Lecture notes: Motives and L-functions

Ildar Gaisin

Abstract

These are lecture notes for the fall semester 2025-26 academic year.

Contents

1 Lecture 1: Weil's Riemann-hypothesis for curves

Grothendieck sent a letter to Serre in 1964 detailing his idea for what a "motive" should be. An extract of this letter can be found in the annéxe of Serre's note on motives [Ser91]. Grothendieck's notion of a "motive" was motivated by proving the Weil conjectures. So what are the Weil conjectures? In 1949 Weil was interested in studying the number of solutions of equations over finite fields and he formulated the following conjecture:

Conjecture 1.1 (Weil Conjectures [Wei49])**.** *Let X be a smooth projective variety over $\mathbb{F}_p$ of dimension n such that $X \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ is irreducible and define the zeta function of X, $z(X,t)$ by*

$$\log z(X,t) := \sum_{m=1}^{\infty} |X(\mathbb{F}_{p^m})| \frac{t^m}{m}.$$

(1) ***Rationality and Riemann-Hypothesis:*** *Then there exists polynomials $P_1(t)$, $P_2(t)$, …,$P_{2n-1}(t) \in \mathbb{Z}[t]$ where $P_i(t)$ factorizes as*

$$P_i(t) = (1 - a_{i1}t)(1 - a_{i2}t)\dots(1 - a_{ib_i}t)$$

where $|a_{ij}| = p^{i/2}$ such that

$$z(X,t) = \frac{P_1(t) \cdot \dots \cdot P_{2n-1}(t)}{(1-t)P_2(t) \cdot \dots \cdot P_{2n-2}(t)(1-p^n t)}.$$

(2) ***Betti numbers:*** *If X comes from reduction modulo p from some integral lift $\tilde{X}/\mathbb{Z}$, then the b_i are the Betti numbers of $\tilde{X}(\mathbb{C})$.*

Example 1.2. *(1) $X = *$, then $|X(\mathbb{F}_{p^m})| = 1$ and so $z(X,t) = \frac{1}{1-t}$.*

(2) $X = \mathbb{P}^1_{\mathbb{F}_p}$, then $|X(\mathbb{F}_{p^m})| = p^m + 1$ and so $z(X,t) = \frac{1}{(1-t)(1-pt)}$.

Remark 1.3. *There is an analogue of Weil's conjecture for Kähler manifolds given by Serre [Ser60]. The latter is a consequence of Hodge theory, while Weil's conjecture is about étale cohomology (and intersection theory as étale cohomology itself is not powerful enough).*

Weil proved these conjectures for the case of a curve a year earlier in [Wei48]. His proof relies on constructing a suitable object from X (which we now call a *pure* motive) and proving it has desirable properties. We now give Weil's proof[1] of the Riemann-Hypothesis following closely the exposition given by Sam Raskin [Ras07].

Proof. Let's relabel X by X_0 and now use X to denote the base change $X := X_0 \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$. Let $Y := X \times_{\overline{\mathbb{F}}_p} X$. **Spoiler: In the case of a curve, the motive attached to X will essentially capture the divisors of Y. So the remainder of the proof proceeds by studying divisors of Y.** Let $\Phi_{X_0} \colon X_0 \to X_0$ be the absolute Frobenius morphism[2] on X_0 and $\mathrm{Fr}_X = \Phi_{X_0} \times \mathrm{id}$ the Frobenius endomorphism of X.

A priori, there are two actions on $X_0(\overline{\mathbb{F}}_p) = X(\overline{\mathbb{F}}_p)$. One is given by the Frobenius endomorphism on X and the other is induced by the Galois action $\Phi_{\mathrm{Spec}(\overline{\mathbb{F}}_p)}$.

Lemma 1.4. *These two actions are the same.*

Proof. Exercise. □

Let $\Delta_X \colon X \to Y$ be the diagonal morphism and $\Gamma_{\mathrm{Fr}_X^n}$ the graph of the Frobenius endomorphism composed n times: Fr_X^n. Both Δ_X and $\Gamma_{\mathrm{Fr}_X^n}$ are closed immersions and cut out divisors in Y. We denote these divisors by $[\Delta_X]$ and $[\Gamma_{\mathrm{Fr}_X^n}]$, respectively.

Lemma 1.5. *We have $[\Gamma_{Fr_X^n}] = [(Fr_X \times \mathrm{id}_X)^*]^n[\Delta_X]$.*

Proof. First note that by functoriality of pullbacks $[(\mathrm{Fr}_X \times \mathrm{id}_X)^*]^n = (\mathrm{Fr}_X^n \times \mathrm{id}_X)^*$. Thus it suffices to show that for an arbitrary endomorphism $\psi \colon X \to X$, we have

$$[\Gamma_\psi] = (\psi \times \mathrm{id}_X)^*[\Delta_X] \tag{1.1}$$

where Γ_ψ is the graph of ψ in Y. We now work locally and assume $X = \mathrm{Spec}(A)$. Take a closed point $x \in X$ and a uniformizer $\pi \in \mathcal{O}_{X,x}$ and assume $\pi \in A$. By pulling back π along the two projections $Y \rightrightarrows X$ we get two global sections π_1, π_2 of Y. Then $\pi_1 - \pi_2$ generates $[\Delta_X]$. But then the LHS of (1.1) is generated[3] by $\psi^*(\pi_1) - \pi_2$.

□

[1] Weil's proof in [Wei48] is slightly different to what is presented here. In particular he relies on the Riemann-Roch theorem for surfaces.

[2] This is the morphism given by identity on the underlying topological space of X_0 and Frobenius on the ring of functions.

[3] To see the last statement, look at the graph morphism at the level of algebras $A \otimes_{\overline{\mathbb{F}}_p} A \to A$. This is given by $x \otimes y \mapsto \psi^*(x)y$ and one sees that the kernel is indeed generated by $\psi^*(\pi_1) - \pi_2$.

Lemma 1.6. *The cardinality of the set $X(\mathbb{F}_{p^n})$ is given by the intersection number*[4] $[\Gamma_{Fr^n_X}]\cdot[\Delta_X]$.

Proof. Before we begin the proof, let us recall what intersection numbers mean in the context of curves on surfaces.

Detour: Intersection numbers of closed curves on surfaces: Let C be a smooth closed curve on a smooth projective surface S and $D \in \mathrm{Div}(S)$. Then one definition of the intersection number is $C \cdot D = \deg(\mathcal{O}_S(D)|_C)$. Unravelling what this means $\mathcal{O}_S(D)|_C$ is a line-bundle on C and its degree is the degree of it's associated divisor.

Now let's go back to the proof of Lemma 1.6. First note that since we are in characteristic p, the differential of Fr^n_X vanishes. Thus if we look at the tangent spaces of Δ_X and $\Gamma_{\mathrm{Fr}^n_X}$, we see that their sum spans all of $T_xX \times T_xX = T_{(x,x)}Y$ at every point of intersection $(x,x) \in Y$. In the literature we say Δ_X and $\Gamma_{\mathrm{Fr}^n_X}$ meet *transversely.* The upshot of transversality is the following proposition[5].

Proposition 1.7. *In the setting of the previous* ***Detour****, suppose also that D is a closed smooth curve. If C and D intersect transversely then*

$$C \cdot D = |C \cap D|$$

Proof. Exercise. □

So Proposition 1.7 says that the intersection number $[\Gamma_{\mathrm{Fr}^n_X}] \cdot [\Delta_X]$ is just the number of (closed) points that Δ_X and $\Gamma_{\mathrm{Fr}^n_X}$ intersect at. Note that the points must indeed be closed as X is irreducible. On the other hand by Hilbert Nullstellensatz, the closed points of Y is just $X(\overline{\mathbb{F}}_p) \times X(\overline{\mathbb{F}}_p)$. The set of points which belong to the intersection of Δ_X and $\Gamma_{\mathrm{Fr}^n_X}$ is precisely $X(\mathbb{F}_{p^n})$ because the set of points fixed by Fr^n_X is the same as those fixed by $\Phi^n_{\mathrm{Spec}(\overline{\mathbb{F}}_p)}$ by Lemma 1.4.

□

We need one more ingredient to finish the proof: the idea of *numerical* equivalence of divisors.

Definition 1.8. We say that two divisors are *numerically* equivalent if their intersection numbers with any third divisor are equal[6]. We define $\mathrm{Num}(Y)$ to be the quotient of $\mathrm{Div}(Y)$ by numerical equivalence. In particular the intersection product descends to a non-degenerate symmetric bilinear form

$$\mathrm{Num}(Y) \times \mathrm{Num}(Y) \to \mathbb{Z}.$$

We will need the Hodge Index Theorem which describes the above linear form [Mum66, Lecture 18]:

Theorem 1.9 (Hodge Index Theorem). *Let S be a smooth projective surface over an algebraically closed field (of arbitrary characteristic). We have a direct sum decomposition*

$$Num(S) \otimes_{\mathbb{Z}} \mathbb{Q} = V \oplus V'$$

such that V has dimension 1 and the intersection form is positive definite on V and negative definite on V'.

[4] For the interested reader, Fulton's book [Ful84] develops intersection theory in rather great generality. In general one has to be careful outside of smooth/projective assumptions.

[5] It's so fundamental that I've labeled it a Proposition, even though we are inside a Lemma.

[6] Technically we haven't defined intersection numbers of divisors in general, but let's assume there is a reasonable definition for now.

Example 1.10. *Consider the quadric surface $S \subset \mathbb{P}^3$ given by the Segre embedding*

$$\begin{aligned}\mathbb{P}^1 \times \mathbb{P}^1 &\hookrightarrow \mathbb{P}^3 \\ [w:x] \times [y:z] &\mapsto [wy:wz:xy:xz]\end{aligned}$$

$S := V(x_0x_3 - x_1x_2)$. Show that $\mathrm{Div}(S) = \mathit{Num}(S) = \mathbb{Z} \oplus \mathbb{Z}$ *and the intersection product is given by*

$$\begin{aligned}\mathrm{Div}(S) \times \mathrm{Div}(S) &\to \mathbb{Z} \\ (a_1, b_1) \times (a_2, b_2) &\mapsto a_1b_2 + b_1a_2.\end{aligned}$$

Verify the Hodge Index theorem in this case.

Lemma 1.11. *We have $|X(\mathbb{F}_{p^n})| = p^n + O(p^{n/2})$.*

Proof. Let $[H]$ and $[V]$ be the divisors in $\mathrm{Div}(Y)$ corresponding to $X \times \{x_0\}$ and $\{x_0\} \times X$ for some closed point $x_0 \in X$, respectively. Since $[H] \cdot [V] = 1$ and $[H] \cdot [H] = 0$, these cannot be equal in $\mathrm{Num}(Y)$. Moreover $U := \mathbb{Q}[H] \oplus \mathbb{Q}[V]$ is a finite-dimensional subspace of $W := \mathrm{Num(Y)} \otimes_{\mathbb{Z}} \mathbb{Q}$, we can write

$$W = U \oplus U'$$

where U' is the orthogonal complement. We claim that the intersection form on U' is negative-definite: Indeed on matrix on U with respect to the basis $\{[H], [V]\}$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has one positive eigenvalue and so by the Hodge Index Theorem, the subspace on which the intersection form us positive-definite is contained in U.

Let

$$\begin{aligned}T\colon W &\to W \\ D &\mapsto (\mathrm{Fr}_X \times \mathrm{id}_X)^* D\end{aligned}$$

Then $T([H]) = p[H]$ and $T([V]) = [V]$. We know by Lemma 1.5 that $T^n[\Delta_X] = [\Gamma_{\mathrm{Fr}_X^n}]$. For the following note that pullback/pushforward of divisors (up to linear equivalence) descends to numerical equivalence (this is essentially the content of the *moving lemma*). Moreover for any $D, E \in \mathrm{Num}(Y)$

$$\begin{aligned}(\mathrm{Fr}_X \times \mathrm{id}_X)^* D \cdot (\mathrm{Fr}_X \times \mathrm{id}_X)^* E &\overset{(1)}{=} D \cdot (\mathrm{Fr}_X \times \mathrm{id}_X)_* (\mathrm{Fr}_X \times \mathrm{id}_X)^* E \\ &\overset{(2)}{=} D \cdot pE \\ &\overset{(3)}{=} pD \cdot E\end{aligned}$$

where (1) follows from Proposition 1.12:

Proposition 1.12. *Given $\varphi\colon Y \to Z$ so that $\varphi^*\colon$ $\mathrm{Div}(Z) \to \mathrm{Div}(Y)$ and $\varphi_*\colon$ $\mathrm{Div}(Y) \to \mathrm{Div}(Z)$ are well-defined, we have*

$$C \cdot \varphi^*(D) = \varphi_*(C) \cdot D$$

for any $C \in \mathrm{Div}(Y)$ and $D \in \mathrm{Div}(Z)$.

Proof. Exercise. □

and (2) follows from Proposition

Proposition 1.13. *In the setting of Proposition 1.12 $\varphi_*\varphi^*\colon \mathrm{Div}(Z) \to \mathrm{Div}(Z)$ is given by $D \mapsto \deg(\varphi)D$.*

Proof. Exercise $\square$

and (3) follows by linearity of the intersection form.

Thus for all $v, w \in W$, we have $Tv \cdot Tw = p(v \cdot w)$. Since $[H] \cdot [\Delta_X] = [V] \cdot [\Delta_X] = 1$, we can write

$$[\Delta_X] = [H] + [V] + u' \tag{1.2}$$

for some $u' \in U'$. We then compute

$$\begin{aligned} |X(\mathbb{F}_{p^n})| &\overset{(a)}{=} [\Gamma_{\mathrm{Fr}_X^n}] \cdot [\Delta_X] \\ &\overset{(b)}{=} T^n[\Delta_X] \cdot [\Delta_X] \\ &\overset{(c)}{=} p^n + 1 + T^n u' \cdot u' \end{aligned}$$

where (a) follows from Lemma 1.6, (b) by Lemma 1.5 and (c) because $T([H]) = p[H]$ and $T([V]) = [V]$.

It's easy to check that $T^n u' \in U'$ and so we can apply the Cauchy-Schwarz inequality to get

$$|T^n u' \cdot u'| \leq \sqrt{|T^n u' \cdot T^n u'||u' \cdot u'|} = p^{n/2}|u' \cdot u'|$$

This completes the proof of Lemma 1.11.

$\square$

A relatively straightforward analysis argument then concludes the proof of the Riemann-Hypothesis for curves. We won't include the details, as it's not what we are after conceptually.

$\square$

2 Lecture 2: Algebraic cycles and adequate equivalence relations

For this lecture, I will be following the book by Murre-Nagel-Peters [MNP13, Chapter 1]. Another good reference is [And04, Chapitre 3]. Let k be an arbitrary field and X a k-variety.

Definition 2.1 (algebraic cycle)**.** An algebraic cycle on X is a formal finite integral linear combination $Z = \sum n_\alpha Z_\alpha$ of irreducible closed subvarieties Z_α of X. If all Z_α have the same dimension i, we say that Z is a dimension i cycle. We denote by $Z_i(X)$ the abelian group of dimension i cycles on X. When considering the codimension point of view we write $Z^{d-i}(X) := Z_i(X)$ if X is of dimension d. We write $Z(X) := \oplus_i Z^i(X)$ and consider it as a group with a graded structure.

Lemma 2.2. *Suppose X is smooth. Then two closed subvarieties V and W of X with codimensions i and j, respectively, have intersection*

$$V \cap W = \cup_\alpha Z_\alpha$$

where each Z_α is an irreducible subvariety of codimension at most i+j.

Proof. We have that $V \cap W = \Delta^{-1}(V \times W)$ where $\Delta\colon X \to X \times X$ is the diagonal map. Since X is smooth, we can write $X \times X = \mathrm{Spec}(A)$ and $X = V(f_1, \ldots, f_c)$ where $f_1, \ldots, f_c$ is a regular sequence in A and $c = \dim(X)$. Then if $V \times W = \mathrm{Spec}(A/\mathfrak{p})$ then $V \cap W = \mathrm{Spec}(A/(\mathfrak{p} + (f_1, \ldots, f_c)))$. Then for $z \in Z_\alpha$ a closed point

$$\dim(V \times W) = \dim \mathcal{O}_{V \times W, z} \quad \text{and} \quad \dim(Z_\alpha) = \dim \mathcal{O}_{Z_\alpha, z} = \dim \mathcal{O}_{V \times W, z}/(f_1, \ldots, f_c).$$

From here one can compare the relevant dimensions by the fact that quotienting a local ring by an element in the maximal ideal, reduces the dimension by at most one. □

Definition 2.3 (proper intersection product of algebraic cycles)**.** In the setting of Lemma 2.2, we say that the intersection $V \cap W$ is *proper* (or V and W intersect *properly*) if the codimension of each Z_α is $i+j$. In this case the *intersection number* is defined by

$$i(V \cdot W; Z) := \sum_r (-1)^r \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathrm{Tor}_r^{\mathcal{O}_{X,Z}}(\mathcal{O}_{V,Z}, \mathcal{O}_{W,Z}))$$

where $A := \mathcal{O}_{?,Z}$ denotes the local ring of ? at the generic point of Z. We define the *intersection product*

$$V \cdot W := \sum_\alpha i(V \cdot W; Z_\alpha) Z_\alpha.$$

Definition 2.4 (proper pushforward)**.** Let $f \colon X \to Y$ be a proper morphism of k-varieties and $Z \subset X$ a k-dimensional closed irreducible subvariety. We define

$$f_* Z = \begin{cases} 0, & \text{if } \dim(f(Z)) < k \\ [R(Z) \colon R(f(Z))] f(Z), & \text{otherwise} \end{cases} \tag{2.1}$$

where $R(?)$ is the field of rational functions[7] on ?. Extending by linearity induces a homomorphism

$$f_* \colon Z_k(X) \to Z_k(Y).$$

In general we say two algebraic cycles $\alpha, \beta \in Z(X)$ intersect properly if each components of α intersects each component of β properly.

Definition 2.5 (flat pullback)**.** Let $f \colon X \to Y$ be a flat morphism of k-varieties and $Z \subset Y$ a k-codimensional closed irreducible subvariety. We define

$$f^* Z = f^{-1}(Z)$$

Because f is flat, $f^{-1}(Z)$ turns out to be of codimension k (assuming it is non-empty). Extending by linearity induces a homomorphism

$$f^* \colon Z^k(Y) \to Z^k(X)$$

Exercise 2.6 (Projection formula)**.** *Prove* $f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta$

Definition 2.7. A *correspondence* from X to Y is a cycle in $X \times Y$. A correspondence $Z \in Z^t(X \times Y)$ acts on cycles on X as follows

$$\begin{aligned} Z \colon Z^i(X) &\to Z^{i+t-\dim(X)}(Y) \\ T &\mapsto pr_{Y*}(Z \cdot (T \times Y)) \end{aligned}$$

whenever defined. We call $t - \dim(X)$ the degree of the correspondence.

Example 2.8. *It turns out the notion of correspondences generalizes the notion of (proper) pushforward and (flat) pullback. Prove this.*

As we see correspondences (or intersection products) are not always defined. This is where the notion of *adequate equivalence* comes in. These are equivalences classes on the groups Z^i such that the intersection product is always defined.

[7] Note that $f|_Z \colon Z \to f(Z)$ is a dominant morphism, so the above degree is well-defined.

2.1 Adequate Equivalence

We now work in the category $\mathrm{SmProj}(k)$ of smooth projective varieties over k.

Definition 2.9 (Adequate Equivalence)**.** We say that an equivalence relation $\sim$ on $Z(X)$ is *adequate* if

(1) (**linearity**) $\sim$ is compatible with addition and graduation.

(2) (**moving lemma**) For all α, $\beta \in Z(X)$, $\exists \alpha' \sim \alpha$ such that α' and β intersect properly.

(3) (**correspondence**) $\sim$ is compatible with correspondences: In the setting of Definition 2.7 if $T \sim 0$ and Z intersects $T \times Y$ properly, then $Z(T) \sim 0$.

We write $Z_\sim(X) := Z(X)/\sim$ and for some field F, $Z_\sim(X)_F := Z(X) \otimes_{\mathbb{Z}} F/\sim$. We also write

$$Z^i_{\sim,0}(X) := \{Z \in Z^i(X) |\ Z \sim 0\}$$

The fact that intersection product is defined on the whole $Z_\sim(X)$ is a straightforward consequence of Definition 2.9 (cf. [Sam58, Proposition 6 and 7]).

Lemma 2.10. *For any adequate equivalence relation $\sim$ on $X \in SmProj(k)$, we have*

(1) $Z_\sim(X)$ is a graded ring with product induced by the intersection product of cycles.

(2) A correspondence Z from X to Y of degree r induces $Z_*\colon Z^i_\sim(X) \to Z^{i+r}_\sim(Y)$ and equivalent correspondences induce the same Z_*.

We now discuss the following adequate equivalence relations

- rational equivalence $\sim_{\mathrm{rat}}$
- algebraic equivalence $\sim_{\mathrm{alg}}$
- smash nilpotence equivalence $\sim_{\otimes\mathrm{nil}}$
- homological equivalence $\sim_{\mathrm{hom}}$
- numerical equivalence $\sim_{\mathrm{num}}$

2.1.1 Rational equivalence

Definition 2.11 (Rational equivalence)**.** A cycle $\alpha \in Z(X)$ is rationally equivalent to 0 ($\alpha \sim_{\mathrm{rat}} 0$) if there exists $\beta \in Z(X \times \mathbb{P}^1)$ such that $\beta(0)$ and $\beta(\infty)$ are well-defined and $\alpha = \beta(0) - \beta(\infty)$.

Lemma 2.12. *Rational equivalence corresponds to linear equivalence for codimension 1 cycles $Z^1(X)$.*

Proof. We first show $\mathrm{div}(f) \sim_{\mathrm{rat}} 0$ for $f \in R(X)$ a rational function. We can think of f as $f\colon U \to \mathbb{P}^1$ for some dense open $U \subset X$. Let $W \subset X \times \mathbb{P}^1$ be the closure of the graph of f. Then W gives a cycle $\beta \in Z(X \times \mathbb{P}^1)$ and essentially by definition $\mathrm{div}(f) = \beta(0) - \beta(\infty)$.

For the converse suppose $\alpha \in Z^1(X)$ and $\alpha \sim_{\mathrm{rat}} 0$. Take a component $Z' \subset X \times \mathbb{P}^1$ of β (with β part of Definition 2.11). Then Z' dominates $\mathbb{P}^1$. Let $Z \subset X$ be the image of Z' under the projection to X. Then $Z \subset X$ is closed (as projection is proper) and $Z' \to Z$ is proper and dominant with fibers of dimension 0 or 1.

There are two cases as to whether $\dim(Z) < \dim(Z')$ or $\dim(Z) = \dim(Z')$.

