetric** if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ implies that $x = y$ for all $x, y \in A$.*

Note that the terms symmetric and antisymmetric are not opposites since a relation can have both of these properties or may lack both of them.

Example 4.8 *With $A = \{1, 2, 3\}$, we have*
$\mathcal{R}_1 = \{(1,2),(2,1),(1,3),(3,1)\}$ *is symmetric but not reflexive on A. Reasons: $(1,1),(2,2),(3,3)$ are not in $\mathcal{R}_1$.*
$\mathcal{R}_2 = \{(1,1),(2,2),(3,3),(2,3)\}$ *is reflexive but not symmetric on A. Reason: $(2,3) \in \mathcal{R}_2$ but $(3,2) \notin \mathcal{R}_2$*
$\mathcal{R}_3 = \{(1,1),(2,2),(3,3)\}$ *and* $\mathcal{R}_4 = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$ *are both reflexive and symmetric.*
$\mathcal{R}_5 = \{(1,1),(2,3),(3,3)\}$ *is neither reflexive nor symmetric on A.*

Example 4.9 *The relation "`divides`" on the set of positive integers is not symmetric since, for instance, 1 divides 2 but 2 does not divide 1. It is antisymmetric, for if a and b are positive integers such that a divides b and b divides a, then $a = b$.*

Definition 4.6 *A relation $\mathcal{R}$ on a set A is said to be **transitive** if whenever $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ then $(x, z) \in \mathcal{R}$ for all $x, y, z \in A$.*

That is, if x "is related to" y and y "is related to" z, then x "is related to" z, with y playing the role of intermediary.

Example 4.10 *The "**divides**" relation on positive integers is transitive, since if a divides b and b divides c, then there exists positive integers k and l such that* $b = ak$ *and* $c = bl$. *Hence* $c = bl = akl$, *so a divides c.*

Example 4.11 *If* $A = \{1, 2, 3, 4\}$, *then* $\mathcal{R}_1 = \{(1, 1), (2, 3), (3, 4), (2, 4)\}$ *is a transitive relation on A, whereas* $\mathcal{R}_2 = \{(1, 3), (3, 2)\}$ *is not transitive because* $(1, 3), (3, 2) \in \mathcal{R}_2$ *but* $(1, 2) \notin \mathcal{R}_2$.

Definition 4.7 *A relation* $\mathcal{R}$ *on a set A is said to be an* ***equivalence relation*** *if it is reflexive, symmetric and transitive.*

Definition 4.8 *A relation* $\mathcal{R}$ *on a set A is said to be a* ***partial order*** *or a* ***partial ordering*** *if it is reflexive, antisymmetric and transitive.*

Example 4.12 *If* $A = \{1, 2, 3\}$, *then*
$\mathcal{R}_1 = \{(1, 1), ((2, 2), (3, 3)\}$
$\mathcal{R}_2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$
$\mathcal{R}_3 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$ *and*
$\mathcal{R}_4 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ *are equivalence relations on A.*

Example 4.13 *Let* $\mathcal{R}$ *be the relation on the set of real numbers* $\mathbb{R}$ *such that* $a\mathcal{R}b$ *iff* $a - b$ *is an integer. Is* $\mathcal{R}$ *an equivalence relation?*

Solution
Since $a - a = 0$ is an integer for all real numbers a , $a\mathcal{R}a$. Hence $\mathcal{R}$ is reflexive.
Now, suppose that $a\mathcal{R}b$. Then $a - b$ is an integer, so $b - a$ is also an integer. Hence, $b\mathcal{R}a$. It follows that $\mathcal{R}$ is symmetric.
If $a\mathcal{R}b$ and $b\mathcal{R}c$, then $a - b$ and $b - c$ are integers. Therefore $a - c = (a - b) + (b - c)$ is also an integer. Hence $a\mathcal{R}c$. Thus, $\mathcal{R}$ is transitive.

4.4 Combining Relations

Since relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined as in Chapter 1.

Example 4.14 *Let* $A = \{1, 2, 3\}$ *and* $B = \{1, 2, 3, 4\}$. *The relations* $\mathcal{R}_1 = \{(1, 1), (2, 2), (3, 3)\}$ *and* $\mathcal{R}_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ *can be combined to obtain*

$$\mathcal{R}_1 \cup \mathcal{R}_2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$$

$$\mathcal{R}_1 \cap \mathcal{R}_2 = \{(1, 1)\}$$

$$\mathcal{R}_2 - \mathcal{R}_1 = \{(1, 2), (1, 3), (1, 4)\}$$

Example 4.15 *Let A and B be the set of students and the set of all courses at a certain University, respectively. Suppose that $\mathcal{R}_1$ consists of all ordered pairs (a, b), where a is a student who has taken course b, and $\mathcal{R}_2$ consists of all ordered pairs (a, b), where a is a student who requires course b to graduate. What are the relations $\mathcal{R}_1 \cup \mathcal{R}_2$, $\mathcal{R}_1 \cap \mathcal{R}_2$, $\mathcal{R}_1 \triangle \mathcal{R}_2$, $\mathcal{R}_1 - \mathcal{R}_2$ and $\mathcal{R}_2 - \mathcal{R}_1$?*

Solution

⊙ The relation $\mathcal{R}_1 \cup \mathcal{R}_2$ consists of all ordered pairs (a, b), where a is a student who either has taken course b or needs course b to graduate.

⊙ The relation $\mathcal{R}_1 \cap \mathcal{R}_2$ is the set of all ordered pairs (a, b) where a is a student who has taken course b and needs this course to graduate.

⊙ The relation $\mathcal{R}_1 \triangle \mathcal{R}_2$ consists of all ordered pairs (a, b) where student a has taken course b but does not need it to graduate or needs course b to graduate but has not taken it.

⊙ The relation $\mathcal{R}_1 - \mathcal{R}_2$ is the set of all ordered pairs (a, b), where student a has taken course b but does not need it to graduate. That is, b is an elective course that a has taken.

⊙ The relation $\mathcal{R}_2 1 \mathcal{R}_1$ consists of all ordered pairs (a, b), where b is a course that student a needs to graduate but has not taken.

4.5 Functions

Definition 4.9 *Let A and B be nonempty sets. A* ***function*** *or a* ***mapping*** *or* ***map*** *f from A to B, denoted $f : A \longrightarrow B$ is a binary relation that associates to each element $a \in A$ a unique element $f(a) \in B$. That is, in which every element of A appears exactly once as the first element of an ordered pair in the relation. That is, for every $a \in A$, there is exactly one $b \in B$ such that $(a, b) \in f$.*
The set A is called the ***domain*** *of f and the set B is called the* ***co-domain*** *of f and is denoted $Dom(f)$.*

If $(a, b) \in f$ the element $b \in B$ is called the **image** of $a \in A$ under f, and is written $b = f(a)$. That is

$$Im(f) = \{b : (a, b) \in f,\ for\ some\ a \in A\}$$

Example 4.16 *For $A = \{1, 2, 3\}$ and $B = \{w, x, , z\}$, $f = \{(1, w), (2, x), (3, x)\}$ is a function from A to B, while $g = \{(1, w), (2, w), (2, x), (3, z)\}$ is a relation but not a function from A to B, because $2 \in A$ appears twice as a first component of the ordered pairs.*

Definition 4.10 *Let $f : A \longrightarrow B$ be a function. The subset of B of those elements that appear as second components in the ordered pairs of f is called the* ***range*** *of f and is denoted by $Ran(f)$ or $f(A)$. Thus, $Ran(f)$ is the set of images of the elements of A under f.*

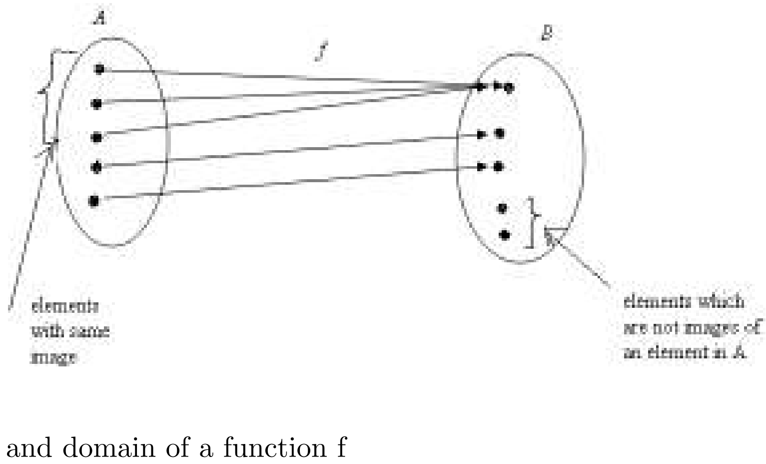


Figure 4.3: Range and domain of a function f

Remark 4.1

Let $f : X \longrightarrow Y$ be a relation. If every value of x is associated with exactly one value of y, then y is said to be a function of x and we write $y = f(x)$. It is customary to use x for what is called the independent variable and y as the dependent variable. This means that y depends on x.

Example 4.17 *From Example 4.16,* $Dom(f) = \{1, 2, 3\}$, $Ran(f) = \{w, x\}$.

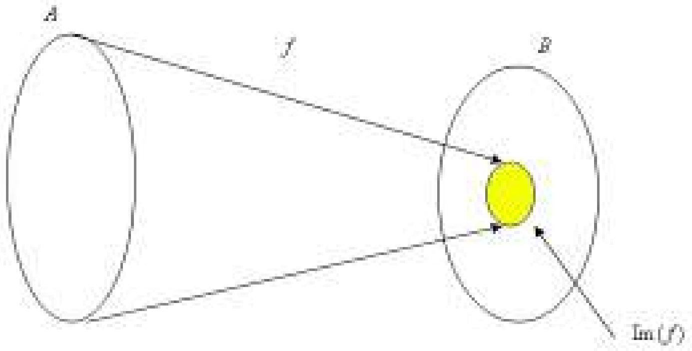


Figure 4.4: Image of a function f

Clearly, the domain of a function $f : X \longrightarrow Y$ is the set of possible values for the independent variable and the set of all possible values for the dependent variable is the range of f. In other words

$$Dom(f) = \{x \in X : y = f(x) \text{ is defined}\}$$

and

$$Ran(f) = \{y \in Y : y = f(x) \text{ for some } x \in Dom(f)\}$$

Example 4.18 *The image set or range of* $f : \mathbb{N} \longrightarrow \mathbb{N}$ *defined by* $f(n) = 2^n$ *is* $\{2, 4, 8, 16, 32, ...\}$.

Example 4.19 *Find the range of* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *defined by* $f(x) = \frac{3x}{x^2+1}$.

Solution

$y \in Ran(f)$ iff $y = \frac{3x}{x^2+1}$ for some $x \in \mathbb{R}$

$\Longleftrightarrow \quad yx^2 + y = 3x$ *or* $yx^2 - 3x + y = 0$. Using the quadratic formula, we have, provided $y \neq 0$

$$x = \frac{3 \pm \sqrt{9 - 4y^2}}{2y}$$

For this to have a real solution we require $y \neq 0$ and $9 - 4y^2 \geq 0$. Hence $y^2 \leq \frac{9}{4}$ $(y \neq 0)$, which means that

$$\frac{-3}{2} \leq y \leq \frac{3}{2} \quad \textit{and } y \neq 0$$

Hence $Ran(f) = [-\frac{3}{2}, \frac{3}{2}] = \{y \in \mathbb{R} : \frac{-3}{2} \leq y \leq \frac{3}{2}\}$.

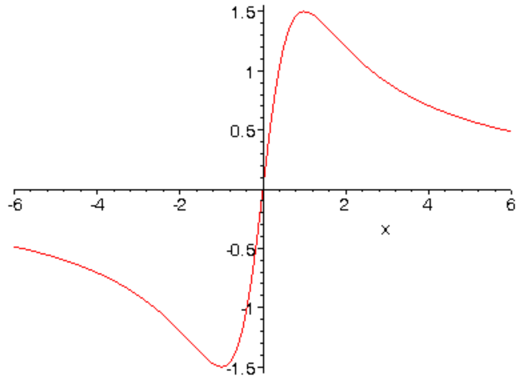


Figure 4.5: Range of f

4.6 Types of Functions

In this section we consider special types of functions.

4.6.1 One-to-one Functions and Many-to-one Functions

Definition 4.11 *Let A and B be sets. A function $f : A \longrightarrow B$ is called **one-to-one** or an **injection** or an **injective map** if each element of B appears at most once as image of an element of A. That is, f is one-to-one if and only if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies that $a_1 = a_2$. Equivalently, taking the contrapositive, $a_1 \neq a_2$ implies that $f(a_1) \neq f(a_2)$.*

Example 4.20 *Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = 3x + 7$ for all $x \in \mathbb{R}$. We find that*

$$f(x_1) = f(x_2) \Longrightarrow 3x_1 + 7 = 3x_2 + 7 \Longrightarrow 3x_1 = 3x_2 \Longrightarrow x_1 = x_2$$

So f is one-to-one.

Example 4.21 *Suppose that $g : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $g(x) = x^4 - x$ for each $x \in \mathbb{R}$. Then $g(0) = 0$ and $g(1) = 0$. Consequently, g is not one-to-one because there exist real numbers x_1, x_2 where*

$$g(x_1) = g(x_2) \not\Longrightarrow x_1 = x_2. \tag{4.1}$$

Functions which satisfy (4.1) are called **many-to-one** functions.

4.6.2 Onto Functions

Definition 4.12 *Let A and B be sets. A function $f : A \longrightarrow B$ is called **onto** or **surjective** or a **surjection** if $f(A) = B$. That is, if for all $b \in B$, there is at least one $a \in A$ with $f(a) = b$. That is, if $Ran(f) = Co - domain\ of\ f$.*

Example 4.22 *Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = x^3$ for all $x \in \mathbb{R}$. Clearly f is onto.*

Example 4.23 *The function $g : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = x^2$ for all $x \in \mathbb{R}$ is not onto. No negative numbers appear in $Ran(f)$. Here $Ran(f) = [0, \infty) \neq \mathbb{R} = Co - dom(f)$.*

4.6.3 Bijective Functions

Definition 4.13 *Let A and B be sets. A function $f : A \longrightarrow B$ is called **bijective** or a **one-to-one correspondence** if f is both one-to-one and onto.*

Example 4.24 *Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = x + 2$ for all $x \in \mathbb{R}$. Clearly f is a bijection, since it is both one-to-one and onto.*

4.7 Composition and Inverses of Functions

Definition 4.14 *Let A be a set. The function $f : A \longrightarrow A$, defined by $f(a) = a$ for all $a \in A$, is called the **identity function**. The identity function on A is usually denoted by 1_A.*

Definition 4.15 *Let A and B be sets and let $f, g : A \longrightarrow B$ be functions. We say that f and g are **equal** and write $f = g$, if $f(a) = g(a)$ for all $a \in A$.*

4.7.1 Composition of Functions

Definition 4.16 *Let A, B and C be sets. If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions, we define the **composite function**, which is denoted by $g \circ f : A \longrightarrow C$, by $(g \circ f)(a) = g\Big(f(a)\Big)$ for each $a \in A$.*

Example 4.25 *Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ and $C = \{w, x, y, z\}$, with $f : A \longrightarrow B$ and $g : B \longrightarrow C$, given by $f = \{(1, a), (2, a), (3, b), (4, c)\}$ and $g = \{(a, x), (b, y), (c, 2)\}$. For each element of A we find that:*
$(g \circ f)(1) = g(f(1)) = g(a) = x$
$(g \circ f)(2) = g(f(2)) = g(a) = x$
$(g \circ f)(3) = g(f(3)) = g(b) = y$
$(g \circ f)(4) = g(f(4)) = g(a) = z$

Note that the composition $f \circ g$ is not defined in Example 4.25.

Example 4.26 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^2$ and $g(x) = x + 5$. Then*

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 5,$$

where as

$$(f \circ g)(x) = f(g(x)) = f(x + 5) = (x + 5)^2 = x^2 + 10x + 25$$

Note that $f \circ g \neq g \circ f$ in general. That is, composition of functions is not, in general, commutative.

Theorem 4.1 *Let A, B and C be sets and $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions.*
(a). If f and g are one-to-one then $g \circ f$ is one-to-one.
(b). If f and g are onto then $g \circ f$ is onto.

Proof
(a). To prove that $g \circ f : A \longrightarrow C$ is one-to-one, let $a_1, a_2 \in A$ with $(g \circ f)(a_1) = (g \circ f)(a_2)$. Then $(g \circ f)(a_1) = (g \circ f)(a_2) = g(f(a_1)) = g(f(a_2)) \Longrightarrow f(a_1) = f(a_2)$, because g is one-to-one.
Also, $f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$, because f is one-to-one. Consequently, $g \circ f$ is one-to-one.
(b). If $g \circ f : A \longrightarrow C$, let $z \in C$. Since g is onto, there exists $y \in B$ with $g(y) = z$. With f onto and $y \in B$, there is $x \in A$ with $f(x) = y$. Hence $z = g(y) = g(f(x)) = (g \circ f)(x)$. So $Ran(g \circ f) = C = co - domain\ of\ g \circ f$. This proves that $g \circ f$ is onto.

Remark 4.2 *Note that function composition is, in general, an associative operation. That is $(h \circ g) \circ f = h \circ (g \circ f)$.*

4.7.2 Inverse of a Function

Definition 4.17 *Let A and B be sets and $f : A \longrightarrow B$ be a function. We say that f is invertible if there is a function $g : B \longrightarrow A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.*

Theorem 4.2 *A function $f : A \longrightarrow B$ is invertible if and only if it is one-to-one and onto. The inverse relation*

$$f^{-1} = \{(b, a) : (a, b) \in f\}$$

from B to A will be called the ***inverse*** *of f and pronounced "f* ***inverse***".

Thus f and g in Definition 4.17 are inverses of each other.

Theorem 4.3 *If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are invertible functions then $g \circ f : A \longrightarrow C$ is invertible and*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Remark 4.3 *To find the inverse of an invertible function $f : X \longrightarrow Y$, defined by*

$$y = f(x) \tag{4.2}$$

We solve (4.2) for x (i.e. we make x the subject of the formula), then swap the roles of x and y.

Example 4.27 *If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z, t\}$, and $f = \{(1, x), (2, y), (3, z), (4, t)\}$ then $f^{-1} = \{(x, 1), (y, 2), (z, 3), (t, 4)\}$.*

Note that $(f^{-1})^{-1} = f$.

Example 4.28 *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = 2x - 3$ and $g(x) = \frac{x+1}{5}$, respectively. Find f^{-1} and g^{-1}.*

Solution
It is easy to verify that both f and g are bijective and so each has an inverse. According to (4.2)
$2x = y+3$. Thus $x = \frac{1}{2}(y+3)$. We now swap the roles of x and y. Thus $f^{-1}(x) = \frac{1}{2}(x+3)$.
Similarly, $x + 1 = 5y$. Thus $x = 5y - 1$. We now swap the roles of x and y to get $g^{-1}(x) = 5x - 1$.

Definition 4.18 *To each function $f : A \longrightarrow B$, there corresponds the relation in $A \times B$ given by $\{(a, f(a)) : a \in A\}$. We call this set the **graph** of f.*

4.7.3 The Vertical Line Test for a Function

A curve in the (x, y)-plane is the graph of some function $f : A \longrightarrow \mathbb{R}$ where $A \subseteq \mathbb{R}$ if and only if the following condition is satisfied.

$\odot$ *`Every vertical line in the plane meets the curve at most once.`*

4.8 Some Special Real Valued Functions

The following are some special functions.

1. Identity Function

$f : R \longrightarrow \mathbb{R}$, defined by $f(x) = x$, for all $x \in \mathbb{R}$.

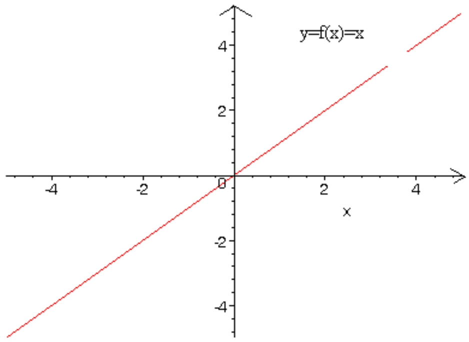


Figure 4.6: Graph of the Identity function

2. Constant Functions

$f : X \longrightarrow \mathbb{R}$ defined by $f(x) = c$, where $x \in X$ and c is a fixed real number.

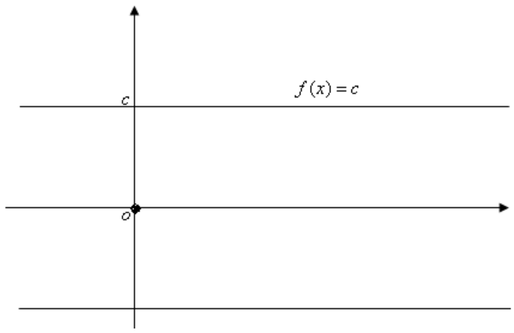


Figure 4.7: Graph of the constant function

3. Linear Functions

$f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = mx + c$, where $m \neq 0$ and $c \in \mathbb{R}$ is a constant. Graphs of such functions are straight lines with gradient m and y-intercept c. The range of f is $\mathbb{R}$ itself. Note that the identity and constant functions are linear functions.

4. Quadratic Functions

$f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = ax^2 + bx + c$, $a \neq 0$, $\quad b, c \in \mathbb{R}$ constants. The graphs of these functions are parabolas. They are very useful in describing supply and demand curves, cost, revenue and profit curves in economics.

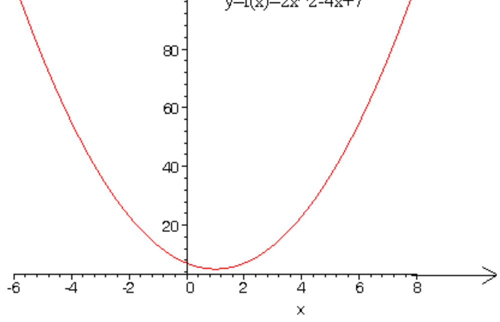


Figure 4.8: Graph of the quadratic function $f(x) = 2x^2 - 4x + 7$

5. Reciprocal Functions

$f(x) = \frac{1}{x}$

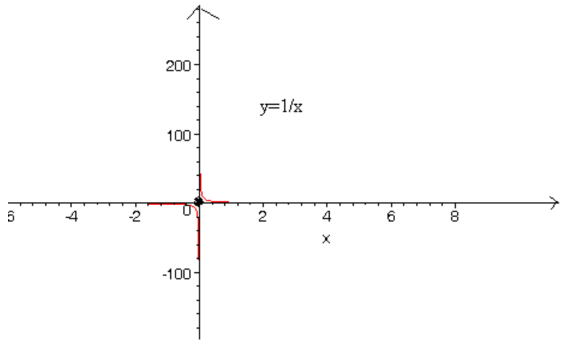


Figure 4.9: Graph of the reciprocal function $f(x) = \frac{1}{x}$

6. Absolute Value Functions

These are functions of the form $f(x) = a|\beta x + c|$. For instance, when $a = 1, \beta = 1$ and $c = 0$, we have

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

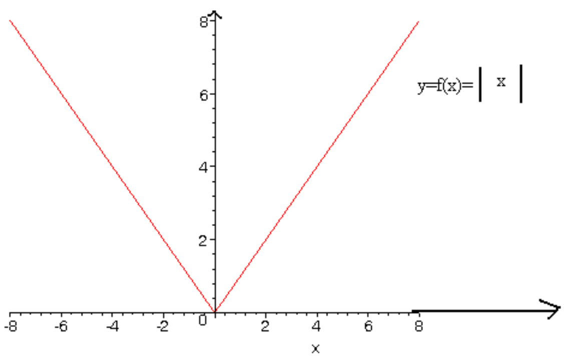


Figure 4.10: Graph of the Absolute Value function $f(x) = |x|$

7. Polynomial Functions

$f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_0, a_1, \cdots, a_n \in \mathbb{R}$ are called `coefficients`, is a real-valued polynomial. If $a_n \neq 0$, we call it a `polynomial of degree` n; $n \geq 0$ integer. For $n = 1$, a polynomial function takes the form $f(x) = a_0 + a_1 x$, a linear function. Thus a linear function is a polynomial of degree 1. A linear form $f(x) = a_0$ for $a \in \mathbb{R}$ is a polynomial function of degree 0 and is a constant function. A polynomial of degree 2 is a quadratic, while a cubic is a polynomial of degree 3.

8. Floor and Ceiling Functions

The function $f : \mathbb{R} \longrightarrow \mathbb{Z}$ defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer less or equal to x is called the **floor function** or the **greatest integer function**. For instance $[0] = 0$, $[0.5] = 0$, $[\pi] = 3$, $[-0.2] = -1$. The greatest integer function is also defined by $f(x) = \llcorner x \lrcorner$.

The function $f : \mathbb{R} \longrightarrow \mathbb{Z}$ defined by $f(x) =]x[$, where $]x[$ denotes the smallest integer greater or equal to x is called the **ceiling function** or the **smallest integer function**. For instance $]0[= 0$, $]0.5[= 1$, $]\pi[= 4$, $]-0.2[= 0$.

The smallest integer function is also defined by $f(x) = \ulcorner x \urcorner$.

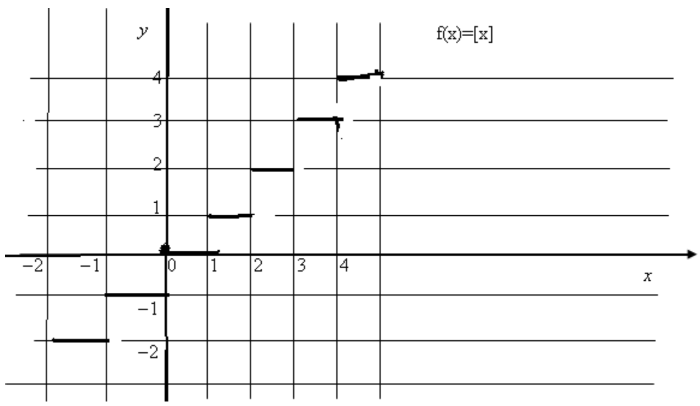


Figure 4.11: Graph of the the Floor function $f(x) = [x]$

Note that the floor and ceiling functions are onto but not one-to-one.

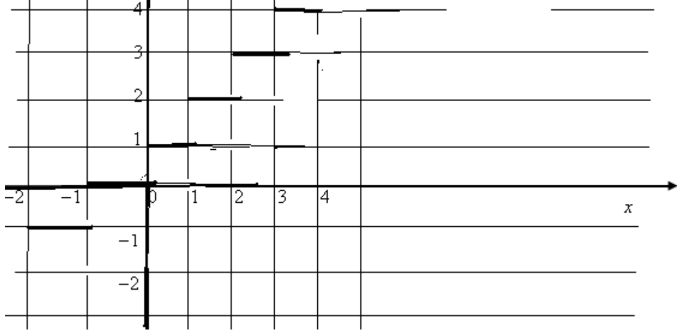


Figure 4.12: Graph of the the Ceiling function $f(x) =]x[$

9. Exponential and Logarithmic Functions

The **exponential function** to base b (for $b > 0,\ b \neq 1$) is the function $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ where $f(x) = b^x$, where $\mathbb{R}^+$ is the set of positive real numbers.

The **logarithm function** to base b (for $b > 0,\ b \neq 1$) is the function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by $g(x) = \log_b x$. The logarithm function to base b is the inverse of the exponential function to base b; that is,

$\log_b x = y$ iff $b^y = x$.

The **common logarithm function** $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by $g(x) = \log_{10} x$, also written log is the inverse of the exponential function to base 10; that is $\log_{10} x = y$ when $10^y = x$.

The **natural logarithm function** or **Naperian function** $g : \mathbb{R}^+ \longrightarrow \mathbb{R}$ defined by $g(x) = \ln(x)$, where is the inverse of the exponential function to base e; that is $\ln x = y$ when $e^y = x$, where $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.718281828459$.

