

Lecture notes: Motives and L-functions

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Abstract

These are lecture notes for the fall semester 2025-26 academic year.

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1 Lecture 1: Weil's Riemann-hypothesis for curves

Grothendieck sent a letter to Serre in 1964 detailing his idea for what a "motive" should be. An extract of this letter can be found in the annéxe of Serre's note on motives [Ser91]. Grothendieck's notion of a "motive" was motivated by proving the Weil conjectures. So what are the Weil conjectures? In 1949 Weil was interested in studying the number of solutions of equations over finite fields and he formulated the following conjecture:

Conjecture 1.1 (Weil Conjectures [Wei49]). *Let X be a smooth projective variety over \mathbb{F}_p of dimension n such that $X \times_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ is irreducible and define the zeta function of X , $z(X, t)$ by*

$$\log z(X, t) := \sum_{m=1}^{\infty} |X(\mathbb{F}_{p^m})| \frac{t^m}{m}.$$

- (1) **Rationality and Riemann-Hypothesis:** *Then there exists polynomials $P_1(t)$, $P_2(t)$, $\dots, P_{2n-1}(t) \in \mathbb{Z}[t]$ where $P_i(t)$ factorizes as*

$$P_i(t) = (1 - a_{i1}t)(1 - a_{i2}t) \dots (1 - a_{ib_i}t)$$

where $|a_{ij}| = p^{i/2}$ such that

$$z(X, t) = \frac{P_1(t) \cdot \dots \cdot P_{2n-1}(t)}{(1-t)P_2(t) \cdot \dots \cdot P_{2n-2}(t)(1-p^n t)}.$$

- (2) **Betti numbers:** *If X comes from reduction modulo p from some integral lift \tilde{X}/\mathbb{Z} , then the b_i are the Betti numbers of $\tilde{X}(\mathbb{C})$.*

Example 1.2. (1) $X = *$, then $|X(\mathbb{F}_{p^m})| = 1$ and so $z(X, t) = \frac{1}{1-t}$.

(2) $X = \mathbb{P}_{\mathbb{F}_p}^1$, then $|X(\mathbb{F}_{p^m})| = p^m + 1$ and so $z(X, t) = \frac{1}{(1-t)(1-pt)}$.

Remark 1.3. *There is an analogue of Weil's conjecture for Kähler manifolds given by Serre [Ser60]. The latter is a consequence of Hodge theory, while Weil's conjecture is about étale cohomology (and intersection theory as étale cohomology itself is not powerful enough).*

Weil proved these conjectures for the case of a curve a year earlier in [Wei48]. His proof relies on constructing a suitable object from X (which we now call a *pure motive*) and proving it has desirable properties. We now give Weil's proof¹ of the Riemann-Hypothesis following closely the exposition given by Sam Raskin [Ras07].

Proof. Let's relabel X by X_0 and now use X to denote the base change $X := X_0 \times_{\mathbb{F}_p} \overline{\mathbb{F}_p}$. Let $Y := X \times_{\overline{\mathbb{F}_p}} X$. **Spoiler: In the case of a curve, the motive attached to X will essentially capture the divisors of Y . So the remainder of the proof proceeds by studying divisors of Y .** Let $\Phi_{X_0}: X_0 \rightarrow X_0$ be the absolute Frobenius morphism² on X_0 and $\text{Fr}_X = \Phi_{X_0} \times \text{id}$ the Frobenius endomorphism of X .

A priori, there are two actions on $X_0(\overline{\mathbb{F}_p}) = X(\overline{\mathbb{F}_p})$. One is given by the Frobenius endomorphism on X and the other is induced by the Galois action $\Phi_{\text{Spec}(\overline{\mathbb{F}_p})}$.

Lemma 1.4. *These two actions are the same.*

Proof. Exercise. □

Let $\Delta_X: X \rightarrow Y$ be the diagonal morphism and $\Gamma_{\text{Fr}_X^n}$ the graph of the Frobenius endomorphism composed n times: Fr_X^n . Both Δ_X and $\Gamma_{\text{Fr}_X^n}$ are closed immersions and cut out divisors in Y . We denote these divisors by $[\Delta_X]$ and $[\Gamma_{\text{Fr}_X^n}]$, respectively.

Lemma 1.5. *We have $[\Gamma_{\text{Fr}_X^n}] = [(\text{Fr}_X \times \text{id}_X)^*]^n [\Delta_X]$.*

¹Weil's proof in [Wei48] is slightly different to what is presented here. In particular he relies on the Riemann-Roch theorem for surfaces.

²This is the morphism given by identity on the underlying topological space of X_0 and Frobenius on the ring of functions.

Proof. First note that by functoriality of pullbacks $[(\mathrm{Fr}_X \times \mathrm{id}_X)^*]^n = (\mathrm{Fr}_X^n \times \mathrm{id}_X)^*$. Thus it suffices to show that for an arbitrary endomorphism $\psi: X \rightarrow X$, we have

$$[\Gamma_\psi] = (\psi \times \mathrm{id}_X)^*[\Delta_X] \quad (1.1)$$

where Γ_ψ is the graph of ψ in Y . We now work locally and assume $X = \mathrm{Spec}(A)$. Take a closed point $x \in X$ and a uniformizer $\pi \in \mathcal{O}_{X,x}$ and assume $\pi \in A$. By pulling back π along the two projections $Y \rightrightarrows X$ we get two global sections π_1, π_2 of Y . Then $\pi_1 - \pi_2$ generates $[\Delta_X]$. But then the LHS of (1.1) is generated³ by $\psi^*(\pi_1) - \pi_2$. \square

Lemma 1.6. *The cardinality of the set $X(\mathbb{F}_{p^n})$ is given by the intersection number⁴ $[\Gamma_{\mathrm{Fr}_X^n}] \cdot [\Delta_X]$.*

Proof. Before we begin the proof, let us recall what intersection numbers mean in the context of curves on surfaces.

Detour: Intersection numbers of closed curves on surfaces: Let C be a smooth closed curve on a smooth projective surface S and $D \in \mathrm{Div}(S)$. Then one definition of the intersection number is $C \cdot D = \deg(\mathcal{O}_S(D)|_C)$. Unravelling what this means $\mathcal{O}_S(D)|_C$ is a line-bundle on C and its degree is the degree of its associated divisor.

Now let's go back to the proof of Lemma 1.6. First note that since we are in characteristic p , the differential of Fr_X^n vanishes. Thus if we look at the tangent spaces of Δ_X and $\Gamma_{\mathrm{Fr}_X^n}$, we see that their sum spans all of $T_x X \times T_x X = T_{(x,x)} Y$ at every point of intersection $(x, x) \in Y$. In the literature we say Δ_X and $\Gamma_{\mathrm{Fr}_X^n}$ meet *transversely*. The upshot of transversality is the following proposition⁵.

Proposition 1.7. *In the setting of the previous **Detour**, suppose also that D is a closed smooth curve. If C and D intersect transversely then*

$$C \cdot D = |C \cap D|$$

Proof. Exercise. \square

So Proposition 1.7 says that the intersection number $[\Gamma_{\mathrm{Fr}_X^n}] \cdot [\Delta_X]$ is just the number of (closed) points that Δ_X and $\Gamma_{\mathrm{Fr}_X^n}$ intersect at. Note that the points must indeed be closed as X is irreducible. On the other hand by Hilbert Nullstellensatz, the closed points of Y is just $X(\overline{\mathbb{F}}_p) \times X(\overline{\mathbb{F}}_p)$. The set of points which belong to the intersection of Δ_X and $\Gamma_{\mathrm{Fr}_X^n}$ is precisely $X(\mathbb{F}_{p^n})$ because the set of points fixed by Fr_X^n is the same as those fixed by $\Phi_{\mathrm{Spec}(\overline{\mathbb{F}}_p)}^n$ by Lemma 1.4. \square

We need one more ingredient to finish the proof: the idea of *numerical* equivalence of divisors.

Definition 1.8. We say that two divisors are *numerically* equivalent if their intersection numbers with any third divisor are equal⁶. We define $\mathrm{Num}(Y)$ to be the quotient of $\mathrm{Div}(Y)$ by numerical equivalence. In particular the intersection product descends to a non-degenerate symmetric bilinear form

$$\mathrm{Num}(Y) \times \mathrm{Num}(Y) \rightarrow \mathbb{Z}.$$

³To see the last statement, look at the graph morphism at the level of algebras $A \otimes_{\overline{\mathbb{F}}_p} A \rightarrow A$. This is given by $x \otimes y \mapsto \psi^*(x)y$ and one sees that the kernel is indeed generated by $\psi^*(\pi_1) - \pi_2$.

⁴For the interested reader, Fulton's book [Ful84] develops intersection theory in rather great generality. In general one has to be careful outside of smooth/projective assumptions.

⁵It's so fundamental that I've labeled it a Proposition, even though we are inside a Lemma.

⁶Technically we haven't defined intersection numbers of divisors in general, but let's assume there is a reasonable definition for now.

We will need the Hodge Index Theorem which describes the above linear form [Mum66, Lecture 18]:

Theorem 1.9 (Hodge Index Theorem). *Let S be a smooth projective surface over an algebraically closed field (of arbitrary characteristic). We have a direct sum decomposition*

$$\text{Num}(S) \otimes_{\mathbb{Z}} \mathbb{Q} = V \oplus V'$$

such that V has dimension 1 and the intersection form is positive definite on V and negative definite on V' .

Example 1.10. *Consider the quadric surface $S \subset \mathbb{P}^3$ given by the Segre embedding*

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\hookrightarrow \mathbb{P}^3 \\ [w : x] \times [y : z] &\mapsto [wy : wz : xy : xz] \end{aligned}$$

$S := V(x_0x_3 - x_1x_2)$. *Show that $\text{Div}(S) = \text{Num}(S) = \mathbb{Z} \oplus \mathbb{Z}$ and the intersection product is given by*

$$\begin{aligned} \text{Div}(S) \times \text{Div}(S) &\rightarrow \mathbb{Z} \\ (a_1, b_1) \times (a_2, b_2) &\mapsto a_1b_2 + b_1a_2. \end{aligned}$$

Verify the Hodge Index theorem in this case.