If $\dim(Z) < \dim(Z')$, then $Z' = Z \times \mathbb{P}^1$ and $[Z'_0] - [Z'_\infty] = [Z] - [Z] = 0$.

If $\dim(Z) = \dim(Z')$, then $Z' \to Z$ is generically finite (i.e. inverse image of generic point is finite). Then I leave it as an exercise[8] to show that $[Z'_0] - [Z'_\infty] = \mathrm{div}(\mathrm{Nm}(f))$ where $f\colon Z' \to \mathbb{P}^1$ viewed as a rational function on Z'. □

[8] The main idea is essentially in the proof of [Ful84, Proposition 1.4(b)]

The technical difficulty in proving that rational equivalence is indeed an adequate equivalence relation lies in proving the *moving lemma*. The proof is roughly as follows: We embed $X \hookrightarrow \mathbb{P}^N$ and given $V, W \subset X$, we need to move V so that it intersects W properly. There are two cases to consider as to whether $X = \mathbb{P}^N$ or not. In the former, there is some general linear transformation that makes V and W intersect properly. In the later case, one considers a linear subspace $L \subset \mathbb{P}^N$ and the cone $C(L, V)$. For the details we refer to [Ful84, Example 11.4.1].

Exercise 2.13. *What goes wrong with the moving lemma for rational equivalence if we relax the smoothness assumption? What if we keep smoothness and relax the projectivity assumption?*

Definition 2.14 (Chow ring)**.** The corresponding graded ring $CH(X) := Z_{\mathrm{rat}}(X)$ is called the Chow ring. We will also denote by $\mathrm{Corr}(X, Y) := CH(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ the *correspondences* from X to Y.

Lemma 2.15. *Among all adequate equivalence relations, rational equivalence is the finest.*

Proof. Let $\sim$ be an adequate equivalence relation. It suffices to prove $[0] \sim [\infty]$ as then by using a correspondence, we get the definition of rational equivalence. Since the point $1 \in \mathbb{P}^1$ does not intersect itself properly, by the moving lemma there exists $\sum_i n_i[x_i] \in Z^1(\mathbb{P}^1) \sim 1$ with $x_i \in \mathbb{P}^1$ such that x_i intersects 1 properly. In other words $x_i \neq 1$. Consider now the correspondence $Z \in Z^1(\mathbb{P}^1 \times \mathbb{P}^1)$ given by the graph of the polynomial

$$1 - \prod_i \left(\frac{x - x_i}{1 - x_i}\right)^{m_i} \tag{2.2}$$

for a collection of $m_i > 0$ and $T = \sum_i n_i[x_i] - 1$. Then $Z(T)$ is just the pushforward of T by (2.2). The pushforward of T is just $mn[1] - m[0]$ where $m = \sum_i m_i$ and $n = \sum_i n_i$. Since this holds for arbitrary m_i, we get $n[1] \sim [0]$. Applying the condition of correspondence to the automorphism $x \mapsto \frac{1}{x}$, we get $n[1] \sim [\infty]$, from which we can conclude. □

2.1.2 Algebraic equivalence

Definition 2.16 (Algebraic Equivalence)**.** This is the same definition as rational equivalence but with $\mathbb{P}^1$ replaced by any smooth projective irreducible curve and the two points 0 and ∞ by any two k-rational points on the curve. In other words $\alpha \in Z(X)$ is $\sim_{\mathrm{alg}} 0$ if there exists a smooth irreducible projective curve C and $\beta \in Z(X \times C)$ and two points $a, b \in C(k)$ such that $\beta(a) = 0$ and $\beta(b) = \alpha$.

Example 2.17 (algebraic equivalence is coarser than rational equivalence)**.** *Take an elliptic curve E over $\mathbb{C}$ and two distinct points $a, b \in |E|$. Then $a - b$ is not a divisor of any rational function. This is because we can make an identification $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ and we see by Cauchy's residue theorem that any rational function has it's sum of residues equal to 0. So any rational function cannot have a simple pole. In summary $a - b \not\sim_{rat} 0$.*

On the other hand E is equipped with a degree 2 cover over $\mathbb{P}^1$ with 4 ramification points (by Hurwitz's theorem). If we take the graph $Z \subset E \times \mathbb{P}^1$ of this cover, then we get that these 4 ramification points must be algebraically equivalent[9]*.*

2.2 Smash Nilpotent equivalence

Definition 2.18 (Smash Nilpotent equivalence)**.** For $Z \in Z(X)$ we say $Z \sim_{\otimes} 0$ iff for some positive integer n, $Z^n \sim_{\mathrm{rat}} 0$ where we view $Z^n \in Z(X^n)$.

Theorem 2.19 ($\sim_{\otimes}$ vs $\sim_{\mathrm{alg}}$)**.** *We have $Z^i_{alg,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \subset Z^i_{\otimes,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

[9]We can just take the graph of the identity map to conclude that any two rational points on any smooth projective curve are algebraically equivalent.

Theorem 2.19 is due independently to Voevodsky [Voe95] and Voisin [Voi96].

Proof. We proceed in several steps as in [MNP13, Appendix B].
Step 0: Reduce to $k = \overline{k}$.

Exercise 2.20. *For any (adequate) equivalence relation $\sim$, there is a natural map*

$$Z_\sim(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to (Z_\sim(X_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{Q})^{\mathrm{Gal}(\overline{k}/k)}$$

given by restricting a closed subvariety over k to one over $\overline{k}$. Prove that this map is a bijection.

Exercise 2.20 allows us to assume k is algebraically closed.
Step 1: Reduce to the case of a smooth projective curve. Take $Z \sim_{\mathrm{alg}} 0$. Then by definition $\exists\, \Gamma \in \mathrm{Corr}(C, X)$ and two points $a, b \in C(k)$ such that $Z = \Gamma_*(a-b)$. Thus taking products gives $Z^n = (\Gamma^n)_*(a-b)^n$ and so it suffices to show $(a-b)^n \sim_{\mathrm{alg}} 0$ on C^n. In fact we shall show

$$(a-b)^n \sim_{\mathrm{alg}} 0 \quad \text{for} \quad n > g,$$

where g is the genus of the curve C.
Step 2: Reducing $(a-b)^n$ as a divisor on the n-fold symmetric product of C. A priori $(a-b)^n \in Z(C^n)$. However the symmetric group S_n induces an action on C^n and clearly $(a-b)^n \in (Z(C^n) \otimes_{\mathbb{Z}} \mathbb{Q})^{S_n}$.

Exercise 2.21. *Show that $(Z(C^n) \otimes_{\mathbb{Z}} \mathbb{Q})^{S_n} \cong Z(C^n/S^n) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Exercise 2.21 allows us to view $(a-b)^n$ in the quotient variety C^n/S^n (the n-fold symmetric product of C).
Step 3: Comparison of C/S^n with the Jacobian[10] $J(C)$. Fix a base point $e \in C(k)$. Denote by

$$\begin{aligned} \pi_n \colon C^n &\to C^n/S^n \\ (x_1, \ldots, x_n) &\mapsto [x_1, \ldots, x_n] \end{aligned}$$

the natural surjection and

$$\begin{aligned} \varphi_n \colon C^n/S^n &\to J(C) \\ [x_1, \ldots, x_n] &\mapsto \sum_i (x_i - e). \end{aligned}$$

Lemma 2.22. *The induced map $(\varphi_n)_* \colon CH_0(C^n/S^n) \to CH_0(J(C))$ is an isomorphism for all $n \geq g$.*

Proof. If $n = g$, then we claim that φ_n is a birational morphism. Indeed by Riemann-Roch

$$\ell(x_1 + \cdots + x_g) = g + 1 - g + \ell(K - x_1 - \cdots - x_g)$$

if none of the x_i are base points of the canonical divisor K, then since $\ell(K) = g$, we get[11] $\ell(K - x_1 - \cdots - x_g) = 0$ and so $\ell(x_1 + \cdots + x_g) = 1$. This means that φ_n is an isomorphism[12] outside of a finite set of points. So it is a birational morphism. By [Ful84, Example 16.1.11], the group CH_0 is invariant for birational morphisms.

[10]The Jacobian variety $J(C)$ is the variety which represents the functor $T \mapsto \{\text{invertible sheaves of degree 0 on } X \times T\}$.
[11]This is related to [Har77, Chapter IV, Proposition 3.1]
[12]Because the fiber of φ_n is just the set of points $[x_1, \ldots, x_n]$ such that $\sum_i x_i$ form a complete linear system. This also means fibers are projective.

Suppose $n > g$ and consider the natural embedding

$$\iota \colon C^g/S^g \to C^n/S^n$$
$$[x_1, \dots, x_g] \mapsto [x_1, \dots, x_g, \underbrace{e, \dots, e}_{n-g}]$$

Then $(\varphi_n)_* \iota_* = (\varphi_g)_*$. Since $(\varphi_g)_*$ is an isomorphism, it follows that ι_* is an injection. It remains to show that it is a surjection. So take $y \in C^n/S^n$ and consider the image $z \in J(C)$ and some point $x \in C^g/S^g$ which maps to z under equivalence. Then $\iota(x)$ and y belong to the fiber $\varphi_n^{-1}(z)$. But the fibers[13] of φ_n are projective and any two points are rationally equivalent. This proves the lemma. □

Step 4: Application of Bloch's theorem. We have that $(\varphi_{g+1})_*((a-b)^{g+1}) = (\varphi_1(a) - \varphi_1(b))^{*(g+1)}$ and this vanishes by [Blo76].

□

Exercise 2.23. *Use ideas from the proof of Theorem 2.19 to show that the cartesian product of two non-zero Chow cycles can be zero.*

2.3 Homological equivalence

To define *homological equivalence* we need to define a *Weil cohomology theory*. Let F be a field of characteristic 0. We denote $\mathrm{GrVect}_F^{\geq 0}$ be the category of finite dimensional graded F-vector spaces, where the grading is concentrated in non-negative degrees.

Definition 2.24. A Weil cohomology theory is a functor

$$H \colon \mathrm{SmProj}(k)^{\mathrm{opp}} \to \mathrm{GrVect}_F^{\geq 0}$$

which satisfies the following axioms:

(1) A one-dimensional F-vector space $F(1)$, which gives rise to *Tate* twists.

(2) $\exists$ a graded cup product $\cup \colon H(X) \times H(X) \to H(X)$ such that if $a \in H^i(X)$, $b \in H^j(X)$, then $a \cup b = (-1)^{ij} b \cup a$.

(3) one has Poincaré duality (assume X has pure dimension d): $\exists$ a trace isomorphism

$$\mathrm{Tr} \colon H^{2d}(X)(d) \xrightarrow{\sim} F$$

such that

$$H^i(X) \times H^{2d-i}(X)(d) \xrightarrow{\cup} H^{2d}(X)(d) \xrightarrow{\mathrm{Tr}} F$$

is a perfect pairing.

(4) A Künneth map

$$H(X) \otimes H(Y) \xrightarrow{(pr_X)^* \otimes (pr_Y)^*} H(X \times Y)$$

which is a (graded) isomorphism.

(5) there are cycle class maps

$$\gamma_X \colon \mathrm{CH}^i(X) \to H^{2i}(X)(i)$$

which satisfy various compatibilities[14].

[13] fiber above a point is a complete linear system

[14] I will state them explicitly when we need them.

(6) If X is pure of dimension d and $\iota\colon Y \hookrightarrow X$ is a smooth hyperplane, then *weak Lefschetz* holds:

$$H^i(X) \xrightarrow{\iota^*} H^i(Y)$$

is an isomorphism if $i < d-1$ and an injection for $i = d-1$.

(7) With the setting as in (6), the Lefschetz operator $L(\alpha) := \alpha \cup \gamma_X(Y)$ induces isomorphisms

$$L^i\colon H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X)$$

for $0 \leq i \leq d$. This is known as *Hard Lefschetz*.

Example 2.25. *For $k = \mathbb{C}$, we get many examples of Weil cohomology theories:*

(1) singular cohomology groups: $H^i(X_{an})$ where X_{an} is the complex manifold attached to X.

(2) classical de Rham cohomology: $H^i_{dR}(X_{an}, \mathbb{C})$.

(3) algebraic de Rham cohomology: $H^i_{dR} := \mathbb{H}(X, \Omega^\bullet_{X/\mathbb{C}})$.

The fact that these are indeed Weil cohomology groups follows from classical reasons together with comparison isomorphisms. On the other hand the fact that $H^i_{ét}(X, \mathbb{Q}_\ell)$ is a Weil cohomology theory (in particular satisfies Hard Lefschetz) is deep work of Deligne [Del80].

Definition 2.26 (Homological equivalence)**.** Fix a Weil cohomology theory. Then for $Z \in Z(X)$ we say $Z \sim_{\mathrm{hom}} 0$ if $\gamma_X(Z) = 0$.

We can compare homological equivalence to algebraic and smash nilpotent equivalence.

Lemma 2.27 ($\sim_\otimes$ and $\sim_{\mathrm{alg}}$ vs $\sim_{\mathrm{hom}}$)**.** *(1) $Z^i_{alg,0}(X) \subset Z^i_{hom,0}(X)$.*

(2) $Z^i_{\otimes,0}(X) \subset Z^i_{hom,0}(X)$.

Proof. For (1), note that $\alpha \sim_{\mathrm{alg}} 0$ means that for some smooth projective curve C, $\alpha = pr_{X*}pr_C^*([a] - [b])$ for two rational points $a, b \in C$. Now cycle map is compatible with push-forward and pullbacks (one of the conditions I didn't state in Definition 2.24(4)). So we can reduce to the case of a curve. We then conclude by Matsusaka's theorem:

Theorem 2.28 (Matsusaka's Theorem)**.**

$$Z^1_{hom,0}(X) = \{D \in Z^1(X) | \quad nD \sim_{alg} 0 \text{ for some } n \in \mathbb{Z}\}$$

For part (2), note that $\alpha \sim_\otimes 0$ means $\alpha^n \sim_{\mathrm{rat}} 0$ for some $n > 0$. Then

$$\gamma_{X^n}(\alpha^n) = \underbrace{\gamma_X(\alpha) \otimes \ldots \otimes \gamma_X(\alpha)}_{n}$$

is zero. So each of $\gamma_X(\alpha) = 0$. ☐

Exercise 2.29. *Find an alternative proof of Lemma 2.27(1) using part (2) and Voevodsky-Voisin Theorem 2.19.*

2.4 Numerical equivalence

Definition 2.30 (Numerical Equivalence)**.** Let X be of pure dimension d. For $Z \in Z^i(X)$, we say $Z \sim_{\text{num}} 0$ if for every $W \in Z^{d-i}(X)$ such that $Z \cdot W$ is defined, we have $\deg(Z \cdot W) = 0$.

We can compare homological equivalence and numerical equivalence.

Lemma 2.31 ($\sim_{\text{hom}}$ vs $\sim_{\text{num}}$)**.** *$Z^i_{hom,0}(X) \subset Z^i_{num,0}(X)$.*

Proof. We will need to use that γ_X (the cycle class map) is compatible with intersection products:

$$\gamma_X(\alpha \cdot \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta) \tag{2.3}$$

and compatible with points P:

$$\operatorname{Tr} \circ \gamma_X = \deg \qquad \text{on } CH^d(X). \tag{2.4}$$

Conditions (2.3) and (2.4) are the remaining conditions I didn't state in Definition 2.24(4)).

By property (2.4), we see that the result holds for $i = d$ (i.e. zero cycles). Suppose now $i < d$ and $Z \in Z^i_{\text{hom},0}(X)$ and $W \in Z^{d-i}(X)$. such that $Z \cdot W$ is defined. Then

$$\begin{aligned} 0 &\overset{(i)}{=} \operatorname{Tr}(\gamma_X(Z) \cup \gamma_X(W)) \\ &\overset{(ii)}{=} \operatorname{Tr}(\gamma_X(Z \cdot W)) \\ &\overset{(iii)}{=} \deg(Z \cdot W) \end{aligned}$$

where (i) holds because $\gamma_X(Z) = 0$, (ii) holds because of (2.3) and (iii) holds because of (2.4). □

Exercise 2.32. *Show that by realizing the degree map as a correspondence, that $Z^i_{\sim,0}(X) \subset Z^i_{num,0}(X)$ for any non-trivial adequate equivalence relation.*

Summarizing Lemmas 2.15, 2.27(1) and 2.31 have shown the following chain

$$Z^i_{\text{rat},0}(X) \subset Z^i_{\text{alg},0}(X) \subset Z^i_{\text{hom},0}(X) \subset Z^i_{\text{num},0}(X)$$

As part of the standard conjectures:

Conjecture 2.33 (Standard Conjecture D)**.** *$Z^i_{hom,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = Z^i_{num,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

3 Lecture 3: Grothendieck's pure motives

For this lecture, I will be following the book by Murre-Nagel-Peters [MNP13, Chapter 2]. Another good reference is [And04, Chapitre 4]. Let k be an arbitrary field and X and Y smooth projective k-varieties.

The next definition is along the same lines as Definitions 2.7 and 2.14.

Definition 3.1 (correspondences and degree r correspondence)**.** For an adequate equivalence relation $\sim$, we denote the graded vector spaces of correspondences:

$$\operatorname{Corr}_\sim(X, Y) := Z_\sim(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and if X is pure of dimension d, we also consider the degree r correspondences by

$$\operatorname{Corr}^r_\sim(X, Y) := Z^{d+r}_\sim(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We also need to know how to compose correspondences:

Definition 3.2 (composition of correspondences)**.** We define composition

$$\begin{aligned}\mathrm{Corr}_\sim(X,Y)\times\mathrm{Corr}_\sim(Y,Z) &\to \mathrm{Corr}_\sim(X,Z)\\ (f,g)&\mapsto g\circ f\end{aligned}$$

by $g\circ f := \mathrm{pr}_{XZ*}\{(f\times Z)\cdot(X\times g)\}$.

Exercise 3.3. *Check that Definition 3.2 restricts to give a composition of degree 0 correspondences. In general composition respects the grading.*

Definition 3.4 (projectors)**.** A projector for X is an element $p\in\mathrm{Corr}_\sim(X,X)$ for which $p\circ p = p$.

Exercise 3.5. *Show that the diagonal Δ_X is an example of a projector.*

We now proceed to give the construction of (pure) motives in several steps. For the following fix an adequate equivalence relation $\sim$.
Construction of (pure) motives:
Step 1: Consider the category $Z_\sim\mathrm{SmProj}(k)$ with

(1) objects: same as $\mathrm{SmProj}(k)^{\mathrm{op}}$

(2) morphisms: degree 0 correspondences. More precisely $\mathrm{Hom}(X,Y) := \mathrm{Corr}^0_\sim(X,Y)$.

We are hoping that the category we construct is abelian and it's formal nonsense[15] to see that one should keep track of idempotent morphisms (i.e. projectors). This leads to
Step 2: Consider the category of *effective* motives $\mathrm{Mot}^{\mathrm{eff}}_\sim(k)$ with

(1) objects: pairs (X,p) with $X\in\mathrm{SmProj}(k)$ and p a projector.

(2) morphisms: $\mathrm{Hom}((X,p),(Y,q)) := q\circ\mathrm{Corr}^0_\sim(X,Y)\circ p$.

Exercise 3.6. *Show that the mapping $X\mapsto(X,\Delta_X)$ realizes $Z_\sim SmProj(k)$ as a full subcategory of $Mot^{\mathrm{eff}}_\sim(k)$.*

Finally we want to include duals (i.e. Tate twists):
Step 3: The category of pure motives $\mathrm{Mot}_\sim(k)$ with

(1) objects: triples (X,p,m) with (X,p) an object of $\mathrm{Mot}^{\mathrm{eff}}_\sim(k)$ and $m\in\mathbb{Z}$.

(2) morphisms: $\mathrm{Hom}((X,p,m),(Y,q,n)) := q\circ\mathrm{Corr}^{n-m}_\sim(X,Y)\circ p$

Exercise 3.7. *Show that the mapping $(X,p)\mapsto(X,p,0)$ realizes $Mot^{\mathrm{eff}}_\sim(k)$ as a full subcategory of $Mot_\sim(k)$.*

The category $\mathrm{Mot}_\sim(k)$ has a natural structure of a symmetric monoidal category with duals. We touch on this in the next example.

Example 3.8. *By Exercises 3.6 and 3.7, we get that*

$$\mathrm{End}_{Mot_\sim(k)}((\mathbb{P}^1,\Delta_{\mathbb{P}^1})) = Corr^0(\mathbb{P}^1,\mathbb{P}^1) = \mathbb{Q}\oplus\mathbb{Q}.$$

Moreover $\Delta_{\mathbb{P}^1} = e_0\oplus e_1$ with $e_0 = \{0\}\times\mathbb{P}^1$ and $e_1 = \mathbb{P}^1\times\{0\}$ and this allows us to write

$$(\mathbb{P}^1,\Delta_{\mathbb{P}^1}) := \mathbf{1}\oplus\mathbb{L} \tag{3.1}$$

[15] The method of passing from **Step 1** to **Step 2** is an instance of a more general idea of taking a pseudo-abelian completion of an additive category.

where $\mathbf{1} = (\mathrm{Spec}(k), \mathrm{id})$ *corresponds to the motive of a point and* $\mathbb{L} = (\mathbb{P}^1, \mathbb{P}^1 \times \{0\})$ *the Lefschetz motive. Note that* (3.1) *is a definition that follows from pseudo-abelian completion. It's then an exercise*[16] *to show that* $\mathbb{L} \cong (\mathrm{Spec}(k), \mathrm{id}, -1) =: \mathbf{1}(-1)$. *The dual* $\mathbf{1}(1) := (\mathrm{Spec}(k), \mathrm{id}, 1)$ *is called the Tate motive. In general the definition of the dual of* (X, Δ_X) *is* $(X, \Delta_X) \otimes \mathbb{L}^{-d}$, *where* d *is the dimension of* X.

Definition 3.9 (symmetric monoidal structure)**.** $(X, p, m) \otimes (Y, q, n) := (X \times Y, p \times q, m + n)$

Exercise 3.10. *Show that* $Z^r_\sim(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}(\mathbb{L}^r, (X, \Delta_X))$.

Under the isomorphism given by Exercise 3.10, for $\alpha \in Z^r_\sim(X)$, we write $\alpha_* \in \mathrm{Hom}(\mathbb{L}^r, (X, \Delta_X))$ for the corresponding morphism. The corresponding dual morphism $\alpha^* \colon (X, \Delta_X) \otimes \mathbb{L}^r \to \mathbb{L}^{\dim(X)}$.

Definition 3.11 (Chow motives and Grothendieck's (numerical) motives)**.** We denote by $\mathrm{CHM}(k) := M_{\mathrm{rat}}(k)$ the category of Chow motives and $\mathrm{NM}(k) := M_{\mathrm{num}}(k)$ the category of Grothendieck motives (or numerical motives).

The next result is due to Scholl [Sch94, Corollary 3.5].