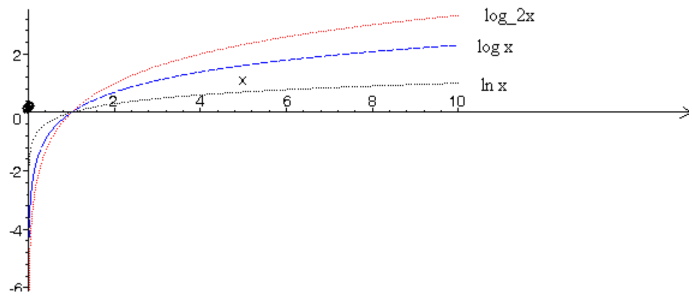


Figure 4.13: Graphs of some logarithm functions

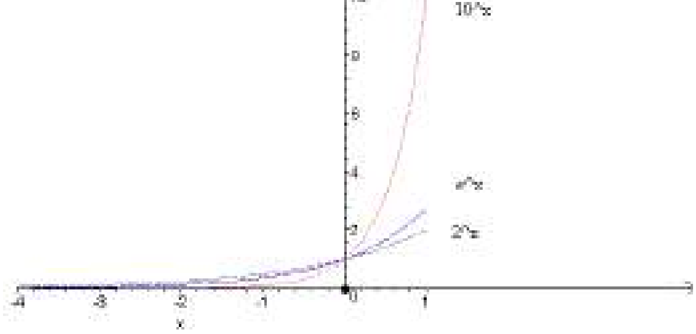


Figure 4.14: Graphs of some exponential functions

Note that

- ⊙ $\log_b(xy) = \log_b x + \log_b y$
- ⊙ $\log_b(x^y) = y \log_b x$
- ⊙ $\log_b(b^x) = x$
- ⊙ $\log_b x = \frac{\log_a x}{\log_a b}$

4.9 Some Classification of Real Valued Functions

4.9.1 Even Functions

Definition 4.19 *An **even function** $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function for which $f(-x) = f(x)$ for every $x \in \mathbb{R}$.*

The graph of an even function is symmetrical about the y-axis.

Example 4.29 $y = f(x) = x^2, \quad y = g(x) = |x|$ *and* $y = h(x) = \cos x$ *are examples of even functions.*

4.9.2 Odd Functions

Definition 4.20 *An **odd function** $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function for which $f(-x) = -f(x)$ for every $x \in \mathbb{R}$.*

Example 4.30 $y = f(x) = x, \quad y = g(x) = x^3$ *and* $y = h(x) = \sin x$ *are examples of odd functions.*

4.9.3 Functions which are Neither Even nor Odd

There exist functions which are neither even nor odd. An example is $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^3 - 3x + 1$.

4.9.4 Algebraic Functions

Functions which are made up of powers of variables and constants connected by the signs $+, -, \times, \div$ are classified as **algebraic functions**.

4.9.5 Irrational Functions and Rational Functions

If radical signs or fractional indices occur in the definition, the function is said to be **irrational** if not **rational**. A function is rational if it can be expressed as $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions. All other functions are **transcendental**. Sines, cosines, tangents, etc are called **trigonometric** or **circular functions**. Functions such as $\sin^{-1} x$, $\tan^{-1} x$, etc are called **inverse trigonometric functions**. Besides these functions we also have the **hyperbolic functions**: $\sinh x$, $\cosh x$, etc.

Example 4.31 *Show that the functions $f : \mathbb{R} \longrightarrow (1, \infty)$ and $g : (1, \infty) \longrightarrow \mathbb{R}$ defined by $f(x) = 10^{2x} + 1$ and $g(x) = \frac{1}{2}\log_{10}(x-1)$ are inverses of each other.*

Solution

We show that $f \circ g(x) = g \circ f(x) = x$.

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f\Big(\tfrac{1}{2}\log_{10}(x-1)\Big) \\ &= 10^{2[\frac{1}{2}\log_{10}(x-1)]} + 1 \\ &= 10^{\log_{10}(x-1)} + 1 \\ &= x - 1 + 1 \\ &= x \end{aligned}$$

and

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g\Big(10^{2x} + 1\Big) \\ &= \tfrac{1}{2}\log_{10}(10^{2x} + 1 - 1) \\ &= \tfrac{1}{2}\log_{10}(10^{2x}) \\ &= \tfrac{1}{2}(2x) \\ &= x \end{aligned}$$

Thus g and f are inverses of each other.

4.10 Solved Problems

1. Let $f, g, h, l(x) : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions defined by $f(x) = x^2$, $g(x) = \frac{-4x}{2x-3}$, $h(x) = \sqrt{6-x}$, $l(x) = \frac{1}{x+3}$. Find the domain and range of each of these functions

Solution

$Dom(f) = \mathbb{R}$,

$Ran(f) = [0, \infty)$

$Dom(g) = \{x \in \mathbb{R} : x \neq \frac{3}{2}\} = \mathbb{R} - \{\frac{3}{2}\}$

$Ran(g) = \{y \in \mathbb{R} : y \neq 0\} = \mathbb{R} - \{0\}$

$Dom(h) = (-\infty, 6]$, since for y to be a real number, $6 - x$ must be non-negative. This happens only when $6 - x \geq 0$ or $6 \geq x$.

$Ran(h) = [0, \infty)$, because $\sqrt{6-x}$ is always non-negative.

$Dom(l) = \mathbb{R} - \{-3\}$

$Ran(l) = \mathbb{R} - \{0\}$

2. Determine whether or not each of the following relations is a function. If a relation is a function, find its range.

(a). $\{(x, y) : x, y \in \mathbb{Z},\ y = x^2 + 7\}$

(b). $\{(x, y) : x, y \in \mathbb{R},\ y^2 = x\}$

(c). $\{(x, y) : x, y \in \mathbb{R},\ y = 3x + 1\}$

(d). $\{(x, y) : x, y \in \mathbb{Q},\ x^2 + y^2 = 1\}$

Solution

(a). It is a relation and a function from $\mathbb{Z}$ to $\mathbb{Z}$; $Ran(f) = \{7, 8, 11, 16, 23, \cdots\}$

(b). This is a relation but not a function.

(c). This is a relation and a function from $\mathbb{R}$ to $\mathbb{R}$; $Ran(f) = \mathbb{R}$

(d). This is a relation from $\mathbb{Q}$ to $\mathbb{Q}$ but not a function.

3. For each of the following functions, determine whether it is one-to-one, onto and determine its range.

(a). $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, defined by $f(x) = 2x + 1$

(b). $f : \mathbb{Q} \longrightarrow \mathbb{Q}$, defined by $f(x) = 2x + 1$

(c). $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, defined by $f(x) = x^3 - x$

(d). $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = e^x$

(e). $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow \mathbb{R}$, defined by $f(x) = \sin x$

Solution

(a). f is one-to-one and not onto; the range of f is the set of odd integers.

(b). f is one-to-one and onto; $Ran(f) = \mathbb{Q}$.

(c). f is not one-to-one and not onto; $Ran(f) = \{0, \pm 6, \pm 24, \pm 60, \cdots\} = \{n^3 - n : n \in \mathbb{Z}\}$.

(d). f is one-to-one but not onto; $Ran(f) = (0, \infty)$.

(e). f is one-to-one and not onto; $Ran(f) = [0, 1]$.

4. Determine whether or not each of the following functions is a bijection. If a bijection, find its inverse.

(a). $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{x-3}{7}$

(b). $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = log_e x$.

Solution

Easy and hence left as an exercise.

4.11 Exercises

1. Find the domain and range of the following functions

(a). $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = \sqrt{2x^2 + 5x - 12}$

(b). $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $f(x) = \frac{x}{x^2-5}$

2. Let f, g and h be the functions defined as follows

$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = x^2 - 5,$

$g : \mathbb{Z} \longrightarrow \mathbb{R}, \quad g(x) = \frac{5x}{x^2-2}$

$h : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = [x].$

Find the following

(a). $f(3)$

(b). $g(h(4.7))$

(c). $(f \circ h)(x)$

(d). $f \circ (h \circ g)(x)$

(e). $f \circ (h \circ g)(-7)$

3. Let $A = \{humans, livingordead\}$ and let f and g be the functions $A \longrightarrow A$ defined by $f(x) = the\ father\ of\ x$ and $g(x) = the\ mother\ of\ x$, respectively. Describe the composite functions $f \circ f$, $f \circ g$, $g \circ f$ and $g \circ g$.

4. Show that each of the following is a bijection and find its inverse.

(a). $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $\frac{5x+3}{8}$

(b). $f : \mathbb{R} - \{-1\} \longrightarrow \mathbb{R} - \{3\}$, defined by $f(x) = \frac{3x}{x+1}$

5. Show that the functions $f : \mathbb{R} \longrightarrow (1, \infty)$ and $g : (1, \infty) \longrightarrow \mathbb{R}$ defined by $f(x) = 3^{2x} + 1$ and $g(x) = \frac{1}{2} \log_3(x - 1)$ are inverses of each other.

6. Let $\mathbb{W}, \mathbb{Z}$ denote the set of whole numbers and integers, respectively. Give an example of:

(a). a one-to-one function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is not onto.

(b). a many-to-one function $g : \mathbb{Z} \longrightarrow \mathbb{W}$ which is onto.

Chapter 5

TRIGONOMETRY

Trigonometry is the branch of mathematics of measuring (or determining) the sides and angles of a triangle by means of information given about some of the sides and angles. An angle can be defined as the amount of rotation between one straight line and another. Trigonometry is based on certain ratios, called trigonometric functions, which are very useful in surveying, navigation and engineering. These functions also play an important role in the study of vibratory phenomena such as sound, light, electricity, etc. In this chapter we concentrate our focus on trigonometric equations, identities and simplification of trigonometric expressions.

5.1 Radian and Degree Measure of an Angle

Units commonly used for measuring angles are radians or degree measures. The radian measure is employed in advanced mathematics and in many branches of science.

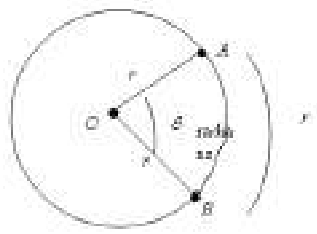


Figure 5.1: Radian Measure of an Angle

Let AB be an arc on the circle of length r. We define the magnitude of angle AOB

which the arc AB subtends at the center as one radian. Since circumference of a circle is $2\pi r$, it subtends at the center an angle of $\frac{Circumference}{Radius\ of\ circle} = 2\pi$ radians. But we know the number of degree in a circle is $360°$. Hence 2π radians $= 360°$, which means that π radians $= 180°$. Thus 1 radian $= \frac{180°}{\pi} \approx 57.3°$. Thus $1° = \frac{\pi}{180}$ radians.

5.2 Trigonometric Ratios

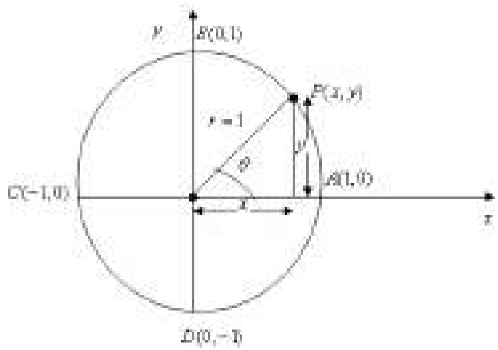


Figure 5.2: Trigonometric ratios

Let (x, y) be the coordinates of point P in Fig 5.2. Define two relations: sine and cosine of angle θ as follows:

$$sine : \theta \longrightarrow y, \qquad \theta \in \mathbb{R}$$

by $y = \sin\theta$ and

$$cosine : \theta \longrightarrow x, \qquad \theta \in \mathbb{R}$$

by $x = \cos\theta$ For any general radius r, we have that $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$, which means that $y = r\sin\theta$ and $x = r\cos\theta$ and $x^2 + y^2 = r^2$.
We define another ratio: $\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$.
We also define other ratios:
cosecant θ (or $\csc\theta$ in abbreviated form)$= \frac{1}{\sin\theta}$.
secant θ (or $\sec\theta$ in abbreviated form)$= \frac{1}{\cos\theta}$.
cotangent θ (or $\cot\theta$ in abbreviated form)$= \frac{1}{\tan\theta}$.

5.3 Trigonometric Identities

5.3.1 Cofunction Identities

Let θ be a certain angle. We have

$$\begin{array}{rclcrcl} \sin\theta & = & \cos(90^\circ - \theta) & & \cos\theta & = & \sin(90^\circ - \theta) \\ \tan\theta & = & \cot(90^\circ - \theta) & & \cot\theta & = & \tan(90^\circ - \theta) \\ \sec\theta & = & csc(90^\circ - \theta) & & \csc\theta & = & \sec(90^\circ - \theta) \end{array}$$

We also have that

$$\begin{array}{rcll} \cos(-\theta) & = & \cos\theta & (cosine\ function\ is\ an\ even\ function) \\ \sin(-\theta) & = & -\sin\theta & (sine\ function\ is\ an\ odd\ function) \\ \tan(-\theta) & = & -\tan\theta & (tangent\ function\ is\ an\ odd\ function) \end{array}$$

5.3.2 Pythagorean Identities

From Figure 5.2, using the right-angled triangle, we have that $x^2+y^2 = 1$ is the equation of the unit circle in the xy-plane. For any point on the unit circle, the coordinates x and y satisfy the equation $x = \cos\theta,\ \ y = \sin\theta$. Substituting in the equation of the unit circle, we have that

$$\cos^2\theta + \sin^2\theta = 1 \tag{5.1}$$

Dividing (5.1) by $\cos^2\theta$, we have

$$1 + tan^2\theta = \sec^2\theta \tag{5.2}$$

Dividing (5.1) by $\sin^2\theta$, we have

$$1 + \cot^2\theta = \csc^2\theta \tag{5.3}$$

We can factor, simplify and manipulate trigonometric expressions in the same way we manipulate strictly algebraic expressions.

Example 5.1 *Simplify* $\cos\theta(\tan\theta - \sec\theta)$.

Solution

$$\begin{aligned}\cos\theta(\tan\theta - \sec\theta) &= \cos\theta\tan\theta - \cos\theta\sec\theta \\ &= \cos\theta\tfrac{\sin\theta}{\cos\theta} - \cos\theta\tfrac{1}{\cos\theta} \\ &= \sin\theta - 1\end{aligned}$$

Remark 5.1 *Note that there is no general procedure for manipulating trigonometric expressions, but it is often helpful to write everything in terms of sines and cosines and apply known identities, where necessary.*

Example 5.2 *Simplify* $\sin^2 x\cos^2 x + \cos^4 x$.

Solution

$$\begin{aligned}\sin^2 x\cos^2 x + \cos^4 x &= \cos^2 x(\sin^2 x + \cos^2 x) \\ &= \cos^2 x(1) \\ &= \cos^2 x\end{aligned}$$

Example 5.3 *Simplify each of the following trigonometric expressions*

(a). $\frac{\cot(-\theta)}{\csc(-\theta)}$.

(b). $\frac{2\sin^2\theta+\sin\theta-3}{1-\cos^2\theta-\sin\theta}$.

Solution

(a).

$$\begin{aligned}\frac{\cot(-\theta)}{\csc(-\theta)} &= \frac{\frac{\cos(-\theta)}{\sin(-\theta)}}{\frac{1}{\sin(-\theta)}} \\ &= \frac{\cos(-\theta)}{\sin(-\theta)}\sin(-\theta) \\ &= \cos(-\theta) \\ &= \cos\theta\end{aligned}$$

(b).

$$\begin{aligned}\frac{2\sin^2\theta+\sin\theta-3}{1-\cos^2\theta-\sin\theta} &= \frac{2\sin^2\theta+\sin\theta-3}{\sin^2\theta-\sin\theta} \\ &= \frac{(2\sin\theta+3)(\sin\theta-1)}{\sin\theta(\sin\theta-1)} \\ &= \frac{2\sin\theta+3}{\sin\theta} \\ &= 2+\frac{3}{\sin\theta} \text{ or } 2+3\csc\theta\end{aligned}$$

5.3.3 Sum and Difference Identities

We now develop some important identities involving sum and difference of two angles. For any given angles A and B, we have

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{5.4}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{5.5}$$

Writing $-B$ for B in (5.4) and (5.5) we get

$$\sin(A-B) = \sin A\cos B - \cos A\sin B \tag{5.6}$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B \tag{5.7}$$

Dividing (5.4) by (5.5) gives

$$\begin{aligned}\tan(A+B) &= \tfrac{\sin(A+B)}{\cos(A+B)} \\ &= \tfrac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B} \\ &= \tfrac{\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}} \\ &= \tfrac{\tan A+\tan B}{1-\tan A\tan B} \qquad (\mathit{on\ dividing\ each\ term\ by}\ \cos A\cos B\ \mathit{and\ simplifying})\end{aligned} \tag{5.8}$$

Replacing B by $-B$ in (5.8) gives

$$\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B} \tag{5.9}$$

Adding (5.5) and (5.7) gives

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

or

$$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)] \tag{5.10}$$

Adding (5.4) and (5.6) gives

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

or

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)] \tag{5.11}$$

Subtracting (5.6) from (5.4) we have

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

or

$$\cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)] \tag{5.12}$$

Subtracting (5.7) from (5.5) gives

$$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$$

or

$$\sin A\sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)] \tag{5.13}$$

Identities (5.10), (5.11), (5.12) and (5.13) are called **products of sines and cosines identities**.

From the above identities, it is easy to derive the following identities.

$$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) \tag{5.14}$$

$$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B) \tag{5.15}$$

$$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B) \tag{5.16}$$

$$\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B) \tag{5.17}$$

Proof

Let $A+B=\alpha$ and $A-B=\beta$ so that $A=\frac{1}{2}(\alpha+\beta)$ and $B=\frac{1}{2}(\alpha-\beta)$. Then

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

Thus

$$\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$$

This proves (5.14).

The other identities can be proved similarly and are left as exercises.

Example 5.4 *Express each of the following as a sum or difference*

(a). $\sin 40^\circ \cos 30^\circ$.

(b). $\cos 110^\circ \sin 55^\circ$

Solution

(a).

$$\begin{aligned} \sin 40^\circ \cos 30^\circ &= \tfrac{1}{2}[\sin(40^\circ + 30^\circ) + \sin(40^\circ - 30^\circ)] \\ &= \tfrac{1}{2}(\sin 70^\circ + \sin 10^\circ) \end{aligned}$$

(b).

$$\begin{aligned} \cos 110^\circ \sin 55^\circ &= \tfrac{1}{2}[\sin(110^\circ + 55^\circ) - \sin(110^\circ - 55^\circ)] \\ &= \tfrac{1}{2}(\sin 165^\circ - \sin 55^\circ) \end{aligned}$$

Example 5.5 *Express each of the following as a product.*

(a). $\sin 50^\circ + \sin 40^\circ$

(b). $\sin 70^\circ - \sin 20^\circ$

Solution

(a).

$$\begin{aligned} \sin 50^\circ + \sin 40^\circ &= 2\sin\tfrac{1}{2}(50^\circ + 40^\circ)\cos\tfrac{1}{2}(50^\circ - 40^\circ) \\ &= 2\sin 45^\circ \cos 5^\circ \end{aligned}$$

(b).

$$\begin{aligned} \sin 70^\circ - \sin 20^\circ &= 2\cos\tfrac{1}{2}(70^\circ + 20^\circ)\sin\tfrac{1}{2}(70^\circ - 20^\circ) \\ &= 2\cos 45^\circ \sin 25^\circ \end{aligned}$$

Example 5.6 *Prove that*

$$\frac{\sin 4A + \sin 2A}{\cos 4A + \cos 2A} = \tan 3A$$

Solution

$$\begin{aligned} \tfrac{\sin 4A+\sin 2A}{\cos 4A+\cos 2A} &= \tfrac{2\sin\frac{1}{2}(4A+2A)\cos\frac{1}{2}(4A-2A)}{2\cos\frac{1}{2}(4A+2A)\cos\frac{1}{2}(4A-2A)} \\ &= \tfrac{\sin 3A}{\cos 3A} \\ &= \tan 3A \end{aligned}$$

Example 5.7 *Prove that*

$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$$

Solution

$$\begin{aligned}
\frac{\sin A-\sin B}{\sin A+\sin B} &= \frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} \\
&= \cot\tfrac{1}{2}(A+B)\tan\tfrac{1}{2}(A-B) \\
&= \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}
\end{aligned}$$

Example 5.8 *Prove that*

$$1+\cos 2x+\cos 4x+\cos 6x = 4\cos x\cos 2x\cos 3x$$

Solution

$$\begin{aligned}
1+\cos 2x+\cos 4x+\cos 6x &= 1+(\cos 2x+\cos 4x)+\cos 6x \\
&= 1+2\cos 3x\cos x+\cos 6x \\
&= (1+\cos 6x)+1+2\cos 3x\cos x \\
&= 2\cos^2 3x+1+2\cos 3x\cos x \\
&= 2\cos 3x(\cos 3x+\cos x) \\
&= 2\cos 3x(2\cos 2x\cos x) \\
&= 4\cos x\cos 2x\cos 3x
\end{aligned}$$

5.3.4 Double-Angle Identities

In (5.4), (5.5) and (5.8), if we let $A = B = x$, we have

$$\begin{aligned}
\sin 2x &= \sin x\cos x+\cos x\sin x \\
&= 2\sin x\cos x
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
\cos 2x &= \cos x\cos x-\sin x\sin x \\
&= \cos^2 x-\sin^2 x
\end{aligned} \tag{5.19}$$

and in a similar manner

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} \tag{5.20}$$

Equation (5.19) can also be written as

$$\begin{aligned}
\cos 2x &= \cos x\cos x-\sin x\sin x \\
&= \cos^2 x-\sin^2 x \\
&= 1-\sin^2 x-\sin^2 x \\
&= 1-2\sin^2 x \\
&= 2\cos^2 x-1
\end{aligned} \tag{5.21}$$

From (5.20) and (5.21), we obtain the following identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \tag{5.22}$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \tag{5.23}$$

Dividing these two we have

$$\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x} \tag{5.24}$$

Example 5.9 *Find an equivalent expression for each of the following:*
(a). $\sin 3\theta$ *in terms of function values of* θ*.*
(b). $\cos^3 x$ *in terms of values of* x *or* $2x$*, raised only to the first power.*

Solution
(a).

$$\begin{aligned}\sin 3\theta &= \sin(2\theta+\theta)\\ &= \sin 2\theta\cos\theta + \cos 2\theta \sin\theta\\ &= (2\sin\theta\cos\theta)\cos\theta + (2\cos^2\theta - 1)\sin\theta\\ &= 2\sin\theta\cos^2\theta + 2\sin\theta\cos^2\theta - \sin\theta\\ &= 4\sin\theta\cos^2\theta - \sin\theta\end{aligned}$$

(b).

$$\begin{aligned}\cos^3 x &= \cos^2 x\cos x\\ &= (\tfrac{1+\cos 2x}{2})\cos x\\ &= \tfrac{\cos x+\cos x\cos 2x}{2}\end{aligned}$$

5.3.5 Half-Angle Identities

In (5.22), (5.23) and (5.24), if we replace x with $\frac{x}{2}$ and take square roots, we get

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \tag{5.25}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \tag{5.26}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\sin x}{\sin x} \tag{5.27}$$

The use of + or − depends on the quadrant in which the angle $\frac{x}{2}$ is.
Also using the formula (5.20) and writing $2x = \theta$ and so $x = \frac{\theta}{2}$, we have

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \quad or \quad \frac{2t}{1-t^2}$$

where $t = \tan\frac{\theta}{2}$.
It is possible to find formulae for $\sin\theta$ and $\cos\theta$ in terms of t using aright-angled triangle. It is easy to show that

$$\sin\theta = \frac{2t}{1+t^2}$$

and

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

Example 5.10 *Find* $\tan\frac{\pi}{8}$

Solution
$\tan\frac{\pi}{8} = \tan\frac{\frac{\pi}{4}}{2} = \frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2+\sqrt{2}} = \sqrt{2}-1$

Example 5.11 *Simplify each of the following*
(a). $\frac{\sin x\cos x}{\frac{1}{2}\cos 2x}$
(b). $2\sin^2\frac{x}{2} + \cos x$

Solution
(a). Multiply the numerator and denominator by $\frac{2}{2}$ and simplify using known identities to get

$$\begin{aligned} \frac{2\sin x\cos}{\cos 2x} &= \frac{\sin 2x}{\cos 2x} \\ &= \tan 2x \end{aligned}$$

(b).
$$\begin{aligned} 2\sin^2\frac{x}{2} + \cos x &= 2(\tfrac{1-\cos x}{2}) + \cos x \\ &= 1 - \cos x + \cos x \\ &= 1 \end{aligned}$$

5.4 Proving Identities

We can use some known identities to prove other identities.

Example 5.12 *Prove the identity*

$$1 + \sin 2\theta = (\sin\theta + \cos\theta)^2$$

Solution

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= 1+2\sin\theta\cos\theta\\ &= 1+\sin 2\theta\end{aligned}$$

We could also begin with the left side and obtain the right side:

$$\begin{aligned}1+\sin 2\theta &= 1+2\sin\theta\cos\theta\\ &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= (\sin\theta+\cos\theta)^2\end{aligned}$$

Example 5.13 *Prove that*

$$\frac{1}{(csc\theta-\sin\theta)(\sec\theta-\cos\theta)}=\tan\theta+\cot\theta$$

Solution

We write the left hand side in terms of $\sin\theta$ and $\cos\theta$:

$$\begin{aligned}LHS &= \frac{1}{(\frac{1}{\sin\theta}-\sin\theta)(\frac{1}{\cos\theta}-\cos\theta)}\\ &= \frac{1}{\frac{(1-\sin^2\theta)}{\sin\theta}\frac{(1-\cos^2\theta)}{\cos\theta}}\\ &= \frac{\sin\theta\cos\theta}{(1-\sin^2\theta)(1-\cos^2\theta)}\\ &= \frac{\sin\theta\cos\theta}{\cos^2\theta\sin^2\theta}\\ &= \frac{1}{\sin\theta\cos\theta}\end{aligned}$$

We now write the right hand side in terms of $\sin\theta$ and $\cos\theta$:

$$\begin{aligned}RHS &= \tan\theta+\cot\theta\\ &= \frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\\ &= \frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\\ &= \frac{\sin\theta\cos\theta}{\cos^2\theta\sin^2\theta}\\ &= \frac{1}{\sin\theta\cos\theta}=LHS\end{aligned}$$

5.5 Solving Trigonometric Equations

Example 5.14 *Solve the equation*

$$12\cos^2\theta+\sin\theta=11$$

on the domain $0^\circ\leq\theta\leq 360^\circ$

Solution

Recall that $\cos^2\theta = 1 - \sin^2\theta$.

$$\begin{aligned} 12(1-\sin^2\theta) + \sin\theta &= 11 \\ 12 - 12\sin^2\theta + \sin\theta &= 11 \\ \textit{or} \ -12\sin^2\theta + \sin\theta &= -1 \\ \textit{or} \ 12\sin^2\theta - \sin\theta = 1 \ \textit{or} \ 12\sin^2\theta - \sin\theta - 1 &= 0 \\ \textit{or} \ (3\sin\theta - 1)(4\sin\theta + 1) &= 0 \end{aligned}$$

$$\implies \quad \sin\theta = \tfrac{1}{3} \ \textit{or} \ \sin\theta = -\tfrac{1}{4}$$

$\implies \quad \theta = 19.47^\circ$ (`principal value`) *or* $\theta = 180^\circ - 19.47^\circ = 160.53^\circ$(`secondary value`) *or*

$\theta = -14.48^\circ$ (`principal value`) *or* -165.52° (`secondary value`)

The general solution is $\theta = -14.48^\circ + 360n^\circ$ or $-165.52^\circ + 360n^\circ$.

On domain $0^\circ \leq \theta \leq 360^\circ$

$\theta = 345.52^\circ$ *or* 194.48°.

So the complete solution on domain $0^\circ \leq \theta \leq 360^\circ$ is

$\theta = 19.47^\circ$ or $\theta = 160.53^\circ$

or

$\theta = 194.48^\circ$ or $\theta = 345.52^\circ$

Example 5.15 *Find the general solution of the equation*

$$12\sec^2\theta - 13\tan\theta - 9 = 0$$

Solution

$$\begin{aligned} 12\sec^2\theta - 13\tan\theta - 9 = 12(1+\tan^2\theta) - 13\tan\theta - 9 &= 0 \\ \textit{i.e.} \ 12 + 12\tan^2\theta - 13\tan\theta - 9 &= 0 \\ \textit{i.e.} \ 12\tan^2\theta - 13\tan\theta + 3 &= 0 \\ \textit{i.e.} \ (4\tan\theta - 3)(3\tan\theta - 1) &= 0 \\ \textit{i.e.} \ \tan\theta = \tfrac{3}{4} \ \textit{or} \ \tan\theta &= \tfrac{1}{3} \end{aligned}$$

giving a general solution

$$\theta = 36.87^\circ + 180n^\circ \ \textit{or} \ \theta = 18.43^\circ + 180n^\circ, \ n \in \mathbb{Z}.$$

Example 5.16 *Find all angles θ in the domain $-360^\circ \leq \theta \leq 360^\circ$ for which* $\sin\theta =$ *0.3.*

Solution

The principal value is $17.46°$. The secondary solution is in the second quadrant. That is, $180° - 17.46° = 162.54°$.