Lemma 1.11. *We have $|X(\mathbb{F}_{p^n})| = p^n + O(p^{n/2})$.*

Proof. Let $[H]$ and $[V]$ be the divisors in $\text{Div}(Y)$ corresponding to $X \times \{x_0\}$ and $\{x_0\} \times X$ for some closed point $x_0 \in X$, respectively. Since $[H] \cdot [V] = 1$ and $[H] \cdot [H] = 0$, these cannot be equal in $\text{Num}(Y)$. Moreover $U := \mathbb{Q}[H] \oplus \mathbb{Q}[V]$ is a finite-dimensional subspace of $W := \text{Num}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$, we can write

$$W = U \oplus U'$$

where U' is the orthogonal complement. We claim that the intersection form on U' is negative-definite: Indeed on matrix on U with respect to the basis $\{[H], [V]\}$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has one positive eigenvalue and so by the Hodge Index Theorem, the subspace on which the intersection form is positive-definite is contained in U .

Let

$$\begin{aligned} T: W &\rightarrow W \\ D &\mapsto (\text{Fr}_X \times \text{id}_X)^* D \end{aligned}$$

Then $T([H]) = p[H]$ and $T([V]) = [V]$. We know by Lemma 1.5 that $T^n[\Delta_X] = [\Gamma_{\text{Fr}_X^n}]$. For the following note that pullback/pushforward of divisors (up to linear equivalence) descends to numerical equivalence (this is essentially the content of the *moving lemma*). Moreover for any $D, E \in \text{Num}(Y)$

$$\begin{aligned} (\text{Fr}_X \times \text{id}_X)^* D \cdot (\text{Fr}_X \times \text{id}_X)^* E &\stackrel{(1)}{=} D \cdot (\text{Fr}_X \times \text{id}_X)_* (\text{Fr}_X \times \text{id}_X)^* E \\ &\stackrel{(2)}{=} D \cdot pE \\ &\stackrel{(3)}{=} pD \cdot E \end{aligned}$$

where (1) follows from Proposition 1.12:

Proposition 1.12. *Given $\varphi: Y \rightarrow Z$ so that $\varphi^*: \text{Div}(Z) \rightarrow \text{Div}(Y)$ and $\varphi_*: \text{Div}(Y) \rightarrow \text{Div}(Z)$ are well-defined, we have*

$$C \cdot \varphi^*(D) = \varphi_*(C) \cdot D$$

for any $C \in \text{Div}(Y)$ and $D \in \text{Div}(Z)$.

Proof. Exercise. □

and (2) follows from Proposition

Proposition 1.13. *In the setting of Proposition 1.12 $\varphi_*\varphi^*: \text{Div}(Z) \rightarrow \text{Div}(Z)$ is given by $D \mapsto \deg(\varphi)D$.*

Proof. Exercise □

and (3) follows by linearity of the intersection form.

Thus for all $v, w \in W$, we have $Tv \cdot Tw = p(v \cdot w)$. Since $[H] \cdot [\Delta_X] = [V] \cdot [\Delta_X] = 1$, we can write

$$[\Delta_X] = [H] + [V] + u' \tag{1.2}$$

for some $u' \in U'$. We then compute

$$\begin{aligned} |X(\mathbb{F}_{p^n})| &\stackrel{(a)}{=} [\Gamma_{\mathbb{F}_X^n}] \cdot [\Delta_X] \\ &\stackrel{(b)}{=} T^n[\Delta_X] \cdot [\Delta_X] \\ &\stackrel{(c)}{=} p^n + 1 + T^n u' \cdot u' \end{aligned}$$

where (a) follows from Lemma 1.6, (b) by Lemma 1.5 and (c) because $T([H]) = p[H]$ and $T([V]) = [V]$.

It's easy to check that $T^n u' \in U'$ and so we can apply the Cauchy-Schwarz inequality to get

$$|T^n u' \cdot u'| \leq \sqrt{|T^n u' \cdot T^n u'| |u' \cdot u'|} = p^{n/2} |u' \cdot u'|$$

This completes the proof of Lemma 1.11. □

A relatively straightforward analysis argument then concludes the proof of the Riemann-Hypothesis for curves. We won't include the details, as it's not what we are after conceptually. □

2 Lecture 2: Algebraic cycles and adequate equivalence relations

For this lecture, I will be following the book by Murre-Nagel-Peters [MNP13, Chapter 1]. Another good reference is [And04, Chapitre 3]. Let k be an arbitrary field and X a k -variety.

Definition 2.1 (algebraic cycle). An algebraic cycle on X is a formal finite integral linear combination $Z = \sum n_\alpha Z_\alpha$ of irreducible closed subvarieties Z_α of X . If all Z_α have the same dimension i , we say that Z is a dimension i cycle. We denote by $Z_i(X)$ the abelian group of dimension i cycles on X . When considering the codimension point of view we write $Z^{d-i}(X) := Z_i(X)$ if X is of dimension d . We write $Z(X) := \oplus_i Z^i(X)$ and consider it as a group with a graded structure.

Lemma 2.2. *Suppose X is smooth. Then two closed subvarieties V and W of X with codimensions i and j , respectively, have intersection*

$$V \cap W = \cup_{\alpha} Z_{\alpha}$$

where each Z_{α} is an irreducible subvariety of codimension at most $i+j$.

Proof. We have that $V \cap W = \Delta^{-1}(V \times W)$ where $\Delta: X \rightarrow X \times X$ is the diagonal map. Since X is smooth, we can write $X \times X = \text{Spec}(A)$ and $X = V(f_1, \dots, f_c)$ where f_1, \dots, f_c is a regular sequence in A and $c = \dim(X)$. Then if $V \times W = \text{Spec}(A/\mathfrak{p})$ then $V \cap W = \text{Spec}(A/(\mathfrak{p} + (f_1, \dots, f_c)))$. Then for $z \in Z_{\alpha}$ a closed point

$$\dim(V \times W) = \dim \mathcal{O}_{V \times W, z} \quad \text{and} \quad \dim(Z_{\alpha}) = \dim \mathcal{O}_{Z_{\alpha}, z} = \dim \mathcal{O}_{V \times W, z} / (f_1, \dots, f_c).$$

From here one can compare the relevant dimensions by the fact that quotienting a local ring by an element in the maximal ideal, reduces the dimension by at most one. \square

Definition 2.3 (proper intersection product of algebraic cycles). In the setting of Lemma 2.2, we say that the intersection $V \cap W$ is *proper* (or V and W intersect *properly*) if the codimension of each Z_{α} is $i + j$. In this case the *intersection number* is defined by

$$i(V \cdot W; Z) := \sum_r (-1)^r \text{length}_{\mathcal{O}_{X, Z}}(\text{Tor}_r^{\mathcal{O}_{X, Z}}(\mathcal{O}_{V, Z}, \mathcal{O}_{W, Z}))$$

where $A := \mathcal{O}_{?, Z}$ denotes the local ring of $?$ at the generic point of Z . We define the *intersection product*

$$V \cdot W := \sum_{\alpha} i(V \cdot W; Z_{\alpha}) Z_{\alpha}.$$

Definition 2.4 (proper pushforward). Let $f: X \rightarrow Y$ be a proper morphism of k -varieties and $Z \subset X$ a k -dimensional closed irreducible subvariety. We define

$$f_* Z = \begin{cases} 0, & \text{if } \dim(f(Z)) < k \\ [R(Z): R(f(Z))] f(Z), & \text{otherwise} \end{cases} \quad (2.1)$$

where $R(?)$ is the field of rational functions⁷ on $?$. Extending by linearity induces a homomorphism

$$f_*: Z_k(X) \rightarrow Z_k(Y).$$

In general we say two algebraic cycles $\alpha, \beta \in Z(X)$ intersect properly if each components of α intersects each component of β properly.

Definition 2.5 (flat pullback). Let $f: X \rightarrow Y$ be a flat morphism of k -varieties and $Z \subset Y$ a k -codimensional closed irreducible subvariety. We define

$$f^* Z = f^{-1}(Z)$$

Because f is flat, $f^{-1}(Z)$ turns out to be of codimension k (assuming it is non-empty). Extending by linearity induces a homomorphism

$$f^*: Z^k(Y) \rightarrow Z^k(X)$$

Exercise 2.6 (Projection formula). *Prove $f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta$*

⁷Note that $f|_Z: Z \rightarrow f(Z)$ is a dominant morphism, so the above degree is well-defined.

Definition 2.7. A *correspondence* from X to Y is a cycle in $X \times Y$. A correspondence $Z \in Z^t(X \times Y)$ acts on cycles on X as follows

$$\begin{aligned} Z: Z^i(X) &\rightarrow Z^{i+t-\dim(X)}(Y) \\ T &\mapsto pr_{Y*}(Z \cdot (T \times Y)) \end{aligned}$$

whenever defined. We call $t - \dim(X)$ the degree of the correspondence.

Example 2.8. *It turns out the notion of correspondences generalizes the notion of (proper) pushforward and (flat) pullback. Prove this.*

As we see correspondences (or intersection products) are not always defined. This is where the notion of *adequate equivalence* comes in. These are equivalence classes on the groups Z^i such that the intersection product is always defined.

2.1 Adequate Equivalence

We now work in the category $\text{SmProj}(k)$ of smooth projective varieties over k .

Definition 2.9 (Adequate Equivalence). We say that an equivalence relation \sim on $Z(X)$ is *adequate* if

- (1) (**linearity**) \sim is compatible with addition and graduation.
- (2) (**moving lemma**) For all $\alpha, \beta \in Z(X)$, $\exists \alpha' \sim \alpha$ such that α' and β intersect properly.
- (3) (**correspondence**) \sim is compatible with correspondences: In the setting of Definition 2.7 if $T \sim 0$ and Z intersects $T \times Y$ properly, then $Z(T) \sim 0$.

We write $Z_\sim(X) := Z(X)/\sim$ and for some field F , $Z_\sim(X)_F := Z(X) \otimes_{\mathbb{Z}} F/\sim$. We also write

$$Z_{\sim,0}^i(X) := \{Z \in Z^i(X) \mid Z \sim 0\}$$

The fact that intersection product is defined on the whole $Z_\sim(X)$ is a straightforward consequence of Definition 2.9 (cf. [Sam58, Proposition 6 and 7]).

Lemma 2.10. *For any adequate equivalence relation \sim on $X \in \text{SmProj}(k)$, we have*

- (1) $Z_\sim(X)$ is a graded ring with product induced by the intersection product of cycles.
- (2) A correspondence Z from X to Y of degree r induces $Z_*: Z_\sim^i(X) \rightarrow Z_\sim^{i+r}(Y)$ and equivalent correspondences induce the same Z_* .

