Numeric to Symbolic to Parametric space of solutions of Polynomial Systems of Equations

Ilay Menahem, Michael Shalyt, Ido Kaminer

December 4, 2025

Abstract

There is a problem shared among many fields of science of solving polynomial systems of equations. The required solution is often a space of solutions, rather than a single solution. Applications include QFT, robotics, cryptography, optimization, chemistry, electrical engineering, and more. Existing approaches include both symbolic (e.g. Gröbner bases) and numeric (e.g. homotopy continuation) methods. While symbolic methods can provide exact representations of solution spaces, they often struggle with scalability and computational complexity for large systems. Numeric methods, on the other hand, can efficiently find approximate solutions but may miss the full structure of the solution space.

This paper presents a scalable pipeline for finding a space of solutions (of full dimension) to polynomial systems of equations. We start from numerical solutions, transitioning to symbolic solutions, and finally parametrize the solution space via structured exploration. We then also discuss other complementary methods and discuss what open questions still remain. We demonstrate these methods on concrete examples from the field of QFT.

Introduction

Solving polynomial systems of equations is a fundamental problem in mathematics and computer science, with applications in fields such as robotics, cryptography, optimization, QFT, chemistry, electrical engineering, economics and more. In QFT, polynomial systems arise in the context of generilizing and unifying theories (e.g. quantum gravity). In chemistry, for example, polynomial systems appear in the study and design of reaction chains. In robotics, for example, polynomial systems are used to model kinematics and motion planning.

In most of these use cases, the required solution is often a space of solutions, rather than a single solution. Specifically, in QFT the proposed theory must follow gauge invariance, which leads to a space of solutions, and of these solutions one may want to find the one represents the simplest theory. In chemistry, reaction chains often have multiple valid pathways, leading to a solution space. In robotics, the solution space may represent all possible configurations of a robotic arm that satisfy certain constraints. one would want to work with the entire solution space to find optimal paths and configurations. This type of problem is universal across many fields of science and engineering. It calls for a comprehensive approach that can efficiently find and represent the solution space of polynomial systems.

Classic methods for solving polynomial systems of equations include Groebner bases [], resultants [], and homotopy continuation []. However, these methods can be computationally expensive and may not scale well with the size of the system. Other approaches, such as gradient [], SMT [], XXX [], are focused on finding specific solutions rather than exploring the entire solution space.

In this paper, classify/group/review/discuss specially made approaches for finding the general ... in efficient ways relevant for. We also demonstrate the methods on concrete examples from QFT. Among those methods, we propose a pipeline that leverages numerical methods to find initial solutions, then refines these solutions into symbolic forms, and finally explores a space of solutions. This approach aims to combine the efficiency of numerical methods, and some insights we may have at how the symbolic solutions may look like, to provide a comprehensive understanding of a solution space of polynomial systems. We also present and analyze a complementary examples from QFT, showcasing the effectiveness of our approach in practical scenarios. We finally outline the open questions and future directions [more concrete] in this field.

Mathematical Background

Let $\mathbb{K}[\mathbf{x}]$ denote the polynomial ring in variables $\mathbf{x} = (x_1, \ldots, x_n)$ over a field $\mathbb{K}$, typically $\mathbb{R}$ or $\mathbb{C}$. A polynomial system is defined by a map $F : \mathbb{K}^n \to \mathbb{K}^m$ with components $f_1, \ldots, f_m \in \mathbb{K}[\mathbf{x}]$. The locus of simultaneous solutions constitutes the *affine variety* $V(F) = \{\mathbf{a} \in \mathbb{K}^n \mid F(\mathbf{a}) = \mathbf{0}\}$.

What is a solution?

