Approximation Algorithms (02360521) - HW2

Submission deadline: 26.01.2026

Guidelines:

- The homework can be submitted either in singles or in pairs.
- You are required to provide complete and formal proofs of your claims.
- Assignments should be typed and may not exceed 20 pages (using a font size of at least 11 and 2.5 cm margins on all sides).
- You may consult others as necessary, but write up your own solutions independently - plagiarism of any kind is not allowed.

1. **20pt.** We saw in class algorithm $\mathsf{Modified\_LS_{Max-Cut}}$ for the weighted Max-Cut problem. Given the parameter $\varepsilon \in (0,1)$ and a graph $G = (V, E)$ with non-negative edge weights:

   (a) Show that the number of local improvement iterations until the algorithm reaches an approximate local optimum is polynomial in the input size.

   (b) Suppose that we change Step 1. of the algorithm as follows.

      1. Let $v^* = \arg\max_{v \in V} w(E(\{v\}, V \backslash \{v\}))$. Select the cut $(S, \bar{S})$ where $S = \{v^*\}$.

   Show that the number of local improvement iterations until the algorithm reaches an approximate local optimum is $O(\frac{n \log n}{\varepsilon})$, where $n = |V|$.

2. **20pt.** The input for a HITTING SET instance is a collection of nonempty sets $C = S_1, ..., S_m$, and a weight function w on the sets' elements $U = \cup_{i=1}^m S_i$. An element $x \in U$ is said to hit a given set $S_i \in C$ if $x \in S_i$. A subset $H \subseteq U$ is said to hit a given set $S_i \in C$ if $H \cap S_i \neq \emptyset$. The objective is to find a minimum-cost subset of U that hits all sets $S_i \in C$. Let $S_{max} = max_i |S_i|$.

   Suggest an S_{max}-approximation algorithm for the HITTING SET problem, using the local ratio technique.

3. **20pt.** In the **Multiple Choice Maximum Coverage Problem** we have a universe of elements Ω, and k collections of subsets $C_i = \{S_{i,1}, \ldots, S_{i,\ell_i}\}$ for $1 \leq i \leq k$, where $S_{i,j} \subseteq \Omega$ for $1 \leq i \leq k$ and $1 \leq j \leq \ell_i$. A solution for the problem is a tuple of k indices $(j_1, \ldots, j_k)$ where $1 \leq j_i \leq \ell_i$ for $1 \leq i \leq k$. The value of a solution $(j_1, \ldots, j_k)$ is $V(j_1, \ldots, j_k) = \left| \bigcup_{i=1}^k S_{i,j_i} \right|$. The objective is to find a solution of maximal value.

   Consider the following local search algorithm for the problem. Initialize $(j_1, \ldots, j_k) = (1, \ldots, 1)$. While there is $(j'_1, \ldots, j'_k)$ which differs from $(j_1, \ldots, j_k)$ in a single index (that is, there is $1 \leq i \leq k$ such that $j'_{i'} = j_{i'}$ for every $i' \neq i$) and

$V(j_1', \ldots, j_k') > V(j_1, \ldots, j_k)$, update $(j_1, \ldots, j_k) \leftarrow (j_1', \ldots, j_k')$. Eventually return $(j_1, \ldots, j_k)$.

Show the algorithm is a $\frac{1}{2}$-approximation for the problem.

Hint: Let $(o_1, \ldots, o_k)$ be an optimal solution. For $1 \leq i \leq k$ define $A_i = S_{i,j_i} \setminus \bigcup_{r \in \{1,2,\ldots,k\} \setminus \{i\}} S_{r,j_r}$ and $B_i = S_{i,o_i} \setminus \bigcup_{r \in \{1,2,\ldots,k\} \setminus \{i\}} S_{r,j_r}$. Show that $|B_i| \leq |A_i|$.

4. **20pt.** Let $G = (V, E)$ be a connected undirected graph and $T_1, T_2, \ldots, T_t \subseteq V$. Let $w : E \to \mathbb{R}_{\geq 0}$ be a non-negative weight function over the edges. A *generalized Steiner forest* is a subset $E' \subseteq E$, such that for every $u, v \in T_i$ $(1 \leq i \leq t)$ there is a path in E' connecting u and v. In the generalized Steiner forest problem the objective is to find a generlized Steiner forest of minimum weight.

   (a) Let $T = \bigcup_{i=1}^{t} T_i$. Assume $|T_i| \geq 2$ for every $1 \leq i \leq t$ and consider $w' : E \to \mathbb{R}$ defined as follows. For an edge (u, v), $w'(u, v) = |\{u, v\} \cap T|$. Show that any minimal generalized Steiner forest of G is a 2-approximation with respect to the weight function w'.

   Note: a generalized Steiner forest E' of G is *minimal* if there is no $F \subsetneq E'$ which is also a generalized Steiner forest of G.

   (b) Suggest a 2-approximation algorithm for the generalized Steiner forest problem using the local ratio technique.

5. **20pt.** Consider the following variant of the Knapsack problem. The input consists of n items $I = \{1, \ldots, n\}$, where item $i \in I$ has a weight $w_i \in \mathbb{Z}^+$ and a value $v_i \in \mathbb{Z}^+$. The items are partitioned into disjoint sets $S_1, S_2, \ldots, S_m$; that is, $\bigcup_{j \in [m]} S_j = I$ and $S_j \cap S_\ell = \emptyset$ for all $j \neq \ell \in [m]$. Moreover, each set S_j has a bound $k(j)$. Also, we are given a knapsack capacity $B \in \mathbb{Z}^+$. A solution for the problem is a subset of items $P \subseteq I$ such that $\sum_{i \in P} w_i \leq B$ and $|P \cap S_j| \leq k(j)$ for all $j \in [m]$. The value of the solution P is $\sum_{i \in P} v_i$. The objective is to find a solution P of maximal value. Obtain an FPTAS for the problem.

   Hint: Use a dynamic programming algorithm generalizing the one we saw in class.