One general solution is $17.46° + 360n°$.

We now try various values of n, starting with $0, \pm1, \pm2$, until we find the solutions exceeding the limits of the given domain.

$n = 0, \quad \theta = 17.46°$

$n = 1, \quad \theta = 377.46° \quad$ *too large*

$n = -1 \quad \theta = -342.54°$

$n = -2 \quad \theta = -702.54° \quad$ *too large on the negative side*

So far the general solution $°17.46° + 360n°$ has yielded solutions

$$\theta = 17.46°, \quad -342.54°.$$

The other general solution is $162.54° + 360n°$

$n = 0, \quad \theta = 162.54°$

$n = 1, \quad \theta = 522.54° \quad$ *too large*

$n = -1 \quad \theta = -197.46°$

$n = -2 \quad \theta = -557.46° \quad$ *too large on the negative side*

So the other general solution $162.54° + 360n°$ has yielded solutions

$$\theta = 162.54°, \quad -197.46°.$$

Hence the solution on the domain $-360° \leq \theta \leq 360°$ is

$$\theta = 17.46° \textit{ or } 162.54° \textit{ or } -197.46°.$$

Example 5.17 *Solve using two methods the equation* $\sin 6x - \sin 2x = 0$ *giving the general solution.*

Solution

This equation can be solved by writing $\sin 6x = \sin 2x$ and hence

$$6x = 2x + 2\pi n$$
$$\textit{or} \;\; 6x = \pi - 2x + 2\pi n, \;\; n \in \mathbb{Z}$$
$$\textit{leading to } x = \tfrac{\pi n}{2}$$
$$\textit{or } x = \tfrac{\pi}{8} + \tfrac{\pi n}{4}, \;\; n \in \mathbb{Z}$$

Alternatively using a difference of sines we have

$$\sin 6x - \sin 2x = 0$$
$$i.e.\quad 2\cos\tfrac{1}{2}(6x+2x)\sin\tfrac{1}{2}(6x-2x) = 0$$
$$i.e.\quad 2\cos 4x \sin 2x = 0$$
$$i.e.\quad \cos 4x = 0 \ or\ \sin 2x = 0$$
$$From\quad \cos 4x = 0$$
$$4x = \pm\tfrac{\pi}{2} + 2\pi n,\quad n\in\mathbb{Z}$$
$$or\qquad x = \pm\tfrac{\pi}{8} + \tfrac{\pi n}{2},\quad n\in\mathbb{Z}$$
$$From\quad \sin 2x = 0$$
$$2x = \pi n, n\in\mathbb{Z}$$
$$OR\qquad x = \tfrac{\pi n}{2}, n\in\mathbb{Z}$$

So the complete general solution is

$$x = \pm\frac{\pi}{8} + \frac{\pi n}{2}$$

or

$$x = \frac{\pi n}{2}, n\in\mathbb{Z}$$

It is a complicated process to show that these two different methods give us the same set of angles.

5.6 Exercises

1. Without using tables or a calculator, evaluate

(a). $\sin 105°$

(b). $\cos 255°$

(c). $\tan(-75°)$

2. Simplify the following expressions

(a). $\frac{\cos x}{1+\sin x} + \tan x$

(b). $\frac{4\tan x\sec x + 2\sec x}{6\tan x\sec x + 2\sec x}$

(c). $\frac{\csc(-x)}{\cot(-x)}$

(d). $\frac{\sin^4 x - \cos^4 x}{\sin^x - \cos^2 x}$

(e). $\sin(\frac{\pi}{2} - x)(\sec x - \cos x)$

(f). $\cos(\pi - x) + \cot x \sin(x - \frac{\pi}{2})$

(g). $\sin\theta \sec\theta \cot\theta$

(h). $\tan\theta + \frac{\cos\theta}{1+\sin\theta}$

(i). $\tan^2\theta \cos^2\theta + \cot^2\theta \sin^2\theta$

3. Find an equivalent expression for each of the following

(a). $\sec(x + \frac{\pi}{2})$

(b). $\cot(x - \frac{\pi}{2})$

(c). $\tan(x - \frac{\pi}{2})$

(d). $\csc(x + \frac{\pi}{2})$

4. Verify the following identities

(a). $\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x} = \sec x$

(b). $\sec^2\theta \csc^2\theta = \sec^2\theta + \csc^2\theta$

(c). $\frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{\sin\theta + 1}{\cos\theta}$

5. If $A + B + C = 180°$,(i.e. A, B, C are angles are angles of a triangle), prove that

(a). $\sin A + \sin B + \sin C = 4\cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C$

(b). $\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C$

(c). $\cos A + \cos B + \cos C = 1 + 4\sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C$

(d). $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$

(e). $\tan\frac{1}{2}A \tan\frac{1}{2}B + \tan\frac{1}{2}B \tan\frac{1}{2}C + \tan\frac{1}{2}C \tan\frac{1}{2}A = 1$

(`Hint`:(b). Write $LHS = \sin 2A + \sin 2B - \sin 2C = 2\sin(A+B)\cos(A-B) - \sin(360° - (2A + 2B))$ (switching out of C and using the fact that if $A + B + C = 180°$ then $2A + 2B + 2C = 360°$)) and that $\sin(360° - (2A + 2B) = -\sin(2A + 2B)$.

Thus $LHS = 2\sin(A+B)\cos(A-B) + \sin(2A+2B) = 2\sin(A+B)\cos(A-B) + 2\sin(A+B)\cos(A+B) = 2\sin(A+B)[\cos(A-B) + \cos(A+B)] = 2\sin(A+B)[2\cos A\cos(-B)] = 2\sin(180° - C).(2\cos A\cos B) = 2\sin C(2\cos A\cos B) = 4\cos A\cos B\sin C = RHS$.

6. Show that

(a). $\sin 40° + \sin 20° = \cos 10°$

(b). $\sin 105° + \sin 15° = \frac{\sqrt{6}}{2}$

(c). $\frac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°} = \frac{\sqrt{3}}{3}$

7. Prove that

(a). $\frac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$

(b). $\frac{\sin 2A + \sin 4A}{\cos 2A + \cos 4A} = \tan 3A$

(c). $\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A+B)}$

(d). $\frac{\cos A+\cos B}{\cos A-\cos B}=-\cot\frac{1}{2}(A-B)\cot\frac{1}{2}(A+B)$

(e). $\sin\theta+\sin 2\theta+\sin 3\theta=\sin 2\theta+(\sin\theta+\sin 3\theta)=\sin 2\theta(1+2\cos\theta)$

(f). $\cos\theta+\cos 2\theta+\cos 3\theta=\cos 2\theta(1+2\cos\theta)$

(g). $\sin 2\theta+\sin 4\theta+\sin 6\theta=(\sin 2\theta+\sin 4\theta)+2\sin 3\theta\cos 3\theta=4\cos\theta\cos 2\theta\sin 3\theta$

(h). $\frac{\sin 3x+\sin 5x+\sin 7x+\sin 9x}{\cos 3x+\cos 5x+\cos 7x+\cos 9x}=\tan 6x$

8. Prove that

(a). $\cos 130^\circ+\cos 110^\circ+\cos 10^\circ=0$

(b). $\cos 220^\circ+\cos 100^\circ+\cos 20^\circ=0$

Solutions to some selected problems

5(a). Note that $A+B+C=180^\circ$

$$\begin{aligned}
\sin A+\sin B+\sin C &= (\sin A+\sin B)+\sin C\\
&= 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)+\sin(180^\circ-(A+B))\\
&= 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)+\sin(A+B)\\
&= 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)+2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A+B)\\
&= 2\sin\tfrac{1}{2}(A+B)[\cos\tfrac{1}{2}(A-B)+\cos\tfrac{1}{2}(A+B)]\\
&= 2\sin\tfrac{1}{2}(A+B)[2\cos\tfrac{A}{2}\cos(\tfrac{-B}{2})]\\
&= 2\sin\tfrac{1}{2}(A+B)[2\cos\tfrac{A}{2}\cos(\tfrac{B}{2})]\\
&= 2\sin(90^\circ-\tfrac{C}{2})[2\cos\tfrac{A}{2}\cos(\tfrac{B}{2})]\\
&= 2\cos\tfrac{C}{2}.2\cos\tfrac{A}{2}.\cos\tfrac{B}{2} \qquad (since\ \ \sin(90^\circ-\tfrac{C}{2})=\cos\tfrac{C}{2})\\
&= 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C
\end{aligned}$$

LEARNING OBJECTIVES

After reading this Chapter a student will be able to understand —

- difference between permutation and combination for the purpose of arranging different objects;
- number of permutations and combinations when r objects are chosen out of n different objects.
- meaning and computational techniques of circular permutation and permutation with restrictions.

5.1 INTRODUCTION

In this chapter we will learn problem of arranging and grouping of certain things, taking particular number of things at a time. It should be noted that (a, b) and (b, a) are two different arrangements, but they represent the same group. In case of arrangements, the sequence or order of things is also taken into account.

The manager of a large bank has a difficult task of filling two important positions from a group of five equally qualified employees. Since none of them has had actual experience, he decides to allow each of them to work for one month in each of the positions before he makes the decision. How long can the bank operate before the positions are filled by permanent appointments?

Solution to above - cited situation requires an efficient counting of the possible ways in which the desired outcomes can be obtained. A listing of all possible outcomes may be desirable, but is likely to be very tedious and subject to errors of duplication or omission. We need to devise certain techniques which will help us to cope with such problems. The techniques of permutation and combination will help in tackling problems such as above.

FUNDAMENTAL PRINCIPLES OF COUNTING

(a) **Multiplication Rule:** If certain thing may be done in 'm' different ways and when it has been done, a second thing can be done in 'n ' different ways then total number of ways of doing both things simultaneously = m × n.

Eg. if one can go to school by 5 different buses and then come back by 4 different buses then total number of ways of going to and coming back from school = 5 × 4 = 20.

(b) **Addition Rule :** It there are two alternative jobs which can be done in 'm' ways and in 'n' ways respectively then either of two jobs can be done in (m + n) ways.

Eg. if one wants to go school by bus where there are 5 buses or to by auto where there are 4 autos, then total number of ways of going school = 5 + 4 = 9.

Note :- 1)

$$\boxed{\begin{array}{l}\textbf{AND} \Rightarrow \textbf{Multiply} \\ \textbf{OR} \Rightarrow \textbf{Add}\end{array}}$$

2) The above fundamental principles may be generalised, wherever necessary.

5.2 THE FACTORIAL

Definition : The factorial n, written as n! or $\underline{|n}$, represents the product of all integers from 1 to n both inclusive. To make the notation meaningful, when n = o, we define o! or $\underline{|o}$ = 1.

Thus, n! = n (n – 1) (n – 2) ….. …3.2.1

Example 1 : Find 5!, 4! and 6!

Solution : $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120; 4! = 4 \times 3 \times 2 \times 1 = 24; 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$

Example 2 : Find 9 ! / 6 ! ; 10 ! / 7 !.

Solution : $\frac{9!}{6!} = \frac{9 \times 8 \times 7 \times 6!}{6!} = 9 \times 8 \times 7 = 504 ; \frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7!}{7!} = 10 \times 9 \times 8 = 720$

Example 3 : Find x if 1/9 ! + 1/10 ! = x/11 !

Solution : $1/9! \, (1 + 1/10) = x/11 \times 10 \times 9!$ or, $11/10 = x/11 \times 10$ i.e., x = 121

Example 4 : Find n if $\underline{|n+1} = 30\underline{|n-1}$

Solution: $\underline{|n+1} = 30\underline{|n-1} \Rightarrow (n+1).n\underline{|n-1} = 30\,\underline{|n-1}$

or, $n^2 + n = 30$ or, $n^2 + n - 30$ or, $n^2 + 6n - 5n - 30 = 0$ or, $(n + 6)(n - 5) = 0$

either n = 5 or n = –6. (Not possible) $\therefore$ n = 5.

5.3 PERMUTATIONS

A group of persons want themselves to be photographed. They approach the photographer and request him to take as many different photographs as possible with persons standing in different positions amongst themselves. The photographer wants to calculate how many films does he need to exhaust all possibilities? How can he calculate the number?

In the situations such as above, we can use permutations to find out the exact number of films.

Definition : The ways of arranging or selecting smaller or equal number of persons or objects from a group of persons or collection of objects with due regard being paid to the order of arrangement or selection, are called permutations.

Let us explain, how the idea of permutation will help the photographer. Suppose the group consists of Mr. Suresh, Mr. Ramesh and Mr. Mahesh. Then how many films does the photographer need? He has to arrange three persons amongst three places with due regard to order. Then the various possibilities are (Suresh, Mahesh, Ramesh), (Suresh, Ramesh, Mahesh), (Ramesh, Suresh, Mahesh), (Ramesh, Mahesh, Suresh), (Mahesh, Ramesh, Suresh) and (Mahesh, Suresh, Ramesh). Thus there are six possibilities. Therefore he needs six films. Each one of these possibilities is called a permutation of three persons taken at a time.

This may also be exhibited as follows :

Alternative	Place 1	Place2	Place 3
1	Suresh..........	Mahesh..........	Ramesh
2	Suresh..........	Ramesh..........	Mahesh
3	Ramesh.........	Suresh............	Mahesh
4	Ramesh.........	Mahesh..........	Suresh
5	Mahesh.........	Ramesh..........	Suresh
6	Mahesh.........	Suresh.............	Ramesh

with this example as a base, we can introduce a general formula to find the number of permutations.

Number of Permutations when r objects are chosen out of n different objects. (Denoted by ^nP_r or $_nP_r$ or $P_{(n,r)}$) :

Let us consider the problem of finding the number of ways in which the first *r* rankings are secured by n students in a class. As any one of the n students can secure the first rank, the number of ways in which the first rank is secured is n.

Now consider the second rank. There are (n – 1) students left and the second rank can be secured by any one of them. Thus the different possibilities are (n – 1) ways. Now, applying fundamental principle, we can see that the first two ranks can be secured in n (n – 1) ways by these n students.

After calculating for two ranks, we find that the third rank can be secured by any one of the remaining (n – 2) students. Thus, by applying the generalized fundamental principle, the first three ranks can be secured in n (n – 1) (n – 2) ways .

Continuing in this way we can visualise that the number of ways are reduced by one as the rank is increased by one. Therefore, again, by applying the generalised fundamental principle for r different rankings, we calculate the number of ways in which the first r ranks are secured by n students as

$^nP_r = n\ \{(n-1)\ldots\left(n-\overline{r-1}\right)\}$

$= n\ (n-1)\ \ldots\ (n-r+1)$

Theorem : The number of permutations of n things chosen r at a time is given by

$$\boxed{^nP_r = n\ (n-1)(n-2)\ldots(n-r+1)}$$

where the product has exactly r factors.

5.4 RESULTS

1 Number of permutations of n different things taken all n things at a time is given by

$^nP_n = n\,(n-1)\,(n-2)\,\ldots.\,(n-n+1)$

$= n\,(n-1)\,(n-2)\,\ldots..\,2.1 = n!$ ...(1)

2. ^nP_r using factorial notation.

$^nP_r = n.\,(n-1)\,(n-2)\,\ldots..\,(n-r+1)$

$$= n\,(n-1)\,(n-2)\,\ldots..\,(n-r+1) \times \frac{(n-r)\,(n-r-1)\,2.1}{1.2\,\ldots(n-r-1)\,(n-r)}$$

$= n!/(\,n-r\,)!$ ...(2)

Thus

$$\boxed{^nP_r = \frac{n!}{(n-r)!}}$$

3. Justification for 0! = 1. Now applying r = n in the formula for ^nP_r, we get

$^nP_n = n!/\,(n-n)! = n!/0!$

But from Result 1 we find that $^nP_n = n!$. Therefore, by applying this

we derive, $0! = n!\,/\,n! = 1$

Example 1 : Evaluate each of $^5P_3$, $^{10}P_2$, $^{11}P_5$.

Solution : $^5P_3 = 5{\times}4{\times}\,(5{-}3{+}1) = 5 \times 4 \times 3 = 60$,

$^{10}P_2 = 10 \times \ldots. \times (10{-}2{+}1) = 10 \times 9 = 90$,

$^{11}P_5 = 11!\,/\,(11-5)! = 11 \times 10 \times 9 \times 8 \times 7 \times 6!\,/\,6! = 11 \times 10 \times 9 \times 8 \times 7 = 55440$.

Example 2 : How many three letters words can be formed using the letters of the words (a) square and (b) hexagon?

(Any arrangement of letters is called a word even though it may or may not have any meaning or pronunciation).

Solution :

(a) Since the word 'square' consists of 6 different letters, the number of permutations of choosing 3 letters out of six equals $^6P_3 = 6 \times 5 \times 4 = 120$.

(b) Since the word 'hexagon' contains 7 different letters, the number of permutations is $^7P_3 = 7 \times 6 \times 5 = 210$.

Example 3 : In how many different ways can five persons stand in a line for a group photograph?

Solution : Here we know that the order is important. Hence, this is the number of permutations of five things taken all at a time. Therefore, this equals

$^5P_5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

Example 4 : First, second and third prizes are to be awarded at an engineering fair in which 13 exhibits have been entered. In how many different ways can the prizes be awarded?

Solution : Here again, order of selection is important and repetitions are not meaningful as no exhibit can receive more than one prize. Hence , the answer is the number of permutations of 13 things taken three at a time. Therefore, we find ${}^{13}P_3 = 13!/10! = 13{\times}12{\times}11 = 1{,}716$ ways.

Example 5 : In how many different ways can 3 students be associated with 4 chartered accountants, assuming that each chartered accountant can take at most one student?

Solution : This equals the number of permutations of choosing 3 persons out of 4. Hence , the answer is ${}^4P_3 = 4{\times}3{\times}2 = 24$.

Example 6 : If six times the number permutations of n things taken 3 at a time is equal to seven times the number of permutations of (n – 1) things taken 3 at a time, find n.

Solution : We are given that $6 \times {}^nP_3 = 7 \times {}^{n-1}P_3$ and we have to solve this equality to find the value of n. Therefore,

$$6\frac{\lfloor n}{\lfloor n-3} = 7\frac{\lfloor n-1}{\lfloor n-4}$$

or, $6\,n\,(n-1)\,(n-2) = 7\,(n-1)\,(n-2)\,(n-3)$

or, $6\,n = 7\,(n-3)$

or, $6\,n = 7n - 21$

or, $n = 21$

Therefore, the value of n equals 21.

Example 7 : Compute the sum of 4 digit numbers which can be formed with the four digits 1, 3, 5, 7, if each digit is used only once in each arrangement.

Solution : The number of arrangements of 4 different digits taken 4 at a time is given by ${}^4P_4 = 4! = 24$. All the four digits will occur equal number of times at each of the positions, namely ones, tens, hundreds, thousands.

Thus, each digit will occur 24 / 4 = 6 times in each of the positions. The sum of digits in one's position will be $6 \times (1 + 3 + 5 + 7) = 96$. Similar is the case in ten's, hundred's and thousand's places. Therefore, the sum will be $96 + 96 \times 10 + 96 \times 100 + 96 \times 1000 = 106656$.

Example 8 : Find n if ${}^nP_3 = 60$.

Solution : ${}^nP_3 = \dfrac{n!}{(n-3)!} = 60$ (given)

i.e., $n\,(n-1)\,(n-2) = 60 = 5 \times 4 \times 3$

Therefore, n = 5.

Example 9 : If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, find r.

Solution : We know ${}^np_r = \dfrac{n!}{(n-r)!}$;

$$\therefore {}^{56}P_{r+6} = \frac{56!}{[56-(r+6)]!} = \frac{56\,!}{(50-r)!}$$

Similarly, $^{54}P_{r+3} = \frac{54!}{[54-(r+3)]!} = \frac{54!}{(51-r)!}$

$$\text{Thus, } \frac{^{56}P_{r+6}}{^{54}P_{r+3}} = \frac{56!}{(50-r!)} \times \frac{(51-r)!}{54!}$$

$$\frac{56\times55\times54!}{(50-r)!} \times \frac{(51-r)(50-r)!}{54!} = \frac{56\times55\times(51-r)}{1}$$

But we are given the ratio as 30800 : 1 ; therefore

$$\frac{56\times55\times(51-r)}{1} = \frac{30800}{1}$$

$$\text{or, } (51-r) = \frac{30800}{56\times55} = 10 \quad \therefore r = 41$$

Example 10 : Prove the following

$(n + 1)! - n! = \Rightarrow n.n!$

Solution : By applying the simple properties of factorial, we have

$(n +1)! - n! = (n+1)\ n! - n! = n!.\ (n+1-1) = n.\ n!$

Example 11 : In how many different ways can a club with 10 members select a President, Secretary and Treasurer, if no member can hold two offices and each member is eligible for any office?

Solution : The answer is the number of permutations of 10 persons chosen three at a time. Therefore, it is $^{10}p_3 = 10\times9\times8 = 720$.

Example 12 : When Jhon arrives in New York, he has eight shops to see, but he has time only to visit six of them. In how many different ways can he arrange his schedule in New York?

Solution : He can arrange his schedule in $^8P_6 = 8\times 7 \times 6 \times 5 \times 4 \times 3 = 20160$ ways.

Example 13 : When Dr. Ram arrives in his dispensary, he finds 12 patients waiting to see him. If he can see only one patient at a time, find the number of ways, he can schedule his patients (a) if they all want their turn, and (b) if 3 leave in disgust before Dr. Ram gets around to seeing them.

Solution : (a) There are 12 patients and all 12 wait to see the doctor. Therefore the number of ways = $^{12}P_{12} = 12! = 479{,}001{,}600$

(b) There are 12–3 = 9 patients. They can be seen $^{12}P_9 = 79{,}833{,}600$ ways.

Exercise 5 (A)

Choose the most appropriate option (a) (b) (c) or (d)

1. $^4P_3$ is evaluated as
   a) 43 b) 34 c) 24 d) None of these

2. ${}^{4}P_{4}$ is equal to
   a) 1 b) 24 c) 0 d) none of these
3. $\underline{|7}$ is equal to
   a) 5040 b) 4050 c) 5050 d) none of these
4. $\underline{|0}$ is a symbol equal to
   a) 0 b) 1 c) Infinity d) none of these
5. In ${}^{n}P_{r}$, n is always
   a) an integer b) a fraction c) a positive integer d) none of these
6. In ${}^{n}P_{r}$, the restriction is
   a) $n > r$ b) $n \geq r$ c) $n \leq r$ d) none of these
7. In ${}^{n}P_{r} = n\ (n-1)\ (n-2)\ \ldots\ldots\ldots\ldots\ldots\ldots(n-r+1)$, the number of factors is
   a) n b) r–1 c) n–r d) r
8. ${}^{n}P_{r}$ can also written as
   a) $\dfrac{\underline{|n}}{\underline{|n-r}}$ b) $\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$ c) $\dfrac{\underline{|r}}{\underline{|n-r}}$ d) none of these
9 If ${}^{n}P_{4} = 12 \times {}^{n}P_{2}$, the n is equal to
   a) –1 b) 6 c) 5 d) none of these
10. If . ${}^{n}P_{3} : {}^{n}P_{2} = 3 : 1$, then n is equal to
   a) 7 b) 4 c) 5 d) none of these
11. ${}^{m+n}P_{2} = 56$, ${}^{m-n}P_{2} = 30$ then
   a) m =6, n = 2 b) m = 7, n= 1 c) m=4,n=4 d) None of these
12. if ${}^{5}P_{r} = 60$, then the value of r is
   a) 3 b) 2 c) 4 d) none of these
13. If ${}^{n_1+n_2}P_{2} = 132$, ${}^{n_1-n_2}P_{2} = 30$ then,
   a) $n_1=6, n_2=6$ b) $n_1 = 10, n_2 = 2$ c) $n_1 = 9, n_2 = 3$ d) none of these
14. The number of ways the letters of the word COMPUTER can be rearranged is
   a) 40320 b) 40319 c) 40318 d) none of these
15. The number of arrangements of the letters in the word FAILURE, so that vowels are always coming together is
   a) 576 b) 575 c) 570 d) none of these
16. 10 examination papers are arranged in such a way that the best and worst papers never come together. The number of arrangements is
   a) $9\underline{|8}$ b) $\underline{|10}$ c) $8\underline{|9}$ d) none of these
17. n articles are arranged in such a way that 2 particular articles never come together. The number of such arrangements is
   a) $(n-2)\ \underline{|n-1}$ b) $(n-1)\ \underline{|n-2}$ c) $\underline{|n}$ d) none of these

18. If 12 school teams are participating in a quiz contest, then the number of ways the first, second and third positions may be won is

a) 1230 b) 1320 c) 3210 d) none of these

19. The sum of all 4 digit number containing the digits 2, 4, 6, 8, without repetitions is

a) 133330 b) 122220 c) 213330 d) 133320

20 The number of 4 digit numbers greater than 5000 can be formed out of the digits 3,4,5,6 and 7(no. digit is repeated). The number of such is

a) 72 b) 27 c) 70 d) none of these

21. 4 digit numbers to be formed out of the figures 0, 1, 2, 3, 4 (no digit is repeated) then number of such numbers is

(a) 120 (b) 20 (c) 96. (d) none of these

22. The number of ways the letters of the word "Triangle" to be arranged so that the word 'angle' will be always present is

(a) 20 (b) 60 (c) 24 (d) 32

23. If the letters word 'DAUGHTER' are to be arranged so that vowels occupy the odd places, then number of different words are

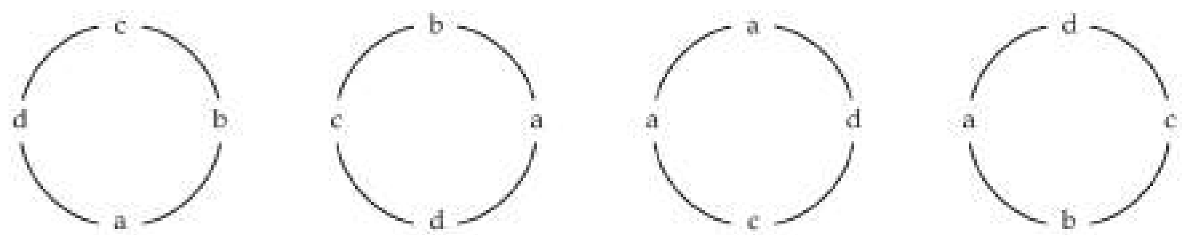
(a) 2880 (b) 676 (c) 625 (d) 2880

5.5 CIRCULAR PERMUTATIONS

So for we have discussed arrangements of objects or things in a row which may be termed as linear permutation. But if we arrange the objects along a closed curve viz., a circle, the permutations are known as circular permutations.

The number of circular permutations of n different things chosen at a time is (n–1)!.

Proof : Let any one of the permutations of n different things taken. Then consider the rearrangement of this permutation by putting the last thing as the first thing. Eventhough this



abcd dabc cdab bcda

is a different permutation in the ordinary sense, it will not be different in all *n* things are arranged in a circle. Similarly, we can consider shifting the last two things to the front and so on. Specially, it can be better understood, if we consider *a,b,c,d*. If we place *a,b,c,d* in order, then we also get *abcd, dabc, cdab, bcda* as four ordinary permutations. These four words in circular case are one and same thing. See above circles.

Thus we find in above illustration that four ordinary permutations equals one in circular.

Therefore, n ordinary permutations equal one circular permutation.

Hence there are ${}^nP_n / n$ ways in which all the n things can be arranged in a circle. This equals $(n-1)!$.

Example 1 : In how many ways can 4 persons sit at a round table for a group discussions?

Solution : The answer can be get from the formula for circular permutations. The answer is $(4-1)! = 3! = 6$ ways.

NOTE : These arrangements are such that every person has got the same two neighbours. The only change is that right side neighbour and vice-versa.