We now discuss the following adequate equivalence relations

- rational equivalence \sim_{rat}
- algebraic equivalence \sim_{alg}
- smash nilpotence equivalence $\sim_{\otimes \text{nil}}$
- homological equivalence \sim_{hom}
- numerical equivalence \sim_{num}

2.1.1 Rational equivalence

Definition 2.11 (Rational equivalence). A cycle $\alpha \in Z(X)$ is rationally equivalent to 0 ($\alpha \sim_{\text{rat}} 0$) if there exists $\beta \in Z(X \times \mathbb{P}^1)$ such that $\beta(0)$ and $\beta(\infty)$ are well-defined and $\alpha = \beta(0) - \beta(\infty)$.

Lemma 2.12. *Rational equivalence corresponds to linear equivalence for codimension 1 cycles $Z^1(X)$.*

Proof. We first show $\text{div}(f) \sim_{\text{rat}} 0$ for $f \in R(X)$ a rational function. We can think of f as $f: U \rightarrow \mathbb{P}^1$ for some dense open $U \subset X$. Let $W \subset X \times \mathbb{P}^1$ be the closure of the graph of f . Then W gives a cycle $\beta \in Z(X \times \mathbb{P}^1)$ and essentially by definition $\text{div}(f) = \beta(0) - \beta(\infty)$.

For the converse suppose $\alpha \in Z^1(X)$ and $\alpha \sim_{\text{rat}} 0$. Take a component $Z' \subset X \times \mathbb{P}^1$ of β (with β part of Definition 2.11). Then Z' dominates \mathbb{P}^1 . Let $Z \subset X$ be the image of Z' under the projection to X . Then $Z \subset X$ is closed (as projection is proper) and $Z' \rightarrow Z$ is proper and dominant with fibers of dimension 0 or 1.

There are two cases as to whether $\dim(Z) < \dim(Z')$ or $\dim(Z) = \dim(Z')$.

If $\dim(Z) < \dim(Z')$, then $Z' = Z \times \mathbb{P}^1$ and $[Z'_0] - [Z'_\infty] = [Z] - [Z] = 0$.

If $\dim(Z) = \dim(Z')$, then $Z' \rightarrow Z$ is generically finite (i.e. inverse image of generic point is finite). Then I leave it as an exercise⁸ to show that $[Z'_0] - [Z'_\infty] = \text{div}(\text{Nm}(f))$ where $f: Z' \rightarrow \mathbb{P}^1$ viewed as a rational function on Z' . \square

The technical difficulty in proving that rational equivalence is indeed an adequate equivalence relation lies in proving the *moving lemma*. The proof is roughly as follows: We embed $X \hookrightarrow \mathbb{P}^N$ and given $V, W \subset X$, we need to move V so that it intersects W properly. There are two cases to consider as to whether $X = \mathbb{P}^N$ or not. In the former, there is some general linear transformation that makes V and W intersect properly. In the later case, one considers a linear subspace $L \subset \mathbb{P}^N$ and the cone $C(L, V)$. For the details we refer to [Ful84, Example 11.4.1].

Exercise 2.13. *What goes wrong with the moving lemma for rational equivalence if we relax the smoothness assumption? What if we keep smoothness and relax the projectivity assumption?*

Definition 2.14 (Chow ring). The corresponding graded ring $CH(X) := Z_{\text{rat}}(X)$ is called the Chow ring. We will also denote by $\text{Corr}(X, Y) := CH(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ the *correspondences* from X to Y .

Lemma 2.15. *Among all adequate equivalence relations, rational equivalence is the finest.*

Proof. Let \sim be an adequate equivalence relation. It suffices to prove $[0] \sim [\infty]$ as then by using a correspondence, we get the definition of rational equivalence. Since the point $1 \in \mathbb{P}^1$ does not intersect itself properly, by the moving lemma there exists $\sum_i n_i [x_i] \in Z^1(\mathbb{P}^1) \sim 1$ with $x_i \in \mathbb{P}^1$ such that x_i intersects 1 properly. In other words $x_i \neq 1$. Consider now the correspondence $Z \in Z^1(\mathbb{P}^1 \times \mathbb{P}^1)$ given by the graph of the polynomial

$$1 - \prod_i \left(\frac{x - x_i}{1 - x_i} \right)^{m_i} \quad (2.2)$$

for a collection of $m_i > 0$ and $T = \sum_i n_i [x_i] - 1$. Then $Z(T)$ is just the pushforward of T by (2.2). The pushforward of T is just $mn[1] - m[0]$ where $m = \sum_i m_i$ and $n = \sum_i n_i$. Since this holds for arbitrary m_i , we get $n[1] \sim [0]$. Applying the condition of correspondence to the automorphism $x \mapsto \frac{1}{x}$, we get $n[1] \sim [\infty]$, from which we can conclude. \square

⁸The main idea is essentially in the proof of [Ful84, Proposition 1.4(b)]

2.1.2 Algebraic equivalence

Definition 2.16 (Algebraic Equivalence). This is the same definition as rational equivalence but with \mathbb{P}^1 replaced by any smooth projective irreducible curve and the two points 0 and ∞ by any two k -rational points on the curve. In other words $\alpha \in Z(X)$ is $\sim_{\text{alg}} 0$ if there exists a smooth irreducible projective curve C and $\beta \in Z(X \times C)$ and two points $a, b \in C(k)$ such that $\beta(a) = 0$ and $\beta(b) = \alpha$.

Example 2.17 (algebraic equivalence is coarser than rational equivalence). *Take an elliptic curve E over \mathbb{C} and two distinct points $a, b \in |E|$. Then $a - b$ is not a divisor of any rational function. This is because we can make an identification $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ and we see by Cauchy's residue theorem that any rational function has its sum of residues equal to 0. So any rational function cannot have a simple pole. In summary $a - b \not\sim_{\text{rat}} 0$.*

On the other hand E is equipped with a degree 2 cover over \mathbb{P}^1 with 4 ramification points (by Hurwitz's theorem). If we take the graph $Z \subset E \times \mathbb{P}^1$ of this cover, then we get that these 4 ramification points must be algebraically equivalent⁹.

2.2 Smash Nilpotent equivalence

Definition 2.18 (Smash Nilpotent equivalence). For $Z \in Z(X)$ we say $Z \sim_{\otimes} 0$ iff for some positive integer n , $Z^n \sim_{\text{rat}} 0$ where we view $Z^n \in Z(X^n)$.

Theorem 2.19 (\sim_{\otimes} vs \sim_{alg}). *We have $Z_{\text{alg},0}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \subset Z_{\otimes,0}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Theorem 2.19 is due independently to Voevodsky [Voe95] and Voisin [Voi96].

Proof. We proceed in several steps as in [MNP13, Appendix B].

Step 0: Reduce to $k = \bar{k}$.

Exercise 2.20. *For any (adequate) equivalence relation \sim , there is a natural map*

$$Z_{\sim}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow (Z_{\sim}(X_{\bar{k}}) \otimes_{\mathbb{Z}} \mathbb{Q})^{\text{Gal}(\bar{k}/k)}$$

given by restricting a closed subvariety over k to one over \bar{k} . Prove that this map is a bijection.

Exercise 2.20 allows us to assume k is algebraically closed.

Step 1: Reduce to the case of a smooth projective curve. Take $Z \sim_{\text{alg}} 0$. Then by definition $\exists \Gamma \in \text{Corr}(C, X)$ and two points $a, b \in C(k)$ such that $Z = \Gamma_*(a - b)$. Thus taking products gives $Z^n = (\Gamma^n)_*(a - b)^n$ and so it suffices to show $(a - b)^n \sim_{\text{alg}} 0$ on C^n . In fact we shall show

$$(a - b)^n \sim_{\text{alg}} 0 \quad \text{for } n > g,$$

where g is the genus of the curve C .

Step 2: Reducing $(a - b)^n$ as a divisor on the n -fold symmetric product of C . A priori $(a - b)^n \in Z(C^n)$. However the symmetric group S_n induces an action on C^n and clearly $(a - b)^n \in (Z(C^n) \otimes_{\mathbb{Z}} \mathbb{Q})^{S_n}$.

Exercise 2.21. *Show that $(Z(C^n) \otimes_{\mathbb{Z}} \mathbb{Q})^{S_n} \cong Z(C^n/S^n) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Exercise 2.21 allows us to view $(a - b)^n$ in the quotient variety C^n/S^n (the n -fold symmetric product of C).

⁹We can just take the graph of the identity map to conclude that any two rational points on any smooth projective curve are algebraically equivalent.

Step 3: Comparison of C/S^n with the Jacobian¹⁰ $J(C)$. Fix a base point $e \in C(k)$. Denote by

$$\begin{aligned}\pi_n: C^n &\rightarrow C^n/S^n \\ (x_1, \dots, x_n) &\mapsto [x_1, \dots, x_n]\end{aligned}$$

the natural surjection and

$$\begin{aligned}\varphi_n: C^n/S^n &\rightarrow J(C) \\ [x_1, \dots, x_n] &\mapsto \sum_i (x_i - e).\end{aligned}$$

Lemma 2.22. *The induced map $(\varphi_n)_*: CH_0(C^n/S^n) \rightarrow CH_0(J(C))$ is an isomorphism for all $n \geq g$.*

Proof. If $n = g$, then we claim that φ_n is a birational morphism. Indeed by Riemann-Roch

$$\ell(x_1 + \dots + x_g) = g + 1 - g + \ell(K - x_1 - \dots - x_g)$$

if none of the x_i are base points of the canonical divisor K , then since $\ell(K) = g$, we get¹¹ $\ell(K - x_1 - \dots - x_g) = 0$ and so $\ell(x_1 + \dots + x_g) = 1$. This means that φ_n is an isomorphism¹² outside of a finite set of points. So it is a birational morphism. By [Ful84, Example 16.1.11], the group CH_0 is invariant for birational morphisms.

Suppose $n > g$ and consider the natural embedding

$$\begin{aligned}\iota: C^g/S^g &\rightarrow C^n/S^n \\ [x_1, \dots, x_g] &\mapsto [x_1, \dots, x_g, \underbrace{e, \dots, e}_{n-g}]\end{aligned}$$

Then $(\varphi_n)_* \iota_* = (\varphi_g)_*$. Since $(\varphi_g)_*$ is an isomorphism, it follows that ι_* is an injection. It remains to show that it is a surjection. So take $y \in C^n/S^n$ and consider the image $z \in J(C)$ and some point $x \in C^g/S^g$ which maps to z under equivalence. Then $\iota(x)$ and y belong to the fiber $\varphi_n^{-1}(z)$. But the fibers¹³ of φ_n are projective and any two points are rationally equivalent. This proves the lemma. \square

Step 4: Application of Bloch's theorem. We have that $(\varphi_{g+1})_*((a - b)^{g+1}) = (\varphi_1(a) - \varphi_1(b))^{*(g+1)}$ and this vanishes by [Blo76]. \square

Exercise 2.23. *Use ideas from the proof of Theorem 2.19 to show that the cartesian product of two non-zero Chow cycles can be zero.*

2.3 Homological equivalence

To define *homological equivalence* we need to define a *Weil cohomology theory*. Let F be a field of characteristic 0. We denote $\text{GrVect}_F^{\geq 0}$ be the category of finite dimensional graded F -vector spaces, where the grading is concentrated in non-negative degrees.