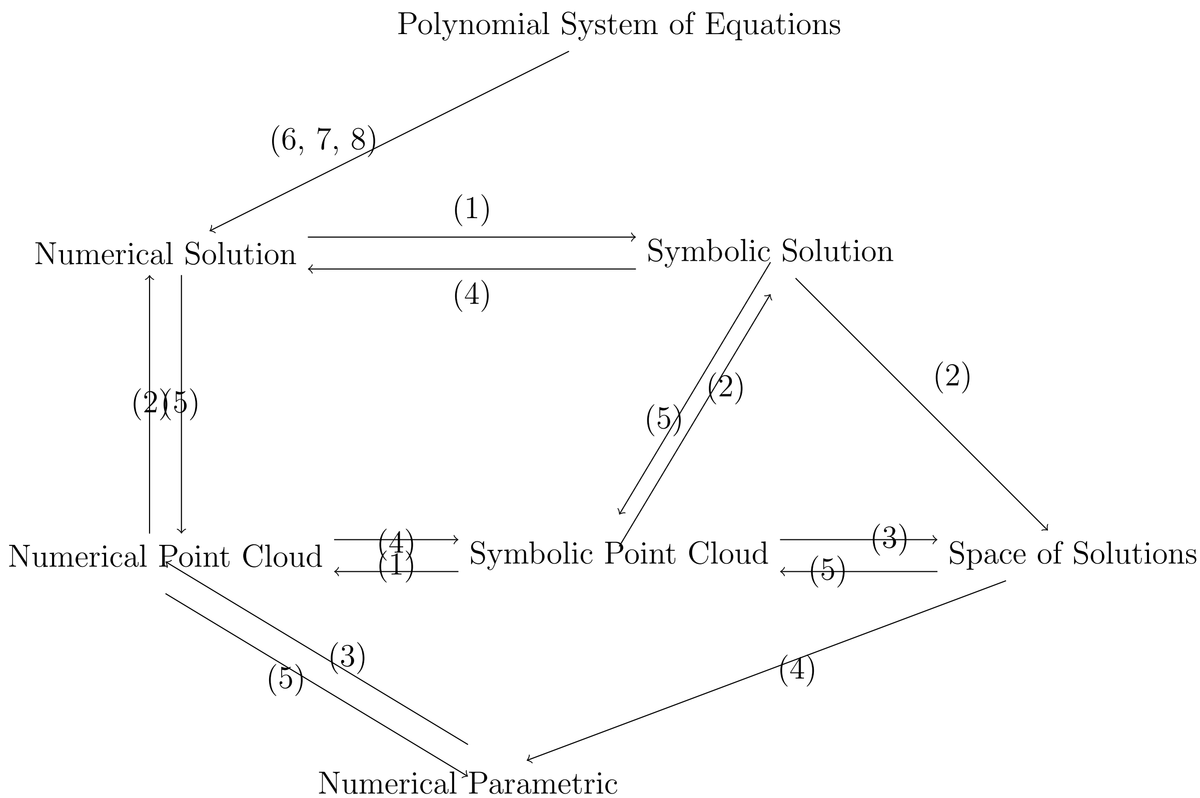
there are a few ways to interpret the meaning of the solution to a polynomial system of equations.

- **numerical solutions:** specific numerical values for the variables that satisfy all equations in the system (with very high precision).
- **symbolic solutions:** specific symbolic expressions of constants that satisfy all equations in the system.
- **lowest norm solution:** a solution numerical or symbolic, that minimizes some norm, such as the Zero norm.
- **parametric expression of a variety:** a set of symbolic or numerical approximated expressions, that provide a parametric representation of the set of all possible solutions to the system. the set of system solutions is also known as the variety, and it's not always possible to express the variety in a parametric form such as in the case of $y^3 - x^2 = 0$ over $\mathbb{R}$.
- **point cloud sampling of the variety:** a large set of numerical or symbolic solutions that sample the variety, which can be used to understand its structure and properties.

something of note, if one has one solution to any system of equations, one can derive a neighborhood of solutions around that point by going in the direction that is in the solution space of the Jacobian. a more in depth treatment of this idea is provided in the appendix.

Moving between types of solutions

one can note that it is fairly straightforward to find certain types of solutions such as numerical solutions via Newton-Raphson, but sometimes it is more desirable to have symbolic solutions or parametric expressions of the variety. in the diagram below, we illustrate the different types of solutions and the methods we will use to move between them.



1. **PSLQ and numerical fitting:** in this method, we use the PSLQ algorithm to find integer relations in the numerical solutions, replace numerical results with symbolic expressions, refit the solution and repeat until we have replaced all numerical values with symbolic expressions.