Proposition 3.12. *Assume that* k *is not contained in the algebraic closure of a finite field. Then the category of Chow motives CHM*(k) *is not an abelian category.*

Proof. Given the conditions on k, there exists an elliptic curve E/k of positive rank[17]. Let $P \in E(k)$ be a point of infinite order. Then by writing the Δ_E as in 1.2, we obtain a decomposition (again by definition of pseudo-abelian completion)

$$(E, \Delta_E) = \mathbf{1} \oplus \mathbb{L} \oplus h^1 E.$$

Then the divisor $[P] - [0]$ is a point on the Jacobian $J(E)$ and determines a non-zero morphism $\eta_* \colon \mathbb{L} \to h^1(E)$ by

Lemma 3.13. *We have an isomorphism* $\mathrm{Hom}(\mathbb{L}, h^1(E)) \cong J(E)(k) \otimes \mathbb{Q}$.

Proof. By Exercise 3.10, we have $\mathrm{Hom}(\mathbb{L}, (E, \Delta_E)) \cong Z^1_{\mathrm{rat}}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. On the other hand

$$\mathrm{Hom}(\mathbb{L}, \mathbf{1}) = \mathrm{Hom}((\mathrm{Spec}(k), \mathrm{id}, -1), (\mathrm{Spec}(k), \mathrm{id}, 0)) \subset \mathrm{Corr}^1_{\mathrm{rat}}(k, k) = 0,$$

and

$$\mathrm{Hom}(\mathbb{L}, \mathbb{L}) = \mathrm{Hom}(\mathbf{1}, \mathbf{1}) = \mathbb{Q}.$$

The projection morphism $Z^1_{\mathrm{rat}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ is just the degree map. Thus $\mathrm{Hom}(\mathbb{L}, h^1(E))$ identifies as the kernel of this map which gives the result. □

The composite $\eta_* \circ \eta^* \colon h^1(E) \otimes \mathbb{L} \to h^1(E)$. Note that

$$\mathrm{Hom}(h^1(E) \otimes \mathbb{L}, h^1(E)) \subset \mathrm{Hom}((E, \Delta_E, -1), (E, \Delta_E, 0)) \subset \mathrm{Corr}^1(E, E) = Z^2_{\mathrm{rat}}(E \times E).$$

and it is a check to see that $\eta_* \circ \eta^*$ corresponds to zero-cycle $c = (P, P) + (0, 0) - (P, 0) - (0, P)$. Assume $P = 2Q$ for $Q \in E(k)$. Then in $Z^2_{\mathrm{rat}}(E \times E)$ we can write

$$c = [(P, P) + (0, 0) - 2(Q, Q)] + [2(Q, Q) - (P, 0) - (0, P)]$$

and this is rationally equivalent to zero. Thus $\eta_* \circ \eta^* = 0$. This means η_* is not a monomorphism. If $\mathrm{CHM}(k)$ were abelian, then $\ker(\eta_*)$ would be a proper subobject of $\mathbb{L}$. Tensoring by the Tate motive would give a proper suboject of $\mathbf{1}$. But the unit object in an abelian category with a symmetric monoidal structure is completely decomposable and this gives a contradiction since $\mathrm{End}(\mathbf{1}) = \mathbb{Q}$ (the only idempotents of $\mathbb{Q}$ are 0 and 1). □

[16] Take a look at [Sta18, Tage 0FGD].

[17] I can't find a reference for this, unless k is a number field.

Proposition 3.14. *A Weil cohomology theory*

$$H\colon \mathit{SmProj}(k)^{opp} \to \mathit{GrVect}_F$$

factorizes as

$$\begin{aligned} H\colon \mathit{SmProj}(k)^{opp} \to \mathit{Mot}_{rat}(k) &\xrightarrow{G} \mathit{GrVect}_F \\ X \mapsto (X, \Delta_X, 0) \end{aligned}$$

Furthermore G precisely corresponds to the datum of H iff $G(\mathbf{1}(1))$ is non-zero only in degree -2.

The next Theorem is an important result due to Jannsen [Jan92] and arguably the most important result concerning pure motives:

Theorem 3.15. *Let $\sim$ be any adequate equivalence relation. TFAE*

(1) $\mathit{Mot}_\sim(k)$ is an abelian semi-simple category.

(2) $\sim$ is numerical equivalence

(3) For all $X \in \mathit{SmProj}(k)$ of pure dimension, $\mathit{Corr}^0(X,X)$ is a finite-dimensional semi-simple $\mathbb{Q}$-algebra.

Proof. (1) $\implies$ (2): Suppose for the sake of contradiction that $\mathrm{Mot}_\sim(k)$ is abelian and semi-simple but $Z_{\sim,0}(X) \neq Z_{\mathrm{num},0}(X)$. On the other hand we know that by Exercise 2.32: $Z_{\sim,0}(X) \subset Z_{\mathrm{num},0}(X)$. So take $Z \in Z^i_{\mathrm{num},0}(X)$ but $Z \notin Z^i_{\sim,0}(X)$. This Z gives a non-zero morphism

$$f\colon \mathbf{1} = (\mathrm{Spec}(k), \mathrm{id}, 0) \to (X, \mathrm{id}, i)$$

in $\mathrm{Mot}_\sim(k)$. Since $\mathrm{Mot}_\sim(k)$ is abelian and semi-simple, there is a morphism

$$g\colon (X, \mathrm{id}, i) \to \mathbf{1}$$

such that $g \circ f = \mathrm{id}_{\mathbf{1}}$. Such a g is given by $W \in Z^{d-i}_\sim(X)$. Then by the definition of composition of correspondences

$$\begin{aligned} g \circ f &= \mathrm{pr}_{\mathrm{Spec}(k)\times\mathrm{Spec}(k),*}((\mathrm{Spec}(k) \times Z \times \mathrm{Spec}(k)) \cdot (\mathrm{Spec}(k) \times W \times \mathrm{Spec}(k)) \\ &= \deg(Z \cdot W)\,\mathrm{Spec}(k) \times \mathrm{Spec}(k) \end{aligned}$$

where the second equality is by defintion of degree as pushforward onto a point. But $g \circ f = \mathrm{id}_{\mathbf{1}}$ and so $\deg(Z \cdot W) = 1$. But this contradicts $Z \in Z^i_{\mathrm{num},0}(X)$.

(2) $\implies$ (3): Fix a Weil cohomology theory (in this case we take étale cohomology with coefficients $\mathbb{Q}_\ell$ where $\ell \neq \mathrm{char}(k)$) and recall the cycle map (ignoring Tate twist)

$$\gamma_X\colon \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{2i}(X)$$

and define $A^i(X) := \mathrm{im}(\gamma_X) \subset H^{2i}(X)$ and set $B^i(X) := Z^i_{\mathrm{num}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. By Lemma 2.31 or Exercise 2.32, we get a surjection

$$A^i(X) \twoheadrightarrow B^i(X).$$

Let $d = \dim(X)$. We need to show $B^d(X \times X)$ is a finite-dimensional semi-simple $\mathbb{Q}$-algebra. By the above surjectivity statement, since $A^d(X \times X)$ is finite-dimensional, so is $B^d(X \times X)$. It remains to show it is semi-simple.

Lemma 3.16. *$B^d(X \times X)$ is a semi-simple $\mathbb{Q}$-algebra.*

Proof. By standard results of non-commutative algebra, it suffices to show $J(B^d(X \times X)) = 0$ where $J(R)$ is the Jacobson radical[18]. Since $J(B^d(X \times X)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = J(B^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$, we reduce to showing

$$J(B^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) = 0.$$

So put $A = A^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, $B = B^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ and $J_A = J(A)$ and $J_B = J(B)$. We have a surjection

$$\Phi \colon A \twoheadrightarrow B$$

and we need to show $J_B = 0$. By formal arguments one shows $\Phi(J_A) = J_B$. So take $f_B \in J_B$, which lifts to $f_A \in J_A$ and so f_A is nilpotent in A (as J_A is nilpotent ideal). Then for any $g \in A$ the Lefschetz trace formula[19] gives

$$\operatorname{Tr}(f_A \cup g) = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}_{H^i_{\text{ét}}(X, \mathbb{Q}_\ell)}(f_A \circ g) \tag{3.2}$$

Since the Jacobson radical is a two-sided ideal, we get $f_A \circ g \in J_A$. So the RHS of (3.2) vanishes. But $\operatorname{Tr}(f_A \cup g) = \deg(f_A \cdot g)$. So this means f_A is numerically 0 and so $f_B = 0$, as desired. □

(3) $\implies$ (1). We won't prove this. □

3.1 Manin's identity principle and Lieberman's lemma

We now change gears and ask ourselves how to tell whether a correspondence is trivial or not. The next example shows something funny can happen.

Example 3.17 (Detection of trivial correspondences)**.** *Consider an elliptic curve E/k and four different points $a, b, c, d \in E(k)$. Then consider $p = \{a - b\} \times \{c - d\} \in CH^2(E \times E)$ is not zero. Viewing $p \in Corr^1_{rat}(E, E)$, we get an induced map*

$$\begin{aligned} p_* \colon CH^i(E) &\to CH^{i+1}(E) \\ T &\mapsto pr_{E*}(p \cdot (T \times E)) \end{aligned}$$

for every $i \geq 0$. Clearly $p_* = 0$ (recall that any two points are algebraically equivalent so $a \cdot T = b \cdot T$).

Manin's identity principle [Man68, pg. 450] gives some characterization of detecting non-trivial correspondences. To state it, we need to think of a correspondence as a functor of points (just like schemes). For the following assume we are working with the rational adequate equivalence (for simplicity):

Definition 3.18 (correspondence as a functor of points)**.** Given $T \in \operatorname{SmProj}(k)$, we put $X(T) := \operatorname{Corr}(T, X)$. Then for $f \in \operatorname{Corr}(X, Y)$, we get the induced morphism

$$\begin{aligned} f_T \colon X(T) &\to Y(T) \\ \alpha &\mapsto f \circ \alpha \end{aligned}$$

Theorem 3.19 (Manin's identity principle)**.** *Let $f, g \in Corr(X, Y)$. TFAE*

(1) $f = g$

(2) $f_T = g_T$ *for all* $T \in SmProj(k)$

(3) $f_X = g_X$

[18] The Jacobson radical of a ring R, $J(R) := \{r \in R| \quad rM = 0 \text{ for all } M \text{ simple}\}$.

[19] Such a formula is a formal consequence of the Weil cohomology theory axioms.

Proof. The only non-trivial direction is (3) $\implies$ (1). But one can check that $f = f \circ \Delta_X$, which gives the resut. $\square$

In practice we need Lieberman's lemma to actually make use of Manin's identity principle.

Lemma 3.20 (Lieberman's lemma)**.** *In the setting of Definition 3.18 $f \circ \alpha = (\Delta_T \times f)_*(\alpha)$.*

Proof. By definition of action of correspondences:

$$(\Delta_T \times f)_*(\alpha) = p^{TTXY}_{TY*}((\Delta_T \times f) \cdot (p^{TTXY*}_{TX}(\alpha))) \tag{3.3}$$

where

$$p^{TTXY}_{TY} : T \times \underline{T} \times X \times \underline{Y} \to \underline{T} \times \underline{Y} \text{ and } p^{TTXY}_{TX} : \underline{T} \times T \times \underline{X} \times Y \to \underline{T} \times \underline{X}.$$

We can rewrite (3.3) as

$$\begin{aligned} p^{TTXY}_{TY*}((\Delta_T \times f) \cdot (p^{TTXY*}_{TX}(\alpha))) &= p^{TTXY}_{TY*}((\Delta_T \times f) \cdot (T \times \alpha \times Y)) \\ &= p^{TTXY}_{TY*}((\Delta_T \times X \times Y) \cdot (T \times T \times f) \cdot (T \times \alpha \times Y)) \end{aligned}$$

Now writing p^{TTXY}_{TY} as the composition:

$$p^{TTXY}_{TY} : T \times T \times X \times Y \xrightarrow{p} T \times X \times Y \xrightarrow{q} T \times Y$$

we get

$$\begin{aligned} p^{TTXY}_{TY*}((\Delta_T \times X \times Y) \cdot (T \times T \times f) \cdot (T \times \alpha \times Y)) &= q_* p_*((\Delta_T \times X \times Y) \cdot p^*(T \times f) \cdot (T \times \alpha \times Y)) \\ &= q_*(p_*((\Delta_T \times X \times Y) \cdot (T \times \alpha \times Y))) \cdot (T \times f)) \\ &= q_*(((\alpha \circ \Delta_T) \times Y) \cdot (T \times f)) \\ &= f \circ \alpha \circ \Delta_T \\ &= f \circ \alpha. \end{aligned}$$

where the second isomorphism follows from the projection formula (cf. Exercise 2.6) and the third/fourth follow from defnition of composition of correspondences. The final equality is just that Δ_T acts as identity when composing. $\square$

Corollary 3.21. *In the context of Manin's identity principle (cf. Theorem 3.19), we get $f = g$ iff*

$$(\mathrm{id}_T \times f)_* = (\mathrm{id}_T \times g)_*$$

considered as maps on the Chow groups

$$CH(T \times X) \to CH(T \times Y) \quad \forall\, T$$

As an application of Manin's identity principle, we sketch the proof of the following:

Lemma 3.22. *Let $\mathscr{E}$ be a locally free sheaf of rank $(m+1)$ on $S \in SmProj(k)$ and let $\pi \colon \mathbb{P}_S(\mathscr{E}) \to S$ be the associated projective bundle. Then there is an isomorphism of motives in $Mot_{rat}(k)$*

$$(\mathbb{P}_S(\mathscr{E}), \Delta_{\mathbb{P}_S(E)}, 0) \xrightarrow{\sim} \bigoplus_{i=0}^{m} (S, \Delta_S, -i)$$

Proof. Let $\xi = \mathcal{O}(1)$ be the tautological line bundle on $\mathbb{P}_S(\mathscr{E})$. Then there is a *projective space bundle formula* [Sta18, Tag 0ERV]:

$$\lambda \colon CH(\mathbb{P}_S(\mathscr{E})) \xrightarrow{\sim} \bigoplus_{i=0}^{m} CH(S)[\xi^i].$$

Moreover the isomorphism λ (and it's inverse μ) are induced by correspondences. Also the morphism λ and μ are compatible with base change $T \to \mathrm{Spec}(k)$.

So this means that $(\mathrm{id}_T \times \lambda) \circ (\mathrm{id}_T \times \mu) = \mathrm{id}$ for all T. The result then follows by Corollary 3.21. $\square$

3.2 $M_{\mathbf{rat}}(k)$ vs category of abelian varieties up to isogeny

We prove that the category of Chow motives contains as a full subcategory the category of abelian varieties up to isogeny.

Recall from the proof of Proposition 3.12 for any curve $X \in \mathrm{SmProj}(k)$, we can write

$$(X, \Delta_X) = \mathbf{1} \oplus \mathbb{L} \oplus h^1 X. \tag{3.4}$$

Then in the spirit of Lemma 3.13 we have

Proposition 3.23. *Given two curves $X, X' \in SmProj(k)$ we have*

$$\mathrm{Hom}(h^1 X, h^1 X') = \mathrm{Hom}_{AV}(J(X), J(X')) \otimes_{\mathbb{Z}} \mathbb{Q}$$

.

Proof. By Weil's theorem [Wei71, Theorem 22, Chapitre VI]

$$Z^1_{\mathrm{rat}}(X \times X') \otimes_{\mathbb{Z}} \mathbb{Q} = (Z^1_{\mathrm{rat}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (Z^1_{\mathrm{rat}}(X') \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus \mathrm{Hom}_{\mathrm{AV}}(J(X), J(X')) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Note that $Z^1_{\mathrm{rat}}(X \times X') \otimes_{\mathbb{Z}} \mathbb{Q} = Hom((X, \Delta_X), (X', \Delta_{X'}))$. By using the decomposition (3.4), it's a check to get the result. □

To get the result one needs the Poincaré reducibility theorem [Mum74, Chapter IV, §19, Theorem 1]:

$$\{\text{category of AV}\}/\text{isogeny} = \text{pseudo-abelian completion of } \{J(C)|C \text{ curve}\}.$$

4 Lecture 4: Grothendieck's standard conjectures

Up to this point we have defined motives. Motives are expected to have good properties, but it turns out that these are still open. In this lecture, we will discuss the so-called *standard conjectures* concerning motives. These were originally formulated by Grothendieck in [Gro69]. In this lecture we will discuss some results in [Kle68] and [Kle94]. We have already seen standard conjecture D in (cf. Conjecture 2.33). In this lecture we will take a look at the remaining standard conjectures:

(1) Standard Conjectures C (Künneth Conjecture)

(2) Standard Conjectures A and B (Conjectures of Lefschetz type)

(3) Standard Conjecture H (Conjecture of Hodge type)

Let $X \in \mathrm{SmProj}(k)$. We fix a Weil cohomology $H(X)$ over a characteristic 0 field and recall we have

$$\gamma_X \colon \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{2i}(X)$$

and define $A^i(X) := \mathrm{im}(\gamma_X) \subset H^{2i}(X)$. We call the elements of $A^i(X)$ the *algebraic* classes.

4.1 Künneth conjecture (Standard conjecture C)

Assume X is pure of dimension d. Let $\Delta_X \in \mathrm{CH}^d(X \times X)$ be the diagonal and consider its class

$$\gamma_{X \times X}(\Delta_X) \in H^{2d}(X \times X) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X)$$

where the equality is the Künneth decomposition (cf. axiom (4) in Definition 2.24). So we can write

$$\gamma_{X \times X}(\Delta_X) = \pi_0 + \pi_1 + \ldots + \pi_i + \ldots + \pi_{2d}$$

with $\pi_i \in H^{2d-i}(X) \otimes H^i(X)$.

Conjecture 4.1 (Künneth conjecture). *The Künneth components π_i are algebraic: $\exists$ cycle classes $\Delta_i \in CH^d(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\gamma_{X\times X}(\Delta_i) = \pi_i$.*

Exercise 4.2. *Let X be a scheme with a* cellular decomposition*: that is there exists a filtration*

$$X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$$

by closed subschemes with each $X_i \backslash X_{i-1}$ a disjoint union of schemes U_{ij} isomorphic to affine spaces $\mathbb{A}^{n_{ij}}$. Then $CH^k(X)$ is finitely generated by $\{[V_{ij}]\}$, where V_{ij} is the closure of U_{ij} in X. Show in this case one has

$$CH(X \times X) \cong CH(X) \otimes_{\mathbb{Z}} CH(X).$$

Exercise 4.3. *Show that any $X \in SmProj(k)$ which satisifes the* Chow-Künneth *decomposition:*

$$CH(X \times X) \cong CH(X) \otimes_{\mathbb{Z}} CH(X)$$

implies that γ_X is in fact an isomorphism. Use this to show that for such X, the Künneth conjecture (trivially) holds.

Remark 4.4. *Projective space $\mathbb{P}^n$ satisfies the condition of Exercise 4.2. In general if X is a* linear *scheme, then it satisfies the conditions of Exercise 4.3 (cf. [Tot14, Proposition 1]).*

The next proposition is less trivial and is due to Katz-Messing [KM74, Theorem 2 part 1)].

Proposition 4.5. *Suppose $k = \mathbb{F}_q$ and $X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ is irreducible. Then the Künneth conjecture holds for X.*

Proof. Fix a prime $\ell \neq p = \mathrm{char}(\mathbb{F}_q)$ and let Fr be the relative Frobenius morphism of X over $\mathbb{F}_q$. Deligne has proved (cf. [Del74a, Théorème I.6]), as part of his proof of the Weil conjectures that the polynomial in T

$$\det(1 - TFr \mid H^i_{\text{ét}}(X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Q}_\ell)) \tag{4.1}$$

lies in $\mathbb{Z}[T]$ and its reciprocal zeros all have complex absolute value $q^{i/2}$ for every $i \geq 0$. As a first step Katz-Messing (cf. [KM74, Theorem 1]) show that the term (4.1) is independent of the Weil cohomology theory, that is:

Lemma 4.6. *We have $\det(1 - TFr \mid H^i_{\acute{e}t}(X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Q}_\ell)) = \det(1 - TFr \mid H^i(X))$ where $H^i(X)$ is our chosen Weil cohomology theory from the start of this lecture.*

Proof. We won't prove this, but let me mention that it relies on Poincaré duality and the weak Lefschetz axiom. □

It follows that the polynomials $G_i(T) = \det(1 - TFr \mid H^i(X))$ are pairwise relatively prime in $\mathbb{Q}[T]$ because their roots have different absolute value. Let $\Pi_i(T) \in \mathbb{Q}[T]$ be a polynomial such that

$$G_j(T) | \Pi_i(T) \text{ for all } j \neq i \text{ and } \Pi_i(T) = 1 \bmod G_i(T).$$

Such a polynomial exists by the Chinese remainder theorem. By the Cayley-Hamilton theorem, it follows that the operator

$$\Pi_i(\mathrm{Fr}^{-1})\colon \bigoplus_{j=0}^{2d} H^j(X) \to H^i(X) \tag{4.2}$$

is exactly the projection operator and these are algebraic. But note that by Poincaré duality we can rewrite the Künneth formula as

$$H^{2d}(X \times X) = \mathrm{Hom}_{\mathrm{GrVect}_F^{\geq 0}}(H(X), H(X)).$$

This means that by (4.2) $\gamma_{X\times X}(\Delta_X) = p_1 + \ldots + p_{2d}$, where $p_i \in \mathrm{Corr}^0(X)$ corresponding to (the graph of) $\Pi_i(\mathrm{Fr}^{-1})$. □

Exercise 4.7. *Using decomposition* (3.4)*, show that the Künneth conjecture holds for curves.*

4.2 Conjectures of Lefschetz type (Standard conjectures A and B)

Assume X is pure of dimension d and let $Y \hookrightarrow X$ be a smooth hyperplane section. Recall the Lefschetz operator

$$\begin{aligned} L\colon H^i(X) &\to H^{i+2}(X) \\ \alpha &\mapsto \alpha \cup \gamma_X(Y). \end{aligned}$$

Recall for H a Weil cohomology theory, we assume hard Lefschetz (cf. Definition 2.24(7))

$$L^i\colon H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X).$$

Lemma 4.8. *The Lefschetz operator L is algebraic. More precisely it is represented by the algebraic cycle $\Delta_*(Y) \in A^{d+1}(X \times X)$ and $\Delta\colon X \to X \times X$ is the diagonal map.*

Proof. For $u = \Delta_*(Y)$, it suffices to show

$$\gamma_X(p_{2*}(p_1^*\alpha \cdot u)) = \alpha \cup \gamma_X(Y).$$

Indeed we have

$$\begin{aligned} \gamma_X(p_{2*}(p_1^*\alpha \cdot u)) &= \gamma_X(p_{2*}(p_1^*\alpha \cdot \Delta_* Y)) \\ &= p_{2*}(p_1^*\alpha \cup \Delta_* Y) \\ &= \alpha \cup p_{2*}\Delta_* Y \end{aligned}$$

where the second equality follows by compatibility of γ_X and pushforwards (and intersection products), the third equality by a version of the projection formula (cf. Exercise 2.6). But note that $p_2 \circ \Delta = \mathrm{id}_X$, so we are done. $\square$

Using Hard Lefschetz we can define a unique linear map $\Lambda\colon H^i(X) \to H^{i-2}(X)$ for each $2 \le i \le 2d$ as follows:

1) For $2 \le i \le d$ which makes the following diagram commutative:

$$\begin{tikzcd} H^i(X) \arrow[r, "L^{d-i}"] \arrow[d, dotted, "\Lambda"'] & H^{2d-i}(X) \arrow[d, "L"] \\ H^{i-2}(X) \arrow[r, "L^{d-i+2}"] & H^{2d-i+2}(X). \end{tikzcd}$$

2) For $i = d+1$, $\Lambda := L^{-1}$ where $L\colon H^{d-1}(X) \xrightarrow{\sim} H^{d+1}(X)$.