Thus the number of ways of arranging n persons along a round table so that no person has the same two neighbours is $= \frac{1}{2}\underline{|n-1}$

Similarly, in forming a necklace or a garland there is no distinction between a clockwise and anti clockwise direction because we can simply turn it over so that clockwise becomes anti clockwise and vice versa. **Hence, the number of necklaces formed with n beads of different colours**$= \frac{1}{2}\underline{|n-1}$

5.6 PERMUTATION WITH RESTRICTIONS

In many arrangements there may be number of restrictions. in such cases, we are to arrange or select the objects or persons as per the restrictions imposed. The total number of arrangements in all cases, can be found out by the application of fundamental principle.

Theorem 1. Number of permutations of n distinct objects taken r at a time when a particular object is not taken in any arrangement is ${}^{n-1}p_r$.

Proof : Since a particular object is always to be excluded, we have to place $n - 1$ objects at r places. Clearly this can be done in ${}^{n-1}p_r$ ways.

Theorem 2. Number of permutations of r objects out of n distinct objects when a particular object is always included in any arrangement is $r.{}^{n-1}p_{r-1}$

Proof : If the particular object is placed at first place, remaining $r - 1$ places can be filled from $n - 1$ distinct objects in ${}^{n-1}p_{r-1}$ ways. Similarly, by placing the particular object in 2nd, 3rd, ..., rth place, we find that in each case the number of permutations is ${}^{n-1}p_{r-1}$. This the total number of arrangements in which a particular object always occurs is $r.\ {}^{n-1}p_{r-1}$

The following examples will enlighten further:

Example 1 : How many arrangements can be made out of the letters of the word DRAUGHT, the vowels never beings separated?

Solution : The word DRAUGHT consists of 7 letters of which 5 are consonants and two are vowels. In the arrangement we are to take all the 7 letters but the restriction is that the two vowels should not be separated.

We can view the two vowels as one letter. The two vowels A and U in this one letter can be arranged in 2! = 2 ways. (i) AU or (ii) UA. Further, we can arrange the six letters : 5 consonants and one letter compound letter consisting of two vowels. The total number of ways of arranging them is ${}^6P_6$ = 6! = 720 ways.

Hence, by the fundamental principle, the total number of arrangements of the letters of the word DRAUGHT, the vowels never being separated = 2 × 720 = 1440 ways.

Example 2 : Show that the number of ways in which n books can be arranged on a shelf so that two particular books are not together.The number is $(n-2).(n-1)!$

Solution: We first find the total number of arrangements in which all n books can be arranged on the shelf without any restriction. The number is, ${}^nP_n = n!$ ..... (1)

Then we find the total number of arrangements in which the two particular books are together.

The books can be together in ${}^2P_2$ = 2! = 2 ways. Now we consider those two books which are kept together as one composite book and with the rest of the $(n-2)$ books from $(n-1)$ books which are to be arranged on the shelf; the number of arrangements = ${}^{n-1}P_{n-1} = (n-1)$!. Hence by the Fundamental Principle, the total number of arrangements on which the two particular books are together equals = $2 \times (n-1)!$ ....(2)

the required number of arrangements of n books on a shelf so that two particular books are not together

$$= (1) - (2)$$
$$= n! - 2 \times (n-1)!$$
$$= n.(n-1)! - 2 . (n-1)!$$
$$= (n-1)! . (n-2)$$

Example 3 : There are 6 books on Economics, 3 on Mathematics and 2 on Accountancy. In how many ways can these be placed on a shelf if the books on the same subject are to be together?

Solution : Consider one such arrangement. 6 Economics books can be arranged among themselves in 6! Ways, 3 Mathematics books can be arranged in 3! Ways and the 2 books on Accountancy can be arranged in 2! ways. Consider the books on each subject as one unit. Now there are three units. These 3 units can be arranged in 3! Ways.

Total number of arrangements = 3! × 6! × 3! × 2!

= 51,840.

Example 4 : How many different numbers can be formed by using any three out of five digits 1, 2, 3, 4, 5, no digit being repeated in any number?

How many of these will (i) begin with a specified digit? (ii) begin with a specified digit and end with another specified digit?

Solution : Here we have 5 different digits and we have to find out the number of permutations of them 3 at a time. Required number is ${}^5P_3$ = 5.4.3 = 60.

(i) If the numbers begin with a specified digit, then we have to find the number of Permutations of the remaining 4 digits taken 2 at a time. Thus, desire number is ${}^4P_2 = 4.3 = 12$.

(ii) Here two digits are fixed; first and last; hence, we are left with the choice of finding the number of permutations of 3 things taken one at a time i.e., ${}^3P_1 = 3$.

Example 5 : How many four digit numbers can be formed out of the digits 1,2,3,5,7,8,9, if no digit is repeated in any number? How many of these will be greater than 3000?

Solution : We are given 7 different digits and a four-digit number is to be formed using any 4 of these digits. This is same as the permutations of 7 different things taken 4 at a time.

Hence, the number of four-digit numbers that can be formed = ${}^7P_4 = 7 \times 6 \times 5 \times 4 \times = 840$ ways.

Next, there is the restriction that the four-digit numbers so formed must be greater than 3,000. Thus, it will be so if the first digit-that in the thousand's position, is one of the five digits 3, 5, 7, 8, 9. Hence, the first digit can be chosen in 5 different ways; when this is done, the rest of the 3 digits are to be chosen from the rest of the 6 digits without any restriction and this can be done in ${}^6P_3$ ways.

Hence, by the Fundamental principle, we have the number of four-digit numbers greater than 3,000 that can be formed by taking 4 digits from the given 7 digits = $5 \times {}^6P_3 = 5 \times 6 \times 5 \times 4 = 5 \times 120 = 600$.

Example 6 : Find the total number of numbers greater than 2000 that can be formed with the digits 1, 2, 3, 4, 5 no digit being repeated in any number.

Solution : All the 5 digit numbers that can be formed with the given 5 digits are greater than 2000. This can be done in

$${}^5P_5 = 5! = 120 \text{ ways} \;......................................(1)$$

The four digited numbers that can be formed with any four of the given 5 digits are greater than 2000 if the first digit, i.e.,the digit in the thousand's position is one of the four digits 2, 3, 4, 5. this can be done in ${}^4P_1 = 4$ ways. When this is done, the rest of the 3 digits are to be chosen from the rest of 5–1 = 4 digits. This can be done in ${}^4P_3 = 4 \times 3 \times 2 = 24$ ways.

Therefore, by the Fundamental principle, the number of four-digit numbers greater than 2000

$$= 4 \times 24 = 96 \;....\; (2)$$

Adding (1) and (2), we find the total number greater than 2000 to be 120 + 96 = 216.

Example 7 : There are 6 students of whom 2 are Indians, 2 Americans, and the remaining 2 are Russians. They have to stand in a row for a photograph so that the two Indians are together, the two Americans are together and so also the two Russians. Find the number of ways in which they can do so.

Solution : The two Indians can stand together in ${}^2P_2 = 2! = 2$ ways. So is the case with the two Americans and the two Russians.

Now these 3 groups of 2 each can stand in a row in ${}^3P_3 = 3 \times 2 = 6$ ways. Hence by the generalized fundamental principle, the total number of ways in which they can stand for a photograph under given conditions is

$$6 \times 2 \times 2 \times 2 = 48$$

Example 8 : A family of 4 brothers and three sisters is to be arranged for a photograph in one row. In how many ways can they be seated if (i) all the sisters sit together, (ii) no two sisters sit together?

Solution :

(i) Consider the sisters as one unit and each brother as one unit. 4 brothers and 3 sisters make 5 units which can be arranged in 5! ways. Again 3 sisters may be arranged amongst themselves in 3! Ways

Therefore, total number of ways in which all the sisters sit together = $5! \times 3! = 720$ ways.

(ii) In this case, each sister must sit on each side of the brothers. There are 5 such positions as indicated below by upward arrows :

$$\uparrow \text{B1} \quad \uparrow \quad \text{B2} \quad \uparrow \quad \text{B3} \quad \uparrow \quad \text{B4} \quad \uparrow$$

4 brothers may be arranged among themselves in 4! ways. For each of these arrangements 3 sisters can sit in the 5 places in ${}^5P_3$ ways.

Thus the total number of ways = ${}^5P_3 \times 4! = 60 \times 24 = 1{,}440$

Example 9 : In how many ways can 8 persons be seated at a round table? In how many cases will 2 particular persons sit together?

Solution : This is in form of circular permutation. Hence the number of ways in which eight persons can be seated at a round table is $(n - 1)! = (8 - 1)! = 7! = 5040$ ways.

Consider the two particular persons as one person. Then the group of 8 persons becomes a group of 7 (with the restriction that the two particular persons be together) and seven persons can be arranged in a circular in 6! Ways.

Hence, by the fundamental principle, we have, the total number of cases in which 2 particular persons sit together in a circular arrangement of 8 persons = $2! \times 6! = 2 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 1{,}440$.

Example 10 : Six boys and five girls are to be seated for a photograph in a row such that no two girls sit together and no two boys sit together. Find the number of ways in which this can be done.

Solution : Suppose that we have 11 chairs in a row and we want the 6 boys and 5 girls to be seated such that no two girls and no two boys are together. If we number the chairs from left to right, the arrangement will be possible if and only if boys occupy the odd places and girls occupy the even places in the row. The six odd places from 1 to 11 may filled in by 6 boys in ${}^6P_6$ ways. Similarly, the five even places from 2 to 10 may be filled in by 5 girls in ${}^5P_5$ ways.

Hence, by the fundamental principle, the total number of required arrangements = ${}^6P_6 \times {}^5P_5 = 6! \times 5! = 720 \times 120 = 86400$.

Exercise 5 (B)

Choose the most appropriate option (a) (b) (c) or (d)

1 The number of ways in which 7 girls form a ring is
(a) 700 (b) 710 (c) 720 (d) none of these

2. The number of ways in which 7 boys sit in a round table so that two particular boys may sit together is
(a) 240 (b) 200 (c) 120 (d) none of these

3. If 50 different jewels can be set to form a necklace then the number of ways is
(a) $\frac{1}{2}\lfloor 50$ (b) $\frac{1}{2}\lfloor 49$ (c) $\lfloor 49$ (d) none of these

4. 3 ladies and 3 gents can be seated at a round table so that any two and only two of the ladies sit together. The number of ways is
(a) 70 (b) 27 (c) 72 (d) none of these

5. The number of ways in which the letters of the word DOGMATIC can be arranged is
(a) 40319 (b) 40320 (c) 40321 (d) none of these

6. The number of arrangements of 10 different things taken 4 at a time in which one particular thing always occurs is
(a) 2015 (b) 2016 (c) 2014 (d) none of these

7. The number of permutations of 10 different things taken 4 at a time in which one particular thing never occurs is
(a) 3020 (b) 3025 (c) 3024 (d) none of these

8. Mr. X and Mr. Y enter into a railway compartment having six vacant seats. The number of ways in which they can occupy the seats is
(a) 25 (b) 31 (c) 32 (d) 30

9. The number of numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5, 6, 7 is
(a) 210 (b) 200 (c) 110 (d) none of these

10. The number of numbers lying between 10 and 1000 can be formed with the digits 2,3,4,0,8,9 is
(a) 124 (b) 120 (c) 125 (d) none of these

11. In a group of boys the number of arrangement of 4 boys is 12 times the number of arrangements of 2 boys. The number of boys in the group is
(a) 10 (b) 8 (c) 6 (d) none of these

12. The value of $\sum_{r=1}^{10} r.{}^{r}P_{r}$ is
(a) ${}^{11}P_{11}$ (b) ${}^{11}P_{11} - 1$ (c) ${}^{11}P_{11} + 1$ (d) none of these

13. The total number of 9 digit numbers of different digits is

(a) $10\lfloor\underline{9}$ (b) $8\lfloor\underline{9}$ (c) $9\lfloor\underline{9}$ (d) none of these

14. The number of ways in which 6 men can be arranged in a row so that the particular 3 men sit together, is

(a) ${}^{4}P_{4}$ (b) ${}^{4}P_{4} \times {}^{3}P_{3}$ (c) $(\lfloor\underline{3})^{2}$ (d) none of these

15. There are 5 speakers A, B, C, D and E. The number of ways in which A will speak always before B is

(a) 24 (b) $\lfloor\underline{4} \times \lfloor\underline{2}$ (c) $\lfloor\underline{5}$ (d) none of these

16. There are 10 trains plying between Calcutta and Delhi. The number of ways in which a person can go from Calcutta to Delhi and return by a different train is

(a) 99 (b) 90 (c) 80 (d) none of these

17. The number of ways in which 8 sweats of different sizes can be distributed among 8 persons of different ages so that the largest sweat always goes to be younger assuming that each one of then gets a sweat is

(a) $\lfloor\underline{8}$ (b) 5040 (c) 5039 (d) none of these

18. The number of arrangements in which the letters of the word MONDAY be arranged so that the words thus formed begin with M and do not end with N is

(a) 720 (b) 120 (c) 96 (d) none of these

19. The total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together is

(a) $\lfloor\underline{7} / \lfloor\underline{3}$ (b) $\lfloor\underline{6} \times \lfloor\underline{7} / \lfloor\underline{3}$ (c) 35 (d) none of these

20. The number of ways in which the letters of the word MOBILE be arranged so that consonants always occupy the odd places is

(a) 36 (b) 63 (c) 30 (d) none of these.

21. 5 persons are sitting in a round table in such way that Tallest Person is always on the right–side of the shortest person; the number of such arrangements is

(a) 6 (b) 8 (c) 24 (d) none of these

5.7 COMBINATIONS

We have studied about permutations in the earlier section. There we have said that while arranging, we should pay due regard to order. There are situations in which order is not important. For example, consider selection of 5 clerks from 20 applicants. We will not be concerned about the order in which they are selected. In this situation, how to find the number of ways of selection? The idea of combination applies here.

Definition : The number of ways in which smaller or equal number of things are arranged or selected from a collection of things where the order of selection or arrangement is not important, are called combinations.

The selection of a poker hand which is a combination of five cards selected from 52 cards is an example of combination of 5 things out of 52 things.

Number of combinations of n different things taken r at a time. (denoted by ^nC_r C(n,r), $C_{n,r}$)

Let ^nC_r denote the required number of combinations. Consider any one of those combinations. It will contain r things. Here we are not paying attention to order of selection. Had we paid attention to this, we will have permutations or r items taken r at a time. In other words, every combination of r things will have ^rP_r permutations amongst them. Therefore, ^nC_r combinations will give rise to ^nC_r. ^rP_r permutations of r things selected from n things. From the earlier section, we can say that $^nC_r \cdot {}^rP_r = {}^nP_r$ as ^nP_r denotes the number of permutations of r things chosen out of n things.

Since, $\quad {}^nC_r \cdot {}^rP_r = {}^nP_r$

$$^nC_r = {}^nP_r / {}^rP_r \quad = n!/(n-r)! \div r!/(r-r)!$$

$$= n!/(n-r)! \times 0!/r!$$

$$= n! / r!(n-r)!$$

$$\therefore {}^nC_r = n!/r!(n-r)!$$

Remarks: Using the above formula, we get

(i) $^nC_0 = n! / 0!(n-0)! = n!/n! = 1$. [As $0! = 1$]

$^nC_n = n! / n!(n-n)! = n! / n!\,0! = 1$ [Applying the formula for ^nC_r with $r = n$]

Example 1 : Find the number of different poker hands in a pack of 52 playing cards.

Solution : This is the number of combinations of 52 cards taken five at a time. Now applying the formula,

$$^{52}C_5 = 52!/5!\,(52-5)! \ = \ 52!/5!\,47! = \frac{52\times51\times50\times49\times48\times47!}{5\times4\times3\times2\times1\times47!}$$

$$= 2{,}598{,}960$$

Example 2 : Let S be the collection of eight points in the plane with no three points on the straight line. Find the number of triangles that have points of S as vertices.

Solution : Every choice of three points out of S determines a unique triangle. The order of the points selected is unimportant as whatever be the order, we will get the same triangle. Hence, the desired number is the number of combinations of eight things taken three at a time. Therefore, we get

$^8C_3 = 8!/3!5! = 8\times7\times6/3\times2\times1 = 56$ choices.

Example 3 : A committee is to be formed of 3 persons out of 12. Find the number of ways of forming such a committee.

Solution : We want to find out the number of combinations of 12 things taken 3 at a time and this is given by

$^{12}C_3$ $= 12!/3!(12-3)!$ [by the definition of ^nC_r]

$= 12!/3!9! = 12\times11\times10\times9!/3!9! = 12\times11\times10/3\times2 = 220$

Example 4 : A committee of 7 members is to be chosen from 6 Chartered Accountants, 4 Economists and 5 Cost Accountants. In how many ways can this be done if in the committee, there must be at least one member from each group and at least 3 Chartered Accountants?

Solution : The various methods of selecting the persons from the various groups are shown below:

Committee of 7 members			
	C.A.s	**Economists**	**Cost Accountants**
Method 1	3	2	2
Method 2	4	2	1
Method 3	4	1	2
Method 4	5	1	1
Method 5	3	3	1
Method 6	3	1	3

Number of ways of choosing the committee members by

Method 1 = $^6C_3\times{}^4C_2\times{}^5C_2 = \dfrac{6\times5\times4}{3\times2\times1}\times\dfrac{4\times3}{2\times1}\times\dfrac{5\times4}{2\times1} = 20\times6\times10 = 1{,}200.$

Method 2 = $^6C_4\times{}^4C_2\times{}^5C_1 = \dfrac{6\times5}{2\times1}\times\dfrac{4\times3}{2\times1}\times\dfrac{5}{1} = 15\times6\times5 = 450$

Method 3 = $^6C_4\times{}^4C_1\times{}^5C_2 = \dfrac{6\times5}{2\times1}\times4\times\dfrac{5\times4}{2\times1} = 15\times4\times10 = 600.$

Method 4 = $^6C_5\times{}^4C_1\times{}^5C_1 = 6\times4\times5 = 120.$

Method 5 = $^6C_3\times{}^4C_3\times{}^5C_1 = \dfrac{6\times5\times4}{3\times2\times1}\times\dfrac{4\times3\times2}{3\times2\times1}\times5 = 20\times4\times5 = 400.$

Method 6 = $^6C_3\times{}^4C_1\times{}^5C_3 = \dfrac{6\times5\times4}{3\times2\times1}\times4\times\dfrac{5\times4}{2\times1} = 20\times4\times10 = 800.$

Therefore, total number of ways = 1,200 + 450 + 600 + 120 + 400 + 800 = 3,570

Example 5: A person has 12 friends of whom 8 are relatives. In how many ways can he invite 7 guests such that 5 of them are relatives?

Solution : Of the 12 friends, 8 are relatives and the remaining 4 are not relatives. He has to invite 5 relatives and 2 friends as his guests. 5 relatives can be chosen out of 8 in $^8C_5$ ways; 2 friends can be chosen out of 4 in $^4C_2$ ways.

Hence, by the fundamental principle, the number of ways in which he can invite 7 guests such that 5 of them are relatives and 2 are friends.

$= {}^{8}C_{5} \times {}^{4}C_{2}$

$= \{8! / 5! (8-5)!\} \times \{4! / 2! (4-2)!\} = [(8 \times 7 \times 6 \times 5!)/5! \times 3!] \times \dfrac{4 \times 3 \times 2 \times !}{2!\,2!} = 8 \times 7 \times 6$

$= 336.$

Example 6 : A Company wishes to simultaneously promote two of its 6 department heads to assistant managers. In how many ways these promotions can take place?

Solution : This is a problem of combination. Hence, the promotions can be done in

${}^{6}C_{2} = 6 \times 5 / 2 = 15$ ways

Example 7 : A building contractor needs three helpers and ten men apply. In how many ways can these selections take place?

Solution : There is no regard for order in this problem. Hence, the contractor can select in any of ${}^{10}C_{3}$ ways i.e.,

$(10 \times 9 \times 8) / (3 \times 2 \times 1) = 120$ ways.

Example 8: In each case, find n:

Solution : (a) $4.\ {}^{n}C_{2} = {}^{n+2}C_{3}$; (b) ${}^{n+2}C_{n} = 45$.

(a) We are given that $4.\ {}^{n}C_{2} = {}^{n+2}C_{3}$. Now applying the formula,

$$4 \times \frac{n!}{2!(n-2)!} = \frac{(n+2)!}{3!(n+2-3)!}$$

or, $$\frac{4 \times n.(n-1)(n-2)!}{2!(n-2)!} = \frac{(n+2)\,(n+1)\,.\,n\,.\,(n-1)!}{3!\,(n-1)!}$$

$4n(n-1)\ /2 = (n+2)\ (n+1)n\ /3!$

or, $4n(n-1)\ /\ 2 = (n+2)(n+1)n\ /3 \times 2 \times 1$

or, $12(n-1) = (n+2)\ (n+1)$

or, $12n-12 = n^2 + 3n + 2$

or, $n^2 - 9n + 14 = 0.$

or, $n^2 - 2n - 7n + 14 = 0.$

or, $(n-2)\ (n-7) = 0$

$\therefore$ $n=2$ or 7.

(b) We are given that ${}^{n+2}C_{n} = 45$. Applying the formula,

$(n+2)!/\{n!(n+2-n)!\} = 45$

or, $(n+2)\ (n+1)\ n!\ /\ n!\ 2! = 45$

or, $(n+1)(n+2) = 45\times2! = 90$

or, $n^2+3n-88 = 0$

or, $n^2+11n-8n-88 = 0$

or, $(n+11)(n-8) = 0$

Thus, n equals either – 11 or 8. But negative value is not possible. Therefore we conclude that n=8.

Example 9 : A box contains 7 red, 6 white and 4 blue balls. How many selections of three balls can be made so that (a) all three are red, (b) none is red, (c) one is of each colour?

Solution : (a) All three balls will be of red colour if they are taken out of 7 red balls and this can be done in

$^7C_3 = 7! / 3!(7-3)!$

$= 7! / 3!4! = 7\times6\times5\times4! / (3\times2\times4!) = 7\times6\times5 / (3\times2) = 35$ ways

Hence, 35 selections (groups) will be there such that all three balls are red.

(b) None of the three will be red if these are chosen from (6 white and 4 blue balls) 10 balls and this can be done in

$^{10}C_3 = 10!/\{3!(10-3)!\} = 10! / 3!7!$

$= 10\times9\times8\times7! / (3\times2\times1\times7!) = 10\times9\times8 /(3\times2) = 120$ ways.

Hence, the selections (or groups) of three such that none is a red ball are 120 in number.

One red ball can be chosen from 7 balls in $^7C_1 = 7$ ways. One white ball can be chosen from 6 white balls in $^6C_1$ ways. One blue ball can be chosen from 4 blue balls in $^4C_1 = 4$ ways. Hence, by generalized fundamental principle, the number of groups of three balls such that one is of each colour $= 7\times6\times4 = 168$ ways.

Example 10 : If $^{10}P_r = 604800$ and $^{10}C_r = 120$; find the value of r,

Solution : We know that $^nC_r \cdot {^rP_r} = {^nP_r}$. We will us this equality to find r.

$^{10}P_r = {^{10}C_r}\,.r!$

or, $604800 = 120 \times r!$

or, $r! = 604800 \div 120 = 5040$

But $r! = 5040 = 7\times6\times4\times3\times2\times1 = 7!$

Therefore, r=7.

Properties of ^nC_r :

1. $^nC_r = {^nC_{n-r}}$

   We have $^nC_r = n! / \{r!(n-r)!\}$ and $^nC_{n-r} = n! / [(n-r)!\,\{n-(n-r)\}!] = n! / \{(n-r)!(n-n+r)!\}$

   Thus $^nC_{n-r} = n! / \{(n-r)!\,(n-n+r)!\} = n! / \{(n-r)!r!\} = {^nC_r}$

2. $^{n+1}C_r = {^nC_r} + {^nC_{r-1}}$

By definition,

${}^nC_{r-1} + {}^nC_r = n! / \{(r-1)!\,(n-r+1)!\} + n! / \{r!(n-r)!\}$

But $r! = r\times(r-1)!$ and $(n-r+1)! = (n-r+1) \times (n-r)!$. Substituting these in above, we get

$$ {}^nC_{r-1} + {}^nC_r = n! \left\{\frac{1}{(r-1)!(n-r+1)(n-r)!} + \frac{1}{r(r-1)!\,(n-r)!}\right\} $$

$= \{n! / (r-1)!\,(n-r)!\}\,\{(1 / n-r+1) + (1/r)\}$

$= \{n! / (r-1)!\,(n-r)!\}\,\{(r+n-r+1) / r(n-r+1)\}$

$= (n+1)\,n! / \{r\,.\,(r-1)!\,(n-r)!\,.\,(n-r+1)\}$

$= (n+1)! / \{r!(n+1-r)!\} = {}^{n+1}C_r$

3. ${}^n\mathbf{C}_0 = n!/\{0!\,(n-0)!\} = n! / n! = \mathbf{1}$.

4. ${}^n\mathbf{C}_n = n!/\{n!\,(n-n)!\} = n! / n!\,.\,0! = \mathbf{1}$.

Note

(a) ${}^nC_r$ has a meaning only when r and n are integers $0 \le r \le n$ and ${}^nC_{n-r}$ has a meaning only when $0 \le n - r \le n$.

(b) ${}^nC_r$ and ${}^nC_{n-r}$ are called complementary combinations, for if we form a group of r things out of n different things, (n–r) remaining things which are not included in this group form another group of rejected things. The number of groups of n different things, taken r at a time should be equal to the number of groups of n different things taken (n–r) at a time.

Example 11 : Find r if ${}^{18}C_r = {}^{18}C_{r+2}$

Solution : As ${}^nC_r = {}^nC_{n-r}$, we have ${}^{18}C_r = {}^{18}C_{18-r}$

But it is given, ${}^{18}C_r = {}^{18}C_{r+2}$

$\therefore {}^{18}C_{18-r} = {}^{18}C_{r+2}$

or, $18 - r = r+2$

Solving, we get

$2r = 18 - 2 = 16$ i.e., $r=8$.

Example 12 : Prove that

${}^nC_r = {}^{n-2}C_{r-2} + 2\,{}^{n-2}C_{r-1} + {}^{n-2}C_r$

Solution : R.H.S $= {}^{n-2}C_{r-2} + {}^{n-2}C_{r-1} + {}^{n-2}C_{r-1} + {}^{n-2}C_r$

$= {}^{n-1}C_{r-1} + {}^{n-1}C_r$ [using Property 2 listed earlier]

$= {}^{(n-1)+1}C_r$ [using Property 2 again]

$= {}^nC_r =$ L.H.S.

Hence, the result

Example 13 : If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$, find r.

Solution : We have ${}^nC_r = n! / \{r!(n-r)!\}$ Now, substituting for n and r, we get

${}^{28}C_{2r} = 28! / \{(2r)!(28-2r)!\}$,

${}^{24}C_{2r-4} = 24! / [(2r-4)!\{24-(2r-4)\}!] = 24! / \{(2r-4)!(28-2r)!\}$

We are given that ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$. Now we calculate,

$$\frac{{}^{28}C_{2r}}{{}^{24}C_{2r-4}} = \frac{28!}{(2r)!(28-2r)!} \div \frac{(2r-4)!(28-2r)!}{24!}$$

$$= \frac{28\times27\times26\times25\times24!}{(2r)(2r-1)(2r-2)(2r-3)(2r-4)!(28-2r)!} \times \frac{(2r-4)!(28-2r)!}{24!}$$

$$= \frac{28\times27\times26\times25}{(2r)(2r-1)(2r-2)(2r-3)} = \frac{225}{11}$$

$$\text{or, } (2r)(2r-1)(2r-2)(2r-3) = \frac{11\times28\times27\times26\times25}{225}$$

$$= 11\times28\times3\times26$$

$$= 11\times7\times4\times3\times13\times2$$

$$= 11\times12\times13\times14$$

$$= 14\times13\times12\times11$$

$$\therefore\ 2r = 14 \quad \text{i.e., } r = 7$$

Example 14 : Find x if ${}^{12}C_5 + 2\,{}^{12}C_4 + {}^{12}C_3 = {}^{14}C_x$

Solution : $\text{L.H.S} = {}^{12}C_5 + 2\,{}^{12}C_4 + {}^{12}C_3$

$= {}^{12}C_5 + {}^{12}C_4 + {}^{12}C_4 + {}^{12}C_3$

$= {}^{13}C_5 + {}^{13}C_4$

$= {}^{14}C_5$

Also ${}^nC_r = {}^nC_{n-r}$. Therefore ${}^{14}C_5 = {}^{14}C_{14-5} = {}^{14}C_9$

Hence, L.H.S $= {}^{14}C_5 = {}^{14}C_9 = {}^{14}C_x =$ R.H.S by the given equality

This implies, either x = 5 or x = 9.