2. **Going penpendicular to Jacobian null space:** given a solution to the polynomial system of equations, and if we will go in a direction that is perpendicular to the Jacobian null space at that point, we will remain in the solution space of the system. by taking small steps in such directions, we can sample more solutions to the system.

3. **Symbolic regression:** given a point cloud of solutions, we can use symbolic regression to find parametric expressions that fit the point cloud.

4. **Evaluating symbolic expressions:** given symbolic expressions, we can evaluate them to get numerical solutions.

5. **Sampling points:** given numerical or symbolic parametric expressions of the variety, we can sample points from them to get point clouds of solutions.

6. **SMT solvers:** SMT solvers can sometimes be used to find symbolic or numerical solutions to systems of equations.

7. **Newton-Raphson:** given a system of equations, we can use the Newton-Raphson method to find numerical solutions to the system.

8. **Homotopy continuation:** Homotopy continuation methods take a system of equations with known solutions and continuously deform it into the target system, tracking the solutions throughout the deformation process to find solutions to the original system.

Linear elimination

even in the case of very large non linear polynomial systems of equations, it is possible to eliminate variables in a linear fashion in the following way: **input:** polynomial system of equations $P = \{p_1, p_2, \ldots, p_k\} \subseteq \mathbb{K}[x_1, x_2, \ldots, x_m]$. **output:** a polynomial system of equations $Q = \{q_1, q_2, \ldots, q_k\}$, which is equivalent to P, and each q_i is a not linear in any variable x_j for $j \in \{1, 2, \ldots, m\}$. **algorithm:**

1. Let $L = \{(x_j, p_i) | p_i \text{ is linear in } x_j\}$ be the list of all variables and polynomials that are linear in those variables.

2. Let $V = \{x_1, x_2, \ldots, x_m\}$ be the set of all variables.

3. Let $Q = P$ be the output polynomial system of equations.

4. While L is not empty:

   (a) Select a variable-polynomial pair (x_j, p_i) from L.

   (b) Remove (x_j, p_i) from L, x_j from V, and p_i from Q.

   (c) Solve p_i for x_j: $x_j = f(\hat{x})$.

(d) For each polynomial p_k in Q substitute x_j with $f(\hat{x})$.

(e) Update L to reflect the changes in Q.

in the example below, we demonstrate that for systems of low degree polynomials, this method can significantly reduce the number of variables and equations in the system, without increasing the degree of any polynomial in the system.

Numerical solutions via Newton-Raphson

Symbolic solutions via iterations of PSLQ and Newton-Raphson

exploration of space of solutions

Numerical exploration of space of solutions

Sybolic exploration of space of solutions

Appendix

Derivation of neighborhood of solutions via Jacobian null space

theorem: let $F = \{f_1, f_2, \ldots, f_n\}$ be a system of smooth functions in variables $x = (x_1, x_2, \ldots, x_m)$. let J be the Jacobian matrix of F with respect to x, x_0 be a point such that $F(x_0) = 0$, and $v \in N(J(x_0))$ be a vector in the null space of J at point x_0. then $\frac{d}{dt}F(x_0 + tv)|_{t=0} = 0$.

proof: By the multivariable chain rule, the derivative of the composition $F(x_0 + tv)$ with respect to t is given by:

$$\frac{d}{dt}F(x_0 + tv) = J(x_0 + tv) \cdot \frac{d}{dt}(x_0 + tv) = J(x_0 + tv) \cdot v$$

Evaluating this expression at $t = 0$:

$$\left.\frac{d}{dt}F(x_0 + tv)\right|_{t=0} = J(x_0) \cdot v$$

Since v is in the null space of $J(x_0)$, we have $J(x_0)v = 0$. Thus:

$$\frac{d}{dt}F(x_0 + tv)\bigg|_{t=0} = 0$$

now to an example system of polynomial equations:

$$f_1 = x^2 + y^2 + z^2 - 1 = 0$$
$$f_2 = x + y + z - 1 = 0$$

Let us choose a solution $x_0 = (1, 0, 0)$. We can verify that $f_1(1, 0, 0) = 1^2 + 0 + 0 - 1 = 0$ and $f_2(1, 0, 0) = 1 + 0 + 0 - 1 = 0$.