3) For $d+2 \le i \le 2d$ which makes the following diagram commutative:

$$\begin{tikzcd} H^{2d-i}(X) \arrow[r, "L^{i-d}"] \arrow[d, "L"'] & H^i(X) \arrow[d, dotted, "\Lambda"] \\ H^{2d-i+2}(X) \arrow[r, "L^{i-d-2}"] & H^{i-2}(X). \end{tikzcd}$$

By Poincaré duality and Künneth formula, we have

$$\Lambda \in \mathrm{Hom}(H^i(X), H^{i-2}(X)) = H^{2d-i}(X) \otimes H^{i-2}(X) \subset H^{2d-2}(X \times X).$$

So we can view Λ canonically as an element in $H^{2d-2}(X \times X)$.

Conjecture 4.9 (Standard conjecture B)**.** *The operator Λ is algebraic: $\exists$ cycle $Z \in CH^{d-1}(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\gamma_{X\times X}(Z) = \Lambda$.*

Lemma 4.10 (Conjecture B implies the Künneth conjecture)**.** *Standard conjecture B implies the Künneth conjecture*

Proof.

Exercise 4.11. *Verify the formula*

$$\pi_i = \Lambda^{d-i}\left(1 - \sum_{j>2d-i} \pi_j\right) L^{d-i}\left(1 - \sum_{j<i} \pi_j\right)$$

where $\Lambda^{d-i}\colon H^{2d-i}(X) \xrightarrow{\sim} H^i(X)$ *(an inverse to* L^{d-i}*).*

We can then proceed by induction as π_0 and π_{2d} are algebraic. $\square$

It turns out that Standard conjecture B is independent of the choice of hyperplane section which defines L (and hence Λ), cf. [Kle94, Theorem 4.1(2)].

Proposition 4.12. *If standard conjecture B holds for one choice of* L*, then it holds for all choices.*

Proof. It suffices to show that standard conjecture B is equivalent to the following statement (which is independent of L):

For each $i \leq d$, there exists an algebraic correspondence

$$\nu_i\colon H^{2d-i}(X) \xrightarrow{\sim} H^i(X). \tag{4.3}$$

Indeed if standard conjecture B is true, then Λ is algebraic and thus by taking a sufficiently large composition Λ^i is also algebraic and induces the above isomorphism.

For the converse suppose (4.3) holds. Then $u := \nu_i \circ L^{d-i}$ is algebraic.

Exercise 4.13. *By looking at the characteristic polynomial of* u*, show that* u^{-1} *is algebraic.*

By exercise 4.13, it follows that $\theta_i := u^{-1} \circ \nu_i$ is an algebraic inverse of L^{d-i}. The result then follows from the following exercise:

Exercise 4.14. *Show that*

$$\Lambda := \sum_{i\leq d} (\pi_{i-1}\theta_i L^{d-i+1}\pi_i + \pi_{2d-i}L^{d-i+1}\theta_{i+2}\pi_{2d-i+2}).$$

$\square$

Remark 4.15. *Standard conjecture B holds true for abelian varities. This result is due to Lieberman-Kleiman [Kle68, Theorem 2A11].*

To state standard conjecture A, note that we have a commutative diagram (for $d \geq 2i$)

$$\begin{tikzcd} A^i(X) \arrow[r] \arrow[d] & A^{d-i}(X) \arrow[d] \\ H^{2i}(X) \arrow[r, "L^{d-2i}"] & H^{2d-2i}(X). \end{tikzcd} \tag{4.4}$$

The top arrow $A^i(X) \to A^{d-i}(X)$ exists because L is algebraic (cf. Lemma 4.8). In fact is injective.

Conjecture 4.16 (Standard Conjecture A)**.** *Hard Lefschetz is true on cycles. That is the top arrow* $A^i(X) \hookrightarrow A^{d-i}(X)$ *in diagram (4.4) is an isomorphism.*

It turns out that Standard conjectures A and B are equivalent, cf. [Kle94, Corollary 4.2]

Proposition 4.17. *Standard conjecture A holds iff standard conjecture B holds.*

Proof. (B) $\implies$ (A): Indeed if (B) is true then we get an algebraic inverse to L^{d-2i} (given by Λ^{d-2i}) and so we get an inverse map $A^{d-i}(X) \to A^i(X)$. Thus (A) is true.

(A) $\implies$ (B): We won't prove this in detail but highlight some steps.

Exercise 4.18. *Show that each $x \in H^i(X)$ has a unique decomposition*

$$x = \sum_{j \geq max(i-d,0)} L^j x_j$$

where $x_j \in \ker(L|H^{i-2j}(X))$.

Given Exercise 4.18 we can define an operator ${}^c\Lambda$ given by:

$${}^c\Lambda(x) := \sum_{j \geq max(i-d,1)} j(n-i+j+1)L^{j-1}x_j$$

Exercise 4.19. *Show that Λ is algebraic iff ${}^c\Lambda$ is algebraic.*

Since we have assumed conjecture A is true, in particular it is true for $X \times X$ equipped with Lefschetz operator $1 \otimes L + L \otimes 1$. Then [Kle68, Proposition 1.4.6(ii) and Proposition 2.1] implies that $1 \otimes^c \Lambda +^c \Lambda \otimes 1$ carries algebraic classes to algebraic classes (a priori it is only defined at the level of cohomology). Moreover Proposition 1.3.4 in loc.cit. shows that it carries Δ_X to $2^c\Lambda$. Therefore ${}^c\Lambda$ is algebraic and we are done by Exercise 4.19.

$\square$

4.3 Hodge standard conjecture

In general by hard Lefschetz

$$L^{d-i} \colon H^i(X) \xrightarrow{\sim} H^{2d-i}(X)$$

is an isomorphism. However in the spirit of Exercise 4.18:

Definition 4.20 (primitive cohomology)**.** We define primitive cohomology as

$$P^i(X) := \ker(L^{d-i+1} \colon H^i(X) \to H^{2d-i+2}(X)).$$

Definition 4.21 (primitive algebraic classes)**.** We define primitive algebraic classes as

$$A^i_{\text{prim}}(X) := A^i(X) \cap P^{2i}(X)$$

For $i \leq d/2$, the cup product gives a pairing

$$\begin{aligned} A^i_{\text{prim}}(X) \times A^i_{\text{prim}}(X) &\to \mathbb{Q} \\ (x, y) &\mapsto (-1)^i \mathrm{Tr} \circ (L^{d-2i}(x) \cup y) \end{aligned}$$

Conjecture 4.22 (Hodge standard conjecture)**.** *The pairing defined above is positive definite.*

The next result can be found in [Kle94, Proposition 5.1].

Proposition 4.23. *Given the Hodge standard conjecture, standard conjecture A is equivalent to standard conjecture D.*

Proof. We need to define a version of the Hodge star operator (appearing in Hodge theory):

$$\begin{aligned} *\colon H^i(X) &\to H^{2d-i}(X) \\ x &\mapsto \sum_{j\geq max(i-d,0)} (-1)^{(i-2j)(i-2j+1)/2} L^{d-i+j} x_j \end{aligned}$$

where the $x_j \in H^{i-2j}(X)$ are those appearing in Exercise 4.18.

Now suppose the Hodge conjecture is true.

Exercise 4.24. *Show that $*^2 = 1$. By using Exercise 4.18 for x, show that the pairing*

$$\begin{aligned} A^i(X) \times A^i(X) &\to \mathbb{Q} \\ (x,y) &\mapsto Tr(x \cup *y) \end{aligned}$$

is positive-definite.

(A) $\implies$ (D): Now standard conjecture A implies the canonical pairing (given by cup product) $A^i(X) \times A^{d-i}(X) \to \mathbb{Q}$ is perfect. Thus if $x \in Z^i_{\text{num},0}(X)$, then $x \in Z^i_{\text{hom},0}(X)$.

(D) $\implies$ (A): In this case we use $A^i(X) \hookrightarrow A^{d-i}(X)$ and again the positive-definiteness of Exercise 4.24.

□

Remark 4.25. *If k is of characteristic zero, then the Hodge standard conjecture is true and is a consequence of Hodge theory.*

5 Lecture 5: Motivic Galois groups

Up to now we have mainly focused on the geometric aspects of motives. In this lecture we start to look at the arithmetic aspects. Grothendieck wanted to build some kind of Galois group coming from a fiber functor[20]

$$\mathrm{Mot}_{\mathrm{num}}(k) \to \{\text{category of finite vector spaces over } k\}.$$

Such a fiber functor can come from a Weil cohomology theory, but the issue is that we don't have standard conjecture D and so a priori one one gets a fiber functor from $\mathrm{Mot}_{\mathrm{hom}}(k)$. Now the issue is that $\mathrm{Mot}_{\mathrm{hom}}(k)$ is no longer abelian and so not Tannakian.

There are several approaches of circumventing conjecture D and modifying the source category $\mathrm{Mot}_{\mathrm{num}}(k)$ just enough so that one has a fiber functor and the category remains abelian. We will study the approach of Deligne-Milne in [DM82].

5.1 Absolute Hodge cycles

Let k be a field of finite transcendence degree over $\mathbb{Q}$ and $X \in \mathrm{SmProj}(k)$. We set

$$H^n_{\text{ét}}(X) := \varprojlim_r H^n_{\text{ét}}(X, \mathbb{Z}/r\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \text{ and } H^n_{\text{ét}}(X)(1) := H^n_{\text{ét}}(X) \otimes_{\mathbb{A}_f} (\varprojlim_r \mu_r \otimes_{\mathbb{Z}} \mathbb{Q})$$

and

$$H^n_{\mathrm{dR}}(X)(m) := H^n_{\mathrm{dR}}(X).$$

Finally we set

$$H^n_{\mathbb{A}}(X)(m) := H^n_{\mathrm{dR}}(X)(m) \times H^n_{\text{ét}}(X)(m).$$

[20]This is the reason that Grothendieck (and his student Saavedra Rivano) introduced the notion of Tannakian category.

Given an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$, there are canonical isomorphisms:

$$\sigma^*_{dR}\colon H^n_{\mathrm{dR}}(X)(m) \otimes_{k,\sigma} \mathbb{C} \xrightarrow{\sim} H^n_{\mathrm{dR}}(\sigma X)(m) \text{ and } \sigma^*_{\text{ét}}\colon H^n_{\text{ét}}(X)(m) \xrightarrow{\sim} H^n_{\text{ét}}(\sigma X)(m)$$

where $\sigma X := X \times_k \mathbb{C}$. We put $\sigma^* := \sigma^*_{\mathrm{dR}} \times \sigma^*_{\text{ét}}$. We now put

$$H^n_B(X)(m) := H^n_B((\sigma X)^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} (2\pi i)^m \mathbb{Q}$$

so that the standard comparison isomorphisms give

$$H^n_B(X)(m) \otimes_{\mathbb{Q}} (\mathbb{C} \times \mathbb{A}_f) \xrightarrow{\sim} H^n_{\mathrm{dR}}(\sigma X)(m) \times H^n_{\text{ét}}(\sigma X)(m)$$

Definition 5.1 (Hodge cycle relative to σ)**.** An element $t \in H^{2p}_{\mathbb{A}}(X)(p)$ is a Hodge cycle relative to σ if

(1) t is rational relative to σ: $\sigma^*(t)$ lies in the rational subspace $H^{2p}_B(X)(p)$ of $H^{2p}_{\mathrm{dR}}(\sigma X)(m) \times H^{2p}_{\text{ét}}(\sigma X)(m)$.

(2) it is of bidegree $(0,0)$.

Definition 5.2 (Absolute Hodge cycle)**.** An element $t \in H^{2p}_{\mathbb{A}}(X)(p)$ is an absolute Hodge cycle if it is a relative Hodge cycle for every embedding $\sigma\colon k \hookrightarrow \mathbb{C}$. We denote by $C^p_{\mathrm{AH}}(X)$ the $\mathbb{Q}$-vector space of absolute Hodge cycles $t \in H^{2p}_{\mathbb{A}}(X)(p)$.

Conjecture 5.3 (Deligne)**.** *Assume k is algebraically closed. A Hodge cycle relative to a single embedding σ is a Hodge cycle for every embedding (i.e. an absolute Hodge cycle).*

Remark 5.4. *Conjecture 5.3 is true for Abelian varieties.*

Example 5.5. *The cycle class maps*

$$\gamma_{dR}\colon CH^p(X) \to H^{2p}_{dR}(X)(p) \text{ and } \gamma_{\acute{e}t}\colon CH^p(X) \to H^{2p}_{\acute{e}t}(X)(p)$$

and we claim that $t := (\gamma_{dR}(Z), \gamma_{\acute{e}t}(Z))$ is an absolute Hodge cycle. Indeed for any $\sigma\colon k \hookrightarrow \mathbb{C}$, $\sigma^*(t) = \gamma_B(\sigma Z)$. This is because the cycle class maps are all compatible via the comparison isomorphisms cf. [Del71, 2.2.5.1]. In addition it is of bidegree (p,p) by a calculation. The Hodge conjecture predicts that there are no other absolute Hodge cycles.

Definition 5.6 (False category of motives)**.** By repeating the procedure of taking the pseudo-abelian completion with morphisms given by $\mathrm{C}^p_{\mathrm{AH}}(X \times Y)$, we get the false category of motives $\dot{M}_k$. More precisely this is the category given by

(1) **objects:** triples (X, p, m) with $X \in \mathrm{SmProj}(k)$, $p \in \mathrm{C}^d_{\mathrm{AH}}(X \times X)$ a projector ($d = \dim(X)$) and $m \in \mathbb{Z}$.

(2) **morphisms:** $\mathrm{Hom}((X,p,m),(Y,q,n)) := q \circ \mathrm{C}^{n-m+d}_{\mathrm{AH}}(X \times Y) \circ p$ with composition given by cup product.

To state the main theorem regarding $\dot{M}_k$, let us recall the notion of a Tannakian category. For a more comprehensive treatment of Tannakian categories, cf. [SR72].

Definition 5.7. Let $\mathcal{C}$ be a (rigid) abelian[21] tensor category with $\mathrm{End}(\mathbf{1}) = k$.

(1) **fibre functor:** A fibre functor on $\mathcal{C}$ with values in a k-algebra R is a k-linear exact faithful tensor functor

$$\eta\colon \mathcal{C} \to \mathrm{Mod}_R$$

that takes values in the subcategory Proj_R.

[21] this essentially means that $\mathcal{C}$ is equipped with a tensor product, has internal homs and duals.

(2) **Tannakian category:** $\mathcal{C}$ is a Tannakian category over k if it admits a fibre functor with values in some nonzero k-algebra. It is a neutral Tannakian category if $R = k$.

The main property of Tannakian categories is that they are essentially representations of some group.

Theorem 5.8. *Let $\mathcal{C}$ be a Tannakian category over k. Then there exists a stack $\mathcal{G}$ in groupoids and a canonical k-linear tensor functor*

$$\mathcal{C} \to \mathrm{Rep}_k(\mathcal{G})$$

which is an equivalence of categories[22]*. If $\mathcal{C}$ is a neutral Tannakian category then $\mathcal{G}$ is represented by an affine group scheme.*

Returning back to our false category of motives $\dot{M}_k$, it turns out for some technical issue, it cannot be Tannakian. To fix this one changes the commutativity constraint[23] as follows: Let

$$\dot{\psi}\colon M \otimes N \to N \otimes M \text{ where } \dot{\psi} = \oplus \dot{\psi}^{r,s} \text{ where } \dot{\psi}^{r,s} := M^r \otimes N^s \to N^s \otimes M^r$$

be the commutativity constraint. An explanation of the notation is in order: the grading $\dot{\psi}^{r,s}$ is coming from the grading induced on morphisms by

$$\mathrm{C}_{\mathrm{AH}}^{p+d}(X \times Y) \subset H_{\mathbb{A}}^{2d+2p}(X \times Y)(p+d) = \oplus_r \mathrm{Hom}(H_{\mathbb{A}}^r(X), H_{\mathbb{A}}^{r+2p}(Y)(p)). \tag{5.1}$$

We now modify the commutativity constraint as

$$\psi\colon M \otimes N \to N \otimes M \text{ where } \psi = \oplus \psi^{r,s} \text{ where } \psi^{r,s} := (-1)^{rs} \dot{\psi}^{r,s} \tag{5.2}$$

Definition 5.9 (True category of motives)**.** We define the true category of motives M_k to be $\dot{M}_k$ with the commutativity constraint ψ.

Proposition 5.10. *The category M_k is a semisimple Tannakian category over $\mathbb{Q}$.*

Proof. By similar ideas to Jannsen's proof that the category of Grothendieck's motives is an abelian semi-simple category (cf. Theorem 3.15), it suffices to show that $C_{\mathrm{AH}}^{\dim(X)}(X \times X)$ is finite-dimensional semi-simple $\mathbb{Q}$-algebra.

Lemma 5.11. *For every $0 \leq r \leq 2d$, there exists $\psi_X \in C_{AH}^{2d-r}(X \times X)$ such that for every $\sigma\colon k \hookrightarrow \mathbb{C}$, the induced morphism (cf.* (5.1)*)*

$$\psi^r\colon H_B^r((\sigma X)^{an}, \mathbb{R}) \times H_B^r((\sigma X)^{an}, \mathbb{R}) \to \mathbb{R}(-r)$$

is a polarization of real Hodge structures[24]*.*

Proof. Recall the $*$-operator appearing in Proposition 4.23

$$*\colon H_B^r((\sigma X)^{\mathrm{an}}, \mathbb{R}) \to H_B^{2d-r}((\sigma X)^{\mathrm{an}}, \mathbb{R})(2d-r).$$

Then we define ψ^r to be the composite (writing X for $(\sigma X)^{\mathrm{an}}$ for the sake of brevity)

$$H_B^r(X, \mathbb{R}) \times H_B^r(X, \mathbb{R}) \xrightarrow{\mathrm{id} \times *} H_B^r(X, \mathbb{R}) \times H_B^{2d-r}(X, \mathbb{R})(2d-r) \xrightarrow{\cup} H_B^{2d}(X)(d-r) \xrightarrow{\mathrm{Tr}} \mathbb{R}(-r)$$

Exercise 5.12. *Show by unraveling the definitions that this is indeed an absolute Hodge cycle.*

[22] Here $\mathrm{Rep}_k(\mathcal{G})$ the category of cartesian functors $\mathcal{G} \to \mathrm{Proj}$, where Proj is the stack such that $\mathrm{Proj}(\mathrm{Spec}\, R) := \mathrm{Proj}_R$.

[23] A commutativity constraint is part of the datum of a tensor category.

[24] A polarization on a real Hodge structure V of weight n is a bilinear form $\phi\colon V \times V \to \mathbb{R}(-n)$ such that the real-valued form $(x, y) \mapsto (2\pi i)^n \phi(x, iy)$ is positive-definite and symmetric.

It is positive-definite due to the Hodge-Riemann bilinear relations (i.e. the Hodge standard conjecture holds true in the complex setting). $\square$

Lemma 5.13. *Let ψ_X (and ψ_Y) be defined as in Lemma 5.11. For any $u \in C_{AH}^{\dim(Y)}(Y \times X)$, there exists $u' \in C_{AH}^{\dim(X)}(X \times Y)$ such that*

$$\psi_X(uy, x) = \psi_Y(y, u'x)$$

for all $x \in H_B^r((\sigma X)^{an}, \mathbb{R})$ and $y \in H_B^r((\sigma Y)^{an}, \mathbb{R})$. Moreover[25]

$$Tr(u \circ u') = Tr(u' \circ u) \in \mathbb{Q} \text{ and } Tr(u \circ u') > 0 \text{ if } u \neq 0.$$

Proof. For the first part just take u' to be the adjoint of u. Such a u' exists because pairing is non-degenerate. The last part follows from formal properties of a polarization for a real Hodge structure (which I will skip). $\square$

The previous lemma, then implies that ψ_X is a *Weil* form on X and this implies that $C_{\mathrm{AH}}^{\dim(X)}(X \times X)$ is a finite-dimensional semi-simple $\mathbb{Q}$-algebra (cf. [DM82, Definition 4.1 and Proposition 4.2]). $\square$

So at this point we know that M_k is a semisimple Tannakian category over $\mathbb{Q}$. Moreover for a fixed embedding $\sigma\colon k \hookrightarrow \mathbb{C}$ we have the Weil cohomology functor (given by rational Betti cohomology)

$$\begin{aligned} H_B\colon \mathrm{SmProj}(k)^{\mathrm{op}} &\to \mathrm{Vect}_{\mathbb{Q}} \\ X &\mapsto \bigoplus_r H_B^r((\sigma X)^{\mathrm{an}}, \mathbb{Q}) \end{aligned}$$

Essentially by formal reasons (cf. Proposition 3.14) this extends to a functor

$$\omega\colon M_k \to \mathrm{Vect}_{\mathbb{Q}} \tag{5.3}$$

and it must be exact because every additive functor from a semi-simple abelian category is exact. It is faithful essentially by (5.1).

Definition 5.14 (Motivic Galois group of Deligne-Milne)**.** The datum of ω in (5.3) together with the main theorem of Tannakian categories (cf. Theorem 5.8) gives rise to affine group scheme $G(\sigma)$ over $\mathbb{Q}$. This is the motivic Galois group of Deligne-Milne.