Example 15 : Prove by reasoning that

(i) ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$

(ii) ${}^nP_r = {}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$

Solution : (i) ${}^{n+1}C_r$ stands for the number of combinations of (n+1) things taken r at a time. As a specified thing can either be included in any combination or excluded from it, the total number of combinations which can be combinations or (n+1) things taken r at a time is the sum of :

(a) combinations of (n+1) things taken r at time in which one specified thing is always included and

(b) the number of combinations of (n+1) things taken r at time from which the specified thing is always excluded.

Now, in case (a), when a specified thing is always included , we have to find the number of ways of selecting the remaining (r–1) things out of the remaining n things which is ${}^{n}C_{r-1.}$

Again, in case (b), since that specified thing is always excluded, we have to find the number of ways of selecting r things out of the remaining n things, which is ${}^{n}C_{r}$.

Thus, ${}^{n+1}C_r = {}^{n}C_{r-1} + {}^{n}C_r$

(i) We devide ${}^{n}P_r$ i.e., the number of permutations of n things take r at a time into two groups:

(a) those which contain a specified thing

(b) those which do not contain a specified thing.

In (a) we fix the particular thing in any one of the r places which can be done in r ways and then fill up the remaining (r–1) places out of (n–1) things which give rise to ${}^{n-1}P_{r-1}$ ways. Thus, the number of permutations in case (a) = r × ${}^{n-1}P_{r-1.}$

In case (b), one thing is to be excluded; therefore, r places are to be filled out of (n–1) things. Therefore, number of permutations = ${}^{n-1}P_r$

Thus, total number of permutations = ${}^{n-1}P_r$ + r. ${}^{n-1}P_{r-1}$

i.e., ${}^{n}P_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$

5.8 STANDARD RESULTS

We now proceed to examine some standard results in permutations and combinations. These results have special application and hence are dealt with separately.

I. Permutations when some of the things are alike, taken all at a time

The number of ways p in which n things may be arranged among themselves, taking them all at a time, when n_1 of the things are exactly alike of one kind , n_2 of the things are exactly alike of another kind, n_3 of the things are exactly alike of the third kind, and the rest all are different is given by,

$$p = \frac{n!}{n_1!n_2!n_3!}$$

Proof : Let there be n things. Suppose n_1 of them are exactly alike of one kind; n_2 of them are exactly alike of another kind; n_3 of them are exactly alike of a third kind; let the rest $(n–n_1–n_2–n_3)$ be all different.

Let p be the required permutations; then if the n things, all exactly alike of one kind were replaced by n_1 different things different from any of the rest in any of the p permutations without altering the position of any of the remaining things, we could form $n_1!$ new permutations. Hence, we should obtain p × $n_1!$ permutations.

Similarly if n_2 things exactly alike of another kind were replaced by n_2 different things different form any of the rest, the number of permutations would be p × $n_1!$ × $n_2!$

Similarly, if n_3 things exactly alike of a third kind were replaced by n_3 different things different from any of the rest, the number of permutations would be $p \times n_1! \times n_2! \times n_3! = n!$

But now because of these changes all the n things are different and therefore, the possible number of permutations when all of them are taken is n!.

Hence, $p \times n_1! \times n_2!\, n_3! = n!$

i.e., $P = \dfrac{n!}{n_1!n_2!n_3!}$

which is the required number of permutations. This results may be extended to cases where there are different number of groups of alike things.

II. Permutations when each thing may be repeated once, twice,...upto r times in any arrangement.

Result: The number of permutations of n things taken r at time when each thing may be repeated r times in any arrangement is n^r.

Proof: There are n different things and any of these may be chosen as the first thing. Hence, there are n ways of choosing the first thing.

When this is done, we are again left with n different things and any of these may be chosen as the second (as the same thing can be chosen again.)

Hence, by the fundamental principle, the two things can be chosen in $n \times n = n^2$ number of ways.

Proceeding in this manner, and noting that at each stage we are to chose a thing from n different things, the total number of ways in which r things can be chosen is obviously equal to $n \times n \times$ ....to r terms = n^r.

III. Combinations of n different things taking some or all of n things at a time

Result : The total number of ways in which it is possible to form groups by taking some or all of n things $(2^n - 1)$.

In symbols, $\sum\limits_{r=1}^{n} {}^{n}C_r = 2^n - 1$

Proof : Each of the n different things may be dealt with in two ways; it may either be taken or left. Hence, by the generalised fundamental principle, the total number of ways of dealing with n things :

$2 \times 2 \times 2 \times$........2, n times i.e., 2^n

But this include the case in which all the things are left, and therefore, rejecting this case, the total number of ways of forming a group by taking some or all of n different things is $2^n - 1$.

IV. Combinations of n things taken some or all at a time when n_1 of the things are alike of one kind, n_2 of the things are alike of another kind n_3 of the things are alike of a third kind. etc.

Result : The total, number of ways in which it is possible to make groups by taking some or all out of n $(=n_1 + n_2 + n_3 +...)$ things, where n_1 things are alike of one kind and so on, is given by

$\{ (n_1 + 1) (n_2 + 1) (n_3 + 1)...\} -1$

Proof : The n_1 things all alike of one kind may be dealt with in $(n_1 + 1)$ ways. We may take 0, 1, 2,....n_1 of them. Similarly n_2 things all alike of a second kind may be dealt with in $(n_2 +1)$ ways and n_3 things all alike of a third kind may de dealt with in $(n_3 +1)$ ways.

Proceeding in this way and using the generalised fundamental principle, the total number of ways of dealing with all n $(= n_1 + n_2 + n_3 +...)$ things, where n_1, things are alike of one kind and so on, is given by

$(n_1 + 1) (n_2 + 1) (n_3 + 1)...$

But this includes the case in which none of the things are taken. Hence, rejecting this case, total number of ways is $\{(n_1 + 1) (n_2 + 1) (n_3 + 1)...\} -1\}$

V. The notion of Independence in Combinations

Many applications of Combinations involve the selection of subsets from two or more independent sets of objects or things. If the combination of a subset having r_1 objects form a set having n_1 objects does not affect the combination of a subset having r_2 objects from a different set having n_2 objects, we call the combinations as being independent. Whenever such combinations are independent, any subset of the first set of objects can be combined with each subset of the second set of the object to form a bigger combination. The total number of such combinations can be found by applying the generalised fundamental principle.

Result : The combinations of selecting r_1 things from a set having n_1 objects and r_2 things from a set having n_2 objects where combination of r_1 things, r_2 things are independent is given by

$$^{n_1}C_{r_1} \times {}^{n_2}C_{r_2}$$

Note : This result can be extended to more than two sets of objects by a similar reasoning.

Example 1 : How many different permutations are possible from the letters of the word CALCULUS?

Solution: The word CALCULUS consists of 8 letters of which 2 are C and 2 are L, 2 are U and the rest are A and S. Hence , by result (I), the number of different permutations from the letters of the word CALCULUS taken all at a time

$$= \frac{8!}{2!2!2!1!1!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{2 \times 2 \times 2} = 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 5040$$

Example 2 : In how many ways can 17 billiard balls be arranged , if 7 of them are black, 6 red and 4 white?

Solution : We have, the required number of different arrangements:

$$= \frac{17!}{7!\,6!\,4!} = 4084080$$

Example 3 : An examination paper with 10 questions consists of 6 questions in Algebra and 4 questions in Geometry. At least one question from each section is to be attempted. In how many ways can this be done?

Solution : A student must answer atleast one question from each section and he may answer all questions from each section.

Consider Section I : Algebra. There are 6 questions and he may answer a question or may not answer it. These are the two alternatives associated with each of the six questions. Hence, by the generalised fundamental principle, he can deal with two questions in 2×2 ....6 factors = 2^6 number of ways. But this includes the possibility of none of the question from Algebra being attempted. This cannot be so, as he must attempt at least one question from this section. Hence, excluding this case, the number of ways in which Section I can be dealt with is $(2^6 - 1)$.

Similarly, the number of ways in which Section II can be dealt with is $(2^4 - 1)$.

Hence, by the Fundamental Principle, the examination paper can be attempted in $(2^6 - 1)\,(2^4 - 1)$ number of ways.

Example 4 : A man has 5 friends. In how many ways can he invite one or more of his friends to dinner?

Solution : By result, (III) of this section, as he has to select one or more of his 5 friends, he can do so in $2^5 - 1 = 31$ ways.

Note : This can also be done in the way, outlines below. He can invite his friends one by one, in twos, in threes, etc. and hence the number of ways.

$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$= 5 + 10 + 10 + 5 + 1 = 31$ ways.

Example 5 : There are 7 men and 3 ladies. Find the number of ways in which a committee of 6 can be formed of them if the committee is to include atleast two ladies?

Solution : The committee of six must include at least 2 ladies, i.e., two or more ladies. As there are only 3 ladies, the following possibilities arise:

The committee of 6 consists of (i) 4 men and 2 ladies (ii) 3 men and 3 ladies.

The number of ways for (i) = ${}^7C_4 \times {}^3C_2 = 35 \times 3 = 105$;

The number of ways for (ii) = ${}^7C_3 \times {}^3C_3 = 35 \times 1 = 35$.

Hence the total number of ways of forming a committee so as to include at least two ladies = $105 + 35 = 140$.

Example 6 : Find the number of ways of selecting 4 letters from the word EXAMINATION.

Solution : There are 11 letters in the word of which A, I, N are repeated twice.

2

SEQUENCES AND SERIES OF REAL NUMBERS

Mathematics is the Queen of Sciences, and arithmetic is the Queen of Mathematics - C.F.Gauss

- Introduction
- Sequences
- Arithmetic Progression (A.P.)
- Geometric Progression (G.P.)
- Series

Leonardo Pisano (Fibonacci)

(1170-1250)

Italy

Fibonacci played an important role in reviving ancient mathematics. His name is known to modern mathematicians mainly because of a number sequence named after him, known as the 'Fibonacci numbers', which he did not discover but used as an example.

2.1 Introduction

In this chapter, we shall learn about sequences and series of real numbers. Sequences are fundamental mathematical objects with a long history in mathematics. They are tools for the development of other concepts as well as tools for mathematization of real life situations.

Let us recall that the letters $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and real numbers respectively.

Let us consider the following real-life situations.

(i) A team of ISRO scientists observes and records the height of a satellite from the sea level at regular intervals over a period of time.

(ii) The Railway Ministry wants to find out the number of people using Central railway station in Chennai on a daily basis and so it records the number of people entering the Central Railway station daily for 180 days.

(iii) A curious 9th standard student is interested in finding out all the digits that appear in the decimal part of the irrational number $\sqrt{5} = 2.236067978\cdots$ and writes down as

$2, 3, 6, 0, 6, 7, 9, 7, 8, \cdots$.

(iv) A student interested in finding all positive fractions with numerator 1, writes $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$.

(v) A mathematics teacher writes down the marks of her class according to alphabetical order of the students' names as 75, 95, 67, 35, 58, 47, 100, 89, 85, 60..

(vi) The same teacher writes down the same data in an ascending order as

35, 47, 58, 60, 67, 75, 85, 89, 95, 100.

In each of the above examples, some sets of real numbers have been listed in a specific order.

Note that in (iii) and (iv) the arrangements have infinite number of terms. In (i), (ii), (v) and (vi) there are only finite number of terms; but in (v) and (vi) the same set of numbers are written in different order.

2.2 Sequences

Definition

A sequence of real numbers is an arrangement or a list of real numbers in a specific order.

(i) If a sequence has only finite number of terms, then it is called a finite sequence.

(ii) If a sequence has infinitely many terms, then it is called an infinite sequence.

We denote a finite sequence as $S : a_1, a_2, a_3, \cdots, a_n$ or $S = \{a_j\}_{j=1}^{n}$ and an infinite sequence as $S : a_1, a_2, a_3, \cdots, a_n, \cdots$ or $S = \{a_j\}_{j=1}^{\infty}$ where a_k denotes the k^{th} term of the sequence. For example, a_1 denotes the first term and a_7 denotes the seventh term in the sequence.

Note that in the above examples, (i), (ii), (v) and (vi) are finite sequences, whereas (iii) and (iv) are infinite sequences

Observe that, when we say that a collection of numbers is listed in a sequence, we mean that the sequence has an identified first member, second member, third member and so on. We have already seen some examples of sequences. Let us consider some more examples below.

(i) 2, 4, 6, 8, $\cdots$, 2010. (finite number of terms)

(ii) 1, -1, 1, -1, 1, -1, 1, $\cdots$. (terms just keep oscillating between 1 and -1)

(iii) $\pi, \pi, \pi, \pi, \pi.$ (terms are same; such sequences are constant sequences)

(iv) 2, 3, 5, 7, 11, 13, 17, 19, 23, $\cdots$. (list of all prime numbers)

(v) 0.3, 0.33, 0.333, 0.3333, 0.33333, $\cdots$. (infinite number of terms)

(vi) $S = \{a_n\}_1^{\infty}$ where $a_n = 1$ or 0 according to the outcome head or tail in the n^{th} toss of a coin.

From the above examples, (i) and (iii) are finite sequences and the other sequences are infinite sequences. One can easily see that some of them, i.e., (i) to (v) have a definite pattern or rule in the listing and hence we can find out any term in a particular position in the sequence. But in (vi), we cannot predict what a particular term is, however, we know it must be either 1 or 0. Here, we have used the word "pattern" to mean that the n^{th} term of a sequence is found based on the knowledge of its preceding elements in the sequence. In general, sequences can be viewed as functions.

2.2.1 Sequences viewed as functions

A finite real sequence $a_1, a_2, a_3, \cdots, a_n$ or $S = \{a_j\}_{j=1}^{n}$ can be viewed as a function $f : \{1, 2, 3, 4, \cdots, n\} \to \mathbb{R}$ defined by $f(k) = a_k$, $k = 1, 2, 3, \cdots, n$.

An infinite real sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ or $S = \{a_j\}_{j=1}^{\infty}$ can be viewed as a function $g : \mathbb{N} \to \mathbb{R}$ defined by $g(k) = a_k$, $\forall\, k \in \mathbb{N}$.

The symbol $\forall$ means "for all". If the general term a_k of a sequence $\{a_k\}_1^{\infty}$ is given, we can construct the whole sequence. Thus, a sequence is a function whose domain is the set{ 1, 2, 3, $\cdots$, }of natural numbers, or some subset of the natural numbers and whose range is a subset of real numbers.

Remarks

A function is not necessarily a sequence. For example, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 2x + 1, \forall\, x \in \mathbb{R}$ is not a sequence since the required listing is not possible. Also, note that the domain of f is not $\mathbb{N}$ or a subset $\{1, 2, \cdots, n\}$ of $\mathbb{N}$.

Example 2.1

Write the first three terms in a sequence whose n^{th} term is given by

$$c_n = \frac{n(n+1)(2n+1)}{6}, \ \forall\, n \in \mathbb{N}$$

Solution Here, $c_n = \dfrac{n(n+1)(2n+1)}{6}$, $\forall\, n \in \mathbb{N}$

For $n = 1$, $c_1 = \dfrac{1(1+1)(2(1)+1)}{6} = 1.$

For $n = 2$, $c_2 = \dfrac{2(2+1)(4+1)}{6} = \dfrac{2(3)(5)}{6} = 5.$

Finally $n = 3$, $c_3 = \dfrac{3(3+1)(7)}{6} = \dfrac{(3)(4)(7)}{6} = 14.$

Hence, the first three terms of the sequence are 1, 5, and 14.

In the above example, we were given a formula for the general term and were able to find any particular term directly. In the following example, we shall see another way of generating a sequence.

Example 2.2

Write the first five terms of each of the following sequences.

(i) $a_1 = -1$, $a_n = \dfrac{a_{n-1}}{n+2}$, $n > 1$ and $\forall\, n \in \mathbb{N}$

(ii) $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, \cdots.$

Solution

(i) Given $a_1 = -1$ and $a_n = \dfrac{a_{n-1}}{n+2},\ n > 1$

$$a_2 = \frac{a_1}{2+2} = -\frac{1}{4}$$

$$a_3 = \frac{a_2}{3+2} = \frac{-\frac{1}{4}}{5} = -\frac{1}{20}$$

$$a_4 = \frac{a_3}{4+2} = \frac{-\frac{1}{20}}{6} = -\frac{1}{120}$$

$$a_5 = \frac{a_4}{5+2} = \frac{-\frac{1}{120}}{7} = -\frac{1}{840}$$

$\therefore$ The required terms of the sequence are $-1, -\frac{1}{4}, -\frac{1}{20}, -\frac{1}{120}$ and $-\frac{1}{840}$.

(ii) Given that $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n = 3, 4, 5, \cdots$.

Now, $F_1 = 1, F_2 = 1$

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$

$$F_4 = F_3 + F_2 = 2 + 1 = 3$$

$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

$\therefore$ The first five terms of the sequence are 1, 1, 2, 3, 5.

Remarks

The sequence given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, \cdots$ is called the Fibonacci sequence. Its terms are listed as 1, 1, 2, 3, 5, 8, 13, 21, 34, $\cdots$. The Fibonacci sequence occurs in nature, like the arrangement of seeds in a sunflower. The number of spirals in the opposite directions of the seeds in a sunflower are consecutive numbers of the Fibonacci sequence.

Exercise 2.1

1. Write the first three terms of the following sequences whose n^{th} terms are given by

   (i) $a_n = \dfrac{n(n-2)}{3}$ (ii) $c_n = (-1)^n 3^{n+2}$ (iii) $z_n = \dfrac{(-1)^n n(n+2)}{4}$

2. Find the indicated terms in each of the sequences whose n^{th} terms are given by

   (i) $a_n = \dfrac{n+2}{2n+3};\ a_7, a_9$ (ii) $a_n = (-1)^n 2^{n+3}(n+1);\ a_5, a_8$

   (iii) $a_n = 2n^2 - 3n + 1;\ a_5, a_7$ (iv) $a_n = (-1)^n(1 - n + n^2);\ a_5, a_8$

3. Find the 18th and 25th terms of the sequence defined by

$$a_n = \begin{cases} n(n+3), & \text{if } n \in \mathbb{N} \text{ and } n \text{ is even} \\ \dfrac{2n}{n^2+1}, & \text{if } n \in \mathbb{N} \text{ and } n \text{ is odd.} \end{cases}$$

4. Find the 13th and 16th terms of the sequence defined by

$$b_n = \begin{cases} n^2, & \text{if } n \in \mathbb{N} \text{ and } n \text{ is even} \\ n(n+2), & \text{if } n \in \mathbb{N} \text{ and } n \text{ is odd.} \end{cases}$$

5. Find the first five terms of the sequence given by

$$a_1 = 2, a_2 = 3 + a_1 \text{ and } a_n = 2a_{n-1} + 5 \quad \text{for } n > 2.$$

6. Find the first six terms of the sequence given by

$$a_1 = a_2 = a_3 = 1 \text{ and } a_n = a_{n-1} + a_{n-2} \quad \text{for } n > 3.$$

2.3 Arithmetic sequence or Arithmetic Progression (A.P.)

In this section we shall see some special types of sequences.

Definition

A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called an arithmetic sequence if $a_{n+1} = a_n + d$, $n \in \mathbb{N}$ where d is a constant. Here a_1 is called the first term and the constant d is called the common difference. An arithmetic sequence is also called an Arithmetic Progression (A.P.).

Examples

(i) $2, 5, 8, 11, 14, \cdots$ is an A.P. because $a_1 = 2$ and the common difference $d = 3$.

(ii) $-4, -4, -4, -4, \cdots$ is an A.P. because $a_1 = -4$ and $d = 0$.

(iii) $2,\ 1.5,\ 1,\ 0.5,\ 0,\ -0.5,\ -1.0,\ -1.5, \cdots$ is an A.P. because $a_1 = 2$ and $d = -0.5$.

The general form of an A.P.

Let us understand the general form of an A.P. Suppose that a is the first term and d is the common difference of an arithmetic sequence $\{a_k\}_{k=1}^{\infty}$. Then, we have

$$a_1 = a \text{ and } a_{n+1} = a_n + d, \ \forall\, n \in \mathbb{N}.$$

For $n = 1, 2, 3$ we get,

$$a_2 = a_1 + d = a + d = a + (2-1)d$$
$$a_3 = a_2 + d = (a+d) + d = a + 2d = a + (3-1)d$$
$$a_4 = a_3 + d = (a+2d) + d = a + 3d = a + (4-1)d$$

Following the pattern, we see that the n^{th} term a_n as

$$a_n = a_{n-1} + d = [a + (n-2)d] + d = a + (n-1)d.$$

Thus, we have $a_n = a + (n-1)d$ for every $n \in \mathbb{N}$.

So, a typical arithmetic sequence or A.P. looks like

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,\ a+(n-1)d,\ a+nd, \cdots$$

Also, the formula for the general term of an Arithmetic sequence is of the form

$$t_n = a + (n-1)d \quad \text{for every } n \in \mathbb{N}.$$

Note

(i) Remember a sequence may also be a finite sequence. So, if an A.P. has only n terms, then the last term l is given by $l = a + (n-1)d$

(ii) $l = a + (n-1)d$ can also be rewritten as $n = \left(\frac{l-a}{d}\right) + 1$. This helps us to find the number of terms when the first, the last term and the common difference are given.

(iii) Three consecutive terms of an A.P. may be taken as $m-d,\ m,\ m+d$

(iv) Four consecutive terms of an A.P. may be taken as $m-3d,\ m-d,\ m+d,\ m+3d$ with common difference $2d$.

(v) An A.P. remains an A.P. if each of its terms is added or subtracted by a same constant.

(vi) An A.P. remains an A.P. if each of its terms is multiplied or divided by a non-zero constant.

Example 2.3

Which of the following sequences are in an A.P.?

(i) $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \cdots$. (ii) $3m-1, 3m-3, 3m-5, \cdots$.

Solution

(i) Let t_n, $n \in \mathbb{N}$ be the n^{th} term of the given sequence.

$\therefore$ $t_1 = \frac{2}{3},\ t_2 = \frac{4}{5},\ t_3 = \frac{6}{7}$

So $t_2 - t_1 = \frac{4}{5} - \frac{2}{3} = \frac{12-10}{15} = \frac{2}{15}$

$t_3 - t_2 = \frac{6}{7} - \frac{4}{5} = \frac{30-28}{35} = \frac{2}{35}$

Since $t_2 - t_1 \neq t_3 - t_2$, the given sequence is not an A.P.

(ii) Given $3m-1, 3m-3, 3m-5, \cdots$.

Here $t_1 = 3m-1,\ t_2 = 3m-3,\ t_3 = 3m-5, \cdots$.

$\therefore$ $t_2 - t_1 = (3m-3) - (3m-1) = -2$

Also, $t_3 - t_2 = (3m-5) - (3m-3) = -2$

Hence, the given sequence is an A.P. with first term $3m-1$ and the common difference -2.

Example 2.4

Find the first term and common difference of the A.P.

(i) $5, 2, -1, -4, \cdots$. (ii) $\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \cdots, \frac{17}{6}$

Solution

(i) First term $a = 5$, and the common difference $d = 2-5 = -3$.

(ii) $a = \frac{1}{2}$ and the common difference $d = \frac{5}{6} - \frac{1}{2} = \frac{5-3}{6} = \frac{1}{3}$.

Example 2.5

Find the smallest positive integer n such that t_n of the arithmetic sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, \cdots$ is negative.?

Solution Here we have $a = 20$, $d = 19\frac{1}{4} - 20 = -\frac{3}{4}$.

We want to find the first positive integer n such that $t_n < 0$.

This is same as solving $a + (n-1)d < 0$ for smallest $n \in \mathbb{N}$.

That is solving $20 + (n-1)\left(-\frac{3}{4}\right) < 0$ for smallest $n \in \mathbb{N}$.

Now, $(n-1)\left(-\frac{3}{4}\right) < -20$

$\Longrightarrow (n-1)\times\frac{3}{4} > 20$ (The inequality is reversed on multiplying both sides by -1)

$\therefore \quad n - 1 > 20 \times \frac{4}{3} = \frac{80}{3} = 26\frac{2}{3}$.

This implies $n > 26\frac{2}{3} + 1$. That is, $n > 27\frac{2}{3} = 27.66$

Thus, the smallest positive integer $n \in \mathbb{N}$ satisfying the inequality is $n = 28$.

Hence, the 28th term, t_{28} is the first negative term of the A.P.

Example 2.6

In a flower garden, there are 23 rose plants in the first row, 21 in the second row, 19 in the third row and so on. There are 5 rose plants in the last row. How many rows are there in the flower garden?

Solution Let n be the number of rows in the flower garden .

The number of rose plants in the 1st, 2nd, 3rd $,\cdots, n^{th}$ rows are $23, 21, 19, \cdots, 5$ respectively.

Now, $t_k - t_{k-1} = -2$ for $k = 2, \cdots, n$.

Thus, the sequence $23, 21, 19, \cdots, 5$ is in an A.P.

We have $a = 23, \quad d = -2$, and $l = 5$.

$$\therefore \quad n = \frac{l-a}{d} + 1 = \frac{5-23}{-2} + 1 = 10.$$

So, there are 10 rows in the flower garden.

Example 2.7

If a person joins his work in 2010 with an annual salary of ₹30,000 and receives an annual increment of ₹600 every year, in which year, will his annual salary be ₹39,000?

Solution Suppose that the person's annual salary reaches ₹39,000 in the n^{th} year.

Annual salary of the person in 2010, 2011, 2012, $\cdots$, $[2010 + (n-1)]$ will be

₹30,000, ₹30,600, ₹31,200, $\cdots$, ₹39000 respectively.

First note that the sequence of salaries form an A.P.

To find the required number of terms, let us divide each term of the sequence by a fixed constant 100. Now, we get the new sequence 300, 306, 312, $\cdots$, 390.

Here $a = 300$, $d = 6$, $l = 390$.

So, $$n = \frac{l-a}{d} + 1$$

$$= \frac{390-300}{6} + 1 = \frac{90}{6} + 1 = 16$$

Thus, 16^{th} annual salary of the person will be ₹39,000.

$\therefore$ His annual salary will reach ₹39,000 in the year 2025.

Example 2.8

Three numbers are in the ratio $2:5:7$. If the first number, the resulting number on the substraction of 7 from the second number and the third number form an arithmetic sequence, then find the numbers.

Solution Let the numbers be $2x, 5x$ and $7x$ for some unknown $x, (x \neq 0)$

By the given information, we have that $2x, 5x-7, 7x$ are in A.P.

$\therefore \; (5x-7) - 2x = 7x - (5x-7) \quad \Longrightarrow 3x - 7 = 2x + 7$ and so $x = 14$.

Thus, the required numbers are 28, 70, 98.

Exercise 2.2

1. The first term of an A.P. is 6 and the common difference is 5. Find the A.P. and its general term.

2. Find the common difference and 15^{th} term of the A.P. 125, 120, 115, 110, $\cdots$.

3. Which term of the arithmetic sequence $24, 23\frac{1}{4}, 22\frac{1}{2}, 21\frac{3}{4}, \cdots$. is 3?

4. Find the 12^{th} term of the A.P. $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, \cdots$.

5. Find the 17^{th} term of the A.P. 4, 9, 14, $\cdots$.

6. How many terms are there in the following Arithmetic Progressions?

   (i) $-1, -\frac{5}{6}, -\frac{2}{3}, \cdots, \frac{10}{3}$. (ii) 7, 13, 19, $\cdots$, 205.

7. If 9^{th} term of an A.P. is zero, prove that its 29^{th} term is double (twice) the 19^{th} term.

8. The 10^{th} and 18^{th} terms of an A.P. are 41 and 73 respectively. Find the 27^{th} term.

9. Find n so that the n^{th} terms of the following two A.P.'s are the same.

   1, 7, 13, 19,$\cdots$ and 100, 95, 90, $\cdots$.

10. How many two digit numbers are divisible by 13?

11. A TV manufacturer has produced 1000 TVs in the seventh year and 1450 TVs in the tenth year. Assuming that the production increases uniformly by a fixed number every year, find the number of TVs produced in the first year and in the 15^{th} year.

12. A man has saved ₹640 during the first month, ₹720 in the second month and ₹800 in the third month. If he continues his savings in this sequence, what will be his savings in the 25^{th} month?