¹⁰The Jacobian variety $J(C)$ is the variety which represents the functor $T \mapsto \{\text{invertible sheaves of degree 0 on } X \times T\}$.

¹¹This is related to [Har77, Chapter IV, Proposition 3.1]

¹²Because the fiber of φ_n is just the set of points $[x_1, \dots, x_n]$ such that $\sum_i x_i$ form a complete linear system. This also means fibers are projective.

¹³fiber above a point is a complete linear system

Definition 2.24. A Weil cohomology theory is a functor

$$H: \text{SmProj}(k)^{\text{opp}} \rightarrow \text{GrVect}_{\bar{F}}^{\geq 0}$$

which satisfies the following axioms:

- (1) A one-dimensional F -vector space $F(1)$, which gives rise to *Tate* twists.
- (2) \exists a graded cup product $\cup: H(X) \times H(X) \rightarrow H(X)$ such that if $a \in H^i(X)$, $b \in H^j(X)$, then $a \cup b = (-1)^{ij} b \cup a$.
- (3) one has Poincaré duality (assume X has pure dimension d): \exists a trace isomorphism

$$\text{Tr}: H^{2d}(X)(d) \xrightarrow{\sim} F$$

such that

$$H^i(X) \times H^{2d-i}(X)(d) \xrightarrow{\cup} H^{2d}(X)(d) \xrightarrow{\text{Tr}} F$$

is a perfect pairing.

- (4) A Künneth map

$$H(X) \otimes H(Y) \xrightarrow{(pr_X)^* \otimes (pr_Y)^*} H(X \times Y)$$

which is a (graded) isomorphism.

- (5) there are cycle class maps

$$\gamma_X: \text{CH}^i(X) \rightarrow H^{2i}(X)(i)$$

which satisfy various compatibilities¹⁴.

- (6) If X is pure of dimension d and $\iota: Y \hookrightarrow X$ is a smooth hyperplane, then *weak Lefschetz* holds:

$$H^i(X) \xrightarrow{\iota^*} H^i(Y)$$

is an isomorphism if $i < d - 1$ and an injection for $i = d - 1$.

- (7) With the setting as in (6), the Lefschetz operator $L(\alpha) := \alpha \cup \gamma_X(Y)$ induces isomorphisms

$$L^i: H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X)$$

for $0 \leq i \leq d$. This is known as *Hard Lefschetz*.

Example 2.25. For $k = \mathbb{C}$, we get many examples of Weil cohomology theories:

- (1) *singular cohomology groups*: $H^i(X_{an})$ where X_{an} is the complex manifold attached to X .
- (2) *classical de Rham cohomology*: $H_{dR}^i(X_{an}, \mathbb{C})$.
- (3) *algebraic de Rham cohomology*: $H_{dR}^i := \mathbb{H}(X, \Omega_{X/\mathbb{C}}^\bullet)$.

The fact that these are indeed Weil cohomology groups follows from classical reasons together with comparison isomorphisms. On the other hand the fact that $H_{\acute{e}t}^i(X, \mathbb{Q}_\ell)$ is a Weil cohomology theory (in particular satisfies Hard Lefschetz) is deep work of Deligne [Del80].

Definition 2.26 (Homological equivalence). Fix a Weil cohomology theory. Then for $Z \in \mathcal{Z}(X)$ we say $Z \sim_{\text{hom}} 0$ if $\gamma_X(Z) = 0$.

We can compare homological equivalence to algebraic and smash nilpotent equivalence.

¹⁴I will state them explicitly when we need them.

Lemma 2.27 (\sim_\otimes and \sim_{alg} vs \sim_{hom}). (1) $Z_{\text{alg},0}^i(X) \subset Z_{\text{hom},0}^i(X)$.

(2) $Z_{\otimes,0}^i(X) \subset Z_{\text{hom},0}^i(X)$.

Proof. For (1), note that $\alpha \sim_{\text{alg}} 0$ means that for some smooth projective curve C , $\alpha = pr_{X*} pr_C^*([a] - [b])$ for two rational points $a, b \in C$. Now cycle map is compatible with push-forward and pullbacks (one of the conditions I didn't state in Definition 2.24(4)). So we can reduce to the case of a curve. We then conclude by Matsusaka's theorem:

Theorem 2.28 (Matsusaka's Theorem).

$$Z_{\text{hom},0}^1(X) = \{D \in Z^1(X) \mid nD \sim_{\text{alg}} 0 \text{ for some } n \in \mathbb{Z}\}$$

For part (2), note that $\alpha \sim_\otimes 0$ means $\alpha^n \sim_{\text{rat}} 0$ for some $n > 0$. Then

$$\gamma_{X^n}(\alpha^n) = \underbrace{\gamma_X(\alpha) \otimes \dots \otimes \gamma_X(\alpha)}_n$$

is zero. So each of $\gamma_X(\alpha) = 0$. □

Exercise 2.29. Find an alternative proof of Lemma 2.27(1) using part (2) and Voevodsky-Voisin Theorem 2.19.

2.4 Numerical equivalence

Definition 2.30 (Numerical Equivalence). Let X be of pure dimension d . For $Z \in Z^i(X)$, we say $Z \sim_{\text{num}} 0$ if for every $W \in Z^{d-i}(X)$ such that $Z \cdot W$ is defined, we have $\deg(Z \cdot W) = 0$.

We can compare homological equivalence and numerical equivalence.

Lemma 2.31 (\sim_{hom} vs \sim_{num}). $Z_{\text{hom},0}^i(X) \subset Z_{\text{num},0}^i(X)$.

Proof. We will need to use that γ_X (the cycle class map) is compatible with intersection products:

$$\gamma_X(\alpha \cdot \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta) \tag{2.3}$$

and compatible with points P :

$$\text{Tr} \circ \gamma_X = \deg \quad \text{on } CH^d(X). \tag{2.4}$$

Conditions (2.3) and (2.4) are the remaining conditions I didn't state in Definition 2.24(4).

By property (2.4), we see that the result holds for $i = d$ (i.e. zero cycles). Suppose now $i < d$ and $Z \in Z_{\text{hom},0}^i(X)$ and $W \in Z^{d-i}(X)$ such that $Z \cdot W$ is defined. Then

$$\begin{aligned} 0 &\stackrel{(i)}{=} \text{Tr}(\gamma_X(Z) \cup \gamma_X(W)) \\ &\stackrel{(ii)}{=} \text{Tr}(\gamma_X(Z \cdot W)) \\ &\stackrel{(iii)}{=} \deg(Z \cdot W) \end{aligned}$$

where (i) holds because $\gamma_X(Z) = 0$, (ii) holds because of (2.3) and (iii) holds because of (2.4). □

Exercise 2.32. Show that by realizing the degree map as a correspondence, that $Z_{\sim,0}^i(X) \subset Z_{\text{num},0}^i(X)$ for any non-trivial adequate equivalence relation.

Summarizing Lemmas 2.15, 2.27(1) and 2.31 have shown the following chain

$$Z_{\text{rat},0}^i(X) \subset Z_{\text{alg},0}^i(X) \subset Z_{\text{hom},0}^i(X) \subset Z_{\text{num},0}^i(X)$$

As part of the standard conjectures:

Conjecture 2.33 (Standard Conjecture D). $Z_{\text{hom},0}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} = Z_{\text{num},0}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

3 Lecture 3: Grothendieck's pure motives

For this lecture, I will be following the book by Murre-Nagel-Peters [MNP13, Chapter 2]. Another good reference is [And04, Chapitre 4]. Let k be an arbitrary field and X and Y smooth projective k -varieties.

The next definition is along the same lines as Definitions 2.7 and 2.14.

Definition 3.1 (correspondences and degree r correspondence). For an adequate equivalence relation \sim , we denote the graded vector spaces of correspondences:

$$\mathrm{Corr}_\sim(X, Y) := Z_\sim(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and if X is pure of dimension d , we also consider the degree r correspondences by

$$\mathrm{Corr}_\sim^r(X, Y) := Z_\sim^{d+r}(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We also need to know how to compose correspondences:

Definition 3.2 (composition of correspondences). We define composition

$$\begin{aligned} \mathrm{Corr}_\sim(X, Y) \times \mathrm{Corr}_\sim(Y, Z) &\rightarrow \mathrm{Corr}_\sim(X, Z) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

by $g \circ f := \mathrm{pr}_{XZ*}\{(f \times Z) \cdot (X \times g)\}$.

Exercise 3.3. Check that Definition 3.2 restricts to give a composition of degree 0 correspondences. In general composition respects the grading.

Definition 3.4 (projectors). A projector for X is an element $p \in \mathrm{Corr}_\sim(X, X)$ for which $p \circ p = p$.

Exercise 3.5. Show that the diagonal Δ_X is an example of a projector.

We now proceed to give the construction of (pure) motives in several steps. For the following fix an adequate equivalence relation \sim .

Construction of (pure) motives:

Step 1: Consider the category $Z_\sim \mathrm{SmProj}(k)$ with

- (1) objects: same as $\mathrm{SmProj}(k)^{\mathrm{op}}$
- (2) morphisms: degree 0 correspondences. More precisely $\mathrm{Hom}(X, Y) := \mathrm{Corr}_\sim^0(X, Y)$.

We are hoping that the category we construct is abelian and it's formal nonsense¹⁵ to see that one should keep track of idempotent morphisms (i.e. projectors). This leads to

Step 2: Consider the category of *effective* motives $\mathrm{Mot}_\sim^{\mathrm{eff}}(k)$ with

- (1) objects: pairs (X, p) with $X \in \mathrm{SmProj}(k)$ and p a projector.
- (2) morphisms: $\mathrm{Hom}((X, p), (Y, q)) := q \circ \mathrm{Corr}_\sim^0(X, Y) \circ p$.

Exercise 3.6. Show that the mapping $X \mapsto (X, \Delta_X)$ realizes $Z_\sim \mathrm{SmProj}(k)$ as a full subcategory of $\mathrm{Mot}_\sim^{\mathrm{eff}}(k)$.