Proposition 5.15. *The group $G(\sigma)$ is a pro-reductive affine group scheme over $\mathbb{Q}$.*

Proof. Let $X \in \mathrm{ob}(M_k)$ and let $\mathcal{C}_X$ be the abelian tensor subcategory of M_k generated by X, $X^\vee$, $\mathbb{L}$ and $\mathbb{L}^\vee$, where $\mathbb{L}$ is the Lefschetz motive. Then $\omega|_{\mathcal{C}_X}$ is again a fibre functor and applying the main theorem of Tannakian categories gives $\mathcal{C}_X = \mathrm{Rep}_k(G_X)$ for some affine group scheme over $\mathbb{Q}$. Moreover standard yoga involving Tannakian categories gives

$$G(\sigma) = \varprojlim_X G_X.$$

The fact that G_X are reductive follows from the following fact (cf. [DM82, 6.9]), which we won't prove:

Lemma 5.16. *Let G be a connected affine group scheme over k (a field of characteristic 0). Then G is reductive if and only if*

[25] To make sense of this $u \circ u'$ is another correspondence or simply an element of $C_{\mathrm{AH}}^{\dim(X)}(X \times X)$ and this acts on the graded space $\oplus_i H^i_{\mathbb{A}}(X)$ and the trace is taken there.

(1) $Rep_k(G)$ has a tensor generator (in our case we can take $X \oplus \mathbb{L}$)

(2) $Rep_k(G)$ has no non-trivial object X such that $\langle X \rangle$ is stable under $\otimes$ (this is the reason we took $\mathbb{L}$ in $\mathcal{C}_X$). Here $\langle X \rangle$ is the full subcategory of $\mathcal{C}_X$ which is a subquotient of powers of X and $X^\vee$ (cf. Definition 6.12 in loc.cit.)

(3) $Rep_k(G)$ is semisimple (which is the case as M_k and hence $\mathcal{C}_X$ is semisimple).

This shows that $G(\sigma)$ is pro-reductive. □

5.2 Structure of the motivic Galois group

To describe the motivic Galois group $G(\sigma)$ and compare it to the traditional Galois group $\mathrm{Gal}(\overline{k}/k)$ we need the notion of *Artin* motives:

Definition 5.17 (Artin motives)**.** Let $V_k^0 \subset M_k$ be the image of zero-dimensional varieties over k and let M_k^0 be the Tannakian subcategory of M_k generated by V_k^0. The category M_k^0 is called the category of Artin motives.

It turns out that V_k^0 is already Tannakian:

Proposition 5.18. *We have that $M_k^0 = V_k^0$ and $M_k^0 \cong Rep_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k))$.*

Proof. Let X be a zero-dimensional variety over k. Then $X(\overline{k})$ is a finite set on which $\mathrm{Gal}(\overline{k}/k)$ acts continuously (cf. [Sta18, Tag 03QR]). Thus $\mathbb{Q}^{X(\overline{k})}$ is a finite-dimensional $\mathbb{Q}$-representation of $\mathrm{Gal}(\overline{k}/k)$. Let X and Y be zero-dimensional varieties over k. We compute

$$\begin{aligned} \mathrm{Hom}_{M_k}(X,Y) &= C^0_{\mathrm{AH}}(X \times Y) \\ &= (\mathbb{Q}^{X(\overline{k}) \times Y(\overline{k})})^{\mathrm{Gal}(\overline{k}/k)} \\ &= \mathrm{Hom}_{\mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k))}(\mathbb{Q}^{X(\overline{k})}, \mathbb{Q}^{Y(\overline{k})}) \end{aligned}$$

Thus

$$\begin{aligned} V_k^0 &\to \mathrm{Rep}_{\mathbb{Q}}(\mathrm{Gal}(\overline{k}/k)) \\ X &\mapsto \mathbb{Q}^{X(\overline{k})} \end{aligned}$$

is fully-faithful. It is essentially surjective by loc.cit. Therefore V_k^0 is abelian and thus Tannakian. □

Now fix an embedding $\sigma\colon \overline{k} \hookrightarrow \mathbb{C}$. By standard Tannakian yoga the inclusion $M_k^0 \to M_k$ defines a homomorphism $\pi\colon G(\sigma) \to \mathrm{Gal}(\overline{k}/k)$. Similarly the functor $M_k \to M_{\overline{k}}$ defines a homomorphism $\iota\colon G^0(\sigma) \to G(\sigma)$, where $G^0(\sigma)$ is the group such that $\mathrm{Rep}_{\overline{k}}(G^0(\sigma)) \cong M_{\overline{k}}$.

Proposition 5.19. *The sequence*

$$1 \to G^0(\sigma) \xrightarrow{\iota} G(\sigma) \xrightarrow{\pi} \mathrm{Gal}(\overline{k}/k) \to 1$$

is exact.

Proof. First as $M_k^0 \to M_k$ is fully faithful the morphism π is faithfully flat (this is also standard Tannakian yoga, but see [DM82, Corollary 5.2] for a proof). Similarly to prove ι is an injection by [DM82, Corollary 5.1] it suffices to prove the following: every motive (attached to) $X \in \mathrm{SmProj}(\overline{k})$ is a subquotient of a motive $X' \otimes_k \overline{k}$ for some $X' \in \mathrm{SmProj}(k)$. But X has a model X_0 over a finite extension k' of k and one can take $X' = \mathrm{Res}_{k'/k} X_0$.

Exercise 5.20. *Try to show exactness at $G(\sigma)$.*

$\square$

Example 5.21. *Recall from the proof of Proposition 5.15 that* $G(\sigma) = \varprojlim_X G_X$*. Then one can compute* G_X *for various* $X \in M_k$*:*

(1) $G_{\mathbb{L}} = \mathbb{G}_m$*.*

(2) If X *is an elliptic curve and has no CM over* $\overline{k}$ *(i.e.* $End_{\overline{k}} X \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$*, then* $G_X = \mathrm{GL}_2$*).*

Remark 5.22. *One can think of the motivic Galois group* $G(\sigma)$ *as a generalization of [Sta18, Tag 03QR] to higher dimensions.*

6 Lecture 6: Various enriched realizations of motives

Recall that a Weil cohomology theory defined over a field F of characteristic 0 gives rise to a functor

$$H\colon \mathrm{Mot}_{\mathrm{rat}}(k) \to \mathrm{GrVect}_F. \tag{6.1}$$

This lands in the category of graded finite-dimensional vector spaces GrVect_F. However several Weil cohomology theories carry additional structure (e.g. étale cohomology carries an action of Galois). It is often the case that there is some Tannakian category $\mathcal{A}$ such that (6.1) actually lands in

$$H_{\mathcal{A}}\colon \mathrm{Mot}_{\mathrm{rat}}(k) \to \text{Gr-}\mathcal{A}$$

where Gr-$\mathcal{A}$ is a rigid tensor category of $\mathcal{A}$ with a $\mathbb{Z}$-graduation[26]. Similarly we get a functor

$$H_{\mathcal{A}}\colon \mathrm{Mot}_{\mathrm{hom}}(k) \to \text{Gr-}\mathcal{A} \tag{6.2}$$

It will be more convenient to drop the grading on both sides. For this we will need to suppose a weakened version of the standard conjecture C:

Conjecture 6.1 (Sign conjecture)**.** *In the context of the Künneth conjecture, cf. §4.1, the Künneth sum of even-degree projections* $\sum_i \pi_{2i}$ *is algebraic.*

6.1 Detour on Tannakian categories

Let $(C, \otimes)$ be a rigid tensor category. For each object $X \in C$ there is a canonical trace map:

$$\mathrm{Tr}_X\colon \mathrm{End}(X) \to \mathrm{End}(\mathbf{1})$$

defined by sending $f\colon X \to X$ to the composite

$$\mathbf{1} \to X \otimes X^\vee \xrightarrow{f \otimes X^\vee} X \otimes X^\vee \xrightarrow{\text{comm. constraint}} X^\vee \otimes X \to \mathbf{1}.$$

We let $\dim(X)$ denote the trace of id_X.

Theorem 6.2. *Assuming* k *is of characteristic 0,* $(C, \otimes)$ *is Tannakian iff for all objects* X*,* $\dim(X) \geq 0$*.*

Exercise 6.3. *Assuming the sign conjecture, show that we get a tensor functor which still respects the grading*

$$H_{\mathcal{A}}\colon Mot_{hom}(k) \to Gr\text{-}\mathcal{A}$$

where now the grading on both sides is a $\mathbb{Z}/2$*-grading (where one groups together all even graded parts and groups together all odd graded parts). Show that for* X *smooth projective,* $\dim(X) = \chi(X)$*. Moreover by modifying the commutativity constraint on* $Mot_{hom}(k)$*, show that we get an exact faithful tensor functor*

$$H_{\mathcal{A}}\colon Mot_{hom}(k) \to \mathcal{A}.$$

[26] We refer the reader to [SR72, Chapitre IV] for a formal definition.

6.2 Back to enriched realizations

From now on we assume the sign conjecture and assume we have a functor

$$H_{\mathcal{A}}\colon \mathrm{Mot}_{\mathrm{hom}}(k) \to \mathcal{A}.$$

We are interested in situations where $H_{\mathcal{A}}$ is full and whether the objects in the image are semi-simple[27]. In fact these two properties are related:

Proposition 6.4. *Suppose $H_{\mathcal{A}}$ is full. TFAE:*

(1) The objects in the image of $H_{\mathcal{A}}$ are semi-simple.

(2) Standard conjecture D holds and

$$H_{\mathcal{A}}\colon Mot_{hom}(k) = Mot_{num}(k) \to \mathcal{A}$$

makes $Mot_{num}(k)$ a tannakian sub-category of $\mathcal{A}$.

Proof. (1) $\implies$ (2): $\mathrm{Mot}_{\mathrm{hom}}(k)$ is therefore a semi-simple category. But a semi-simple subcategory of a Tannakian category is abelian and so by Theorem 3.15, Standard conjecture D holds. It is also Tannakian (by standard Tannakian yoga). (2) $\implies$ (1) : This again follows by Theorem 3.15. □

We now consider four different Weil cohomology theories and their associated enrichments: Betti, étale, crystalline and de Rham.

6.3 Hodge realisation (Betti cohomology)

In this setting we suppose $\sigma\colon k \hookrightarrow \mathbb{C}$. By Hodge theory, the Betti cohomology $H^i_B((\sigma X)^{\mathrm{an}}, \mathbb{Q})$ is equipped with a rational Hodge structure:

Definition 6.5 (rational Hodge structure)**.** A rational Hodge structure is a finite-dimensional vector space V over $\mathbb{Q}$ together with a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q} \tag{6.3}$$

such that $\overline{V^{p,q}} = V^{q,p}$. Morphisms are graded morphisms. We denote by $\mathrm{HS}_{\mathbb{Q}}$ the category of rational Hodge structures.

The forgetful functor $\mathrm{HS}_{\mathbb{Q}} \to \mathrm{Vect}_{\mathbb{Q}}$ is a fibre functor and so $\mathrm{HS}_{\mathbb{Q}}$ is a neutral Tannakian category. We get the corresponding *Hodge realisation*

$$H_{\mathrm{Hodge}}\colon \mathrm{Mot}_{\mathrm{hom}}(k) \to \mathrm{HS}_{\mathbb{Q}}.$$

Exercise 6.6 (Mumford-Tate group - an incarnation of the motivic galois group)**.** *The bigraduation on $V \otimes_{\mathbb{Q}} \mathbb{C}$ given by (6.3) corresponds to a homomorphism $\mu\colon \mathbb{G}_m^2 \to \mathrm{GL}(V \otimes_{\mathbb{Q}} \mathbb{C})$. For $V \in HS_{\mathbb{Q}}$, the corresponding Tannakian subcategory generated by V has an associated affine group scheme G_V (by Theorem 5.8). Show that G_V is the Mumford-Tate group of V, that is the smallest closed subgroup $MT(V) \subset \mathrm{GL}(V)$ such that $MT(V)(\mathbb{C})$ contains the image of μ.*

Conjecture 6.7 (fullness of the Hodge realization)**.** *Assume k is algebraically closed. The Hodge realization H_{Hodge} is a full functor.*

[27] Recall that an object X in an abelian category $\mathcal{C}$ is simple, if it only has (at most) two subobjects: itself and the zero object. An object is semi-simple if it is a sum of simple objects.

Remark 6.8 (Relationship with the Hodge conjecture). *Conjecture 6.7 is equivalent to the Hodge conjecture. Recall the Hodge conjecture states that all elements of bidegree* $(0,0)$ *in the Hodge decomposition of* $\oplus_r H_B^{2r}((\sigma X)^{an}, \mathbb{Q})(r)$ *are algebraic.*

Remark 6.9 (Polarizations and semi-simplicity). *All objects in the image of* H_{Hodge} *are semi-simple. This follows from the fact that all objects in the image are polarizable rational Hodge structures and polarizable rational Hodge structures are semi-simple.*

6.4 Tate realisation (étale cohomology)

In this setting k can be arbitrary and let $\overline{k}$ be the separable closure of k. For $\ell \neq \mathrm{char}(k)$, the étale cohomology groups $H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$ are continuous representations of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. Let $\mathrm{Rep}_{\mathbb{Q}_\ell} \mathrm{Gal}(\overline{k}/k)$ be the category of continuous finite-dimensional $\mathbb{Q}_\ell$-representations of $\mathrm{Gal}(\overline{k}/k)$. The forgetful functor $\mathrm{Rep}_{\mathbb{Q}_\ell} \mathrm{Gal}(\overline{k}/k) \to \mathrm{Vect}_{\mathbb{Q}_\ell}$ is a fibre functor and so $\mathrm{Rep}_{\mathbb{Q}_\ell} \mathrm{Gal}(\overline{k}/k)$ is a Tannakian category. We get the corresponding *Tate realisation*

$$H_{\mathrm{Tate}} \colon \mathrm{Mot}_{\mathrm{hom}}(k) \to \mathrm{Rep}_{\mathbb{Q}_\ell} \mathrm{Gal}(\overline{k}/k)$$

Conjecture 6.10. *Suppose* k *is a finite extension of* $\mathbb{Q}$ *or* $\mathbb{F}_p$. *Then*

(1) H_{Tate} *is a full functor.*

(2) every object in the image of H_{Tate} *is semi-simple.*

Remark 6.11 (Relationship with the Tate conjecture). *An element of* $\oplus_r H^{2r}_{ét}(X_{\overline{k}}, \mathbb{Q}_\ell)(r)$ *which is* $\mathrm{Gal}(\overline{k}/k)$*-invariant is called an* ℓ*-adic Tate cycle. The Tate conjecture says that all* ℓ*-adic Tate cycles are a* $\mathbb{Q}_\ell$*-linear combination of algebraic cycles. Conjecture 6.10(1) is equivalent to the Tate conjecture.*

6.5 Ogus realisation (cristalline cohomology)

Let $\Omega^\bullet_{X/k}$ be the complex of differential forms of X over k:

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \xrightarrow{d} \dots .$$

Then (algebraic) de Rham cohomology is defined to be the hypercohomology of this complex:

$$H^i_{\mathrm{dR}}(X) := \mathbb{H}^i(\Omega^\bullet_{X/k}).$$

Now let k be a finite extension of $\mathbb{Q}$. Let v be an unramified prime of k where X has good reduction and let k_v be the completion of k at v. Then by Berthelot-Ogus's comparison theorem (cf. [BO83]) $H^i_{\mathrm{dR}}(X) \otimes_k k_v$ is equipped with a Frobenius-semi-linear bijection:

Theorem 6.12 (Berthelot-Ogus). *Let* A *be a complete discrete valuation ring, with fraction field* k *and perfect residue field* k_0. *We assume that* k *is of characteristic 0 and* k_0 *of characteristic* $p > 0$. *Let* $\mathfrak{X}$ *be a projective and smooth scheme over* A. *Then, there is an isomorphism:*

$$H^i_{dR}(\mathfrak{X} \otimes_A k) \cong H^i_{cris}(\mathfrak{X} \otimes_A k_0) \otimes_{W(k_0)[\frac{1}{p}]} k.$$

Definition 6.13 (Ogus category). Let $\mathrm{Og}(k)$ be the category whose objects are finite-dimensional vector spaces over k such that at almost all places v of k, the v-adic completion $V \otimes_k k_v$ is equipped with a Frobenius-semi-linear bijection. Morphisms are commutative diagrams in the obvious way.

It turns out that $\mathrm{Og}(k)$ is actually a Tannakian category over $\mathbb{Q}$ (one has to show that $\mathrm{End}_{\mathrm{Og}(k)}(\mathbf{1}) = \mathbb{Q}$). We get the corresponding *Ogus realisation*

$$H_{Ogus} \colon \mathrm{Mot}_{\mathrm{hom}}(k) \to \mathrm{Og}(k).$$

Conjecture 6.14. *Suppose k is a finite extension of $\mathbb{Q}$. Then*

(1) H_{Ogus} is a full functor.

(2) every object in the image of H_{Ogus} is semi-simple.

Remark 6.15 (Relationship with the Ogus conjecture)**.** *An element of $\oplus_r H^{2r}_{dR}(X)(r)$ which is Frobenius-invariant at almost all places unramified of k is called an Ogus cycle. The Ogus conjecture says that all Ogus cycles are algebraic cycles. Conjecture 6.14(1) is equivalent to the Ogus conjecture.*

6.6 Betti-de Rham realisation (The Grothendieck period conjecture)

In this setting we suppose $\sigma \colon k \hookrightarrow \mathbb{C}$. As for the Hodge realization, we are motivated by the de-Rham-Betti comparison theorem (also known as the *period* isomorphism proven by Grothendieck in 1966 [Gro66])

$$H^i_{dR}(X) \otimes_k \mathbb{C} \cong H^i_B((\sigma X)^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}. \tag{6.4}$$

Definition 6.16 ($\mathrm{Vect}_{k,\mathbb{Q}}$)**.** Let $\mathrm{Vect}_{k,\mathbb{Q}}$ be the category whose objects are triples (W, V, ω) where $W \in \mathrm{Vect}_k$, $V \in \mathrm{Vect}_{\mathbb{Q}}$ and

$$\omega \colon W \otimes_k \mathbb{C} \to V \otimes_{\mathbb{Q}} \mathbb{C}$$

is an isomorphism. Given two objects (W_1, V_1, ω_1) and (W_2, V_2, ω_2). The group

$$\mathrm{Hom}_{\mathrm{Vect}_{k,\mathbb{Q}}}((W_1, V_1, \omega_1), (W_2, V_2, \omega_2))$$

is the subgroup $\mathrm{Hom}_k(W_1, W_2) \oplus \mathrm{Hom}_{\mathbb{Q}}(V_1, V_2)$ of pairs $(\phi_{\mathrm{dR}}, \phi_B)$ such that the following diagram is commutative:

$$\begin{tikzcd}
W_1 \otimes_k \mathbb{C} \arrow[r, "\phi_{\mathrm{dR}} \otimes_k \mathrm{id}_{\mathbb{C}}"] \arrow[d, "\omega_1"] & W_2 \otimes_k \mathbb{C} \arrow[d, "\omega_2"] \\
V_1 \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r, "\phi_B \otimes_{\mathbb{Q}} \mathrm{id}_{\mathbb{C}}"] & V_2 \otimes_{\mathbb{Q}} \mathbb{C}.
\end{tikzcd}$$

Exercise 6.17. *Show that the forgetful functor*

$$\begin{aligned} \mathit{Vect}_{k,\mathbb{Q}} &\to \mathit{Vect}_{\mathbb{Q}} \\ (W, V, \omega) &\mapsto V \end{aligned}$$

is a fibre functor and thus $\mathit{Vect}_{k,\mathbb{Q}}$ is a neutral Tannakian category over $\mathbb{Q}$.

Grothendieck's period isomorphism then gives us the de Rham-Betti realisation:

$$H_{\text{dR-B}} \colon \mathrm{Mot}_{\mathrm{hom}}(k) \to \mathrm{Vect}_{k,\mathbb{Q}}.$$

Conjecture 6.18. *Suppose k is a finite extension of $\mathbb{Q}$ (or $k = \overline{\mathbb{Q}}$). Then*

(1) $H_{dR\text{-}B}$ is a full functor.

(2) every object in the image of $H_{dR\text{-}B}$ is semi-simple.

Example 6.19. *The isomorphism* (6.4) *is equivalent to giving a matrix of periods* $\Omega_X \in \mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{C})/\mathrm{GL}_n(k)$.

Remark 6.20 (Relationship to the Grothendieck period conjecture)**.** *Assume again that* k *is a number field or* $\overline{\mathbb{Q}}$. *A de Rham-Betti cycle is an element of* $\oplus_r H_B^{2r}((\sigma X)^{an}, \mathbb{Q})(r)$ *which corresponds to an element of* $\oplus_r H_{dR}^{2r}(X)(r)$ *via the isomorphism* (6.4)*. Then Conjecture 6.18(1) is equivalent to the following statement: All de Rham-Betti cycles are algebraic. When fixing a single* r*, we denote this conjecture by* $GPC^r(X)$*. In [Bos16], the authors refer to the latter statement as the Grothendieck period conejecture.*

In [Gro66, note (10), p. 102], the original formulation of the Grothendieck period conjecture is stated differently. It is also related but not equivalent to the formulation in Remark 6.20. Bost shows that $\mathrm{GPC}^1(X)$ is true in the case of an abelian variety [Bos13, Theorem 5.1]. We give a sketch of the proof:

Theorem 6.21. *For* X *an abelian variety over* $\overline{\mathbb{Q}}$*,* $GPC^1(X)$ *is true.*

Proof. We want to show that the realisation functor $H^1_{\text{dR-B}}$ restricted to the category of abelian varieties over $\overline{\mathbb{Q}}$ is fully faithful.

Step 1: identify morphisms between abelian varieties as morphisms of the associated universal vector extensions. For an abelian variety $A/\overline{\mathbb{Q}}$, let $\mathbb{E}_A$ be the $\overline{\mathbb{Q}}$-vector space $\Gamma(A, \Omega^1_{A/\overline{\mathbb{Q}}})$. Then there is an extension

$$0 \to \mathbb{E}_{A^\vee} \to E(A) \to A \to 0$$

of commutative algebraic groups over $\overline{\mathbb{Q}}$. It is called the *universal vector extension* of A. The formation of $E(A)$ is functorial, that is for a morphism $A \to B$, we get a morphism $E(A) \to E(B)$.