The Jacobian matrix J is given by:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix}$$

Evaluating J at $x_0 = (1, 0, 0)$:

$$J(x_0) = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

To find the null space, we solve $J(x_0)v = 0$ for $v = (v_x, v_y, v_z)^T$:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From the first row, $2v_x = 0 \implies v_x = 0$. From the second row, $v_x + v_y + v_z = 0 \implies 0 + v_y + v_z = 0 \implies v_y = -v_z$. Let $v_z = 1$, then $v_y = -1$. Thus, a vector in the null space is $v = (0, -1, 1)^T$.

Moving from x_0 in the direction of v keeps the system approximately solved to the first order. moving the direction of v can be done numerically by choosing a small t, evaluating $x(t) = x_0 + tv$, and doing a Newton-Raphson step to refine the solution.

To perform the symbolic walk, we calculate the null space of J symbolically. The null space direction $v(x, y, z)$ must satisfy $Jv = 0$, which means v is orthogonal to the gradients of f_1 and f_2. This direction is given by the cross product of the gradients (ignoring the scalar factor 2 from ∇f_1):

$$v(x, y, z) = \frac{1}{2}\nabla f_1 \times \nabla f_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} y - z \\ z - x \\ x - y \end{pmatrix}$$

This yields the system of differential equations:

$$\begin{aligned}
\dot{x} &= y - z \\
\dot{y} &= z - x \\
\dot{z} &= x - y
\end{aligned}$$

Solving this system with initial condition $x(0) = (1, 0, 0)$ yields the parametric solution for the variety.

Adding the three equations gives $\dot{x} + \dot{y} + \dot{z} = 0$, implying $x + y + z = C_1$. From the initial condition, $1 + 0 + 0 = 1$, so $x + y + z = 1$. Differentiating $\dot{x}$ gives $\ddot{x} = \dot{y} - \dot{z} = (z - x) - (x - y) = z + y - 2x$. Substituting $y + z = 1 - x$, we get $\ddot{x} = (1 - x) - 2x = 1 - 3x$. This is a linear ODE $\ddot{x} + 3x = 1$. The homogeneous solution is $A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t)$, and the particular solution is $x_p = 1/3$. So $x(t) = A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t) + 1/3$. Using $x(0) = 1$, we get $A + 1/3 = 1 \implies A = 2/3$. Using $\dot{x}(0) = y(0) - z(0) = 0$, we get $\sqrt{3}B = 0 \implies B = 0$. Thus $x(t) = \frac{2}{3}\cos(\sqrt{3}t) + \frac{1}{3}$. By symmetry and the cyclic nature of the equations, the solutions for y and z are phase-shifted or can be derived similarly. Since $\dot{x} = y - z$ and $y + z = 1 - x$, we have a system for y, z: $y - z = -\frac{2}{\sqrt{3}}\sin(\sqrt{3}t)$ and $y + z = 1 - (\frac{2}{3}\cos(\sqrt{3}t) + \frac{1}{3}) = \frac{2}{3} - \frac{2}{3}\cos(\sqrt{3}t)$. Adding these: $2y = \frac{2}{3} - \frac{2}{3}\cos(\sqrt{3}t) - \frac{2}{\sqrt{3}}\sin(\sqrt{3}t) \implies y(t) = \frac{1}{3} - \frac{1}{3}\cos(\sqrt{3}t) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}t)$. Subtracting: $2z = \frac{2}{3} - \frac{2}{3}\cos(\sqrt{3}t) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}t) \implies z(t) = \frac{1}{3} - \frac{1}{3}\cos(\sqrt{3}t) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t)$.

The parametric solution is:

$$\begin{aligned}
x(t) &= \frac{1}{3} + \frac{2}{3}\cos(\sqrt{3}t) \\
y(t) &= \frac{1}{3} - \frac{1}{3}\cos(\sqrt{3}t) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) \\
z(t) &= \frac{1}{3} - \frac{1}{3}\cos(\sqrt{3}t) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t)
\end{aligned}$$