Finally we want to include duals (i.e. Tate twists):

Step 3: The category of pure motives $\mathrm{Mot}_\sim(k)$ with

¹⁵The method of passing from **Step 1** to **Step 2** is an instance of a more general idea of taking a pseudo-abelian completion of an additive category.

- (1) objects: triples (X, p, m) with (X, p) an object of $\text{Mot}_{\sim}^{\text{eff}}(k)$ and $m \in \mathbb{Z}$.
- (2) morphisms: $\text{Hom}((X, p, m), (Y, q, n)) := q \circ \text{Corr}_{\sim}^{n-m}(X, Y) \circ p$

Exercise 3.7. Show that the mapping $(X, p) \mapsto (X, p, 0)$ realizes $\text{Mot}_{\sim}^{\text{eff}}(k)$ as a full subcategory of $\text{Mot}_{\sim}(k)$.

The category $\text{Mot}_{\sim}(k)$ has a natural structure of a symmetric monoidal category with duals. We touch on this in the next example.

Example 3.8. By Exercises 3.6 and 3.7, we get that

$$\text{End}_{\text{Mot}_{\sim}(k)}((\mathbb{P}^1, \Delta_{\mathbb{P}^1})) = \text{Corr}^0(\mathbb{P}^1, \mathbb{P}^1) = \mathbb{Q} \oplus \mathbb{Q}.$$

Moreover $\Delta_{\mathbb{P}^1} = e_0 \oplus e_1$ with $e_0 = \{0\} \times \mathbb{P}^1$ and $e_1 = \mathbb{P}^1 \times \{0\}$ and this allows us to write

$$(\mathbb{P}^1, \Delta_{\mathbb{P}^1}) := \mathbf{1} \oplus \mathbb{L} \tag{3.1}$$

where $\mathbf{1} = (\text{Spec}(k), \text{id})$ corresponds to the motive of a point and $\mathbb{L} = (\mathbb{P}^1, \mathbb{P}^1 \times \{0\})$ the Lefschetz motive. Note that (3.1) is a definition that follows from pseudo-abelian completion. It's then an exercise¹⁶ to show that $\mathbb{L} \cong (\text{Spec}(k), \text{id}, -1) =: \mathbf{1}(-1)$. The dual $\mathbf{1}(1) := (\text{Spec}(k), \text{id}, 1)$ is called the Tate motive. In general the definition of the dual of (X, Δ_X) is $(X, \Delta_X) \otimes \mathbb{L}^{-d}$, where d is the dimension of X .

Definition 3.9 (symmetric monoidal structure). $(X, p, m) \otimes (Y, q, n) := (X \times Y, p \times q, m + n)$

Exercise 3.10. Show that $Z_{\sim}^r(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{Hom}(\mathbb{L}^r, (X, \Delta_X))$.

Under the isomorphism given by Exercise 3.10, for $\alpha \in Z_{\sim}^r(X)$, we write $\alpha_* \in \text{Hom}(\mathbb{L}^r, (X, \Delta_X))$ for the corresponding morphism. The corresponding dual morphism $\alpha^*: (X, \Delta_X) \otimes \mathbb{L}^r \rightarrow \mathbb{L}^{\dim(X)}$.

Definition 3.11 (Chow motives and Grothendieck's (numerical) motives). We denote by $\text{CHM}(k) := M_{\text{rat}}(k)$ the category of Chow motives and $\text{NM}(k) := M_{\text{num}}(k)$ the category of Grothendieck motives (or numerical motives).

The next result is due to Scholl [Sch94, Corollary 3.5].

Proposition 3.12. Assume that k is not contained in the algebraic closure of a finite field. Then the category of Chow motives $\text{CHM}(k)$ is not an abelian category.

Proof. Given the conditions on k , there exists an elliptic curve E/k of positive rank¹⁷. Let $P \in E(k)$ be a point of infinite order. Then by writing the Δ_E as in 1.2, we obtain a decomposition (again by definition of pseudo-abelian completion)

$$(E, \Delta_E) = \mathbf{1} \oplus \mathbb{L} \oplus h^1 E.$$

Then the divisor $[P] - [0]$ is a point on the Jacobian $J(E)$ and determines a non-zero morphism $\eta_*: \mathbb{L} \rightarrow h^1(E)$ by

Lemma 3.13. We have an isomorphism $\text{Hom}(\mathbb{L}, h^1(E)) \cong J(E)(k) \otimes \mathbb{Q}$.

Proof. By Exercise 3.10, we have $\text{Hom}(\mathbb{L}, (E, \Delta_E)) \cong Z_{\text{rat}}^1(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. On the other hand

$$\text{Hom}(\mathbb{L}, \mathbf{1}) = \text{Hom}((\text{Spec}(k), \text{id}, -1), (\text{Spec}(k), \text{id}, 0)) \subset \text{Corr}_{\text{rat}}^1(k, k) = 0,$$

and

$$\text{Hom}(\mathbb{L}, \mathbb{L}) = \text{Hom}(\mathbf{1}, \mathbf{1}) = \mathbb{Q}.$$

The projection morphism $Z_{\text{rat}}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q}$ is just the degree map. Thus $\text{Hom}(\mathbb{L}, h^1(E))$ identifies as the kernel of this map which gives the result. \square

¹⁶Take a look at [Sta18, Tage 0FGD].

¹⁷I can't find a reference for this, unless k is a number field.

The composite $\eta_* \circ \eta^*: h^1(E) \otimes \mathbb{L} \rightarrow h^1(E)$. Note that

$$\mathrm{Hom}(h^1(E) \otimes \mathbb{L}, h^1(E)) \subset \mathrm{Hom}((E, \Delta_E, -1), (E, \Delta_E, 0)) \subset \mathrm{Corr}^1(E, E) = Z_{\mathrm{rat}}^2(E \times E).$$

and it is a check to see that $\eta_* \circ \eta^*$ corresponds to zero-cycle $c = (P, P) + (0, 0) - (P, 0) - (0, P)$. Assume $P = 2Q$ for $Q \in E(k)$. Then in $Z_{\mathrm{rat}}^2(E \times E)$ we can write

$$c = [(P, P) + (0, 0) - 2(Q, Q)] + [2(Q, Q) - (P, 0) - (0, P)]$$

and this is rationally equivalent to zero. Thus $\eta_* \circ \eta^* = 0$. This means η_* is not a monomorphism. If $\mathrm{CHM}(k)$ were abelian, then $\ker(\eta_*)$ would be a proper subobject of \mathbb{L} . Tensoring by the Tate motive would give a proper subobject of $\mathbf{1}$. But the unit object in an abelian category with a symmetric monoidal structure is completely decomposable and this gives a contradiction since $\mathrm{End}(\mathbf{1}) = \mathbb{Q}$ (the only idempotents of \mathbb{Q} are 0 and 1). \square

Proposition 3.14. *A Weil cohomology theory*

$$H: \mathrm{SmProj}(k)^{\mathrm{opp}} \rightarrow \mathrm{GrVect}_F$$

factorizes as

$$\begin{aligned} H: \mathrm{SmProj}(k)^{\mathrm{opp}} &\rightarrow \mathrm{Mot}_{\mathrm{rat}}(k) \xrightarrow{G} \mathrm{GrVect}_F \\ X &\mapsto (X, \Delta_X, 0) \end{aligned}$$

Furthermore G precisely corresponds to the datum of H iff $G(\mathbf{1}(1))$ is non-zero only in degree -2 .

The next Theorem is an important result due to Jannsen [Jan92] and arguably the most important result concerning pure motives:

Theorem 3.15. *Let \sim be any adequate equivalence relation. TFAE*

- (1) $\mathrm{Mot}_{\sim}(k)$ is an abelian semi-simple category.
- (2) \sim is numerical equivalence
- (3) For all $X \in \mathrm{SmProj}(k)$ of pure dimension, $\mathrm{Corr}^0(X, X)$ is a finite-dimensional semi-simple \mathbb{Q} -algebra.

Proof. (1) \implies (2): Suppose for the sake of contradiction that $\mathrm{Mot}_{\sim}(k)$ is abelian and semi-simple but $Z_{\sim,0}(X) \neq Z_{\mathrm{num},0}(X)$. On the other hand we know that by Exercise 2.32: $Z_{\sim,0}(X) \subset Z_{\mathrm{num},0}(X)$. So take $Z \in Z_{\mathrm{num},0}^i(X)$ but $Z \notin Z_{\sim,0}^i(X)$. This Z gives a non-zero morphism

$$f: \mathbf{1} = (\mathrm{Spec}(k), \mathrm{id}, 0) \rightarrow (X, \mathrm{id}, i)$$

in $\mathrm{Mot}_{\sim}(k)$. Since $\mathrm{Mot}_{\sim}(k)$ is abelian and semi-simple, there is a morphism

$$g: (X, \mathrm{id}, i) \rightarrow \mathbf{1}$$

such that $g \circ f = \mathrm{id}_{\mathbf{1}}$. Such a g is given by $W \in Z_{\sim}^{d-i}(X)$. Then by the definition of composition of correspondences

$$\begin{aligned} g \circ f &= \mathrm{pr}_{\mathrm{Spec}(k) \times \mathrm{Spec}(k),*}((\mathrm{Spec}(k) \times Z \times \mathrm{Spec}(k)) \cdot (\mathrm{Spec}(k) \times W \times \mathrm{Spec}(k))) \\ &= \deg(Z \cdot W) \mathrm{Spec}(k) \times \mathrm{Spec}(k) \end{aligned}$$

where the second equality is by definition of degree as pushforward onto a point. But $g \circ f = \mathrm{id}_{\mathbf{1}}$ and so $\deg(Z \cdot W) = 1$. But this contradicts $Z \in Z_{\mathrm{num},0}^i(X)$.

(2) \implies (3): Fix a Weil cohomology theory (in this case we take étale cohomology with coefficients \mathbb{Q}_ℓ where $\ell \neq \text{char}(k)$) and recall the cycle map (ignoring Tate twist)

$$\gamma_X: \text{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H^{2i}(X)$$

and define $A^i(X) := \text{im}(\gamma_X) \subset H^{2i}(X)$ and set $B^i(X) := Z_{\text{num}}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. By Lemma 2.31 or Exercise 2.32, we get a surjection

$$A^i(X) \twoheadrightarrow B^i(X).$$

Let $d = \dim(X)$. We need to show $B^d(X \times X)$ is a finite-dimensional semi-simple \mathbb{Q} -algebra. By the above surjectivity statement, since $A^d(X \times X)$ is finite-dimensional, so is $B^d(X \times X)$. It remains to show it is semi-simple.