Lemma 6.22. *For any two abelian varieties* A *and* B *over* $\overline{\mathbb{Q}}$*, the morphism of* $\mathbb{Z}$*-modules* $\mathrm{Hom}(A, B) \to \mathrm{Hom}(E(A), E(B))$ *is an isomorphism.*

Step 2: group of periods and Lie algebra of universal vector extension. The universal vector extension $E(A)$ has a lie algebra $\mathrm{Lie}\, E(A)$ and we can consider the exponential map of the complexified Lie algebra:

$$\exp_{E(A)_\mathbb{C}} \colon \mathrm{Lie}\, E(A)_\mathbb{C} \to E(A)^{\mathrm{an}}_\mathbb{C}$$

We denote by $\mathrm{Per}E(A)_\mathbb{C} := \ker \exp_{E(A)_\mathbb{C}}$, the group of periods. The exponential map is a covering of $E(A)$ by it's fundamental group. In this case $\mathrm{Per}E(A)_\mathbb{C} = \mathbb{Z}^{2\dim A}$. Moreover by standard results on universal extensions $\mathrm{Lie}\, E(A)_\mathbb{C} = \mathbb{C}^{2\dim A}$. Thus the inclusion $\mathrm{Per}E(A)_\mathbb{C} \hookrightarrow \mathrm{Lie}\, E(A)_\mathbb{C}$ extends to an isomorphism

$$c^{-1} \colon \mathrm{Per}E(A)_\mathbb{C} \otimes_\mathbb{Z} \mathbb{C} \xrightarrow{\sim} \mathrm{Lie}\, E(A)_\mathbb{C}$$

Therefore the triple $\mathrm{LiePer}E(A) := (\mathrm{Lie}\, E(A), \mathrm{Per}E(A)_\mathbb{C} \otimes_\mathbb{Z} \mathbb{Q}, c)$ gives rise to an object in $\mathrm{Vect}_{\overline{\mathbb{Q}},\mathbb{Q}}$. Moreover the construction of $\mathrm{LiePer}E(A)$ is functorial in $E(A)$, that is for a morphism $E(A) \to E(B)$ we get a map on the corresponding LiePer pairs. As in Lemma 6.22, we have

Lemma 6.23. *For any two abelian varieties* A *and* B *over* $\overline{\mathbb{Q}}$*, the morphism of* $\mathbb{Z}$*-modules*

$$\mathrm{Hom}(E(A), E(B)) \to \mathrm{Hom}_{Vect_{\overline{\mathbb{Q}},\mathbb{Q}}}(LiePerE(A), LiePerE(B))$$

is an isomorphism.

Step 3: comparing de Rham-Betti cohomology to LiePer. Let

$$H^i_{\text{dR-B}}(A) := (H^i_{dR}(A), H^i_A((\sigma X)^{\text{an}}, \mathbb{Q}), \omega)$$

where ω is the period isomorphism (6.4). Also we denote by $H_{i,\text{dR-B}}(A) := H^i_{\text{dR-B}}(A)^\vee$ the dual of $H^i_{\text{dR-B}}(A)$ (in the Tannakian category $\text{Vect}_{\overline{\mathbb{Q}},\mathbb{Q}}$).

Lemma 6.24. *There is a canonical functorial ismorphism*

$$LiePerE(A) \xrightarrow{\sim} H_{1,dR\text{-}B}(A).$$

Combining Lemmas 6.22-6.24, shows that the realisation functor $H_{1,\text{dR-B}}$ is fully-faithful on the category of abelian varieties over $\overline{\mathbb{Q}}$.

Step 4: Connection with the Néron-Severi group. Bost completes the proof as follows: From **Step 3**, we get an isomorphism

$$H_{1,\text{dR-B}}\colon \text{Hom}(A, A^\vee) \xrightarrow{\sim} \text{Hom}_{\text{Vect}_{\overline{\mathbb{Q}},\mathbb{Q}}}(H_{1,\text{dR-B}}(A), H_{1,\text{dR-B}}(A^\vee)) \tag{6.5}$$

On the other hand by the duality of homology-cohomology, there is a canonical isomorphism

$$H_{1,\text{dR-B}}(A) \xrightarrow{\sim} H^1_{\text{dR-B}}(A^\vee) \otimes \mathbb{Z}(1) \tag{6.6}$$

Substituting (6.6) into (6.5) gives an isomorphism

$$\text{Hom}(A, A^\vee) \xrightarrow{\sim} \text{Hom}_{\text{Vect}_{\overline{\mathbb{Q}},\mathbb{Q}}}(\mathbb{Z}(0), H^1_{\text{dR-B}}(A) \otimes H^1_{\text{dR-B}}(A) \otimes \mathbb{Z}(1)). \tag{6.7}$$

Recall that the Neron-Sévri group $N(A)$ is the group of divisors of A module algebraic equivalence. It can be shown that $N(A) \hookrightarrow \text{Hom}(A, A^\vee)$ and isomorphism (6.7) induces an isomorphism

$$N(A) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \text{de Rham-Betti cycles}$$

and this completes the sketch. □

7 Lecture 7: Filtrations on the Chow ring

As usual let $X \in \text{SmProj(k)}$ where k is a field.

7.1 Bloch-Beilinson Conjecture

In his Duke 1979 lectures [Blo10] Bloch and independently Beilinson [Bei87] both conjectured that there exists a descending filtration on the rational Chow groups $\text{CH}^i(X)_{\mathbb{Q}} := \text{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The underlying idea is that Grothendieck's theory of motives should not only be used as an *universal cohomology* theory, but also for studying the Chow groups of an algebraic variety.

The precise version of these conjectures was formulated by Jannsen in [Jan94, §2], which we will now review. For a fixed Weil cohomology theory $\sim_{\text{hom}}$, we also need the following subgroups:

$$\text{CH}^i_{\text{hom}}(X) := \{\alpha \in \text{CH}^i(X) \mid \text{cl}(\alpha) = 0\} \text{ and } \text{CH}^i_{\text{hom}}(X)_{\mathbb{Q}} := \text{CH}^i_{\text{hom}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Conjecture 7.1 (Bloch-Beilinson filtration). *For each $i \geq 0$, there exists a descending filtration $F^\bullet_{BB}$ on $CH^i(X)_{\mathbb{Q}}$ such that*

(1) $F^0_{BB}CH^i(X)_{\mathbb{Q}} = CH^i(X)_{\mathbb{Q}}$ and $F^1_{BB}CH^i(X)_{\mathbb{Q}} = CH^i_{hom}(X)_{\mathbb{Q}}$.

(2) $F^r_{BB}CH^i(X)_{\mathbb{Q}} \cdot F^s_{BB}CH^j(X)_{\mathbb{Q}} \subseteq F^{r+s}_{BB}CH^{i+j}(X)_{\mathbb{Q}}$.

(3) *For a morphism $f\colon X \to Y$, the induced morphisms f_* and f^* on the Chow ring $\bigoplus_i CH^i(X)_{\mathbb{Q}}$ respect the grading induced by $F^\bullet_{BB}$.*

(4) *Assuming the Künneth Conjecture (Standard Conjecture C), the jth Künneth component π_j acts*[28] *on $Gr^\nu_{BB} CH^i(X)_{\mathbb{Q}}$ as $\delta_{j,2i-\nu} \cdot \mathrm{id}$.*

(5) $F^{i+1}_{BB} CH^i(X)_{\mathbb{Q}} = 0$.

Remark 7.2. *The way to make sense of (4) is as follows: Firstly by parts (2) and (3), as an action of a correspondence involves only operations of type (2) and (3), we get an induced action of Δ_i on the graded pieces. Each of the π_i are (honest) projections, in the sense that they either send an element to 0 or keep it the same (at least on the cohomological level). By part (1), the induced action of Δ_i are again projections on the cycle level and (4) is cutting out the contributions of each of these projectors.*

Moreover, assuming the existence of an abelian category of *mixed motives $MM(k)$* a more precise conjecture is formulated as follows

Conjecture 7.3 (A stronger version of Conjecture 7.1)**.** *Keep (1), (2), (3) and (5) the same and replace (4) by:*

$$Gr^\nu_{BB} CH^i(X)_{\mathbb{Q}} = Ext^\nu_{MM(k)}(\mathbf{1}, (X, \Delta_{2i-\nu}, i)) \tag{7.1}$$

In Conjecture (7.3), formula (7.1) is called the *Beilinson formula.* Conjecture 7.1/7.3 is called the Bloch-Beilinson conjecture.

7.2 Conjectures of Murre

We now review 4 conjectures of Murre [Mur93] and then study their relation to the above Bloch-Beilinson conjecture. Assume X is pure of dimension d.

Conjecture 7.4 (Chow-Künneth Conjecture)**.** *X has a Chow-Künneth decomposition over k defined below.*

Definition 7.5 (Chow-Künneth decomposition)**.** We say that X admits a Chow-Künneth decomposition if there exists $p_i \in \mathrm{CH}^d(X \times X)_{\mathbb{Q}} = \mathrm{Corr}^0(X, X)$ for $0 \le i \le 2d$ such that

(1) $\Delta_X = \sum_{i=0}^{2d} p_i$.

(2) $p_i \circ p_j = 0$ if $i \ne j$ and each p_i is a projector.

(3) $\gamma_{X\times X}(p_i) = \pi_i$, where π_i is the ith Künneth component of Δ_X (cf. Conjecture 4.1).

Assuming Conjecture 7.4, each of the projectors p_i clearly[29] operate on the Chow groups $\mathrm{CH}^i(X)_{\mathbb{Q}}$.

Conjecture 7.6 (Vanishing conjecture)**.** *For every $0 \le i \le d$, the projectors $p_{2d}, p_{2d-1}, \dots, p_{2i+1}$ and $p_0, p_1, \dots, p_{i-1}$ operate as zero on $CH^i(X)_{\mathbb{Q}}$.*

Assuming Conjectures 7.4 and 7.6 we get the following descending filtration on $\mathrm{CH}^i(X)_{\mathbb{Q}}$:

$$\begin{aligned} F^0\mathrm{CH}^i(X)_{\mathbb{Q}} &= \mathrm{CH}^i(X)_{\mathbb{Q}} \\ F^1\mathrm{CH}^i(X)_{\mathbb{Q}} &= \ker(p_{2i}) \\ F^2\mathrm{CH}^i(X)_{\mathbb{Q}} &= \ker(p_{2i}) \cap \ker(p_{2i-1}) \end{aligned}$$

and in general

$$F^\nu\mathrm{CH}^i(X)_{\mathbb{Q}} = \ker(p_{2i}) \cap \ldots \cap \ker(p_{2i+1-\nu}). \tag{7.2}$$

[28] or rather Δ_i where $\gamma_{X\times X}(\Delta_i) = \pi_i$.

[29] As they are degree 0 correspondences.

Lemma 7.7 (Murre vs Bloch-Beilinson (1) and (5)). *Assuming Conjectures 7.4 and 7.6, we have*

(1) $F^{i+1}CH^i(X)_{\mathbb{Q}} = 0$.

(2) $F^1 CH^i(X)_{\mathbb{Q}} \subseteq CH^i_{hom}(X)_{\mathbb{Q}}$.

Proof. We have $F^{i+1}\mathrm{CH}^i(X)_{\mathbb{Q}} = \ker(p_{2i}) \cap \ldots \cap \ker(p_i)$. On the other hand Conjecture 7.6 implies $\mathrm{id}_X = p_{2i} + \ldots + p_i$.This proves the first statement.

For the second statement we have the commutative diagram

$$\begin{tikzcd} \mathrm{CH}^i(X)_{\mathbb{Q}} \arrow[r, "p_{2i}"] \arrow[d, "\gamma_X"] & \mathrm{CH}^i(X)_{\mathbb{Q}} \arrow[d, "\gamma_X"] \\ H^{2i}(X) \arrow[r, "\pi_{2i}"] & H^{2i}(X). \end{tikzcd}$$

But by definition of the Künneth components the bottom arrow is identity and so $\ker p_{2i} \subseteq \ker \gamma_X = \mathrm{CH}^i_{\mathrm{hom}}(X)_{\mathbb{Q}}$. □

We now come to the third conjecture of Murre.

Conjecture 7.8 (Third conjecture of Murre). *We have* $F^1 CH^i(X)_{\mathbb{Q}} = CH^1_{hom}(X)_{\mathbb{Q}}$.

Finally the fourth conjecture of Murre:

Conjecture 7.9 (Fourth conjecture of Murre). *The filtration $F^\bullet$ defined in* (7.2) *is independent of the choice of the projectors p_i.*

7.3 Bloch-Beilinson vs Murre

It turns out that the Bloch-Beilinson conjecutures and the four conjectures of Murre are equivalent (cf. [Jan94, Theorem 5.2])

Theorem 7.10 (Jannsen). *Fix a field k. Then the Bloch-Beilinson conjectures (cf. Conjecture 7.1) are true for all smooth projective varieties over k iff the four conjectures of Murre (Conjectures 7.4-7.9) are true. Moreover, if these conjectures are true, then the Bloch-Beilinson filtration coincides with Murre's filtration.*

The crucial part of the proof of Theorem 7.10 is given by the following proposition cf. [Jan94, Proposition 5.8]:

Proposition 7.11. *Let X and Y be smooth projective varieties (of pure dimensions d and e, respectively). Suppose that X and Y have Chow-Künneth decompositions given by*

$$ch^i(X) = (X, p_i(X), 0) \text{ for } 0 \le i \le 2d \text{ and } ch^j(Y) = (Y, p_j(Y), 0) \text{ for } 0 \le j \le 2e.$$

Then the product variety $Z = X \times Y$ has a Chow-Künneth decomposition with projectors

$$p_m(Z) := \sum_{r+s=m} p_r(X) \times p_s(Y) \text{ for } 0 \le m \le 2d + 2e. \tag{7.3}$$

Moreover the following holds:

(1) *If, with the projectors* (7.3), *Conjecture 7.6 holds then*

$$\mathrm{Hom}_{Mot_{rat}(k)}(ch^i(X), ch^j(Y)) = 0 \text{ for } i < j.$$

(2) If, with the projectors (7.3)*, Conjecture 7.8 holds then*

$$\mathrm{Hom}_{Mot_{rat}(k)}(ch^i(X), ch^i(Y)) \xrightarrow{\sim} \mathrm{Hom}_{Mot_{hom}(k)}(h^i(X), h^i(Y))$$

Proof. For the claim on the Chow-Künneth decomposition of Z, note that

$$\begin{aligned}\Delta_Z &= \Delta_X \times \Delta_Y \\ &= (\sum_{i=0}^{2d} p_i(X)) \times (\sum_{j=0}^{2e} p_j(Y)) \\ &= \sum_{m=0}^{2(d+e)} p_m(Z)\end{aligned}$$

It's easy to check that $p_{rs}(Z) := p_r(X) \times p_s(Y)$ are themselves projectors and mutually orthogonal. By Künneth decomposition, the relevant sums lift the Künneth projectors.

To simplify the proof of the next two statements, we assume in addition that the Chow-Künneth decomposition is *self-dual*[30] and for this we need the notion of the *transpose* of a correspondence:

Definition 7.12 (Transpose of a correspondence from X to Y)**.** Given a correspondence C from X to Y, the transpose ${}^TC$ is the same cycle C, but considered as a subvariety of $Y \times X$.

With Definition 7.12, we assume in addition that $p_{2d-i}(X) = {}^Tp_i(X)$. Now by definition

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}(k)}(\mathrm{ch}^i(X), \mathrm{ch}^j(Y)) = \left\{p_j(Y) \circ R \circ p_i(X) |\, R \in \mathrm{CH}^d(X \times Y)_{\mathbb{Q}}\right\}.$$

The action of $p_m(Z)$ on such a correspondence is given by

$$p_m(Z)_*(p_j(Y) \circ R \circ p_i(X)) = \sum_{r+s=m} (p_r(X) \times p_s(Y))_*(p_j(Y) \circ R \circ p_i(X)) \tag{7.4}$$

To simplify (7.4), we need Lieberman's Lemma (we stated a weakened version in Lemma 3.20 of Lecture 3, but didn't have enough time to prove it):

Exercise 7.13 (Lieberman's Lemma)**.** *Let $f \in Corr(X, Y)$, $\alpha \in Corr(X, X')$, $\beta \in Corr(Y, Y')$, then $(\alpha \times \beta)_*(f) = \beta \circ f \circ^T \alpha$.*

we get

$$\begin{aligned}\sum_{r+s=m} (p_r(X) \times p_s(Y))_*(p_j(Y) \circ R \circ p_i(X)) &= \sum_{r+s=m} p_s(Y) \circ p_j(Y) \circ R \circ p_i(X) \circ {}^Tp_r(X) \\ &= p_j(Y) \circ R \circ p_i(X)\end{aligned}$$

because the only terms that survive are with $s = j$ and by the self-dual assumption $r = 2d - i$. Moreover $m = r + s = 2d - i + j$. Therefore if Conjecture 7.6 holds, then the above action is 0 for $m > 2d$ or equivalently $i < j$.

If Conjecture 7.8 holds, then ker $p_{2d}(Z) = \mathrm{CH}^d_{\mathrm{hom}}(X \times Y)_{\mathbb{Q}}$. Now suppose $p_i(Y) \circ R \circ p_i(X)$ is homologically equivalent to 0. Then

$$0 = p_{2d}(Z)_*(p_i(Y) \circ R \circ p_i(X)).$$

But again by Lieberman's lemma $p_{2d}(Z)_*(p_i(Y) \circ R \circ p_i(X)) = p_i(Y) \circ R \circ p_i(X)$.

□

[30] It is conjecturally expected that there exists a Chow-Künneth decomposition which is self-dual

7.4 Some evidence for the conjectures in the case of a 3-fold

In this section we sketch the following result due to Murre in [Mur93].

Proposition 7.14. *Suppose $X = S \times C$ where S is a surface and C is a curve. Assume that S and C both admit k-rational points. Then X satisfies the four conjectures of Murre.*

Proof. We first show that X has a Chow-Künneth decomposition (i.e. Conjecture 7.4 is satisfied). By the first part of Proposition 7.11, it suffices to show that both C and S have a Chow-Künneth decomposition. The fact that C has a Chow-Künneth is trivial:

Exercise 7.15. *Consider the Chow-Künneth decomposition of C given by the decomposition $p_0(C) = \{e\} \times C$, $p_2(C) = C \times \{e\}$ for e a rational point of C and $p_1(C) = \Delta_C - p_0(C) - p_2(C)$. Show that $p_1(C)$ acts as identity on $CH^1_{hom}(C)_{\mathbb{Q}}$.*

For the surface S, the result is due to Murre [Mur90]. We won't go into his proof but let us mention that the Chow-Künneth decomposition he gives for S is self-dual in the sense of the proof of Proposition 7.11. We also restrict ourselves in studying $CH^1(X)_{\mathbb{Q}}$. Thus it remains to check Conjectures 7.6 and 7.8 are satisfied for the seven projectors $p_0(X), p_1(X), \ldots, p_6(X)$. Conjecture 7.9, then follows as the induced filtration on $\mathrm{CH}^1(X)_{\mathbb{Q}}$ coincides with $\mathrm{CH}^1(X)_{\mathbb{Q}} \supset \mathrm{CH}^1_{\mathrm{hom}}(X)_{\mathbb{Q}} \supset \{0\}$.

Lemma 7.16. *Conjectures 7.6 and 7.8 hold for $CH^1(S \times C)_{\mathbb{Q}}$*

Proof. First we claim that if $D \in \mathrm{CH}^1_{\mathrm{hom}}(S \times C)_{\mathbb{Q}}$ then $D = D_1 \times C + S \times D_2$ with $D_1 \in \mathrm{CH}^1_{\mathrm{hom}}(S)_{\mathbb{Q}}$ and $D_2 \in \mathrm{CH}^1_{\mathrm{hom}}(C)_{\mathbb{Q}}$. By Matsusaka's Theorem, we can replace D by an integral multiple if necessary and assume $D \in \mathrm{CH}^1_{\mathrm{alg}}(S \times C) \subset \mathrm{CH}^1_{\mathrm{hom}}(S \times C)$ where

$$\mathrm{CH}^i_{\mathrm{alg}}(S \times C) := \{\alpha \in \mathrm{CH}^i(S \times C) \mid \alpha \sim_{\mathrm{alg}} 0\}.$$

The claim then follows from the following exercise:

Exercise 7.17. *Let $X, Y \in SmProj(k)$ and D a divisor on $X \times Y$ such that $D \sim_{alg} 0$. Then for some integer $m \neq 0$, we have $mD = D_1 \times Y + X \times D_2$ with D_1 (resp. D_2) a divisor on X (resp. Y).*

Now consider the projector

$$p_1(X) = p_1(S) \times p_0(C) + p_0(S) \times p_1(C)$$

Murre in [Mur90] also shows that $p_1(S)$ acts as identity on $\mathrm{CH}^1_{\mathrm{hom}}(S)_{\mathbb{Q}}$. By the previous claim together with Exercise 7.15, it follows that $p_1(X)$ acts as identity on $\mathrm{CH}^1_{\mathrm{hom}}(X)_{\mathbb{Q}}$.

Now let $i \neq 1$ and $D \in \mathrm{CH}^1_{\mathrm{hom}}(X)_{\mathbb{Q}}$. Then

$$0 = (p_i(X) \circ p_1(X))_*(D) = p_i(X)_*(D). \tag{7.5}$$

Hence for $i \neq 1$ the projector $p_i(X)$ acts as zero on $\mathrm{CH}^1_{\mathrm{hom}}(X)_{\mathbb{Q}}$. In particular $\ker p_2(X) = \mathrm{CH}^1_{\mathrm{hom}}(X)_{\mathbb{Q}}$ by Lemma 7.7(2). Thus Conjecture 7.8 holds. To prove Conjecture 7.6, we have to show that for $i \neq 1, 2$, $p_i(X)$ acts as 0 on $\mathrm{CH}^1(X)_{\mathbb{Q}}$. Take $D \in \mathrm{CH}^1(X)_{\mathbb{Q}}$. Then

$$0 = (p_2(X) \circ p_i(X))_*(D) = p_2(X)_*(D_i)$$

where $D_i = p_i(X)(D)$. Thus $D_i \in \mathrm{CH}^1_{\mathrm{hom}}(X)_{\mathbb{Q}}$. Therefore

$$p_i(X)_*(D) = (p_i(X) \circ p_i(X))_*(D) = p_i(X)_*(D_i) = 0.$$

where the last equality follows from (7.5). Thus Conjecture 7.6 follows. □

□

Exercise 7.18. *Conjectures 7.6 and 7.8 hold for $CH^3(S \times C)_{\mathbb{Q}}$*

8 Lecture 8: Voevodsky's derived category of mixed motives

So far we have attached motives to smooth projective varieties over a field. We now want to attach motives to arbitrary varieties giving a theory of *mixed* motives. No such theory exists, but there are various constructions of *the* derived category of mixed motives[31]. We shall study Voevodsky's construction in [Voe00]. We shall follow closely the lecture notes of Mazza-Voevodsky-Weibel[32] [MVW06]. Crucially the construction no longer relies on the *moving lemma*, where smoothness and projectivity assumptions are important.

Remark 8.1. *Deligne was motivated by the yoga: pure motives $\longrightarrow$ mixed motives , when he constructed a theory of mixed Hodge structures for non-smooth and non-proper varieties in [Del71] and [Del74b].*

8.1 Finite Correspondences

To solve the problem of partially defined intersections, Voevodsky introduced the notion of finite correspondences. Let Sm_k be the category of smooth separated schemes over k.

Definition 8.2 ($\mathrm{Corr}_{\mathrm{fin}}(X,Y)$)**.** Let $X, Y \in \mathrm{Sm}_k$. The group $\mathrm{Corr}_{\mathrm{fin}}(X,Y)$ of finite correspondences from X to Y is the abelian subgroup of $Z(X \times_k Y)$ generated by integral[33] closed subschemes $W \subset X \times_k Y$ such that

(1) the projection $p_1 \colon W \to X$ is finite

(2) the image $p_1(W) \subset X$ is an irreducible component of X.