13. The sum of three consecutive terms in an A.P. is 6 and their product is –120. Find the three numbers.

14. Find the three consecutive terms in an A. P. whose sum is 18 and the sum of their squares is 140.

15. If m times the m^{th} term of an A.P. is equal to n times its n^{th} term, then show that the $(m+n)^{th}$ term of the A.P. is zero.

16. A person has deposited ₹25,000 in an investment which yields 14% simple interest annually. Do these amounts (principal + interest) form an A.P.? If so, determine the amount of investment after 20 years.

17. If a, b, c are in A.P. then prove that $(a-c)^2 = 4(b^2 - ac)$.

18. If a, b, c are in A.P. then prove that $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are also in A.P.

19. If a^2, b^2, c^2 are in A.P. then show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also in A.P.

20. If $a^x = b^y = c^z, x \neq 0, y \neq 0, z \neq 0$ and $b^2 = ac$, then show that $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

2.4 Geometric Sequence or Geometric Progression (G.P.)

Definition

A sequence $a_1,\ a_2,\ a_3, \cdots,\ a_n,\ \cdots$ is called a geometric sequence if $a_{n+1} = a_n r\ ,\ n \in \mathbb{N}$, where r is a non-zero constant. Here, a_1 is the first term and the constant r is called the common ratio. A geometric sequence is also called a Geometric Progression (G.P.).

Let us consider some examples of geometric sequences.

(i) $3,\ 6,\ 12,\ 24, \cdots$.

A sequence $\{a_n\}_1^\infty$ is a geometric sequence if $\dfrac{a_{n+1}}{a_n} = r \neq 0,\ \ n \in \mathbb{N}$.

Now, $\dfrac{6}{3} = \dfrac{12}{6} = \dfrac{24}{12} = 2 \neq 0$. So the given sequence is a geometric sequence.

(ii) $\dfrac{1}{9},\ -\dfrac{1}{27},\ \dfrac{1}{81},\ -\dfrac{1}{243}, \cdots$.

Here, we have $\dfrac{-\frac{1}{27}}{\frac{1}{9}} = \dfrac{\frac{1}{81}}{-\frac{1}{27}} = \dfrac{-\frac{1}{243}}{\frac{1}{81}} = -\dfrac{1}{3} \neq 0$.

Thus, the given sequence is a geometric sequence.

The general form of a G.P.

Let us derive the general form of a G.P. Suppose that a is the first term and r is the common ratio of a geometric sequence $\{a_k\}_{k=1}^\infty$. Then, we have

$$a_1 = a \quad \text{and} \quad \frac{a_{n+1}}{a_n} = r \quad \text{for } n \in \mathbb{N}.$$

Thus, $\qquad a_{n+1} = r\, a_n \quad \text{for } n \in \mathbb{N}$.

For $n = 1, 2, 3$ we get,

$$a_2 = a_1 r = ar = ar^{2-1}$$
$$a_3 = a_2 r = (ar)r = ar^2 = ar^{3-1}$$
$$a_4 = a_3 r = (ar^2)r = ar^3 = ar^{4-1}$$

Following the pattern, we have

$$a_n = a_{n-1} r = (ar^{n-2})r = ar^{n-1}.$$

Thus, $\qquad a_n = ar^{n-1}$ for every $n \in \mathbb{N}$, gives n^{th} term of the G.P.

So, a typical geometric sequence or G.P. looks like

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1},\ ar^n, \cdots.$$

Thus , the formula for the general term of a geometric sequence is

$$t_n = ar^{n-1},\ \ n = 1, 2, 3, \cdots.$$

Suppose we are given the first few terms of a sequence, how can we determine if the given sequence is a geometric sequence or not?

If $\frac{t_{n+1}}{t_n} = r, \forall\, n \in \mathbb{N}$, where r is a non-zero constant, then $\{t_n\}_1^{\infty}$ is in G.P.

Note

(i) If the ratio of any term other than the first term to its preceding term of a sequence is a non-zero constant, then it is a geometric sequence.

(ii) A geometric sequence remains a geometric sequence if each term is multiplied or divided by a non zero constant.

(iii) Three consecutive terms in a G.P may be taken as $\frac{a}{r}, a, ar$ with common ratio r.

(iv) Four consecutive terms in a G.P may be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$. (here, the common ratio is r^2 not r as above)

Example 2.9

Which of the following sequences are geometric sequences

(i) $5, 10, 15, 20, \cdots$. (ii) $0.15, 0.015, 0.0015, \cdots$. (iii) $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, 3\sqrt{21}, \cdots$.

Solution

(i) Considering the ratios of the consecutive terms, we see that $\frac{10}{5} \neq \frac{15}{10}$.

Thus, there is no common ratio. Hence it is not a geometric sequence.

(ii) We see that $\frac{0.015}{0.15} = \frac{0.0015}{0.015} = \cdots = \frac{1}{10}$.

Since the common ratio is $\frac{1}{10}$, the given sequence is a geometric sequence.

(iii) Now, $\frac{\sqrt{21}}{\sqrt{7}} = \frac{3\sqrt{7}}{\sqrt{21}} = \frac{3\sqrt{21}}{3\sqrt{7}} = \cdots = \sqrt{3}$. Thus, the common ratio is $\sqrt{3}$.

Therefore, the given sequence is a geometric sequence.

Example 2.10

Find the common ratio and the general term of the following geometric sequences.

(i) $\frac{2}{5}, \frac{6}{25}, \frac{18}{125}, \cdots$. (ii) $0.02, 0.006, 0.0018, \cdots$.

Solution

(i) Given sequence is a geometric sequence.

The common ratio is given by $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \cdots$.

Thus, $$r = \frac{\frac{6}{25}}{\frac{2}{5}} = \frac{3}{5}.$$

The first term of the sequence is $\frac{2}{5}$. So, the general term of the sequence is

$$t_n = ar^{n-1}, \ n = 1, 2, 3, \cdots.$$

$$\Longrightarrow \qquad t_n = \frac{2}{5}\left(\frac{3}{5}\right)^{n-1}, \quad n = 1,2,3,\cdots$$

(ii) The common ratio of the given geometric sequence is

$$r = \frac{0.006}{0.02} = 0.3 = \frac{3}{10}.$$

The first term of the geometric sequence is 0.02

So, the sequence can be represented by

$$t_n = (0.02)\left(\frac{3}{10}\right)^{n-1}, \quad n = 1,2,3,\cdots$$

Example 2.11

The 4th term of a geometric sequence is $\frac{2}{3}$ and the seventh term is $\frac{16}{81}$. Find the geometric sequence.

Solution Given that $t_4 = \frac{2}{3}$ and $t_7 = \frac{16}{81}$.

Using the formula $t_n = ar^{n-1}, \ n = 1, 2, 3, \cdots$. for the general term we have,

$$t_4 = ar^3 = \frac{2}{3} \quad \text{and} \quad t_7 = ar^6 = \frac{16}{81}.$$

Note that in order to find the geometric sequence, we need to find a and r.

By dividing t_7 by t_4 we obtain,

$$\frac{t_7}{t_4} = \frac{ar^6}{ar^3} = \frac{\frac{16}{81}}{\frac{2}{3}} = \frac{8}{27}.$$

Thus, $\qquad r^3 = \frac{8}{27} = \left(\frac{2}{3}\right)^3$ which implies $r = \frac{2}{3}$.

Now, $\qquad t_4 = \frac{2}{3} \Longrightarrow ar^3 = \left(\frac{2}{3}\right)$.

$$\Longrightarrow \qquad a\left(\frac{8}{27}\right) = \frac{2}{3}. \qquad \therefore \quad a = \frac{9}{4}.$$

Hence, the required geometric sequence is $a, \ ar, \ ar^2, \ ar^3, \ \cdots, \ ar^{n-1}, \ ar^n, \cdots$

That is, $\qquad \frac{9}{4}, \ \frac{9}{4}\left(\frac{2}{3}\right), \ \frac{9}{4}\left(\frac{2}{3}\right)^2, \ \cdots.$

Example 2.12

The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture initially, how many bacteria will be present at the end of 14th hour?

Solution Note that the number of bacteria present in the culture doubles at the end of successive hours.

Number of bacteria present initially in the culture $= 30$

Number of bacteria present at the end of first hour $= 2(30)$

Number of bacteria present at the end of second hour $= 2(2(30)) = 30(2^2)$

Continuing in this way, we see that the number of bacteria present at the end of every hour forms a G.P. with the common ratio $r = 2$.

Thus, if t_n denotes the number of bacteria after n hours,

$$t_n = 30\,(2^n) \text{ is the general term of the G.P.}$$

Hence, the number of bacteria at the end of 14th hour is given by $t_{14} = 30\,(2^{14})$.

Example 2.13

An amount ₹500 is deposited in a bank which pays annual interest at the rate of 10% compounded annually. What will be the value of this deposit at the end of 10th year?

Solution

The principal is ₹500. So, the interest for this principal for one year is $500\left(\frac{10}{100}\right) = 50$.

Thus, the principal for the 2nd year = Principal for 1st year + Interest

$$= 500 + 500\left(\frac{10}{100}\right) = 500\left(1 + \frac{10}{100}\right)$$

Now, the interest for the second year $= \left(500\left(1 + \frac{10}{100}\right)\right)\left(\frac{10}{100}\right)$.

So, the principal for the third year $= 500\left(1 + \frac{10}{100}\right) + 500\left(1 + \frac{10}{100}\right)\frac{10}{100}$

$$= 500\left(1 + \frac{10}{100}\right)^2$$

Continuing in this way we see that the principal for the n^{th} year $= 500\left(1 + \frac{10}{100}\right)^{n-1}$.

The amount at the end of $(n-1)^{th}$ year = Principal for the n^{th} year.

Thus, the amount in the account at the end of n^{th} year.

$$= 500\left(1 + \frac{10}{100}\right)^{n-1} + 500\left(1 + \frac{10}{100}\right)^{n-1}\left(\frac{10}{100}\right) = 500\left(\frac{11}{10}\right)^n.$$

The amount in the account at the end of 10th year

$$= ₹\ 500\left(1 + \frac{10}{100}\right)^{10} = ₹\ 500\left(\frac{11}{10}\right)^{10}.$$

Remarks

By using the above method, one can derive a formula for finding the total amount for compound interest problems. Derive the formula:

$$A = P(1 + i)^n$$

where A is the amount, P is the principal, $i = \frac{r}{100}$, r is the annual interest rate and n is the number of years.

Example 2.14

The sum of first three terms of a geometric sequence is $\frac{13}{12}$ and their product is -1. Find the common ratio and the terms.

Solution We may take the first three terms of the geometric sequence as $\frac{a}{r}, a, ar$.

Then, $$\frac{a}{r} + a + ar = \frac{13}{12}$$

$$a\left(\frac{1}{r} + 1 + r\right) = \frac{13}{12} \implies a\left(\frac{r^2 + r + 1}{r}\right) = \frac{13}{12} \qquad (1)$$

Also,

$$\left(\frac{a}{r}\right)(a)(ar) = -1$$

$$\implies a^3 = -1 \qquad \therefore \quad a = -1$$

Substituting $a = -1$ in (1) we obtain,

$$(-1)\left(\frac{r^2 + r + 1}{r}\right) = \frac{13}{12}$$

$$\implies 12r^2 + 12r + 12 = -13r$$

$$12r^2 + 25r + 12 = 0$$

$$(3r + 4)(4r + 3) = 0$$

Thus, $r = -\frac{4}{3}$ or $-\frac{3}{4}$

When $r = -\frac{4}{3}$ and $a = -1$, the terms are $\frac{3}{4}, -1, \frac{4}{3}$.

When $r = -\frac{3}{4}$ and $a = -1$, we get $\frac{4}{3}, -1, \frac{3}{4}$, which is in the reverse order.

Example 2.15

If a, b, c, d are in geometric sequence, then prove that

$$(b - c)^2 + (c - a)^2 + (d - b)^2 = (a - d)^2$$

Solution Given a, b, c, d are in a geometric sequence.

Let r be the common ratio of the given sequence. Here, the first term is a.

Thus, $b = ar, \quad c = ar^2, \quad d = ar^3$

Now, $(b - c)^2 + (c - a)^2 + (d - b)^2$

$$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$$

$$= a^2[(r - r^2)^2 + (r^2 - 1)^2 + (r^3 - r)^2]$$

$$= a^2[r^2 - 2r^3 + r^4 + r^4 - 2r^2 + 1 + r^6 - 2r^4 + r^2]$$

$$= a^2[r^6 - 2r^3 + 1] = a^2[r^3 - 1]^2$$

$$= (ar^3 - a)^2 = (a - ar^3)^2 = (a - d)^2$$

Exercise 2.3

1. Find out which of the following sequences are geometric sequences. For those geometric sequences, find the common ratio.

   (i) $0.12,\ 0.24,\ 0.48,\cdots.$ (ii) $0.004,\ 0.02,\ 0.1,\cdots.$ (iii) $\frac{1}{2},\ \frac{1}{3},\ \frac{2}{9},\ \frac{4}{27},\ \cdots.$

   (iv) $12,\ 1,\ \frac{1}{12},\cdots.$ (v) $\sqrt{2},\ \frac{1}{\sqrt{2}},\ \frac{1}{2\sqrt{2}},\cdots.$ (vi) $4, -2, -1, -\frac{1}{2},\cdots.$

2. Find the 10^{th} term and common ratio of the geometric sequence $\frac{1}{4}, -\frac{1}{2}, 1, -2,\cdots.$

3. If the 4^{th} and 7^{th} terms of a G.P. are 54 and 1458 respectively, find the G.P.

4. In a geometric sequence, the first term is $\frac{1}{3}$ and the sixth term is $\frac{1}{729}$, find the G.P.

5. Which term of the geometric sequence,

   (i) $5,\ 2,\ \frac{4}{5},\ \frac{8}{25},\cdots$, is $\frac{128}{15625}$? (ii) $1,\ 2,\ 4,\ 8,\cdots$, is 1024 ?

6. If the geometric sequences $162, 54, 18,\cdots.$ and $\frac{2}{81}, \frac{2}{27}, \frac{2}{9},\cdots$ have their n^{th} term equal, find the value of n.

7. The fifth term of a G.P. is 1875. If the first term is 3, find the common ratio.

8. The sum of three terms of a geometric sequence is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

9. If the product of three consecutive terms in G.P. is 216 and sum of their products in pairs is 156, find them.

10. Find the first three consecutive terms in G.P. whose sum is 7 and the sum of their reciprocals is $\frac{7}{4}$

11. The sum of the first three terms of a G.P. is 13 and sum of their squares is 91. Determine the G.P.

12. If ₹1000 is deposited in a bank which pays annual interest at the rate of 5% compounded annually, find the maturity amount at the end of 12 years .

13. A company purchases an office copier machine for ₹50,000. It is estimated that the copier depreciates in its value at a rate of 15% per year. What will be the value of the copier after 15 years?

14. If a, b, c, d are in a geometric sequence, then show that

    $$(a-b+c)(b+c+d) = ab+bc+cd.$$

15. If a, b, c, d are in a G.P., then prove that $a+b,\ \ b+c,\ \ c+d,$ are also in G.P.

2.5 Series

Let us consider the following problem:

A person joined a job on January 1, 1990 at an annual salary of ₹25,000 and received an annual increment of ₹500 each year. What is the total salary he has received upto January 1, 2010?

First of all note that his annual salary forms an arithmetic sequence

$$25000,\ 25500,\ 26000,\ 26500, \cdots,\ (25000 + 19(500)).$$

To answer the above question, we need to add all of his twenty years salary. That is,

$$25000 + 25500 + 26000 + 26500 + \cdots + (25000 + 19(500)).$$

So, we need to develop an idea of summing terms of a sequence.

Definition

An expression of addition of terms of a sequence is called a series.
If a series consists only a finite number of terms, it is called a finite series.
If a series consists of infinite number of terms of a sequence, it is called an infinite series.

Consider a sequence $S = \{a_n\}_{n=1}^{\infty}$ of real numbers. For each $n \in \mathbb{N}$ we define the partial sums by $S_n = a_1 + a_2 + , \cdots + a_n$, $n = 1, 2, 3, \cdots$. Then $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the given sequence $\{a_n\}_{n=1}^{\infty}$.

The ordered pair $(\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty})$ is called an infinite series of terms of the sequence $\{a_n\}_1^{\infty}$. The infinite series is denoted by $a_1 + a_2 + a_3 + \cdots$, or simply $\sum_{n=1}^{\infty} a_n$ where the symbol $\sum$ stands for summation and is pronounced as sigma.

Well, we can easily understand finite series (adding finite number of terms). It is impossible to add all the terms of an infinite sequence by the ordinary addition, since one could never complete the task. How can we understand (or assign a meaning to) adding infinitely many terms of a sequence? We will learn about this in higher classes in mathematics. For now we shall focus mostly on finite series.

In this section , we shall study Arithmetic series and Geometric series.

2.5.1 Arithmetic series

An arithmetic series is a series whose terms form an arithmetic sequence.

Sum of first n terms of an arithmetic sequence

Consider an arithmetic sequence with first term a and common difference d given by $a,\ a + d,\ a + 2d, ...,\ a + (n-1)d, \cdots$.

Let S_n be the sum of first n terms of the arithmetic sequence.

Thus, $S_n = a + (a+d) + (a+2d) + \cdots \; (a + (n-1)d)$

$\Longrightarrow \; S_n = na + (\, d + 2d + 3d + \cdots + (n-1)d\,)$

$= na + d(1 + 2 + 3 + \cdots + (n-1))$

So, we can simplify this formula if we can find the sum $1 + 2 + \cdots + (n-1)$.

This is nothing but the sum of the arithmetic sequence $1, 2, 3, \cdots, (n-1)$.

So, first we find the sum $1 + 2 + \cdots + (n-1)$ below.

Now, let us find the sum of the first n positive integers.

Let $S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$. (1)

We shall use a small trick to find the above sum. Note that we can write S_n also as

$$S_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1. \quad (2)$$

Adding (1) and (2) we obtain,

$$2S_n = (n+1) + (n+1) + \cdots + (n+1) + (n+1). \quad (3)$$

Now, how many $(n+1)$ are there on the right hand side of (3)?

There are n terms in each of (1) and (2). We merely added corresponding terms from (1) and (2).

Thus, there must be exactly n such $(n+1)$'s.

Therefore, (3) simplifies to $2S_n = n(n+1)$.

Hence, the sum of the first n positive integers is given by

$$S_n = \frac{n(n+1)}{2}. \qquad \text{So, } 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \quad (4)$$

This is a useful formula in finding the sums.

Remarks

The above method was first used by the famous German mathematician **Carl Fredrick Gauss,** known as **Prince of Mathematics,** to find the sum of positive integers upto 100. This problem was given to him by his school teacher when he was just five years old. When you go to higher studies in mathematics, you will learn other methods to arrive at the above formula.

Carl Fredrick Gauss
(1777 – 1855)

Now, let us go back to summing first n terms of a general arithmetic sequence. We have already seen that

$$S_n = na + [d + 2d + 3d + \cdots + (n-1)d]$$

$$= na + d[1 + 2 + 3 + \cdots + (n-1)]$$

$$= na + d\frac{n(n-1)}{2} \quad \text{using (4)}$$

$$= \frac{n}{2}[2a + (n-1)d] \quad (5)$$

Hence, we have

$$S_n = \frac{n}{2}[a + (a + (n - 1)d)] = \frac{n}{2}\ (\text{first term} + \text{last term})$$

$$= \frac{n}{2}(a + l).$$

The sum S_n of the first n terms of an arithmetic sequence with first term a is given by

(i) $S_n = \frac{n}{2}[2a + (n - 1)d]$ if the common difference d is given.

(ii) $S_n = \frac{n}{2}(a + l)$, if the last term l is given.

Example 2.16

Find the sum of the arithmetic series $5 + 11 + 17 + \cdots + 95$.

Solution Given that the series $5 + 11 + 17 + \cdots + 95$ is an arithmetic series.

Note that $a = 5$, $d = 11 - 5 = 6$, $l = 95$.

Now, $$n = \frac{l - a}{d} + 1$$

$$= \frac{95 - 5}{6} + 1 = \frac{90}{6} + 1 = 16.$$

Hence, the sum $$S_n = \frac{n}{2}[l + a]$$

$$S_{16} = \frac{16}{2}[95 + 5] = 8(100) = 800.$$

Example 2.17

Find the sum of the first $2n$ terms of the following series.

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots.$$

Solution We want to find $1^2 - 2^2 + 3^2 - 4^2 + \cdots$ to $2n$ terms

$= 1 - 4 + 9 - 16 + 25 - \cdots$ to $2n$ terms

$= (1 - 4) + (9 - 16) + (25 - 36) + \cdots$ to n terms. (after grouping)

$= -3 + (-7) + (-11) + \cdots$ n terms

Now, the above series is in an A.P. with first term $a = -3$ and common difference $d = -4$

Therefore, the required sum $$= \frac{n}{2}[2a + (n - 1)d]$$

$$= \frac{n}{2}[2(-3) + (n - 1)(-4)]$$

$$= \frac{n}{2}[-6 - 4n + 4] = \frac{n}{2}[-4n - 2]$$

$$= \frac{-2n}{2}(2n + 1) = -n(2n + 1).$$

Example 2.18

In an arithmetic series, the sum of first 14 terms is -203 and the sum of the next 11 terms is –572. Find the arithmetic series.

Solution Given that

$$
\begin{aligned}
S_{14} &= -203 \\
\Longrightarrow \quad \frac{14}{2}[2a+13d] &= -203 \\
\Longrightarrow \quad 7[2a+13d] &= -203 \\
\Longrightarrow \quad 2a+13d &= -29. \qquad (1)
\end{aligned}
$$

Also, the sum of the next 11 terms $= -572$.

$$
\begin{aligned}
\text{Now,} \quad S_{25} &= S_{14} + (-572) \\
\text{That is,} \quad S_{25} &= -203 - 572 = -775. \\
\Longrightarrow \quad \frac{25}{2}[2a+24d] &= -775 \\
\Longrightarrow \quad 2a+24d &= -31 \times 2 \\
\Longrightarrow \quad a+12d &= -31 \qquad (2)
\end{aligned}
$$

Solving (1) and (2) we get, $a = 5$ and $d = -3$.

Thus, the required arithmetic series is $5+(5-3)+(5+2(-3))+\cdots$.

That is, the series is $5+2-1-4-7-\cdots$.

Example 2.19

How many terms of the arithmetic series $24+21+18+15+\cdots$, be taken continuously so that their sum is -351.

Solution In the given arithmetic series, $a = 24$, $d = -3$.

Let us find n such that $S_n = -351$

$$
\begin{aligned}
\text{Now,} \quad S_n &= \frac{n}{2}[2a+(n-1)d] = -351 \\
\text{That is,} \quad \frac{n}{2}[2(24)+(n-1)(-3)] &= -351 \\
\Longrightarrow \quad \frac{n}{2}[48-3n+3] &= -351 \\
\Longrightarrow \quad n(51-3n) &= -702 \\
\Longrightarrow \quad n^2-17n-234 &= 0 \\
(n-26)(n+9) &= 0 \\
\therefore \quad n = 26 &\text{ or } n = -9
\end{aligned}
$$

Here n, being the number of terms needed, cannot be negative.

Thus, 26 terms are needed to get the sum -351.

Example 2.20

Find the sum of all 3 digit natural numbers, which are divisible by 8.

Solution

The three digit natural numbers divisible by 8 are 104, 112, 120, $\cdots$, 992.

Let S_n denote their sum. That is, $S_n = 104 + 112 + 120 + 128 + , \cdots + 992$.

Now, the sequence 104, 112, 120, $\cdots$, 992 forms an A.P.

Here, $a = 104$, $d = 8$ and $l = 992$.

$$\therefore \quad n = \frac{l-a}{d} + 1 = \frac{992-104}{8} + 1$$

$$= \frac{888}{8} + 1 = 112.$$

Thus, $S_{112} = \frac{n}{2}[a + l] = \frac{112}{2}[104 + 992] = 56(1096) = 61376$.

Hence, the sum of all three digit numbers, which are divisible by 8 is equal to 61376.

Example 2.21

The measures of the interior angles taken in order of a polygon form an arithmetic sequence. The least measurement in the sequence is 85°. The greatest measurement is 215°. Find the number of sides in the given polygon.

Solution Let n denote the number of sides of the polygon.

Now, the measures of interior angles form an arithmetic sequence.

Let the sum of the interior angles of the polygon be

$$S_n = a + (a + d) + (a + 2d) + \cdots + l \text{ , where } a = 85 \text{ and } l = 215.$$

We have, $\quad S_n = \frac{n}{2}[l + a] \qquad (1)$

We know that the sum of the interior angles of a polygon is $(n-2)\times 180^0$.

Thus, $\quad S_n = (n-2)\times 180$

From (1), we have $\quad \frac{n}{2}[l + a] = (n-2)\times 180$

$$\Longrightarrow \quad \frac{n}{2}[215 + 85] = (n-2)\times 180$$

$$150n = 180(n-2) \quad \Longrightarrow \quad n = 12..$$

Hence, the number of sides of the polygon is 12.

Exercise 2.4

1. Find the sum of the first (i) 75 positive integers (ii) 125 natural numbers.
2. Find the sum of the first 30 terms of an A.P. whose n^{th} term is $3 + 2n$.
3. Find the sum of each arithmetic series

   (i) $38 + 35 + 32 + \cdots + 2$. (ii) $6 + 5\frac{1}{4} + 4\frac{1}{2} + \cdots$ 25 terms.

4. Find the S_n for the following arithmetic series described.

   (i) $a = 5, \quad n = 30, \quad l = 121$ (ii) $a = 50, \quad n = 25, \quad d = -4$

5. Find the sum of the first 40 terms of the series $1^2 - 2^2 + 3^2 - 4^2 + \cdots$.

6. In an arithmetic series, the sum of first 11 terms is 44 and that of the next 11 terms is 55. Find the arithmetic series.

7. In the arithmetic sequence 60, 56, 52, 48,$\cdots$, starting from the first term, how many terms are needed so that their sum is 368?

8. Find the sum of all 3 digit natural numbers, which are divisible by 9.

9. Find the sum of first 20 terms of the arithmetic series in which 3rd term is 7 and 7th term is 2 more than three times its 3rd term.

10. Find the sum of all natural numbers between 300 and 500 which are divisible by 11.

11. Solve: $1 + 6 + 11 + 16 + \cdots + x = 148$.

12. Find the sum of all numbers between 100 and 200 which are not divisible by 5.

13. A construction company will be penalised each day for delay in construction of a bridge. The penalty will be ₹4000 for the first day and will increase by ₹1000 for each following day. Based on its budget, the company can afford to pay a maximum of ₹1,65,000 towards penalty. Find the maximum number of days by which the completion of work can be delayed

14. A sum of ₹1000 is deposited every year at 8% simple interest. Calculate the interest at the end of each year. Do these interest amounts form an A.P.? If so, find the total interest at the end of 30 years.

15. The sum of first n terms of a certain series is given as $3n^2 - 2n$. Show that the series is an arithmetic series.

16. If a clock strikes once at 1 o'clock, twice at 2 o'clock and so on, how many times will it strike in a day?

17. Show that the sum of an arithmetic series whose first term is a, second term b and the last term is c is equal to $\dfrac{(a+c)(b+c-2a)}{2(b-a)}$.

18. If there are $(2n+1)$ terms in an arithmetic series, then prove that the ratio of the sum of odd terms to the sum of even terms is $(n+1):n$.

19. The ratio of the sums of first m and first n terms of an arithmetic series is $m^2 : n^2$ show that the ratio of the m^{th} and n^{th} terms is $(2m-1):(2n-1)$

20. A gardener plans to construct a trapezoidal shaped structure in his garden. The longer side of trapezoid needs to start with a row of 97 bricks. Each row must be decreased by 2 bricks on each end and the construction should stop at 25th row. How many bricks does he need to buy?

2.5.2 Geometric series

A series is a geometric series if the terms of the series form a geometric sequence.

Let $a, ar, ar^2, \cdots, ar^{n-1}, ar^n, \cdots$ be a geometric sequence where $r \neq 0$ is the common ratio. We want to find the sum of the first n terms of this sequence.

Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ (1)

If $r = 1$, then from (1) it follows that $S_n = na$.