Lemma 3.16. *$B^d(X \times X)$ is a semi-simple \mathbb{Q} -algebra.*

Proof. By standard results of non-commutative algebra, it suffices to show $J(B^d(X \times X)) = 0$ where $J(R)$ is the Jacobson radical¹⁸. Since $J(B^d(X \times X)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = J(B^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$, we reduce to showing

$$J(B^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) = 0.$$

So put $A = A^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, $B = B^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ and $J_A = J(A)$ and $J_B = J(B)$. We have a surjection

$$\Phi: A \twoheadrightarrow B$$

and we need to show $J_B = 0$. By formal arguments one shows $\Phi(J_A) = J_B$. So take $f_B \in J_B$, which lifts to $f_A \in J_A$ and so f_A is nilpotent in A (as J_A is nilpotent ideal). Then for any $g \in A$ the Lefschetz trace formula¹⁹ gives

$$\text{Tr}(f_A \cup g) = \sum_{i=0}^{2d} (-1)^i \text{Tr}_{H_{\text{ét}}^i(X, \mathbb{Q}_\ell)}(f_A \circ g) \quad (3.2)$$

Since the Jacobson radical is a two-sided ideal, we get $f_A \circ g \in J_A$. So the RHS of (3.2) vanishes. But $\text{Tr}(f_A \cup g) = \deg(f_A \cdot g)$. So this means f_A is numerically 0 and so $f_B = 0$, as desired. \square

(3) \implies (1). We won't prove this. \square

3.1 Manin's identity principle and Lieberman's lemma

We now change gears and ask ourselves how to tell whether a correspondence is trivial or not. The next example shows something funny can happen.

Example 3.17 (Detection of trivial correspondences). *Consider an elliptic curve E/k and four different points $a, b, c, d \in E(k)$. Then consider $p = \{a - b\} \times \{c - d\} \in \text{CH}^2(E \times E)$ is not zero. Viewing $p \in \text{Corr}_{\text{rat}}^1(E, E)$, we get an induced map*

$$\begin{aligned} p_*: \text{CH}^i(E) &\rightarrow \text{CH}^{i+1}(E) \\ T &\mapsto pr_{E*}(p \cdot (T \times E)) \end{aligned}$$

for every $i \geq 0$. Clearly $p_* = 0$ (recall that any two points are algebraically equivalent so $a \cdot T = b \cdot T$).

Manin's identity principle [Man68, pg. 450] gives some characterization of detecting non-trivial correspondences. To state it, we need to think of a correspondence as a functor of points (just like schemes). For the following assume we are working with the rational adequate equivalence (for simplicity):

¹⁸The Jacobson radical of a ring R , $J(R) := \{r \in R \mid rM = 0 \text{ for all } M \text{ simple}\}$.

¹⁹Such a formula is a formal consequence of the Weil cohomology theory axioms.

Definition 3.18 (correspondence as a functor of points). Given $T \in \text{SmProj}(k)$, we put $X(T) := \text{Corr}(T, X)$. Then for $f \in \text{Corr}(X, Y)$, we get the induced morphism

$$\begin{aligned} f_T : X(T) &\rightarrow Y(T) \\ \alpha &\mapsto f \circ \alpha \end{aligned}$$

Theorem 3.19 (Manin's identity principle). *Let $f, g \in \text{Corr}(X, Y)$. TFAE*

- (1) $f = g$
- (2) $f_T = g_T$ for all $T \in \text{SmProj}(k)$
- (3) $f_X = g_X$

Proof. The only non-trivial direction is (3) \implies (1). But one can check that $f = f \circ \Delta_X$, which gives the result. \square

In practice we need Lieberman's lemma to actually make use of Manin's identity principle.

Lemma 3.20 (Lieberman's lemma). *In the setting of Definition 3.18 $f \circ \alpha = (\Delta_T \times f)_*(\alpha)$.*

Proof. By definition of action of correspondences:

$$(\Delta_T \times f)_*(\alpha) = p_{TY*}^{TTXY}((\Delta_T \times f) \cdot (p_{TX}^{TTXY*}(\alpha))) \quad (3.3)$$

where

$$p_{TY}^{TTXY} : T \times \underline{T} \times X \times \underline{Y} \rightarrow \underline{T} \times \underline{Y} \text{ and } p_{TX}^{TTXY} : \underline{T} \times T \times \underline{X} \times Y \rightarrow \underline{T} \times \underline{X}.$$

We can rewrite (3.3) as

$$\begin{aligned} p_{TY*}^{TTXY}((\Delta_T \times f) \cdot (p_{TX}^{TTXY*}(\alpha))) &= p_{TY*}^{TTXY}((\Delta_T \times f) \cdot (T \times \alpha \times Y)) \\ &= p_{TY*}^{TTXY}((\Delta_T \times X \times Y) \cdot (T \times T \times f) \cdot (T \times \alpha \times Y)) \end{aligned}$$

Now writing p_{TY}^{TTXY} as the composition:

$$p_{TY}^{TTXY} : T \times T \times X \times Y \xrightarrow{p} T \times X \times Y \xrightarrow{q} T \times Y$$

we get

$$\begin{aligned} p_{TY*}^{TTXY}((\Delta_T \times X \times Y) \cdot (T \times T \times f) \cdot (T \times \alpha \times Y)) &= q_* p_*((\Delta_T \times X \times Y) \cdot p^*(T \times f) \cdot (T \times \alpha \times Y)) \\ &= q_*(p_*((\Delta_T \times X \times Y) \cdot (T \times \alpha \times Y))) \cdot (T \times f)) \\ &= q_*(((\alpha \circ \Delta_T) \times Y) \cdot (T \times f)) \\ &= f \circ \alpha \circ \Delta_T \\ &= f \circ \alpha. \end{aligned}$$

where the second isomorphism follows from the projection formula (cf. Exercise 2.6) and the third/fourth follow from definition of composition of correspondences. The final equality is just that Δ_T acts as identity when composing. \square

Corollary 3.21. *In the context of Manin's identity principle (cf. Theorem 3.19), we get $f = g$ iff*

$$(\text{id}_T \times f)_* = (\text{id}_T \times g)_*$$

considered as maps on the Chow groups

$$CH(T \times X) \rightarrow CH(T \times Y) \quad \forall T$$

As an application of Manin's identity principle, we sketch the proof of the following:

Lemma 3.22. *Let \mathcal{E} be a locally free sheaf of rank $(m+1)$ on $S \in \text{SmProj}(k)$ and let $\pi: \mathbb{P}_S(\mathcal{E}) \rightarrow S$ be the associated projective bundle. Then there is an isomorphism of motives in $\text{Mot}_{\text{rat}}(k)$*

$$(\mathbb{P}_S(\mathcal{E}), \Delta_{\mathbb{P}_S(\mathcal{E})}, 0) \xrightarrow{\sim} \bigoplus_{i=0}^m (S, \Delta_S, -i)$$

Proof. Let $\xi = \mathcal{O}(1)$ be the tautological line bundle on $\mathbb{P}_S(\mathcal{E})$. Then there is a *projective space bundle formula* [Sta18, Tag 0ERV]:

$$\lambda: CH(\mathbb{P}_S(\mathcal{E})) \xrightarrow{\sim} \bigoplus_{i=0}^m CH(S)[\xi^i].$$

Moreover the isomorphism λ (and it's inverse μ) are induced by correspondences. Also the morphism λ and μ are compatible with base change $T \rightarrow \text{Spec}(k)$.

So this means that $(\text{id}_T \times \lambda) \circ (\text{id}_T \times \mu) = \text{id}$ for all T . The result then follows by Corollary 3.21. \square

3.2 $M_{\text{rat}}(k)$ vs category of abelian varieties up to isogeny

We prove that the category of Chow motives contains as a full subcategory the category of abelian varieties up to isogeny.

Recall from the proof of Proposition 3.12 for any curve $X \in \text{SmProj}(k)$, we can write

$$(X, \Delta_X) = \mathbf{1} \oplus \mathbb{L} \oplus h^1 X. \quad (3.4)$$

Then in the spirit of Lemma 3.13 we have

Proposition 3.23. *Given two curves $X, X' \in \text{SmProj}(k)$ we have*

$$\text{Hom}(h^1 X, h^1 X') = \text{Hom}_{\text{AV}}(J(X), J(X')) \otimes_{\mathbb{Z}} \mathbb{Q}$$

.

Proof. By Weil's theorem [Wei71, Theorem 22, Chapitre VI]

$$Z_{\text{rat}}^1(X \times X') \otimes_{\mathbb{Z}} \mathbb{Q} = (Z_{\text{rat}}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (Z_{\text{rat}}^1(X') \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus \text{Hom}_{\text{AV}}(J(X), J(X')) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Note that $Z_{\text{rat}}^1(X \times X') \otimes_{\mathbb{Z}} \mathbb{Q} = \text{Hom}((X, \Delta_X), (X', \Delta_{X'}))$. By using the decomposition (3.4), it's a check to get the result. \square

To get the result one needs the Poincaré reducibility theorem [Mum74, Chapter IV, §19, Theorem 1]:

$$\{\text{category of AV}\}/\text{isogeny} = \text{pseudo-abelian completion of } \{J(C)|C \text{ curve}\}.$$

4 Lecture 4: Grothendieck's standard conjectures

Up to this point we have defined motives. Motives are expected to have good properties, but it turns out that these are still open. In this lecture, we will discuss the so-called *standard conjectures* concerning motives. These were originally formulated by Grothendieck in [Gro69]. In this lecture we will discuss some results in [Kle68] and [Kle94]. We have already seen standard conjecture D in (cf. Conjecture 2.33). In this lecture we will take a look at the remaining standard conjectures:

- (1) Standard Conjectures C (Künneth Conjecture)
- (2) Standard Conjectures A and B (Conjectures of Lefschetz type)
- (3) Standard Conjecture H (Conjecture of Hodge type)

Let $X \in \text{SmProj}(k)$. We fix a Weil cohomology $H(X)$ over a characteristic 0 field and recall we have

$$\gamma_X: CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H^{2i}(X)$$

and define $A^i(X) := \text{im}(\gamma_X) \subset H^{2i}(X)$. We call the elements of $A^i(X)$ the *algebraic* classes.