Example 8.3. *For $X, Y \in Sm_k$, the graph Γ_f of a morphism $f \colon X \to Y$ is a finite correspondence from $X \to Y$. Indeed $\Gamma_f \to X$ is an isomorphism and Γ_f is closed due to separated assumption.*

Exercise 8.4 (Construction of composition of finite correspondences)**.** *Let $X, Y, Z \in Sm_k$. In this exercise you will construct a composition law:*

$$\circ \colon Corr_{fin}(Y,Z) \times Corr_{fin}(X,Y) \to Corr_{fin}(X,Z)$$

(1) Given closed subsets $V \subset X \times Y$ and $W \subset Y \times Z$ which are finite and surjective over X and Y, respectively, show that $V \times Z$ and $X \times W$ intersect properly. This defines a cycle $[T]$ in $X \times Y \times Z$.

(2) Show that[34] $p_*([T])$ where $p \colon X \times Y \times Z \to X \times Z$ is finite and surjective over X.

This allows us to define $W \circ V := p_*[(V \times Z) \cdot (X \times W)]$.

Definition 8.5 (Category of finite correspondences $\mathrm{Corr}_{\mathrm{fin}}(k)$)**.** The category $\mathrm{Corr}_{\mathrm{fin}}(k)$ is the category with the same objects as Sm_k and morphisms

$$\mathrm{Hom}_{\mathrm{Corr}_{\mathrm{fin}}(k)}(X,Y) := \mathrm{Corr}_{\mathrm{fin}}(X,Y)$$

with composition given by Exercise 8.4.

[31] With the hope that there is a suitable t-structure whose heart would give the sought-after category.

[32] The category of effective geometric motives is constructed in a different way to [Voe00].

[33] Recall a scheme is integral iff it is reduced and irreducible.

[34] Strictly speaking the pushforward in Definition 2.4 is defined only for the proper case. However we can define it as the action of the graph of p viewed as a correspondence. In any case, what is important is that the compoents of $p_*([T])$ are just the supports of $p(T_i)$ for every irreducible component $T_i \subset T$.

Exercise 8.6. *Show that* $\Gamma_f \circ \Gamma_g = \Gamma_{f \circ f}$ *and hence the functor*

$$\begin{aligned} Sm_k &\to Corr_{fin}(k) \\ X &\mapsto X \\ (f\colon X \to Y) &\mapsto \Gamma_f. \end{aligned}$$

is faithful.

Remark 8.7 (Category of finite correspondences over a general base)**.** *In [MVW06, Appendix 1A], for S a Noetherian scheme, a category $Corr_{fin}(S)$ is constructed. It's objects are schemes of finite type over S.*

8.2 The category of effective geometric motives

The category $\mathrm{Corr}_{\mathrm{fin}}(k)$ has hom-sets abelian groups and composition of morphisms is bilinear. Thus it is a preadditive category. Furthermore if $X = \coprod_i X_i$, then $\mathrm{Corr}_{\mathrm{fin}}(X, Y) = \oplus_i \mathrm{Corr}_{\mathrm{fin}}(X_i, Y)$. Thus it is an additive category (with finite disjoint union as a finitary (co)product).

Definition 8.8 ($\mathrm{Corr}_{\mathrm{fin}}(k)$ as a symmetric monoidal category)**.** If X and Y are two objects in $\mathrm{Corr}_{\mathrm{fin}}(k)$, we define the tensor product

$$X \otimes Y = X \times_k Y$$

It follows that the bounded homotopy category $K^b(\mathrm{Corr}_{\mathrm{fin}}(k))$ is a triangulated tensor category. One defines triangles to be triangles that are isomorphic (in $K^b(\mathrm{Corr}_{\mathrm{fin}}(k))$) to the cone sequence

$$A \xrightarrow{f} B \to C(f) \to A[1]$$

Definition 8.9 (Construction of the triangulated tensor category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$)**.**

(1) Localize $K^b(\mathrm{Corr}_{\mathrm{fin}}(k))$ with respect to the thick subcategory generated by complexes of the form

  (a) **Homotopy:** $X \times \mathbb{A}^1 \to X$.

  (b) **Mayer-Vietoris** $U \cap V \to U \oplus V \to X$, where U and V are Zariski open subsets of X such $X = U \cup V$. In other words invert the induced[35] map $C(U \cap V \to U \oplus V) \to X$.

(2) Take the pseudo-abelian completion of the resulting quotient category[36].

The resulting category is the category of *effective geometric motives* $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

Definition 8.10 (Motive of a smooth scheme)**.** We get a symmetric monoidal covariant[37] functor

$$M\colon \mathrm{Sm}_k \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k).$$

Example 8.11 (Immediate consequences from Definition 8.9)**.**

(1) ***Homotopy-invariance:*** $M(X \times \mathbb{A}^1) \cong M(X)$

(2) ***Mayer-Vietoris triangle:***

$$M(U \cap V) \to M(U) \oplus M(V) \to M(X) \to M(U \cap V)[1].$$

[35]Such an induced map comes from the universal property of the mapping cone: Given a moprhism of chain of complexes $X \to Y \to Z$, if the composition is null-homotopic, then we get a map $C(X \to Y) \to Z$.

[36]Voevodsky only does this step, in order to make comparisons with the category of pure motives.

[37]As opposed to the associated functor for pure motives which is contravariant.

8.3 The category of geometric motives

Definition 8.12 (reduced motive $\widetilde{X}$)**.** For $X \in \mathrm{Sm}_k$ and a rational point $e \in X(k)$ we can consider the triangle

$$M(\operatorname{Spec} k) \xrightarrow{e} M(X) \to \widetilde{X} \to \cdot$$

Similarly the structure morphism $p\colon X \to \operatorname{Spec} k$, gives rise to a triangle

$$\widetilde{Y} \to M(X) \xrightarrow{p} M(\operatorname{Spec} k) \to \cdot$$

Exercise 8.13. *Since $p \circ e = \mathrm{id}_k$, show by using the axioms of triangulated categories that $\widetilde{X} \cong \widetilde{Y}$ and $M(X) \cong M(\operatorname{Spec} k) \oplus \widetilde{X}$ (i.e. the triangle splits).*

Definition 8.14 (Lefschetz motive)**.** We set $\mathbb{Z}(1) := \widetilde{\mathbb{P}^1}[-2]$ and $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$ for $n \geq 0$.

Example 8.15. *$M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$ and $\mathbb{Z} = M(\operatorname{Spec} k)$.*

Definition 8.16 (Construction of $\mathrm{DM}_{\mathrm{gm}}(k)$)**.** We invert the Lefschetz motive. More formally: the objects are $M(r)$ for $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ and $r \in \mathbb{Z}$ and morphisms

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(r), N(s)) := \varinjlim_{n} \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M \otimes \mathbb{Z}(n+r), N \otimes \mathbb{Z}(n+s)).$$

Remark 8.17. *To compare $DM^{eff}_{gm}(k)$ with $DM_{gm}(k)$ and to the usual category of (Chow) motives, we need to view $DM^{eff}_{gm}(k)$ as a subcategory of motivic complexes $DM^{eff}_{-}(k)$.*

8.4 Nisnevich sheaves with transfers

As mentioned in Remark 8.17, the geometric definition of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ (i.e. Definition 8.9) is not powerful enough (due to a lack of site). This section remedies this and provides a site-theoretic framework for mixed motives.

Definition 8.18 (Presheaf with transfers)**.** A presheaf with transfers is a functor $F\colon \mathrm{Corr}_{\mathrm{fin}}(k)^{\mathrm{op}} \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups. Denote this category by $\mathrm{PSh}(\mathrm{Corr}_{\mathrm{fin}}(k))$.

Such a presheaf is called *homotopy invariant* if the natural map

$$F(p_X)\colon F(X) \to F(X \times \mathbb{A}^1)$$

is an isomorphism for all $X \in \mathrm{Sm}_k$ where $p_X\colon X \times \mathbb{A}^1 \to X$ is the projection morphism.

Definition 8.19 ($\mathbb{Z}_{\mathrm{tr}}(X)$)**.** For $X \in \mathrm{Sm}_k$, let $\mathbb{Z}_{\mathrm{tr}}(X)$ denote the presheaf with transfers represented by X, so that $\mathbb{Z}_{\mathrm{tr}}(X)(U) := \mathrm{Corr}_{\mathrm{fin}}(U, X)$. For a morphism $V \to U$, the map is defined $\mathbb{Z}_{\mathrm{tr}}(X)(U) \to \mathbb{Z}_{\mathrm{tr}}(X)(V)$ to be induced by composition of finite correspondences.

Remark 8.20. *A presheaf with transfers attaches an abelian group $F(X)$ for every $X \in Sm_k$ and for every finite correspondence $Z \in Corr_{fin}(X, Y)$ a transfer map*

$$Tr(Z)\colon F(Y) \to F(X).$$

We need to work with a site that is finer than the Zariski topology but coarser than the étale topology. This is due to Nisnevich [Nis89].

Definition 8.21 (Nisnevich covering)**.** A family of étale morphisms $\{p_i\colon U_i \to X\}_{i \in I}$ is said to be a Nisnevich covering of X if it has the Nisnevich lifting property: for all $x \in X$, there is an $i \in I$ and a $u \in U_i$ so that $p_i(u) = x$ and the induced map on residue fields $k(x) \to k(u)$ is an isomorphism.

Exercise 8.22. *Show that Nisnevich coverings give a Grothendieck topology on Sm_k.*

Example 8.23. *If the characteristic of k is zero, then the two morphisms $j\colon \mathbb{A}^1 - \{a\} \hookrightarrow \mathbb{A}^1$ and $i\colon \mathbb{A}^1 - \{0\} \xrightarrow{z\mapsto z^2} \mathbb{A}^1$ is a Nisnevich covering of $\mathbb{A}^1$ iff $a \in (k^*)^2$. On the other hand it is an étale map for any non-zero a.*

Recall the local rings in the étale topology are strict henselian rings, while for the Nisnevich topology they are just henselian rings.

Definition 8.24 (The category of Nisnevich sheaves with transfers: $\mathrm{Nis}_{\mathrm{tr}}(k)$)**.** A presheaf with transfers is called a Nisnevich sheaf with transfers if it's restriction to $\mathrm{Sm}_k^{\mathrm{op}}$ is a sheaf for the Nisnevich topology.

We'll postpone the proof of the next proposition to the next lecture.

Proposition 8.25. *The presheaf with transfer $\mathbb{Z}_{tr}(X)$ is a Nisnevich sheaf with transfer.*

Theorem 8.26 ($\mathrm{Nis}_{\mathrm{tr}}(k)$ is a Grothendieck topos)**.** *The embedding*

$$Nis_{tr}(k) \hookrightarrow PSh(Corr_{fin}(k))$$

has a left-adjoint which is left-exact.

Proof.

Lemma 8.27. *Let $p\colon U \to Y$ be a Nisnevich covering and $f\colon X \to Y$ a finite correspondence. Then there is a Nisnevich covering $p'\colon V \to X$ and a finite correspondence $f'\colon V \to U$ such that the following diagram commutes in $Corr_{fin}(k)$:*

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "p'"'] & U \arrow[d, "p"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{8.1}$$

Proof. We can assume that f is defined by a closed irreducible $Z \subset X \times Y$ which surjects onto X. Let $Z_U := Z \times_Y U \subset X \times U$. Since the projection $Z_U \to Z$ is a Nisnevich cover and $Z \to X$ is finite, we claim that $Z_U \to Z$ splits Nisnevich locally[38] on X. Indeed we can assume[39] $X = \mathrm{Spec}(R)$ where R is henselian. Then by [Sta18, Tag 04GH], Z is a disjoint union of henselian schemes and so the required map has a section. But then $s(V \times_X Z) \subset V \times U$ is finite over V (as $Z \to X$ is finite) and $V \times_X Z \to V$ is a surjection (again since this is true for $Z \to X$). □

Lemma 8.28. *Let $\mathcal{F}$ be a presheaf with transfers. Denote by $\mathcal{F}_{Nis}$ the sheafification of $\mathcal{F}|_{Sm_k}$ for the Nisnevich topology. Then there exists a unique Nisnevich sheaf with transfers $\mathcal{G}$ such that $\mathcal{G}|_{Sm_k} = \mathcal{F}_{Nis}$ equipped with a morphism of presheaves with transfers $\mathcal{F} \to \mathcal{G}$.*

Proof. We first show uniqueness. Suppose there are two Nisnevich sheaves with transfers $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $\mathcal{G}_1|_{\mathrm{Sm}_k} = \mathcal{G}_2|_{\mathrm{Sm}_k} = \mathcal{F}_{\mathrm{Nis}}$. Given a morphism $f\colon X \to Y \in \mathrm{Corr}_{\mathrm{fin}}(k)$ we need to check that $\mathcal{G}_1(f) = \mathcal{G}_2(f)$ as morphisms $\mathcal{G}_1(Y) \to \mathcal{G}_1(X)$. Let $y \in \mathcal{G}_1(Y) = \mathcal{G}_2(Y)$. Choose a Nisnevich cover $p\colon U \to Y$ such that $y|_U \in \mathcal{G}_1(U)$ is the image of some $u \in \mathcal{F}(U)$. Then applying Lemma 8.27, we get diagram (8.1). We now compute

$$\begin{aligned} \mathcal{G}_1(p')\mathcal{G}_1(f)(y) &= \mathcal{G}_1(f')\mathcal{G}_1(p)(y) \\ &= \mathcal{G}_1(f')(y|_U) \\ &= \mathcal{G}_2(f')(y|_U) \\ &= \mathcal{G}_2(p')\mathcal{G}_2(f)(y) \\ &= \mathcal{G}_1(p')\mathcal{G}_2(f)(y) \end{aligned}$$

[38]That is there is an étale cover $V \to X$ so that $V \times_X Z_U \to V \times_X Z$ has a section s.

[39]by taking the limit across all Nisnevich neighborhoods of some point $x \in X$.

This implies $\mathcal{G}_1(f) = \mathcal{G}_2(f)$ as p' is a covering. This shows uniqueness.

We now prove existence. We need to define a morphism $\mathcal{F}_{\mathrm{Nis}}(Y) \to \mathcal{F}_{\mathrm{Nis}}(X)$ for each $f \colon X \to Y \in \mathrm{Corr}_{\mathrm{fin}}(X, Y)$ □

Lemma 8.29. *Let $X \in Sm_k$ and $U \to X$ a Nisnevich covering of X. The Čech complex*

$$\dots \mathbb{Z}_{tr}(U \times_X U) \to \mathbb{Z}_{tr}(U) \to \mathbb{Z}_{tr}(X) \to 0 \tag{8.2}$$

is exact as a complex of Nisnevich sheaves.

Proof. By [Nis89, Corollary 1.17], the Nisnevich topology on Sm_k has enough points given by henselian localizations. This means that to prove exactness of (8.2), it suffices to prove exactness of

$$\dots \varinjlim_i \mathbb{Z}_{\mathrm{tr}}(U \times_X U)(S_i) \to \varinjlim_i \mathbb{Z}_{\mathrm{tr}}(U)(S_i) \to \varinjlim_i \mathbb{Z}_{\mathrm{tr}}(X)(S_i) \to 0 \tag{8.3}$$

where each $S_i \in \mathrm{Sm}_k$ is affine and $S = \varprojlim_i S_i$ is a henselian scheme. For a closed subscheme $Z \subset X \times S$, which is finite over S, denote by $L(Z/S)$ the free abelian group generated by irreducible components of Z which are finite and surjective over S. Then (8.3) is the colimit of complexes of the form

$$\dots L(Z_U \times_Z Z_U/S) \to L(Z_U/S) \to L(Z/S) \to 0 \tag{8.4}$$

where $Z_U = Z \times_X U$ and the colimit is taken over all Z closed subschemes of $X \times S$ which are finite and surjective over S. Thus it suffices to show exactness of (8.4). Since S is henselian and $Z \to S$ is finite, Z is a finite product of henselian rings and so $Z_U \to Z$ splits. Let $s_1 \colon Z \to Z_U$ be a splitting. Put $(Z_U)^k_Z = Z_U \times_Z \dots \times_Z Z_U$ and set $s_k \colon (Z_U)^k_Z \to (Z_U)^{k+1}_Z$ to be $s_1 \times_Z \mathrm{id}_{(Z_U)^k_Z}$. The s_k induce homomorphisms of abelian groups

$$\sigma_k \colon L((Z_U)^k_Z/S) \to L((Z_U)^{k+1}_Z/S).$$

Exercise 8.30. *Show that the σ_k provide a chain homotopy from the identity morphism of* (8.4) *to the zero morphism of* (8.4).

□

We now produce a morphism

$$\mathcal{F}_{\mathrm{Nis}}(Y) \to \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Sm}_k)}(\mathbb{Z}_{\mathrm{tr}}(Y), \mathcal{F}_{\mathrm{Nis}})$$

where the Hom is in the category of Nisnevich sheaves on Sm_k.

For $y \in \mathcal{F}_{\mathrm{Nis}}(Y)$, choose a Nisnevich cover $p \colon U \to Y$ such that $y|_U \in \mathcal{F}_{\mathrm{Nis}}(U)$ is the image of some $u \in \mathcal{F}(U)$. By Yoneda's lemma ($\mathcal{F}(U) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Corr}_{\mathrm{fin}}(k))}(\mathbb{Z}_{\mathrm{tr}}(U), \mathcal{F})$) and so u determines a morphism $\mathbb{Z}_{\mathrm{tr}}(U) \to \mathcal{F}$ of presheaves with transfer. By shrinking U if necessary, we can assume that the difference map $d \colon \mathcal{F}(U) \to \mathcal{F}(U \times_Y U)$ sends u to 0. Consider now the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Sm}_k)}(\mathbb{Z}_{\mathrm{tr}}(Y), \mathcal{F}_{\mathrm{Nis}}) \arrow[r] & \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Sm}_k)}(\mathbb{Z}_{\mathrm{tr}}(U), \mathcal{F}_{\mathrm{Nis}}) \arrow[r] & \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Sm}_k)}(\mathbb{Z}_{\mathrm{tr}}(U^2_Y), \mathcal{F}_{\mathrm{Nis}}) \\
 & & \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Corr}_{\mathrm{fin}}(k))}(\mathbb{Z}_{\mathrm{tr}}(U), \mathcal{F}) \arrow[r] \arrow[u] & \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Corr}_{\mathrm{fin}}(k))}(\mathbb{Z}_{\mathrm{tr}}(U^2_Y), \mathcal{F}) \arrow[u]
\end{tikzcd}$$

The top row is exact by Lemma 8.29 and so we get $[y] \in \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Sm}_k)}(\mathbb{Z}_{\mathrm{tr}}(Y), \mathcal{F}_{\mathrm{Nis}})$

Exercise 8.31. *Check that $[y]$ is independent of the choice of U and u.*

Finally to define a morphism $\mathcal{F}_{\mathrm{Nis}}(Y) \to \mathcal{F}_{\mathrm{Nis}}(X)$ for each finite correspondence f, it suffices to define a pairing

$$\mathrm{Corr}_{\mathrm{fin}}(X,Y) \times \mathcal{F}_{\mathrm{Nis}}(Y) \to \mathcal{F}_{\mathrm{Nis}}(X).$$

Indeed given $f \in \mathrm{Corr}_{\mathrm{fin}}(X,Y) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Corr}_{\mathrm{fin}}(k))}(\mathbb{Z}_{\mathrm{tr}}(X), \mathbb{Z}_{\mathrm{tr}}(Y))$ and $y \in \mathcal{F}_{\mathrm{Nis}}(Y)$, the previous paragraph gives $[y]\colon \mathbb{Z}_{\mathrm{tr}}(Y) \to \mathcal{F}_{\mathrm{Nis}}$. Thus we get the composition

$$\mathbb{Z}_{\mathrm{tr}}(X)(X) \to \mathbb{Z}_{\mathrm{tr}}(Y)(X) \to \mathcal{F}_{\mathrm{Nis}}(X)$$

Exercise 8.32. *Show that the image of the identity map gives the required pairing (i.e. the constructed morphisms are natural).*

Exactness follows because sheafification is an exact procedure. □

Definition 8.33 (The category of effective motivic complexes: $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$)**.** By Theorem 8.26, we can consider the derived category of bounded above complexes of Nisnevich sheaves with transfers: $D^{-}(\mathrm{Nis}_{\mathrm{tr}}(k))$. The category $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ is the full subcategory of $D^{-}(\mathrm{Nis}_{\mathrm{tr}}(k))$ whose cohomology sheaves are homotopy invariant.

9 Lecture 9: The localization theorem of $\mathbf{DM}^{\mathbf{eff}}_{-}(k)$

The following is the postponed proof of Proposition 8.25 from the previous lecture.

Proposition 9.1. *The presheaf with transfer $\mathbb{Z}_{tr}(X)$ is a Nisnevich sheaf with transfer.*

Proof. We need to check the following two properties

(1) For every surjective Nisnevich morphism of smooth separated schemes $U \to Y$, the sequence

$$0 \to \mathbb{Z}_{\mathrm{tr}}(X)(Y) \to \mathbb{Z}_{\mathrm{tr}}(X)(U) \to \mathbb{Z}_{\mathrm{tr}}(X)(U \times_Y U)$$

is exact

(2) $\mathbb{Z}_{\mathrm{tr}}(X)(U \amalg V) = \mathbb{Z}_{\mathrm{tr}}(X)(U) \oplus \mathbb{Z}_{\mathrm{tr}}(X)(V)$

For (2) note that $\mathbb{Z}_{\mathrm{tr}}(X)(U \amalg V) = \mathrm{Corr}_{\mathrm{fin}}(U \amalg V, X) = \mathrm{Corr}_{\mathrm{fin}}(V,X) \oplus \mathrm{Corr}_{\mathrm{fin}}(V,X)$.

For (1) we can assume Y is connected and therefore irreducible (as it is smooth). But then a finite correspondence in $\mathrm{Corr}_{\mathrm{fin}}(Y,X)$ is dominant and determined by the fiber at the generic point[40] of Y. Thus we get exactness at $\mathbb{Z}_{\mathrm{tr}}(X)(Y)$, as the map $\mathbb{Z}_{\mathrm{tr}}(X)(Y) \to \mathbb{Z}_{\mathrm{tr}}(X)(U)$ is just pullback of cycles. It remains to show exactness at $\mathbb{Z}_{\mathrm{tr}}(X)(U)$. Take $Z_U \in \mathrm{Corr}_{\mathrm{fin}}(U,X)$ whose images in $\mathrm{Corr}_{\mathrm{fin}}(U \times_Y U, X)$ coincide along the two projections.