For $r \neq 1$, using (1) we have

$$rS_n = r(a + ar + ar^2 + \cdots + ar^{n-1}) = ar + ar^2 + ar^3 + \cdots + ar^n. \quad (2)$$

Now subtracting (2) from (1), we get

$$S_n - rS_n = (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + \cdots + ar^n)$$

$$\Longrightarrow \quad S_n(1-r) = a(1-r^n)$$

Hence, we have $S_n = \dfrac{a(1-r^n)}{1-r}$, since $r \neq 1$.

The sum of the first n terms of a geometric series is given by

$$S_n = \begin{cases} \dfrac{a(r^n-1)}{r-1} = \dfrac{a(1-r^n)}{1-r}, & \text{if } r \neq 1 \\ na & \text{if } r = 1. \end{cases}$$

where a is the first term and r is the common ratio.

Remarks

Actually, if $-1 < r < 1$, then the following formula holds:

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1-r}.$$

Note that the sum of infinite number of positive numbers may give a finite value.

Example 2.22

Find the sum of the first 25 terms of the geometric series

$16 - 48 + 144 - 432 + \cdots$.

Solution Here, $a = 16$, $r = -\dfrac{48}{16} = -3 \neq 1$. Now, $S_n = \dfrac{a(1-r^n)}{1-r}$, $r \neq 1$.

So, we have $S_{25} = \dfrac{16(1-(-3)^{25})}{1-(-3)} = \dfrac{16(1+3^{25})}{4} = 4(1+3^{25})$.

Example 2.23

Find S_n for each of the geometric series described below:

(i) $a = 2,\ t_6 = 486,\ n = 6$ (ii) $a = 2400,\ r = -3,\ n = 5$

Solution

(i) Here $a = 2,\ t_6 = 486,\ n = 6$

Now $t_6 = 2(r)^5 = 486$

$$\Longrightarrow \quad r^5 = 243 \quad \therefore \quad r = 3.$$

$$\text{Now,} \quad S_n = \frac{a(r^n - 1)}{r - 1} \text{ if } r \neq 1$$

$$\text{Thus,} \quad S_6 = \frac{2(3^6 - 1)}{3 - 1} = 3^6 - 1 = 728.$$

(ii) Here $a = 2400,\ r = -3,\ n = 5$

$$\text{Thus,} \quad S_5 = \frac{a(r^5 - 1)}{r - 1} \text{ if } r \neq 1$$

$$= \frac{2400[(-3)^5 - 1]}{(-3) - 1}$$

$$\text{Hence,} \quad S_5 = \frac{2400}{4}(1 + 3^5) = 600(1 + 243) = 146400.$$

Example 2.24

In the geometric series $2 + 4 + 8 + \cdots$, starting from the first term how many consecutive terms are needed to yield the sum 1022?

Solution Given the geometric series is $2 + 4 + 8 + \cdots$.

Let n be the number of terms required to get the sum.

Here $a = 2,\ r = 2,\ S_n = 1022$.

To find n, let us consider

$$S_n = \frac{a[r^n - 1]}{r - 1} \text{ if } r \neq 1$$

$$= (2)\left[\frac{2^n - 1}{2 - 1}\right] = 2(2^n - 1).$$

But $S_n = 1022$ and hence $2(2^n - 1) = 1022$

$$\Longrightarrow \quad 2^n - 1 = 511$$

$$\Longrightarrow \quad 2^n = 512 = 2^9. \quad \text{Thus,} \quad n = 9.$$

Example 2.25

The first term of a geometric series is 375 and the fourth term is 192. Find the common ratio and the sum of the first 14 terms.

Solution Let a be the first term and r be the common ratio of the given G.P.

Given that $a = 375$, $t_4 = 192$.

$$\text{Now,} \quad t_n = ar^{n-1}$$

$$\therefore \quad t_4 = 375\,r^3 \implies 375\,r^3 = 192$$

$$r^3 = \frac{192}{375} \implies r^3 = \frac{64}{125}$$

$$r^3 = \left(\frac{4}{5}\right)^3 \implies r = \frac{4}{5}, \text{ which is the required common ratio.}$$

$$\text{Now,} \quad S_n = a\left[\frac{r^n - 1}{r-1}\right] \text{ if } r \neq 1$$

$$\text{Thus,} \quad S_{14} = \frac{375\left[\left(\frac{4}{5}\right)^{14} - 1\right]}{\frac{4}{5} - 1} = (-1)\times 5 \times 375\left[\left(\frac{4}{5}\right)^{14} - 1\right]$$

$$= (375)(5)\left[1 - \left(\frac{4}{5}\right)^{14}\right] = 1875\left[1 - \left(\frac{4}{5}\right)^{14}\right].$$

> *Note*
>
> In the above example, one can use $S_n = a\left[\frac{1-r^n}{1-r}\right]$ if $r \neq 1$ instead of $S_n = a\left[\frac{r^n - 1}{r-1}\right]$ if $r \neq 1$.

Example 2.26

A geometric series consists of four terms and has a positive common ratio. The sum of the first two terms is 8 and the sum of the last two terms is 72. Find the series.

Solution Let the sum of the four terms of the geometric series be $a + ar + ar^2 + ar^3$ and $r > 0$

Given that $a + ar = 8$ and $ar^2 + ar^3 = 72$

$$\text{Now,} \quad ar^2 + ar^3 = r^2(a + ar) = 72$$

$$\implies \quad r^2(8) = 72 \quad \therefore \quad r = \pm 3$$

Since $r > 0$, we have $r = 3$.

$$\text{Now,} \quad a + ar = 8 \implies a = 2$$

Thus, the geometric series is $2 + 6 + 18 + 54$.

Example 2.27

Find the sum to n terms of the series $6 + 66 + 666 + \cdots$

Solution Note that the given series is not a geometric series.

$$\text{We need to find} \quad S_n = 6 + 66 + 666 + \cdots \text{ to } n \text{ terms}$$

$$S_n = 6(1 + 11 + 111 + \cdots \text{ to } n \text{ terms})$$

$$= \frac{6}{9}(9 + 99 + 999 + \cdots \text{ to } n \text{ terms}) \quad \text{(Multiply and divide by 9)}$$

$$= \frac{2}{3}[(10-1) + (100-1) + (1000-1) + \cdots \text{ to } n \text{ terms}]$$

$$= \frac{2}{3}[(10 + 10^2 + 10^3 + \cdots n \text{ terms}) - n]$$

$$\text{Thus,} \quad S_n = \frac{2}{3}\left[\frac{10(10^n - 1)}{9} - n\right].$$

Example 2.28

An organisation plans to plant saplings in 25 streets in a town in such a way that one sapling for the first street, two for the second, four for the third, eight for the fourth street and so on. How many saplings are needed to complete the work?

Solution The number of saplings to be planted for each of the 25 streets in the town forms a G.P. Let S_n be the total number of saplings needed.

Then, $S_n = 1 + 2 + 4 + 8 + 16 + \cdots$ to 25 terms.

Here, $a = 1, \quad r = 2, \quad n = 25$

$$S_n = a\left[\frac{r^n - 1}{r - 1}\right]$$

$$S_{25} = (1)\,\frac{[2^{25} - 1]}{2 - 1}$$

$$= 2^{25} - 1$$

Thus, the number of saplings to be needed is $2^{25} - 1$.

Exercise 2.5

1. Find the sum of the first 20 terms of the geometric series $\frac{5}{2} + \frac{5}{6} + \frac{5}{18} + \cdots$.

2. Find the sum of the first 27 terms of the geometric series $\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$.

3. Find S_n for each of the geometric series described below.

   (i) $a = 3, \; t_8 = 384, \; n = 8$. (ii) $a = 5, \; r = 3, \; n = 12$.

4. Find the sum of the following finite series

   (i) $1 + 0.1 + 0.01 + 0.001 + \cdots + (0.1)^9$ (ii) $1 + 11 + 111 + \cdots$ to 20 terms.

5. How many consecutive terms starting from the first term of the series

   (i) $3 + 9 + 27 + \cdots$ would sum to 1092? (ii) $2 + 6 + 18 + \cdots$ would sum to 728?

6. The second term of a geometric series is 3 and the common ratio is $\frac{4}{5}$. Find the sum of first 23 consecutive terms in the given geometric series.

7. A geometric series consists of four terms and has a positive common ratio. The sum of the first two terms is 9 and sum of the last two terms is 36. Find the series.

8. Find the sum of first n terms of the series

   (i) $7 + 77 + 777 + \cdots$. (ii) $0.4 + 0.94 + 0.994 + \cdots$.

9. Suppose that five people are ill during the first week of an epidemic and each sick person spreads the contagious disease to four other people by the end of the second week and so on. By the end of 15^{th} week, how many people will be affected by the epidemic?

10. A gardener wanted to reward a boy for his good deeds by giving some mangoes. He gave the boy two choices. He could either have 1000 mangoes at once or he could get 1 mango on the first day, 2 on the second day, 4 on the third day, 8 mangoes on the fourth day and so on for ten days. Which option should the boy choose to get the maximum number of mangoes?

11. A geometric series consists of even number of terms. The sum of all terms is 3 times the sum of odd terms. Find the common ratio.

12. If S_1, S_2 and S_3 are the sum of first n, $2n$ and $3n$ terms of a geometric series respectively, then prove that $S_1(S_3 - S_2) = (S_2 - S_1)^2$.

Remarks

The sum of the first n terms of a geometric series with $a = 1$ and common ratio $x \neq 1$, is given by $1 + x + x^2 + \cdots + x^{n-1} = \dfrac{x^n - 1}{x - 1}$, $\quad x \neq 1$.

Note that the left hand side of the above equation is a special polynomial in x of degree $n - 1$. This formula will be useful in finding the sum of some series.

2.5.3 Special series $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$

We have already used the symbol Σ for summation.

Let us list out some examples of finite series represented by sigma notation.

Sl. No.	Notation	Expansion
1.	$\sum_{k=1}^{n} k$ or $\sum_{j=1}^{n} j$	$1 + 2 + 3 + \cdots + n$
2.	$\sum_{n=2}^{6} (n-1)$	$1 + 2 + 3 + 4 + 5$
3.	$\sum_{d=0}^{5} (d+5)$	$5 + 6 + 7 + 8 + 9 + 10$
4.	$\sum_{k=1}^{n} k^2$	$1^2 + 2^2 + 3^3 + \cdots + n^2$
5.	$\sum_{k=1}^{10} 3 = 3\sum_{k=1}^{10} 1$	$3[1 + 1 + \cdots 10 \text{ terms}] = 30.$

We have derived that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$. This can also be obtained using A.P. with $a = 1$, $d = 1$ and $l = n$ as $S_n = \dfrac{n}{2}(a + l) = \dfrac{n}{2}(1 + n)$.

Hence, using sigma notation we write it as $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Let us derive the formulae for

$$\text{(i)}\ \sum_{k=1}^{n}(2k-1), \qquad \text{(ii)}\ \sum_{k=1}^{n}k^2 \quad \text{and} \quad \text{(iii)}\ \sum_{k=1}^{n}k^3.$$

Proof:

(i) Let us find $\sum_{k=1}^{n}(2k-1) = 1+3+5+\cdots+(2n-1)$.

This is an A.P. consisting of n terms with $a = 1,\ d = 2,\ l = (2n-1)$.

$$\therefore \quad S_n = \frac{n}{2}(1+2n-1) = n^2 \qquad \left(S_n = \frac{n}{2}(a+l)\right)$$

Thus, $\sum_{k=1}^{n}(2k-1) = n^2$ (1)

Remarks

1. The formula (1) can also be obtained by the following method

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n}2k - \sum_{k=1}^{n}1 = 2\left(\sum_{k=1}^{n}k\right) - n = \frac{2(n)(n+1)}{2} - n = n^2.$$

2. From (1), $1+3+5+\cdots+l = \left(\frac{l+1}{2}\right)^2$, since $l = 2n-1 \Longrightarrow n = \frac{l+1}{2}$.

(ii) We know that $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

$$\therefore \quad k^3-(k-1)^3 = k^2+k(k-1)+(k-1)^2 \quad (\text{ take } a = k \text{ and } b = k-1)$$

$$\Longrightarrow \quad k^3-(k-1)^3 = 3k^2-3k+1 \qquad (2)$$

When $k = 1$, $1^3 - 0^3 = 3(1)^2 - 3(1) + 1$

When $k = 2$, $2^3 - 1^3 = 3(2)^2 - 3(2) + 1$

When $k = 3$, $3^3 - 2^3 = 3(3)^2 - 3(3) + 1$. Continuing this, we have

when $k = n$, $n^3-(n-1)^3 = 3(n)^2 - 3(n) + 1$.

Adding the above equations corresponding to $k = 1, 2, \cdots, n$ column-wise, we obtain

$$n^3 = 3[1^2+2^2+\cdots+n^2] - 3[1+2+\cdots+n] + n$$

Thus, $3[1^2+2^2+\cdots+n^2] = n^3 + 3[1+2+\cdots+n] - n$

$$3\left[\sum_{k=1}^{n}k^2\right] = n^3 + \frac{3n(n+1)}{2} - n$$

Hence, $$\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}. \qquad (3)$$

(iii) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots n^3$

Let us observe the following pattern.

$$1^3 = 1 = (1)^2$$
$$1^3 + 2^3 = 9 = (1+2)^2$$
$$1^3 + 2^3 + 3^3 = 36 = (1+2+3)^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = 100 = (1+2+3+4)^2.$$

Extending this pattern to n terms, we get

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = [1+2+3+\cdots+n]^2$$
$$= \left[\frac{n(n+1)}{2}\right]^2$$

Thus, $$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 = \left[\frac{n(n+1)}{2}\right]^2. \qquad (4)$$

(i) The sum of the first n natural numbers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

(ii) The sum of the first n odd natural numbers, $\sum_{k=1}^{n} (2k-1) = n^2$.

(iii) The sum of first n odd natural numbers (when the last term l is given) is

$$1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2.$$

(iv) The sum of squares of first n natural numbers,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(v) The sum of cubes of the first n natural numbers,

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Example 2.29

Find the sum of the following series

(i) $26 + 27 + 28 + \cdots + 60$ (ii) $1 + 3 + 5 + \cdots$ to 25 terms (iii) $31 + 33 + \cdots + 53$.

Solution

(i) We have $26 + 27 + 28 + \cdots + 60 = (1+2+3+\cdots+60) - (1+2+3+\cdots+25)$

$$= \sum_{1}^{60} n - \sum_{1}^{25} n$$
$$= \frac{60(60+1)}{2} - \frac{25(25+1)}{2}$$
$$= (30 \times 61) - (25 \times 13) = 1830 - 325 = 1505.$$

(ii) Here, $n = 25$

$\therefore \quad 1 + 3 + 5 + \cdots$ to 25 terms $= 25^2$ $\qquad \left(\sum_{k=1}^{n} (2k-1) = n^2 \right)$

$= 625.$

(iii) $31 + 33 + \cdots + 53$

$= (1 + 3 + 5 + \cdots + 53) - (1 + 3 + 5 + \cdots + 29)$

$= \left(\frac{53+1}{2}\right)^2 - \left(\frac{29+1}{2}\right)^2 \qquad \left(1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2 \right)$

$= 27^2 - 15^2 = 504.$

Example 2.30

Find the sum of the following series

(i) $1^2 + 2^2 + 3^2 + \cdots + 25^2$ (ii) $12^2 + 13^2 + 14^2 + \cdots + 35^2$

(iii) $1^2 + 3^2 + 5^2 + \cdots + 51^2$.

Solution

(i) Now, $1^2 + 2^2 + 3^2 + \cdots + 25^2 = \sum_{1}^{25} n^2$

$= \frac{25(25+1)(50+1)}{6} \qquad \left(\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \right)$

$= \frac{(25)(26)(51)}{6}$

$\therefore \quad 1^2 + 2^2 + 3^2 + \cdots + 25^2 = 5525.$

(ii) Now, $12^2 + 13^2 + 14^2 + \cdots + 35^2$

$= (1^2 + 2^2 + 3^2 + \cdots + 35^2) - (1^2 + 2^2 + 3^2 + \cdots + 11^2)$

$= \sum_{1}^{35} n^2 - \sum_{1}^{11} n^2$

$= \frac{35(35+1)(70+1)}{6} - \frac{11(12)(23)}{6}$

$= \frac{(35)(36)(71)}{6} - \frac{(11)(12)(23)}{6}$

$= 14910 - 506 = 14404.$

(iii) Now, $1^2 + 3^2 + 5^2 + \cdots + 51^2$

$= (1^2 + 2^2 + 3^2 + \cdots + 51^2) - (2^2 + 4^2 + 6^2 + \cdots 50^2)$

$= \sum_{1}^{51} n^2 - 2^2[1^2 + 2^2 + 3^2 + \cdots + 25^2]$

$$= \sum_{1}^{51} n^2 - 4\sum_{1}^{25} n^2$$

$$= \frac{51(51+1)(102+1)}{6} - 4 \times \frac{25(25+1)(50+1)}{6}$$

$$= \frac{(51)(52)(103)}{6} - 4 \times \frac{25(26)(51)}{6}$$

$$= 45526 - 22100 = 23426.$$

Example 2.31

Find the sum of the series.

(i) $1^3 + 2^3 + 3^3 + \cdots + 20^3$ (ii) $11^3 + 12^3 + 13^3 + \cdots + 28^3$

Solution

(i) $$1^3 + 2^3 + 3^3 + \cdots 20^3 = \sum_{1}^{20} n^3$$

$$= \left(\frac{20(20+1)}{2}\right)^2 \quad \text{using } \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

$$= \left(\frac{20 \times 21}{2}\right)^2 = (210)^2 = 44100.$$

(ii) Next we consider $11^3 + 12^3 + \cdots + 28^3$

$$= (1^3 + 2^3 + 3^3 + \cdots + 28^3) - (1^3 + 2^3 + \cdots + 10^3)$$

$$= \sum_{1}^{28} n^3 - \sum_{1}^{10} n^3$$

$$= \left[\frac{28(28+1)}{2}\right]^2 - \left[\frac{10(10+1)}{2}\right]^2$$

$$= 406^2 - 55^2 = (406+55)(406-55)$$

$$= (461)(351) = 161811.$$

Example 2.32

Find the value of k, if $1^3 + 2^3 + 3^3 + \cdots + k^3 = 4356$

Solution Note that k is a positive integer.

Given that $1^3 + 2^3 + 3^3 + \cdots + k^3 = 4356$

$$\Longrightarrow \left(\frac{k(k+1)}{2}\right)^2 = 4356 = 6 \times 6 \times 11 \times 11$$

Taking square root, we get $\dfrac{k(k+1)}{2} = 66$

$$\Longrightarrow k^2 + k - 132 = 0 \quad \Longrightarrow (k+12)(k-11) = 0$$

Thus, $k = 11$, since k is positive.

Example 2.33

(i) If $1 + 2 + 3 + \cdots + n = 120$, find $1^3 + 2^3 + 3^3 + \cdots n^3$.

(ii) If $1^3 + 2^3 + 3^3 + \cdots + n^3 = 36100$, then find $1 + 2 + 3 + \cdots + n$.

Solution

(i) Given $1 + 2 + 3 + \cdots + n = 120$ i.e. $\frac{n(n+1)}{2} = 120$

$$\therefore \quad 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = 120^2 = 14400$$

(ii) Given $1^3 + 2^3 + 3^3 + \cdots + n^3 = 36100$

$$\Longrightarrow \quad \left(\frac{n(n+1)}{2}\right)^2 = 36100 = 19 \times 19 \times 10 \times 10$$

$$\Longrightarrow \quad \frac{n(n+1)}{2} = 190$$

Thus, $1 + 2 + 3 + \cdots + n = 190$.

Example 2.34

Find the total area of 14 squares whose sides are 11 cm, 12 cm, $\cdots$, 24 cm, respectively.

Solution The areas of the squares form the series $11^2 + 12^2 + \cdots + 24^2$

Total area of 14 squares $= 11^2 + 12^2 + 13^2 + \cdots + 24^2$

$$= (1^2 + 2^2 + 3^2 + \cdots + 24^2) - (1^2 + 2^2 + 3^2 + \cdots + 10^2)$$

$$= \sum_1^{24} n^2 - \sum_1^{10} n^2$$

$$= \frac{24(24+1)(48+1)}{6} - \frac{10(10+1)(20+1)}{6}$$

$$= \frac{(24)(25)(49)}{6} - \frac{(10)(11)(21)}{6}$$

$$= 4900 - 385$$

$$= 4515 \text{ sq. cm.}$$

Exercise 2.6

1. Find the sum of the following series.

(i) $1 + 2 + 3 + \cdots + 45$ (ii) $16^2 + 17^2 + 18^2 + \cdots + 25^2$

(iii) $2 + 4 + 6 + \cdots + 100$ (iv) $7 + 14 + 21 \cdots + 490$

(v) $5^2 + 7^2 + 9^2 + \cdots + 39^2$ (vi) $16^3 + 17^3 + \cdots + 35^3$

2. Find the value of k if

(i) $1^3 + 2^3 + 3^3 + \cdots + k^3 = 6084$ (ii) $1^3 + 2^3 + 3^3 + \cdots + k^3 = 2025$

3. If $1 + 2 + 3 + \cdots + p = 171$, then find $1^3 + 2^3 + 3^3 + \cdots + p^3$.

4. If $1^3 + 2^3 + 3^3 + \cdots + k^3 = 8281$, then find $1 + 2 + 3 + \cdots + k$.

5. Find the total area of 12 squares whose sides are 12cm, 13cm, $\cdots$, 23cm. respectively.

6. Find the total volume of 15 cubes whose edges are 16cm, 17cm, 18cm, $\cdots$, 30cm respectively.

Exercise 2.7

Choose the correct answer.

1. Which one of the following is not true?

(A) A sequence is a real valued function defined on $\mathbb{N}$.

(B) Every function represents a sequence.

(C) A sequence may have infinitely many terms.

(D) A sequence may have a finite number of terms.

2. The 8th term of the sequence 1, 1, 2, 3, 5, 8, $\cdots$ is

(A) 25 (B) 24 (C) 23 (D) 21

3. The next term of $\frac{1}{20}$ in the sequence $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \cdots$ is

(A) $\frac{1}{24}$ (B) $\frac{1}{22}$ (C) $\frac{1}{30}$ (D) $\frac{1}{18}$

4. If a, b, c, l, m are in A.P, then the value of $a - 4b + 6c - 4l + m$ is

(A) 1 (B) 2 (C) 3 (D) 0

5. If a, b, c are in A.P. then $\frac{a-b}{b-c}$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{c}$ (C) $\frac{a}{c}$ (D) 1

6. If the n^{th} term of a sequence is $100n+10$, then the sequence is

(A) an A.P. (B) a G.P. (C) a constant sequence (D) neither A.P. nor G.P.

7. If $a_1, a_2, a_3, \cdots$ are in A.P. such that $\frac{a_4}{a_7} = \frac{3}{2}$, then the 13th term of the A.P. is

(A) $\frac{3}{2}$ (B) 0 (C) $12a_1$ (D) $14a_1$

8. If the sequence $a_1, a_2, a_3, \cdots$ is in A.P. , then the sequence $a_5, a_{10}, a_{15}, \cdots$ is

(A) a G.P. (B) an A.P. (C) neither A.P nor G.P. (D) a constant sequence

9. If $k+2$, $4k-6$, $3k-2$ are the three consecutive terms of an A.P, then the value of k is

(A) 2 (B) 3 (C) 4 (D) 5

10. If $a, b, c, l, m. n$ are in A.P., then $3a+7, 3b+7, 3c+7, 3l+7, 3m+7, 3n+7$ form

(A) a G.P. (B) an A.P. (C) a constant sequence (D) neither A.P. nor G.P

11. If the third term of a G.P is 2, then the product of first 5 terms is

(A) 5^2 (B) 2^5 (C) 10 (D) 15

12. If a, b, c are in G.P, then $\frac{a-b}{b-c}$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{a}{c}$ (D) $\frac{c}{b}$

13. If $x,\ 2x+2,\ 3x+3$ are in G.P, then $5x,\ 10x+10,\ 15x+15$ form

(A) an A.P. (B) a G.P. (C) a constant sequence (D) neither A.P. nor a G.P.

14. The sequence $-3, -3, -3, \cdots$ is

(A) an A.P. only (B) a G.P. only (C) neither A.P. nor G.P (D) both A.P. and G.P.

15. If the product of the first four consecutive terms of a G.P is 256 and if the common ratio is 4 and the first term is positive, then its 3rd term is

(A) 8 (B) $\frac{1}{16}$ (C) $\frac{1}{32}$ (D) 16

16. In a G.P, $t_2 = \frac{3}{5}$ and $t_3 = \frac{1}{5}$. Then the common ratio is

(A) $\frac{1}{5}$ (B) $\frac{1}{3}$ (C) 1 (D) 5

17. If $x \neq 0$, then $1 + \sec x + \sec^2 x + \sec^3 x + \sec^4 x + \sec^5 x$ is equal to

(A) $(1+\sec x)(\sec^2 x + \sec^3 x + \sec^4 x)$ (B) $(1+\sec x)(1+\sec^2 x + \sec^4 x)$

(C) $(1-\sec x)(\sec x + \sec^3 x + \sec^5 x)$ (D) $(1+\sec x)(1+\sec^3 x + \sec^4 x)$

18. If the n^{th} term of an A.P. is $t_n = 3 - 5n$, then the sum of the first n terms is

(A) $\frac{n}{2}[1-5n]$ (B) $n(1-5n)$ (C) $\frac{n}{2}(1+5n)$ (D) $\frac{n}{2}(1+n)$

19. The common ratio of the G.P. $a^{m-n},\ a^m,\ a^{m+n}$ is

(A) a^m (B) a^{-m} (C) a^n (D) a^{-n}

20. If $1 + 2 + 3 + \ldots + n = k$ then $1^3 + 2^3 + \cdots + n^3$ is equal to

(A) k^2 (B) k^3 (C) $\frac{k(k+1)}{2}$ (D) $(k+1)^3$

Points to Remember

- A sequence of real numbers is an arrangement or a list of real numbers in a specific order.
- The sequence given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, \cdots$ is called the Fibonacci sequence which is nothing but 1, 1, 2, 3, 5, 8, 13, 21, 34, $\cdots$
- A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called an arithmetic sequence if $a_{n+1} = a_n + d$, $n \in \mathbb{N}$ where d is a constant. Here a_1 is called the first term and the constant d is called the common difference.

  The formula for the general term of an A.P. is $t_n = a + (n-1)d \quad \forall\ n \in \mathbb{N}$.
- A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called a geometric sequence if $a_{n+1} = a_n r$, where $r \neq 0$, $n \in \mathbb{N}$ where r is a constant. Here, a_1 is the first term and the constant r is called the common ratio. The formula for the general term of a G.P. is $t_n = ar^{n-1}$, $n = 1, 2, 3, \cdots$.
- An expression of addition of terms of a sequence is called a series. If the sum consists only finite number of terms, then it is called a finite series. If the sum consists of infinite number of terms of a sequence, then it is called an infinite series.
- The sum S_n of the first n terms of an arithmetic sequence with first term a and common difference d is given by $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$, where l is the last term.
- The sum of the first n terms of a geometric series is given by

$$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r - 1} = \dfrac{a(1 - r^n)}{1 - r}, & \text{if } r \neq 1 \\ na & \text{if } r = 1. \end{cases}$$

  where a is the first term and r is the common ratio.
- The sum of the first n natural numbers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.
- The sum of the first n odd natural numbers, $\sum_{k=1}^{n} (2k-1) = n^2$
- The sum of first n odd natural numbers (when the last term l is given) is

$$1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2.$$

- The sum of squares of first n natural numbers, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
- The sum of cubes of the first n natural numbers, $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Do you know?

A Mersenne number, named after Marin Mersenne, is a positive integer of the form $M = 2^p - 1$, where p is a positive integer. If M is a prime, then it is called a Mersenne prime. Interestingly, if $2^p - 1$ is prime, then p is prime. The largest known prime number $2^{43,112,609} - 1$ is a Mersenne prime.

Chapter 16

The binomial series

16.1 Pascal's triangle

A **binomial expression** is one that contains two terms connected by a plus or minus sign. Thus $(p+q)$, $(a+x)^2$, $(2x+y)^3$ are examples of binomial expression. Expanding $(a+x)^n$ for integer values of n from 0 to 6 gives the results shown at the bottom of the page.

From the results the following patterns emerge:

(i) 'a' decreases in power moving from left to right.