4.1 Künneth conjecture (Standard conjecture C)

Assume X is pure of dimension d . Let $\Delta_X \in CH^d(X \times X)$ be the diagonal and consider its class

$$\gamma_{X \times X}(\Delta_X) \in H^{2d}(X \times X) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X)$$

where the equality is the Künneth decomposition (cf. axiom (4) in Definition 2.24). So we can write

$$\gamma_{X \times X}(\Delta_X) = \pi_0 + \pi_1 + \dots + \pi_i + \dots + \pi_{2d}$$

with $\pi_i \in H^{2d-i}(X) \otimes H^i(X)$.

Conjecture 4.1 (Künneth conjecture). *The Künneth components π_i are algebraic: \exists cycle classes $\Delta_i \in CH^d(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\gamma_{X \times X}(\Delta_i) = \pi_i$.*

Exercise 4.2. *Let X be a scheme with a cellular decomposition: that is there exists a filtration*

$$X = X_n \supset X_{n-1} \supset \dots \supset X_0 \supset X_{-1} = \emptyset$$

by closed subschemes with each $X_i \setminus X_{i-1}$ a disjoint union of schemes U_{ij} isomorphic to affine spaces $\mathbb{A}^{n_{ij}}$. Then $CH^k(X)$ is finitely generated by $\{[V_{ij}]\}$, where V_{ij} is the closure of U_{ij} in X . Show in this case one has

$$CH(X \times X) \cong CH(X) \otimes_{\mathbb{Z}} CH(X).$$

Exercise 4.3. *Show that any $X \in \text{SmProj}(k)$ which satisfies the Chow-Künneth decomposition:*

$$CH(X \times X) \cong CH(X) \otimes_{\mathbb{Z}} CH(X)$$

implies that γ_X is in fact an isomorphism. Use this to show that for such X , the Künneth conjecture (trivially) holds.

Remark 4.4. *Projective space \mathbb{P}^n satisfies the condition of Exercise 4.2. In general if X is a linear scheme, then it satisfies the conditions of Exercise 4.3 (cf. [Tot14, Proposition 1]).*

The next proposition is less trivial and is due to Katz-Messing [KM74, Theorem 2 part 1]).

Proposition 4.5. *Suppose $k = \mathbb{F}_q$ and $X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ is irreducible. Then the Künneth conjecture holds for X .*

Proof. Fix a prime $\ell \neq p = \text{char}(\mathbb{F}_q)$ and let Fr be the relative Frobenius morphism of X over \mathbb{F}_q . Deligne has proved (cf. [Del74a, Théorème I.6]), as part of his proof of the Weil conjectures that the polynomial in T

$$\det(1 - T \text{Fr} \mid H_{\text{ét}}^i(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_{\ell})) \quad (4.1)$$

lies in $\mathbb{Z}[T]$ and its reciprocal zeros all have complex absolute value $q^{i/2}$ for every $i \geq 0$. As a first step Katz-Messing (cf. [KM74, Theorem 1]) show that the term (4.1) is independent of the Weil cohomology theory, that is:

Lemma 4.6. *We have $\det(1 - TFr \mid H_{\acute{e}t}^i(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)) = \det(1 - TFr \mid H^i(X))$ where $H^i(X)$ is our chosen Weil cohomology theory from the start of this lecture.*

Proof. We won't prove this, but let me mention that it relies on Poincaré duality and the weak Lefschetz axiom. \square

It follows that the polynomials $G_i(T) = \det(1 - TFr \mid H^i(X))$ are pairwise relatively prime in $\mathbb{Q}[T]$ because their roots have different absolute value. Let $\Pi_i(T) \in \mathbb{Q}[T]$ be a polynomial such that

$$G_j(T) \mid \Pi_i(T) \text{ for all } j \neq i \text{ and } \Pi_i(T) \equiv 1 \pmod{G_i(T)}.$$

Such a polynomial exists by the Chinese remainder theorem. By the Cayley-Hamilton theorem, it follows that the operator

$$\Pi_i(\text{Fr}^{-1}): \bigoplus_{j=0}^{2d} H^j(X) \rightarrow H^i(X) \quad (4.2)$$

is exactly the projection operator and these are algebraic. But note that by Poincaré duality we can rewrite the Künneth formula as

$$H^{2d}(X \times X) = \text{Hom}_{\text{GrVect}_{\overline{\mathbb{F}}}^{\geq 0}}(H(X), H(X)).$$

This means that by (4.2) $\gamma_{X \times X}(\Delta_X) = p_1 + \dots + p_{2d}$, where $p_i \in \text{Corr}^0(X)$ corresponding to (the graph of) $\Pi_i(\text{Fr}^{-1})$. \square

Exercise 4.7. *Using decomposition (3.4), show that the Künneth conjecture holds for curves.*

4.2 Conjectures of Lefschetz type (Standard conjectures A and B)

Assume X is pure of dimension d and let $Y \hookrightarrow X$ be a smooth hyperplane section. Recall the Lefschetz operator

$$\begin{aligned} L: H^i(X) &\rightarrow H^{i+2}(X) \\ \alpha &\mapsto \alpha \cup \gamma_X(Y). \end{aligned}$$

Recall for H a Weil cohomology theory, we assume hard Lefschetz (cf. Definition 2.24(7))

$$L^i: H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X).$$

Lemma 4.8. *The Lefschetz operator L is algebraic. More precisely it is represented by the algebraic cycle $\Delta_*(Y) \in A^{d+1}(X \times X)$ and $\Delta: X \rightarrow X \times X$ is the diagonal map.*

Proof. For $u = \Delta_*(Y)$, it suffices to show

$$\gamma_X(p_{2*}(p_1^* \alpha \cdot u)) = \alpha \cup \gamma_X(Y).$$

Indeed we have

$$\begin{aligned} \gamma_X(p_{2*}(p_1^* \alpha \cdot u)) &= \gamma_X(p_{2*}(p_1^* \alpha \cdot \Delta_* Y)) \\ &= p_{2*}(p_1^* \alpha \cup \Delta_* Y) \\ &= \alpha \cup p_{2*} \Delta_* Y \end{aligned}$$

where the second equality follows by compatibility of γ_X and pushforwards (and intersection products), the third equality by a version of the projection formula (cf. Exercise 2.6). But note that $p_2 \circ \Delta = \text{id}_X$, so we are done. \square

Using Hard Lefschetz we can define a unique linear map $\Lambda: H^i(X) \rightarrow H^{i-2}(X)$ for each $2 \leq i \leq 2d$ as follows:

1) For $2 \leq i \leq d$ which makes the following diagram commutative:

$$\begin{array}{ccc} H^i(X) & \xrightarrow{L^{d-i}} & H^{2d-i}(X) \\ \Lambda \downarrow & & \downarrow L \\ H^{i-2}(X) & \xrightarrow{L^{d-i+2}} & H^{2d-i+2}(X). \end{array}$$

2) For $i = d + 1$, $\Lambda := L^{-1}$ where $L: H^{d-1}(X) \xrightarrow{\sim} H^{d+1}(X)$.

3) For $d + 2 \leq i \leq 2d$ which makes the following diagram commutative:

$$\begin{array}{ccc} H^{2d-i}(X) & \xrightarrow{L^{i-d}} & H^i(X) \\ L \downarrow & & \downarrow \Lambda \\ H^{2d-i+2}(X) & \xrightarrow{L^{i-d-2}} & H^{i-2}(X). \end{array}$$

By Poincaré duality and Künneth formula, we have

$$\Lambda \in \text{Hom}(H^i(X), H^{i-2}(X)) = H^{2d-i}(X) \otimes H^{i-2}(X) \subset H^{2d-2}(X \times X).$$

So we can view Λ canonically as an element in $H^{2d-2}(X \times X)$.

Conjecture 4.9 (Standard conjecture B). *The operator Λ is algebraic: \exists cycle $Z \in CH^{d-1}(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\gamma_{X \times X}(Z) = \Lambda$.*

Lemma 4.10 (Conjecture B implies the Künneth conjecture). *Standard conjecture B implies the Künneth conjecture*

Proof.

Exercise 4.11. *Verify the formula*

$$\pi_i = \Lambda^{d-i} \left(1 - \sum_{j > 2d-i} \pi_j \right) L^{d-i} \left(1 - \sum_{j < i} \pi_j \right)$$

where $\Lambda^{d-i}: H^{2d-i}(X) \xrightarrow{\sim} H^i(X)$ (an inverse to L^{d-i}).

We can then proceed by induction as π_0 and π_{2d} are algebraic. \square

It turns out that Standard conjecture B is independent of the choice of hyperplane section which defines L (and hence Λ), cf. [Kle94, Theorem 4.1(2)].

Proposition 4.12. *If standard conjecture B holds for one choice of L , then it holds for all choices.*

Proof. It suffices to show that standard conjecture B is equivalent to the following statement (which is independent of L):

For each $i \leq d$, there exists an algebraic correspondence

$$\nu_i: H^{2d-i}(X) \xrightarrow{\sim} H^i(X). \quad (4.3)$$

Indeed if standard conjecture B is true, then Λ is algebraic and thus by taking a sufficiently large composition Λ^i is also algebraic and induces the above isomorphism.

For the converse suppose (4.3) holds. Then $u := \nu_i \circ L^{d-i}$ is algebraic.

Exercise 4.13. By looking at the characteristic polynomial of u , show that u^{-1} is algebraic.

By exercise 4.13, it follows that $\theta_i := u^{-1} \circ \nu_i$ is an algebraic inverse of L^{d-i} . The result then follows from the following exercise:

Exercise 4.14. Show that

$$\Lambda := \sum_{i \leq d} (\pi_{i-1} \theta_i L^{d-i+1} \pi_i + \pi_{2d-i} L^{d-i+1} \theta_{i+2} \pi_{2d-i+2}).$$

□

Remark 4.15. Standard conjecture B holds true for abelian varieties. This result is due to Lieberman-Kleiman [Kle68, Theorem 2A11].

To state standard conjecture A, note that we have a commutative diagram (for $d \geq 2i$)

$$\begin{array}{ccc} A^i(X) & \longrightarrow & A^{d-i}(X) \\ \downarrow & & \downarrow \\ H^{2i}(X) & \xrightarrow{L^{d-2i}} & H^{2d-2i}(X). \end{array} \quad (4.4)$$

The top arrow $A^i(X) \rightarrow A^{d-i}(X)$ exists because L is algebraic (cf. Lemma 4.8). In fact is injective.