Exercise 9.2. *Show that there is a Zariski open $V \subset Y$ and $Z_V \in \mathit{Corr}_{\mathit{fin}}(V,X)$ agreeing with Z_U in $\mathit{Corr}_{\mathit{fin}}(U \times_Y V, X)$.*

Exercise 9.3. *Given an étale surjection $U \to Y$ with Y irreducible, show that there exists an irreducible components $U_1 \subset U$ such that $U_1 \to Y$ is still a surjection.*

By Exercise 9.3 we can assume U is irreducible, and we can write $Z_V = \sum n_i Z_i$ and $Z_U = \sum n_i Z_i'$ so that Z_i and Z_i' agree in $\mathrm{Corr}_{\mathrm{fin}}(U \times_Y V, X)$ (since U and V are irreducible, the same trick as in the first paragraph works to see that one gets the claimed presentation for Z_V and Z_U). Thus we can assume $Z_V = Z_i$ and $Z_U = Z_i'$.

Let Z be the closure of Z_V in $Y \times X$. Since $Z \times_Y V = Z_V$ is irreducible and dominant over V, it means Z is irreducible and dominant over Y.

Exercise 9.4. *Prove that Z maps to Z_U along the pullback of $U \times X \to Y \times X$.*

[40]Think of what is happening along $U \times X \to Y \times X \to Y$.

This means that the components of $Z \times_Y U$ are finite over U. By faithfully flat descent, this means Z is finite over Y. $\square$

We want to construct a functor $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \to \mathrm{DM}^{\mathrm{eff}}_{-}(k)$. For this we need to view $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ as a localization of $D^{-}(\mathrm{Nis}_{\mathrm{tr}}(k))$, just like $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ was defined as a certain localization.

Definition 9.5. Define the cosimplicial scheme $\Delta^{\bullet}$ over k which is defined by

$$\Delta^n = \operatorname{Spec} k[x_0, \dots, x_n]/(\sum_{i=0}^{n} x_i = 1)$$

The ith face map $\delta_i \colon \Delta^n \to \Delta^{n+1}$ is given by $x_i = 0$.

Definition 9.6 (Suslin complex)**.** Let $\mathcal{F}$ be a presheaf with transfers. Define the presheaf with transfers $C_n(\mathcal{F})$ by

$$C_n(\mathcal{F})(X) := \mathcal{F}(X \times \Delta^n)$$

The *Suslin complex* $C_*(\mathcal{F})$ is the complex with differential

$$d_n := \sum_i (-1)^i \delta_i^* \colon C_n(\mathcal{F}) \to C_{n-1}(\mathcal{F}).$$

Proposition 9.7. *The Suslin complex in Definition 9.6 defines a functor*

$$C_* \colon \mathit{Nis}_{tr}(k) \to \mathit{DM}^{\mathit{eff}}_{-}(k)$$

Proof. A priori $C_* \colon \mathrm{Nis}_{\mathrm{tr}}(k) \to D^{-}(\mathrm{Nis}_{\mathrm{tr}}(k))$. We first show that the cohomology *presheaves* of $C_*(\mathcal{F})$ are homotopy-invariant.

Lemma 9.8. *For a rational point $\alpha \in \mathbb{A}^1(k)$, let $i_\alpha \colon X \hookrightarrow X \times \mathbb{A}^1$ be the inclusion $x \mapsto (x, \alpha)$. Then $\mathcal{F}$ is homotopy invariant iff*

$$\mathcal{F}(i_0) = \mathcal{F}(i_1) \colon \mathcal{F}(X \times \mathbb{A}^1) \to \mathcal{F}(X)$$

for all $X \in Sm_k$.

Proof. Denote by $p_X \colon X \times \mathbb{A}^1 \to X$, the projection. Then $p_X \circ i_\alpha = \mathrm{id}_X$ and so if $\mathcal{F}$ is homotopy invariant (i.e. $\mathcal{F}(p_X)$ is an isomorphism), then $\mathcal{F}(i_0) = \mathcal{F}(i_1)$. For the converse suppose $\mathcal{F}(i_0) = \mathcal{F}(i_1)$. Applying $\mathcal{F}$ to the multiplication map $m \colon \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ gives the diagram

$$\begin{tikzcd}
 & \mathcal{F}(X \times \mathbb{A}^1) \arrow[dl, "\mathcal{F}(\mathrm{id}_{X \times \mathbb{A}^1})"'] \arrow[r, "\mathcal{F}(i_0)"] \arrow[d, "\mathcal{F}(\mathrm{id}_X \times m)"] & \mathcal{F}(X) \arrow[d, "\mathcal{F}(p_X)"] \\
\mathcal{F}(X \times \mathbb{A}^1) & \mathcal{F}(X \times \mathbb{A}^1 \times \mathbb{A}^1) \arrow[l, "\mathcal{F}(i_1 \times \mathrm{id}_{\mathbb{A}^1})"] \arrow[r, "\mathcal{F}(i_0 \times \mathrm{id}_{\mathbb{A}^1})"'] & \mathcal{F}(X \times \mathbb{A}^1).
\end{tikzcd}$$

Thus $\mathcal{F}(p_X) \circ \mathcal{F}(i_0) = \mathcal{F}(\mathrm{id}_{X \times \mathbb{A}^1})$. Thus it follows that $\mathcal{F}(p_X)$ is onto. But we know it is injective as it has a section. Thus it is an isomorphism and $\mathcal{F}$ is homotopy invariant. $\square$

Lemma 9.9. *Let $\mathcal{F}$ be a presheaf with transfers. The induced chain maps $i_0^*, i_1^* \colon C_*(\mathcal{F})(X \times \mathbb{A}^1) \to C_*(\mathcal{F})(X)$ are chain homotopic.*

Proof. For each $i = 0, \dots, n$ define $\theta_i \colon \Delta^{n+1} \to \Delta^n \times \mathbb{A}^1$ to be the map that sends the vertex v_j defined by $x_j = 1$ and $x_{j'} = 0$ for $j' \neq j$ to $v_j \times \{0\}$ for $j \leq i$ and to $v_{j-1} \times \{1\}$, otherwise. These θ_i induces maps

$$h_i \colon \mathcal{F}(\mathrm{id}_X \times \theta_i) \colon C_n(\mathcal{F})(X \times \mathbb{A}^1) \to C_{n+1}(\mathcal{F})(X).$$

Exercise 9.10. *Check that* $i_1^* = \delta_0 \circ h_0$ *and* $i_0^* = \delta_{n+1} \circ h_n$ *and that the alternating sum* $\sum(-1)^i h_i$ *is a chain homotopy from* i_1^* *to* i_0^*.

□

By Lemmas 9.8 and 9.9, it follows that the cohomology presheaves of $C_*(\mathcal{F})$ are homotopy-invariant. The Proposition now follows from Theorem 9.11, which we will not prove. □

Theorem 9.11. *Let* $\mathcal{F}$ *be a homotopy invariant presheaf with transfers. Then the Zariski sheaf*[41] *given by the Zariski sheafification on* Sm_k *is homotopy invariant. Moreover* $\mathcal{F}_{Zar} = \mathcal{F}_{Nis}|_{Sm_k}$ *where* $\mathcal{F}_{Nis}$ *is given by Theorem 8.26.*

Before we extend the functor C_* to the full triangulated category $D^-(\mathrm{Nis}_{\mathrm{tr}}(k))$, we need to put a tensor structure on it.

9.1 Tensor structure on $D^-(\mathbf{Nis}_{\mathbf{tr}}(k))$

First we define a tensor product on $\mathrm{Nis}_{\mathrm{tr}}(k)$. We set $\mathbb{Z}_{\mathrm{tr}}(X) \otimes \mathbb{Z}_{\mathrm{tr}}(Y) := \mathbb{Z}_{\mathrm{tr}}(X \times Y)$. For general $\mathcal{F}$, we have a canonical surjection

$$\bigoplus_{(X, s \in \mathcal{F}(X))} \mathbb{Z}_{\mathrm{tr}}(X) \to \mathcal{F}$$

Iterating this construction we get a canonical left resolution $\mathcal{L}(\mathcal{F})$ of $\mathcal{F}$ which consists of direct sums of presheaves of the form $\mathbb{Z}_{\mathrm{tr}}(X)$ for $X \in \mathrm{Sm}_k$. We then set

$$\mathcal{F} \otimes \mathcal{G} := H_0^{\mathrm{Nis}}(\mathcal{L}(F) \otimes \mathcal{L}(\mathcal{G})).$$

This induces a tensor structure on $D^-(\mathrm{Nis}_{\mathrm{tr}}(k))$. The unit $\mathbf{1}$ is $\mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}\, k)$.

For presheaves with transfers $\mathcal{F}, \mathcal{G}$, denote by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ the presheaf with transfers given by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(X) = \mathrm{Hom}(F \otimes \mathbb{Z}_{\mathrm{tr}}(X), \mathcal{G})$$

Exercise 9.12. *Show that* $\mathcal{H}om(-,-)$ *is the internal Hom-object with respect to the tensor product, that is there is a canonical isomorphism*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})) \to \mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}).$$

9.2 Localization Theorem

Proposition 9.13 (Localization Theorem)**.** *The functor* C_* *extends to a functor*

$$RC_* \colon D^-(Nis_{tr}(k)) \to DM_-^{\mathit{eff}}(k).$$

which is left adjoint to the natural embedding. The functor RC_* *identifies* $DM_-^{\mathit{eff}}(k)$ *with the localization of* $D^-(Nis_{tr}(k))$ *with respect to the thick subcategory generated by complexes of the form*

$$\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \xrightarrow{\mathbb{Z}_{tr}(p_X)} \mathbb{Z}_{tr}(X)$$

for all $X \in Sm_k$.

Proof. Denote by A the class of objects in $D^-(\mathrm{Nis}_{\mathrm{tr}}(k))$ of the given form $\mathbb{Z}_{\mathrm{tr}}(X \times \mathbb{A}^1) \to \mathbb{Z}_{\mathrm{tr}}(X)$ and let $\mathcal{A}$ be the minimal triangulated subcategory in $D^-(\mathrm{Nis}_{\mathrm{tr}}(k))$ which contains A which is closed under direct sums and direct summands. Consider the localization $D^-(\mathrm{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$ with respect to the class of morphisms whose cone are in $\mathcal{A}$.

To prove the proposition, we need to prove the following two statements:

[41] viewed as an honest sheaf on Sm_k, without transfers.

(1) For any $\mathcal{F} \in \mathrm{Nis}_{\mathrm{tr}}(k)$, the canonical morphism $\mathcal{F} \to C_*(\mathcal{F})$ is an isomorphism in $D^-(\mathrm{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$.

(2) For any object T of $\mathrm{DM}^{\mathrm{eff}}_-(k)$ and any object B of $\mathcal{A}$ one has $\mathrm{Hom}(B, T) = 0$.

Indeed (1) implies $\mathrm{DM}^{\mathrm{eff}}_-(k) \to D^-(\mathrm{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$ is surjective on isomorphism classes and (2) implies $\mathrm{DM}^{\mathrm{eff}}_-(k) \to D^-(\mathrm{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$ is fully faithful.

proof of (2): We can assume[42] $B := \mathbb{Z}_{\mathrm{tr}}(X \times \mathbb{A}^1) \to \mathbb{Z}_{\mathrm{tr}}(X)$. To get a grip on $\mathrm{Hom}(B, T)$, we need some preparation:

Lemma 9.14. *For any $X \in Sm_k$, $i \in \mathbb{Z}$, $\mathcal{F} \in Nis_{tr}(k)$, there is a canonical isomorphism*

$$\mathrm{Ext}^i_{Nis_{tr}(k)}(\mathbb{Z}_{tr}(X), \mathcal{F}) = H^i_{Nis}(X, \mathcal{F}).$$

Proof. By Theorem 8.26, $\mathrm{Nis}_{\mathrm{tr}}(k)$ is a Grothendieck topos and so it has enough injective objects. The result is true for $i = 0$ and so by taking an injective resolution of $\mathcal{F}$, we have to show that $H^i_{\mathrm{Nis}}(X, \mathcal{I}) = 0$ for any injective Nisnevich sheaf with transfers $\mathcal{I}$ and $i > 0$. It suffices to show higher Čech cohomology vanishes. Let $U \to X$ be a Nisnevich cover and $\alpha \in \check{H}^i_{\mathrm{Nis}}(U/X, \mathcal{I})$. Then α is given by a section a of $\mathcal{I}(U^i_X)$ or equivalently a morphism $\mathbb{Z}_{\mathrm{tr}}(U^i_X) \to \mathcal{I}$ Since a is a cocycle: $a \in \ker(\mathcal{I}(U^i_X) \to \mathcal{I}(U^{i+1}_X))$, this means that the section a is equivalent to a morphism $\mathrm{coker}(\mathbb{Z}_{\mathrm{tr}}(U^{i+1}_X) \to \mathbb{Z}_{\mathrm{tr}}(U^i_X)) \to \mathcal{I}$. But by Lemma 8.29, $\mathrm{Im}(\mathbb{Z}_{\mathrm{tr}}(U^{i+1}_X) \to \mathbb{Z}_{\mathrm{tr}}(U^i_X)) = \ker(\mathbb{Z}_{\mathrm{tr}}(U^i_X) \to \mathbb{Z}_{\mathrm{tr}}(U^{i-1}_X))$ and so $a \in \mathrm{Im}(\mathcal{I}(U^{i-1}_X) \to \mathcal{I}(U^i_X))$ and thus higher Čech cohomology vanishes. □

Lemma 9.15. *For any $X \in Sm_k$, $i \in \mathbb{Z}$ and $K \in D^-(Nis_{tr}(k))$ there is a canonical isomorphism*

$$\mathrm{Hom}_{D^-(Nis_{tr}(k))}(\mathbb{Z}_{tr}(X), K[i]) = \mathbb{H}^i_{Nis}(X, K)$$

where the groups on the RHS are the hypercohomology of K in the Grothendieck over-category Sm_k/X.

Proof. This is just a decorated version of Lemma 9.14. For K concentrated in a single degree, this is precisely Lemma 9.14. We can then extend to bounded complexes K by taking triangles and induction. To extend to bounded above complexes, we need that Nisnevich topology has bounded cohomological dimension by [Nis89, Theorem 1.32]. Essentially by bounded cohomological dimension, we can take injective resolutions by [Sta18, Tag 07K7]. □

Returning to the **proof of (2)**, by Lemma 9.15, it suffices to show that the induced morphism

$$\mathbb{H}^*(X, T) \to \mathbb{H}^*(X \times \mathbb{A}^1, T)$$

is an isomorphism. This is the *derived* version of Theorem 9.11 and similarly we won't prove this.

proof of (1): We need the following lemma:

Lemma 9.16. *The category $\mathcal{A}$ is a $\otimes$-ideal: $A \in \mathcal{A}$, $B \in D^-(Nis_{tr}(k))$ implies $A \otimes B \in \mathcal{A}$.*

Proof. We can assume $A = \mathbb{Z}_{\mathrm{tr}}(X \times \mathbb{A}^1) \to \mathbb{Z}_{\mathrm{tr}}(X)$ and by the existence of the canonical left resolution by direct sums of $\mathbb{Z}_{\mathrm{tr}}(Y)$, we can assume $B = \mathbb{Z}_{\mathrm{tr}}(Y)$. But then $A \otimes B = \mathbb{Z}_{\mathrm{tr}}(X \times Y \times \mathbb{A}^1) \to \mathbb{Z}_{\mathrm{tr}}(X \times Y)$, which is in $\mathcal{A}$. □

and also the notion of $\mathbb{A}^1$-homotopy equivalence:

[42] Since $\mathcal{A}$ is formed by taking shifts, cones and summands of objects of A.

Definition 9.17 ($\mathbb{A}^1$-homotopy equivalence)**.** (1) We say that two morphism $f, g\colon \mathcal{F} \to \mathcal{G}$ of presheaves with transfer are $\mathbb{A}^1$-homotopic if there is a map

$$h\colon \mathcal{F} \otimes \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1) \to \mathcal{G}$$

with $f = h \circ (\mathrm{id}_{\mathcal{F}} \otimes i_0)$ and $g = h \circ (\mathrm{id}_{\mathcal{F}} \otimes i_1)$, where $i_0, i_1\colon \mathbf{1} \to \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1)$ induced by the inclusions $i_0, i_1\colon \operatorname{Spec}(k) \to \mathbb{A}^1$.

(2) A morphism of presheaves with transfer $f\colon \mathcal{F} \to \mathcal{G}$ is an $\mathbb{A}^1$-homotopy equivalence if there is a morphism $g\colon \mathcal{G} \to \mathcal{F}$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity morphisms of $\mathcal{G}$ and $\mathcal{F}$, respectively.

Lemma 9.18. *Let $f\colon \mathcal{F} \to \mathcal{G}$ be a $\mathbb{A}^1$-homotopy equivalence of presheaves with transfer. Then the cone of f belongs to $\mathcal{A}$.*

Proof. The cone of f belonging to $\mathcal{A}$ is equivalent to saying that f becomes an isomorphism in the localized category $D^-(\mathrm{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$. It is enough to show that an endomorphism of a presheaf with transfers which is homotopic to the identity equals the identity morphism in $D^-(\mathrm{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$. Unraveling the definitions it suffices to show that the two morphisms

$$\mathrm{id}_{\mathcal{F}} \otimes i_0\colon \mathcal{F} \to \mathcal{F} \otimes \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1) \text{ and } \mathrm{id}_{\mathcal{F}} \otimes i_1\colon \mathcal{F} \to \mathcal{F} \otimes \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1)$$

are equal in $D^-(\mathrm{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$. Note that $i_0 - i_1\colon \mathbf{1} \to \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1)$ goes to 0 after composition with $\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1) \to \mathbf{1}$. Thus it factors through a map[43] $\phi\colon \mathbf{1} \to (\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1) \to \mathbf{1})$. This $\mathrm{id}_{\mathcal{F}} \otimes i_0 - \mathrm{id}_{\mathcal{F}} \otimes i_1$ factors through $\mathrm{id} \otimes \phi\colon \mathcal{F} \to \mathcal{F} \otimes (\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1) \to \mathbf{1})$. Now by Lemma 9.16, $\mathcal{F} \otimes (\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1) \to \mathbf{1}) \in \mathcal{A}$ and so $\mathrm{id} \otimes \phi$ is the zero map in $D^-(\mathrm{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$. □

Coming back to the **proof of (1)**, let $C_{\geq 1}(\mathcal{F})$ be the cone of our map $\mathcal{F} \to C_*(\mathcal{F})$. We need to show that $C_{\geq 1}(\mathcal{F}) \in \mathcal{A}$. To understand what $C_{\geq 1}(\mathcal{F}) \in \mathcal{A}$ looks like, consider the homomorphisms

$$\eta_n\colon \mathcal{F} \to C_n(\mathcal{F})$$

which on sections X is induced by the projection $X \times \Delta^n \to X$.

Lemma 9.19. *The morphisms η_n are $\mathbb{A}^1$-homotopy equivalences*

Proof. Since Δ^n is isomorphic to $\mathbb{A}^n$, we have

$$C_n(\mathcal{F}) = C_1(C_{n-1}(\mathcal{F}))$$

and so it suffices to show that η_1 is a homotopy equivalence. Let $\alpha\colon C_1(\mathcal{F}) \to \mathcal{F}$ be the morphism which on sections is induced by $X \times \{0\} \to X \times \mathbb{A}^1$. Then $\alpha \circ \eta_1 = \mathrm{id}_{\mathcal{F}}$ and it suffices to show there exists a morphism $h\colon C_1(\mathcal{F}) \otimes \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1) \to C_1(\mathcal{F})$ such that

$$h \circ (\mathrm{id} \otimes i_0) = \eta_1 \circ \alpha \text{ and } h \circ (id \otimes i_1) = \mathrm{id}$$

□

□

Remark 9.20. *Let us mention without proof that $DM_{gm}(k)$ remains a triangulated tensor category and the canonical embedding $DM_{gm}^{\mathit{eff}}(k) \to DM_{gm}(k)$ is fully faithful in the case k is a perfect field (Voevodsky's cancellation theorem). However the reasons for this are less formal than the case for pure (effective) motives.*

[43] To see this consider the triangle $\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1) \to \mathbf{1} \to (\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1) \to \mathbf{1}) \to \cdot$ and consider the induced triangle after taking $\mathrm{Hom}(\mathbf{1}, -)$.

9.3 Motivic Cohomology

Recall that by Exercise 3.10 $Z^r_\sim(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}(\mathbf{1}(-r), (X, \Delta_X))$. In a *similar* spirit motivic cohomology is defined as

Definition 9.21 (Motivic cohomology)**.** For $X \in \mathrm{Sm}_k$ we put

$$H^p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X), \mathbb{Z}(q)[p])$$

9.4 Comparison to Chow Motives

10 Lecture 10: Homotopy category of Morel-Voevodsky

11 Lecture 11: Motives over rigid-analytic varities - part I (after Ayoub)

12 Lecture 12: Motives over rigid-analytic varieties - part II (after Ayoub)

13 Lecture 13: 6-functor formalism of motives over rigid-analytic varieties

The work of Ayoub-Gallauer-Vezzani [AGV22] produced a 6-functor formalism for rigid-analytic varieties.

This in turn relied on the work of Ayoub [Ayo15], where for a quite general adic space S, he constructed a category of (étale) rigid analytic motives over S with rational coefficients $\mathbf{RigDA}_{\text{ét}}(S, \mathbb{Q})$.

In [Ayo15], he extended the work of the theory of motives of over an algebraic variety. Given a scheme S there are two known approaches to constructing a theory of motives over S:

(1) the homotopic approach of Morel-Voevodsky leading to the homotopic category $\mathbf{H}(S)$ (cf. [MV99]) and its stable version $\mathbf{SH}(S)$ (cf. [Jar00]).

(2) the "approach by transfers" by [VSF00].

14 Appendix

14.1 Solutions to exercises

Solution 14.1 (To Example 1.10)**.** *The Segre embedding shows that S is a closed subspace of projective space. Thus it is also projective. It's also smooth (being the product of two smooth varieties), so* $\mathrm{Div}(S)$ *is well-defined. Next we show* $\mathrm{Div}(S) = \mathbb{Z} \oplus \mathbb{Z}$*.*

To show $\mathrm{Div}(S) \cong \mathbb{Z} \oplus \mathbb{Z}$ *we refer to [Har77, Example 6.6.1]. We can take as generators divisors (up to linear equivalence) $p := 0 \times \mathbb{P}^1$ and $q := \mathbb{P}^1 \times 0$. Then $p \cdot q = 1$ (as they meet transversely and they intersect at a single point) and $p \cdot p = 0$ because we can move p to another parallel line with no intersection. Similarly for q. This determines the intersection product claimed formula by [Har77, Theorem 1.1].*

It is easy to see that $\mathrm{Div}(S) =$ *Num*(S)*, since have basically described* $\mathrm{Div}(S)$ *and it's intersection product above.*

The claimed signature of the intersection form then follows.

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