(ii) 'x' increases in power moving from left to right.

(iii) The coefficients of each term of the expansions are symmetrical about the middle coefficient when n is even and symmetrical about the two middle coefficients when n is odd.

(iv) The coefficients are shown separately in Table 16.1 and this arrangement is known as **Pascal's triangle**. A coefficient of a term may be obtained by adding the two adjacent coefficients immediately above in the previous row. This is shown by the triangles in Table 16.1, where, for example, $1+3=4$, $10+5=15$, and so on.

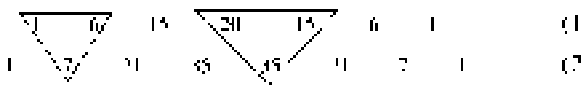
(v) Pasal's triangle method is used for expansions of the form $(a+x)^n$ for integer values of n less than about 8

Problem 1. Use the Pascal's triangle method to determine the expansion of $(a+x)^7$

Table 16.1

$(a+x)^0$							1						
$(a+x)^1$						1		1					
$(a+x)^2$					1		2		1				
$(a+x)^3$				1		3		3		1			
$(a+x)^4$			1		4		6		4		1		
$(a+x)^5$		1		5		10		10		5		1	
$(a+x)^6$	1		6		15		20		15		6		1

From Table 16.1 the row the Pascal's triangle corresponding to $(a+x)^6$ is as shown in (1) below. Adding adjacent coefficients gives the coefficients of $(a+x)^7$ as shown in (2) below.



The first and last terms of the expansion of $(a+x)^7$ and a^7 and x^7 respectively. The powers of 'a' decrease and the powers of 'x' increase moving from left to right. Hence,

$$\mathbf{(a+x)^7 = a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7}$$

$$\begin{aligned}
(a+x)^0 &= && 1\\
(a+x)^1 &= && a+x\\
(a+x)^2 &= (a+x)(a+x) = && a^2+2ax+x^2\\
(a+x)^3 &= (a+x)^2(a+x) = && a^3+3a^2x+3ax^2+x^3\\
(a+x)^4 &= (a+x)^3(a+x) = && a^4+4a^3x+6a^2x^2+4ax^3+x^4\\
(a+x)^5 &= (a+x)^4(a+x) = && a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5\\
(a+x)^6 &= (a+x)^5(a+x) = && a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6
\end{aligned}$$

Problem 2. Determine, using Pascal's triangle method, the expansion of $(2p-3q)^5$

Comparing $(2p-3q)^5$ with $(a+x)^5$ shows that $a=2p$ and $x=-3q$

Using Pascal's triangle method:

$$(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + \cdots$$

Hence

$$\begin{aligned}(2p-3q)^5 &= (2p)^5 + 5(2p)^4(-3q)\\ &\quad + 10(2p)^3(-3q)^2\\ &\quad + 10(2p)^2(-3q)^3\\ &\quad + 5(2p)(-3q)^4 + (-3q)^5\end{aligned}$$

i.e. $\mathbf{(2p-3q)^5 = 32p^5 - 240p^4q + 720p^3q^2 - 1080p^2q^3 + 810pq^4 - 243q^5}$

Now try the following exercise

Exercise 62 Further problems on Pascal's triangle

1. Use Pascal's triangle to expand $(x-y)^7$
$\left[\begin{array}{l}x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3\\ \quad + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7\end{array}\right]$

2. Expand $(2a+3b)^5$ using Pascal's triangle.
$\left[\begin{array}{l}32a^5 + 240a^4b + 720a^3b^2\\ \quad + 1080a^2b^3 + 810ab^4 + 243b^5\end{array}\right]$

16.2 The binomial series

The **binomial series** or **binomial theorem** is a formula for raising a binomial expression to any power without lengthy multiplication. The general binomial expansion of $(a+x)^n$ is given by:

$$\begin{aligned}(a+x)^n &= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2\\ &\quad + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3\\ &\quad + \cdots + x^n\end{aligned}$$

where, for example, 3! denote $3\times 2\times 1$ and is termed 'factorial 3'.

With the binomial theorem n may be a fraction, a decimal fraction or a positive or negative integer.

In the general expansion of $(a+x)^n$ it is noted that the 4th term is:

$$\frac{n(n-1)(n-2)}{3!}a^{n-3}x^3$$

The number 3 is very evident in this expression.

For any term in a binomial expansion, say the r'th term, $(r-1)$ is very evident. It may therefore be reasoned that **the r'th term of the expansion $(a+x)^n$** is:

$$\frac{n(n-1)(n-2)\ldots \text{to}(r-1)\text{terms}}{(r-1)!}a^{n-(r-1)}x^{r-1}$$

If $a=1$ in the binomial expansion of $(a+x)^n$ then:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

which is valid for $-1<x<1$

When x is small compared with 1 then:

$$(1+x)^n \approx 1 + nx$$

16.3 Worked problems on the binomial series

Problem 3. Use the binomial series to determine the expansion of $(2+x)^7$

The binomial expansion is given by:

$$\begin{aligned}(a+x)^n &= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2\\ &\quad + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots\end{aligned}$$

When $a=2$ and $n=7$:

$$\begin{aligned}(2+x)^7 &= 2^7 + 7(2)^6 + \frac{(7)(6)}{(2)(1)}(2)^5x^2\\ &\quad + \frac{(7)(6)(5)}{(3)(2)(1)}(2)^4x^3 + \frac{(7)(6)(5)(4)}{(4)(3)(2)(1)}(2)^3x^4\\ &\quad + \frac{(7)(6)(5)(4)(3)}{(5)(4)(3)(2)(1)}(2)^2x^5\\ &\quad + \frac{(7)(6)(5)(4)(3)(2)}{(6)(5)(4)(3)(2)(1)}(2)x^6\\ &\quad + \frac{(7)(6)(5)(4)(3)(2)(1)}{(7)(6)(5)(4)(3)(2)(1)}x^7\end{aligned}$$

i.e. $\mathbf{(2+x)^7 = 128 + 448x + 672x^2 + 560x^3 + 280x^4 + 84x^5 + 14x^6 + x^7}$

Problem 4. Expand $\left(c - \frac{1}{c}\right)^5$ using the binomial series

$$\left(c-\frac{1}{c}\right)^5 = c^5 + 5c^4\left(-\frac{1}{c}\right) + \frac{(5)(4)}{(2)(1)}c^3\left(-\frac{1}{c}\right)^2 + \frac{(5)(4)(3)}{(3)(2)(1)}c^2\left(-\frac{1}{c}\right)^3 + \frac{(5)(4)(3)(2)}{(4)(3)(2)(1)}c\left(-\frac{1}{c}\right)^4 + \frac{(5)(4)(3)(2)(1)}{(5)(4)(3)(2)(1)}\left(-\frac{1}{c}\right)^5$$

i.e. $\mathbf{\left(c-\frac{1}{c}\right)^5 = c^5 - 5c^4 + 10c - \frac{10}{c} + \frac{5}{c^3} - \frac{1}{c^5}}$

Problem 5. Without fully expanding $(3+x)^7$, determine the fifth term

The r'th term of the expansion $(a+x)^n$ is given by:

$$\frac{n(n-1)(n-2)\ldots \text{to } (r-1) \text{ terms}}{(r-1)!} a^{n-(r-1)}x^{r-1}$$

Substituting $n=7$, $a=3$ and $r-1=5-1=4$ gives:

$$\frac{(7)(6)(5)(4)}{(4)(3)(2)(1)}(3)^{7-4}x^4$$

i.e. the fifth term of $(3+x)^7 = 35(3)^3x^4 = \mathbf{945x^4}$

Problem 6. Find the middle term of $\left(2p - \frac{1}{2q}\right)^{10}$

In the expansion of $(a+x)^{10}$ there are $10+1$, i.e. 11 terms. Hence the middle term is the sixth. Using the general expression for the r'th term where $a=2p$, $x=-\frac{1}{2q}$, $n=10$ and $r-1=5$ gives:

$$\frac{(10)(9)(8)(7)(6)}{(5)(4)(3)(2)(1)}(2p)^{10-5}\left(-\frac{1}{2q}\right)^5 = 252(32p^5)\left(-\frac{1}{32q^5}\right)$$

Hence the middle term of $\left(2q - \frac{1}{2q}\right)^{10}$ is $\mathbf{-252\frac{p^5}{q^5}}$

Problem 7. Evaluate $(1.002)^9$ using the binomial theorem correct to (a) 3 decimal places and (b) 7 significant figures

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$(1.002)^9 = (1+0.002)^9$$

Substituting $x=0.002$ and $n=9$ in the general expansion for $(1+x)^n$ gives:

$$\begin{aligned}(1+0.002)^9 &= 1 + 9(0.002) + \frac{(9)(8)}{(2)(1)}(0.002)^2 + \frac{(9)(8)(7)}{(3)(2)(1)}(0.002)^3 + \cdots \\ &= 1 + 0.018 + 0.000144 + 0.000000672 + \cdots \\ &= 1.018144672\ldots\end{aligned}$$

Hence, $(1.002)^9 =$ **1.018, correct to 3 decimal places**
$=$ **1.018145, correct to 7 significant figures**

Problem 8. Determine the value of $(3.039)^4$, correct to 6 significant figures using the binomial theorem

$(3.039)^4$ may be written in the form $(1+x)^n$ as:

$$\begin{aligned}(3.039)^4 &= (3+0.039)^4 \\ &= \left[3\left(1+\frac{0.039}{3}\right)\right]^4 \\ &= 3^4(1+0.013)^4\end{aligned}$$

$$\begin{aligned}(1+0.013)^4 &= 1 + 4(0.013) + \frac{(4)(3)}{(2)(1)}(0.013)^2 + \frac{(4)(3)(2)}{(3)(2)(1)}(0.013)^3 + \cdots \\ &= 1 + 0.052 + 0.001014 + 0.000008788 + \cdots \\ &= 1.0530228\end{aligned}$$

correct to 8 significant figures

Hence $(3.039)^4 = 3^4(1.0530228)$
$=$ **85.2948, correct to 6 significant figures**

Now try the following exercise

Exercise 63 Further problems on the binomial series

1. Use the binomial theorem to expand $(a+2x)^4$
$\begin{bmatrix} a^4 + 8a^3x + 24a^2x^2 \\ + 32ax^3 + 16x^4 \end{bmatrix}$

2. Use the binomial theorem to expand $(2-x)^6$
$\begin{bmatrix} 64 - 192x + 240x^2 - 160x^3 \\ + 60x^4 - 12x^5 + x^6 \end{bmatrix}$

3. Expand $(2x-3y)^4$
$\begin{bmatrix} 16x^4 - 96x^3y + 216x^2y^2 \\ -216xy^3 + 81y^4 \end{bmatrix}$

4. Determine the expansion of $\left(2x+\frac{2}{x}\right)^5$
$\begin{bmatrix} 32x^5 + 160x^3 + 320x + \dfrac{320}{x} \\ + \dfrac{160}{x^3} + \dfrac{32}{x^5} \end{bmatrix}$

5. Expand $(p+2q)^{11}$ as far as the fifth term
$\begin{bmatrix} p^{11} + 22p^{10}q + 220p^9q^2 \\ + 1320p^8q^3 + 5280p^7q^4 \end{bmatrix}$

6. Determine the sixth term of $\left(3p+\frac{q}{3}\right)^{13}$
$[34\,749\ p^8q^5]$

7. Determine the middle term of $(2a-5b)^8$
$[700\,000\ a^4b^4]$

8. Use the binomial theorem to determine, correct to 4 decimal places:
(a) $(1.003)^8$ (b) $(0.98)^7$
[(a) 1.0243 (b) 0.8681]

9. Evaluate $(4.044)^6$ correct to 3 decimal places.
[4373.880]

16.4 Further worked problems on the binomial series

Problem 9.

(a) Expand $\dfrac{1}{(1+2x)^3}$ in ascending powers of x as far as the term in x^3, using the binomial series.

(b) State the limits of x for which the expansion is valid

(a) Using the binomial expansion of $(1+x)^n$, where $n=-3$ and x is replaced by $2x$ gives:

$$\begin{aligned}\frac{1}{(1+2x)^3} &= (1+2x)^{-3} \\ &= 1 + (-3)(2x) + \frac{(-3)(-4)}{2!}(2x)^2 \\ &\quad + \frac{(-3)(-4)(-5)}{3!}(2x)^3 + \cdots \\ &= \mathbf{1 - 6x + 24x^2 - 80x^3 +}\end{aligned}$$

(b) The expansion is valid provided $|2x| < 1$,

i.e. $\mathbf{|x| < \frac{1}{2}}$ or $\mathbf{-\frac{1}{2} < x < \frac{1}{2}}$

Problem 10.

(a) Expand $\dfrac{1}{(4-x)^2}$ in ascending powers of x as far as the term in x^3, using the binomial theorem.

(b) What are the limits of x for which the expansion in (a) is true?

(a) $$\frac{1}{(4-x)^2} = \frac{1}{\left[4\left(1-\frac{x}{4}\right)\right]^2} = \frac{1}{4^2\left(1-\frac{x}{4}\right)^2} = \frac{1}{16}\left(1-\frac{x}{4}\right)^{-2}$$

Using the expansion of $(1+x)^n$

$$\begin{aligned}\frac{1}{(4-x)^2} &= \frac{1}{16}\left(1-\frac{x}{4}\right)^{-2} \\ &= \frac{1}{16}\left[1 + (-2)\left(-\frac{x}{4}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(-\frac{x}{4}\right)^3 + \ldots\right] \\ &= \mathbf{\frac{1}{16}\left(1 + \frac{x}{2} + \frac{3x^2}{16} + \frac{x^3}{16} + \cdots\right)}\end{aligned}$$

(b) The expansion in (a) is true provided $\left|\dfrac{x}{4}\right| < 1$,

i.e. $\mathbf{|x| < 4}$ or $\mathbf{-4 < x < 4}$

Problem 11. Use the binomial theorem to expand $\sqrt{4+x}$ in ascending powers of x to four terms. Give the limits of x for which the expansion is valid

$$\sqrt{4+x} = \sqrt{4\left(1+\frac{x}{4}\right)} = \sqrt{4}\sqrt{1+\frac{x}{4}} = 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$$

Using the expansion of $(1+x)^n$,

$$2\left(1+\frac{x}{4}\right)^{\frac{1}{2}} = 2\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{4}\right)+\frac{(1/2)(-1/2)}{2!}\left(\frac{x}{4}\right)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{x}{4}\right)^3+\cdots\right]$$

$$= 2\left(1+\frac{x}{8}-\frac{x^2}{128}+\frac{x^3}{1024}-\cdots\right)$$

$$= \mathbf{2+\frac{x}{4}-\frac{x^2}{64}+\frac{x^3}{512}-\cdots}$$

This is valid when $\left|\frac{x}{4}\right| < 1$,

i.e. $\mathbf{\left|\frac{x}{4}\right| < 4}$ or $\mathbf{-4 < x < 4}$

Problem 12. Expand $\frac{1}{\sqrt{1-2t}}$ in ascending powers of t as far as the term in t^3. State the limits of t for which the expression is valid

$$\frac{1}{\sqrt{1-2t}} = (1-2t)^{-\frac{1}{2}}$$

$$= 1+\left(-\frac{1}{2}\right)(-2t)+\frac{(-1/2)(-3/2)}{2!}(-2t)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}(-2t)^3+\cdots$$

using the expansion for $(1+x)^n$

$$= \mathbf{1+t+\frac{3}{2}t^2+\frac{5}{2}t^3+\cdots}$$

The expression is valid when $|2t| < 1$,

i.e. $\mathbf{|t| < \frac{1}{2}}$ or $\mathbf{-\frac{1}{2} < t < \frac{1}{2}}$

Problem 13. Simplify $\dfrac{\sqrt[3]{1-3x}\sqrt{1+x}}{\left(1+\frac{x}{2}\right)^3}$ given that powers of x above the first may be neglected

$$\frac{\sqrt[3]{1-3x}\sqrt{1+x}}{\left(1+\frac{x}{2}\right)^3} = (1-3x)^{\frac{1}{3}}(1+x)^{\frac{1}{2}}\left(1+\frac{x}{2}\right)^{-3}$$

$$\approx \left[1+\left(\frac{1}{3}\right)(-3x)\right]\left[1+\left(\frac{1}{2}\right)(x)\right]\left[1+(-3)\left(\frac{x}{2}\right)\right]$$

when expanded by the binomial theorem as far as the x term only,

$$= (1-x)\left(1+\frac{x}{2}\right)\left(1-\frac{3x}{2}\right)$$

$$= \left(1-x+\frac{x}{2}-\frac{3x}{2}\right)$$ when powers of x higher than unity are neglected

$$= \mathbf{(1-2x)}$$

Problem 14. Express $\dfrac{\sqrt{1+2x}}{\sqrt[3]{1-3x}}$ as a power series as far as the term in x^2. State the range of values of x for which the series is convergent

$$\frac{\sqrt{1+2x}}{\sqrt[3]{1-3x}} = (1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}}$$

$$(1+2x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)(2x) + \frac{(1/2)(-1/2)}{2!}(2x)^2+\cdots$$

$$= 1+x-\frac{x^2}{2}+\cdots$$ which is valid for

$$|2x| < 1, \text{ i.e. } |x| < \frac{1}{2}$$

$$(1-3x)^{-\frac{1}{3}} = 1+(-1/3)(-3x) + \frac{(-1/3)(-4/3)}{2!}(-3x)^2+\cdots$$

$$= 1+x+2x^2+\cdots$$ which is valid for

$$|3x| < 1, \text{ i.e. } |x| < \frac{1}{3}$$

Hence $\dfrac{\sqrt{1+2x}}{\sqrt[3]{1-3x}}$

$$= (1+2x)^{\frac{1}{2}}(1-3x)^{\frac{1}{3}}$$

$$= \left(1 + x - \frac{x^2}{2} + \cdots\right)(1 + x + 2x^2 + \cdots)$$

$$= 1 + x + 2x^2 + x + x^2 - \frac{x^2}{2}$$

neglecting terms of higher power than 2

$$= \mathbf{1 + 2x + \frac{5}{2}x^2}$$

The series is convergent if $\mathbf{-\frac{1}{3} < x < \frac{1}{3}}$

Now try the following exercise

Exercise 64 Further problems on the binomial series

In Problems 1 to 5 expand in ascending powers of x as far as the term in x^3, using the binomial theorem. State in each case the limits of x for which the series is valid.

1. $\dfrac{1}{(1-x)}$ $\quad [1 + x + x^2 + x^3 + \cdots, |x| < 1]$

2. $\dfrac{1}{(1+x)^2}$ $\quad \left[1 - 2x + 3x^2 - 4x^3 + \cdots, \; |x| < 1\right]$

3. $\dfrac{1}{(2+x)^3}$ $\quad \left[\dfrac{1}{8}\left(1 - \dfrac{3x}{2} + \dfrac{3x^2}{2} - \dfrac{5x^3}{4} + \cdots\right) \; |x| < 2\right]$

4. $\sqrt{2+x}$ $\quad \left[\sqrt{2}\left(1 + \dfrac{x}{4} - \dfrac{x^2}{32} + \dfrac{x^3}{128} - \cdots\right) \; |x| < 2\right]$

5. $\dfrac{1}{\sqrt{1+3x}}$ $\quad \left[\left(1 - \dfrac{3}{2}x + \dfrac{27}{8}x^2 - \dfrac{135}{16}x^3 + \cdots\right) \; |x| < \dfrac{1}{3}\right]$

6. Expand $(2+3x)^{-6}$ to three terms. For what values of x is the expansion valid? $\quad \left[\dfrac{1}{64}\left(1 - 9x + \dfrac{189}{4}x^2\right) \; |x| < \dfrac{2}{3}\right]$

7. When x is very small show that:
   (a) $\dfrac{1}{(1-x)^2\sqrt{1-x}} \approx 1 + \dfrac{5}{2}x$
   (b) $\dfrac{(1-2x)}{(1-3x)^4} \approx 1 + 10x$
   (c) $\dfrac{\sqrt{1+5x}}{\sqrt[3]{1-2x}} \approx 1 + \dfrac{19}{6}x$

8. If x very small such that x^2 and higher powers may be neglected, determine the power series for $\dfrac{\sqrt{x+4}\;\sqrt[3]{8-x}}{\sqrt[5]{(1+x)^3}}$ $\quad \left[4 - \dfrac{31}{15}x\right]$

9. Express the following as power series in ascending powers of x as far as the term in x^2. State in each case the range of x for which the series is valid.
   (a) $\sqrt{\dfrac{1-x}{1+x}}$ (b) $\dfrac{(1+x)\;\sqrt[3]{(1-3x)^2}}{\sqrt{1+x^2}}$
   $\left[\begin{array}{l}\text{(a) } 1 - x + \dfrac{1}{2}x^2, \; |x| < 1 \\ \text{(b) } 1 - x - \dfrac{7}{2}x^2, \; |x| < \dfrac{1}{3}\end{array}\right]$

16.5 Practical problems involving the binomial theorem

Binomial expansions may be used for numerical approximations, for calculations with small variations and in probability theory.

Problem 15. The radius of a cylinder is reduced by 4% and its height is increased by 2%. Determine the approximate percentage change in (a) its volume and (b) its curved surface area, (neglecting the products of small quantities)

Volume of cylinder $= \pi r^2 h$

Let r and h be the original values of radius and height.

The new values are $0.96r$ or $(1-0.04)r$ and $1.02\,h$ or $(1+0.02)h$

(a) New volume $=\pi[(1-0.04)r]^2[(1+0.02)h]$

$$=\pi r^2 h(1-0.04)^2(1+0.02)$$

Now $(1-0.04)^2 = 1-2(0.04)+(0.04)^2$

$$=(1-0.08), \text{ neglecting powers of small terms}$$

Hence new volume

$$\approx \pi r^2 h(1-0.08)(1+0.02)$$

$$\approx \pi r^2 h(1-0.08+0.02), \text{ neglecting products of small terms}$$

$$\approx \pi r^2 h(1-0.06) \text{ or } 0.94\pi r^2 h, \text{ i.e. 94\% of the original volume}$$

Hence the volume is reduced by approximately 6%

(b) Curved surface area of cylinder $=2\pi rh$.

New surface area

$$= 2\pi[(1-0.04)r][(1+0.02)h]$$

$$= 2\pi rh(1-0.04)(1+0.02)$$

$$\approx 2\pi rh(1-0.04+0.02), \text{ neglecting products of small terms}$$

$$\approx 2\pi rh(1-0.02) \text{ or } 0.98(2\pi rh), \text{ i.e. 98\% of the original surface area}$$

Hence the curved surface area is reduced by approximately 2%

Problem 16. The second moment of area of a rectangle through its centroid is given by $\dfrac{bl^3}{12}$. Determine the approximate change in the second moment of area if b is increased by 3.5% and l is reduced by 2.5%

New values of b and l are $(1+0.035)b$ and $(1-0.025)l$ respectively.
New second moment of area

$$=\frac{1}{2}[(1+0.035)b][(1-0.025)l]^3$$

$$=\frac{bl^3}{12}(1+0.035)(1-0.025)^3$$

$$\approx \frac{bl^3}{12}(1+0.035)(1-0.075), \text{ neglecting powers of small terms}$$

$$\approx \frac{bl^3}{12}(1+0.035-0.075), \text{ neglecting products of small terms}$$

$$\approx \frac{bl^3}{12}(1-0.040) \text{ or } (0.96)\frac{bl^3}{12}, \text{ i.e. 96\% of the original second moment of area}$$

Hence the second moment of area is reduced by approximately 4%

Problem 17. The resonant frequency of a vibrating shaft is given by: $f=\dfrac{1}{2\pi}\sqrt{\dfrac{k}{I}}$, where k is the stiffness and I is the inertia of the shaft. Use the binomial theorem to determine the approximate percentage error in determining the frequency using the measured values of k and I when the measured value of k is 4% too large and the measured value of I is 2% too small

Let f, k and I be the true values of frequency, stiffness and inertia respectively. Since the measured value of stiffness, k_1, is 4% too large, then

$$k_1 = \frac{104}{100}k = (1+0.04)k$$

The measured value of inertia, I_1, is 2% too small, hence

$$I_1 = \frac{98}{100}I = (1-0.02)I$$

The measured value of frequency,

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{k_1}{I_1}} = \frac{1}{2\pi}k_1^{\frac{1}{2}}I_1^{-\frac{1}{2}}$$

$$=\frac{1}{2\pi}[(1+0.04)k]^{\frac{1}{2}}[(1-0.02)I]^{-\frac{1}{2}}$$

$$=\frac{1}{2\pi}(1+0.04)^{\frac{1}{2}}k^{\frac{1}{2}}(1-0.02)^{-\frac{1}{2}}I^{-\frac{1}{2}}$$

$$=\frac{1}{2\pi}k^{\frac{1}{2}}I^{-\frac{1}{2}}(1+0.04)^{\frac{1}{2}}(1-0.02)^{-\frac{1}{2}}$$

i.e. $$f_1 = f(1+0.04)^{\frac{1}{2}}(1-0.02)^{-\frac{1}{2}}$$
$$\approx f\left[1+\left(\frac{1}{2}\right)(0.04)\right]\left[1+\left(-\frac{1}{2}\right)(-0.02)\right]$$
$$\approx f(1+0.02)(1+0.01)$$

Neglecting the products of small terms,

$$f_1 \approx (1+0.02+0.01)f \approx 1.03f$$

Thus the percentage error in f based on the measured values of k and I is approximately [(1.03)(100) − 100], i.e. **3% too large**

Now try the following exercise

Exercise 65 Further practical problems involving the binomial theorem

1. Pressure p and volume v are related by $pv^3 = c$, where c is a constant. Determine the approximate percentage change in c when p is increased by 3% and v decreased by 1.2%. [0.6% decrease]

2. Kinetic energy is given by $\frac{1}{2}mv^2$. Determine the approximate change in the kinetic energy when mass m is increased by 2.5% and the velocity v is reduced by 3%. [3.5% decrease]

3. An error of +1.5% was made when measuring the radius of a sphere. Ignoring the products of small quantities determine the approximate error in calculating (a) the volume, and (b) the surface area. [(a) 4.5% increase (b) 3.0% increase]

4. The power developed by an engine is given by $I = k$ PLAN, where k is a constant. Determine the approximate percentage change in the power when P and A are each increased by 2.5% and L and N are each decreased by 1.4%. [2.2% increase]

5. The radius of a cone is increased by 2.7% and its height reduced by 0.9%. Determine the approximate percentage change in its volume, neglecting the products of small terms. [4.5% increase]

6. The electric field strength H due to a magnet of length $2l$ and moment M at a point on its axis distance x from the centre is given by:
$$H = \frac{M}{2l}\left\{\frac{1}{(x-l)^2} - \frac{1}{(x+l)^2}\right\}$$
Show that is l is very small compared with x, then $H \approx \frac{2M}{x^3}$

7. The shear stress τ in a shaft of diameter D under a torque T is given by: $\tau = \frac{kT}{\pi D^3}$. Determine the approximate percentage error in calculating τ if T is measured 3% too small and D 1.5% too large. [7.5% decrease]

8. The energy W stored in a flywheel is given by: $W = kr^5N^2$, where k is a constant, r is the radius and N the number of revolutions. Determine the approximate percentage change in W when r is increased by 1.3% and N is decreased by 2%. [2.5% increase]

9. In a series electrical circuit containing inductance L and capacitance C the resonant frequency is given by: $f_r = \frac{1}{2\pi\sqrt{LC}}$. If the values of L and C used in the calculation are 2.6% too large and 0.8% too small respectively, determine the approximate percentage error in the frequency. [0.9% too small]

10. The viscosity η of a liquid is given by: $\eta = \frac{kr^4}{\nu l}$, where k is a constant. If there is an error in r of +2%, in ν of +4% and l of −3%, what is the resultant error in η? [+7%]

11. A magnetic pole, distance x from the plane of a coil of radius r, and on the axis of the coil, is subject to a force F when a current flows in the coil. The force is given by: $F = \frac{kx}{\sqrt{(r^2+x^2)^5}}$, where k is a constant. Use the binomial theorem to show that when x is small compared to r, then $F \approx \frac{kx}{r^5} - \frac{5kx^3}{2r^7}$

12. The flow of water through a pipe is given by: $G = \sqrt{\frac{(3d)^5H}{L}}$. If d decreases by 2% and H by 1%, use the binomial theorem to estimate the decrease in G. [5.5%]