Conjecture 4.16 (Standard Conjecture A). *Hard Lefschetz is true on cycles. That is the top arrow $A^i(X) \hookrightarrow A^{d-i}(X)$ in diagram (4.4) is an isomorphism.*

It turns out that Standard conjectures A and B are equivalent, cf. [Kle94, Corollary 4.2]

Proposition 4.17. *Standard conjecture A holds iff standard conjecture B holds.*

Proof. (B) \implies (A): Indeed if (B) is true then we get an algebraic inverse to L^{d-2i} (given by Λ^{d-2i}) and so we get an inverse map $A^{d-i}(X) \rightarrow A^i(X)$. Thus (A) is true.

(A) \implies (B): We won't prove this in detail but highlight some steps.

Exercise 4.18. Show that each $x \in H^i(X)$ has a unique decomposition

$$x = \sum_{j \geq \max(i-d, 0)} L^j x_j$$

where $x_j \in \ker(L|H^{i-2j}(X))$.

Given Exercise 4.18 we can define an operator ${}^c\Lambda$ given by:

$${}^c\Lambda(x) := \sum_{j \geq \max(i-d, 1)} j(n-i+j+1) L^{j-1} x_j$$

Exercise 4.19. Show that Λ is algebraic iff ${}^c\Lambda$ is algebraic.

Since we have assumed conjecture A is true, in particular it is true for $X \times X$ equipped with Lefschetz operator $1 \otimes L + L \otimes 1$. Then [Kle68, Proposition 1.4.6(ii) and Proposition 2.1] implies that $1 \otimes {}^c\Lambda + {}^c\Lambda \otimes 1$ carries algebraic classes to algebraic classes (a priori it is only defined at the level of cohomology). Moreover Proposition 1.3.4 in loc.cit. shows that it carries Δ_X to $2{}^c\Lambda$. Therefore ${}^c\Lambda$ is algebraic and we are done by Exercise 4.19.

□

4.3 Hodge standard conjecture

In general by hard Lefschetz

$$L^{d-i}: H^i(X) \xrightarrow{\sim} H^{2d-i}(X)$$

is an isomorphism. However in the spirit of Exercise 4.18:

Definition 4.20 (primitive cohomology). We define primitive cohomology as

$$P^i(X) := \ker(L^{d-i+1}: H^i(X) \rightarrow H^{2d-i+2}(X)).$$

Definition 4.21 (primitive algebraic classes). We define primitive algebraic classes as

$$A_{\text{prim}}^i(X) := A^i(X) \cap P^{2i}(X)$$

For $i \leq d/2$, the cup product gives a pairing

$$\begin{aligned} A_{\text{prim}}^i(X) \times A_{\text{prim}}^i(X) &\rightarrow \mathbb{Q} \\ (x, y) &\mapsto (-1)^i \text{Tr} \circ (L^{d-2i}(x) \cup y) \end{aligned}$$

Conjecture 4.22 (Hodge standard conjecture). *The pairing defined above is positive definite.*

The next result can be found in [Kle94, Proposition 5.1].

Proposition 4.23. *Given the Hodge standard conjecture, standard conjecture A is equivalent to standard conjecture D.*

Proof. We need to define a version of the Hodge star operator (appearing in Hodge theory):

$$\begin{aligned} *: H^i(X) &\rightarrow H^{2d-i}(X) \\ x &\mapsto \sum_{j \geq \max(i-d, 0)} (-1)^{(i-2j)(i-2j+1)/2} L^{d-i+j} x_j \end{aligned}$$

where the $x_j \in H^{i-2j}(X)$ are those appearing in Exercise 4.18.

Now suppose the Hodge conjecture is true.

Exercise 4.24. *Show that $*^2 = 1$. By using Exercise 4.18 for x , show that the pairing*

$$\begin{aligned} A^i(X) \times A^i(X) &\rightarrow \mathbb{Q} \\ (x, y) &\mapsto \text{Tr}(x \cup *y) \end{aligned}$$

is positive-definite.

(A) \implies (D): Now standard conjecture A implies the canonical pairing (given by cup product) $A^i(X) \times A^{d-i}(X) \rightarrow \mathbb{Q}$ is perfect. Thus if $x \in Z_{\text{num},0}^i(X)$, then $x \in Z_{\text{hom},0}^i(X)$.

(D) \implies (A): In this case we use $A^i(X) \hookrightarrow A^{d-i}(X)$ and again the positive-definiteness of Exercise 4.24.

□

Remark 4.25. *If k is of characteristic zero, then the Hodge standard conjecture is true and is a consequence of Hodge theory.*

5 Lecture 5: Motivic Galois groups

Up to now we have mainly focused on the geometric aspects of motives. In this lecture we start to look at the arithmetic aspects. Grothendieck wanted to build some kind of Galois group coming from a fiber functor²⁰

$$\mathrm{Mot}_{\mathrm{num}}(k) \rightarrow \{\text{category of finite vector spaces over } k\}.$$

Such a fiber functor can come from a Weil cohomology theory, but the issue is that we don't have standard conjecture D and so a priori one gets a fiber functor from $\mathrm{Mot}_{\mathrm{hom}}(k)$. Now the issue is that $\mathrm{Mot}_{\mathrm{hom}}(k)$ is no longer abelian and so not Tannakian.

There are several approaches of circumventing conjecture D and modifying the source category $\mathrm{Mot}_{\mathrm{num}}(k)$ just enough so that one has a fiber functor and the category remains abelian. We will study the approach of Deligne-Milne in [DM82].

5.1 Absolute Hodge cycles

Let k be a field of finite transcendence degree over \mathbb{Q} and $X \in \mathrm{SmProj}(k)$. We set

$$H_{\mathrm{\acute{e}t}}^n(X) := \varprojlim_r H_{\mathrm{\acute{e}t}}^n(X, \mathbb{Z}/r\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \text{ and } H_{\mathrm{\acute{e}t}}^n(X)(1) := H_{\mathrm{\acute{e}t}}^n(X) \otimes_{\mathbb{A}_f} (\varprojlim_r \mu_r \otimes_{\mathbb{Z}} \mathbb{Q})$$

and

$$H_{\mathrm{dR}}^n(X)(m) := H_{\mathrm{dR}}^n(X).$$

Finally we set

$$H_{\mathbb{A}}^n(X)(m) := H_{\mathrm{dR}}^n(X)(m) \times H_{\mathrm{\acute{e}t}}^n(X)(m).$$

Given an embedding $\sigma: k \hookrightarrow \mathbb{C}$, there are canonical isomorphisms:

$$\sigma_{\mathrm{dR}}^*: H_{\mathrm{dR}}^n(X)(m) \otimes_{k, \sigma} \mathbb{C} \xrightarrow{\sim} H_{\mathrm{dR}}^n(\sigma X)(m) \text{ and } \sigma_{\mathrm{\acute{e}t}}^*: H_{\mathrm{\acute{e}t}}^n(X)(m) \xrightarrow{\sim} H_{\mathrm{\acute{e}t}}^n(\sigma X)(m)$$

where $\sigma X := X \times_k \mathbb{C}$. We put $\sigma^* := \sigma_{\mathrm{dR}}^* \times \sigma_{\mathrm{\acute{e}t}}^*$. We now put

$$H_B^n(X)(m) := H_B^n((\sigma X)^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} (2\pi i)^m \mathbb{Q}$$

so that the standard comparison isomorphisms give

$$H_B^n(X)(m) \otimes_{\mathbb{Q}} (\mathbb{C} \times \mathbb{A}_f) \xrightarrow{\sim} H_{\mathrm{dR}}^n(\sigma X)(m) \times H_{\mathrm{\acute{e}t}}^n(\sigma X)(m)$$

Definition 5.1 (Hodge cycle relative to σ). An element $t \in H_{\mathbb{A}}^{2p}(X)(p)$ is a Hodge cycle relative to σ if

- (1) t is rational relative to σ : $\sigma^*(t)$ lies in the rational subspace $H_B^{2p}(X)(p)$ of $H_{\mathrm{dR}}^{2p}(\sigma X)(m) \times H_{\mathrm{\acute{e}t}}^{2p}(\sigma X)(m)$.
- (2) it is of bidegree $(0, 0)$.

Definition 5.2 (Absolute Hodge cycle). An element $t \in H_{\mathbb{A}}^{2p}(X)(p)$ is an absolute Hodge cycle if it is a relative Hodge cycle for every embedding $\sigma: k \hookrightarrow \mathbb{C}$. We denote by $C_{\mathrm{AH}}^p(X)$ the \mathbb{Q} -vector space of absolute Hodge cycles $t \in H_{\mathbb{A}}^{2p}(X)(p)$.

Conjecture 5.3 (Deligne). Assume k is algebraically closed. A Hodge cycle relative to a single embedding σ is a Hodge cycle for every embedding (i.e. an absolute Hodge cycle).

²⁰This is the reason that Grothendieck (and his student Saavedra Rivano) introduced the notion of Tannakian category.

Remark 5.4. Conjecture 5.3 is true for Abelian varieties.

Example 5.5. The cycle class maps

$$\gamma_{dR}: CH^p(X) \rightarrow H_{dR}^{2p}(X)(p) \text{ and } \gamma_{\acute{e}t}: CH^p(X) \rightarrow H_{\acute{e}t}^{2p}(X)(p)$$

and we claim that $t := (\gamma_{dR}(Z), \gamma_{\acute{e}t}(Z))$ is an absolute Hodge cycle. Indeed for any $\sigma: k \hookrightarrow \mathbb{C}$, $\sigma^*(t) = \gamma_B(\sigma Z)$. This is because the cycle class maps are all compatible via the comparison isomorphisms cf. [Del71, 2.2.5.1]. In addition it is of bidegree (p, p) by a calculation. The Hodge conjecture predicts that there are no other absolute Hodge cycles.

Definition 5.6 (False category of motives). By repeating the procedure of taking the pseudoabelian completion with morphisms given by $C_{\text{AH}}^p(X \times Y)$, we get the false category of motives \dot{M}_k . More precisely this is the category given by

- (1) **objects:** triples (X, p, m) with $X \in \text{SmProj}(k)$, $p \in C_{\text{AH}}^d(X \times X)$ a projector ($d = \dim(X)$) and $m \in \mathbb{Z}$.
- (2) **morphisms:** $\text{Hom}((X, p, m), (Y, q, n)) := q \circ C_{\text{AH}}^{n-m+d}(X \times Y) \circ p$ with composition given by cup product.

To state the main theorem regarding \dot{M}_k , let us recall the notion of a Tannakian category. For a more comprehensive treatment of Tannakian categories, cf. [SR72].

Definition 5.7. Let \mathcal{C} be a (rigid) abelian²¹ tensor category with $\text{End}(\mathbf{1}) = k$.

- (1) **fibre functor:** A fibre functor on \mathcal{C} with values in a k -algebra R is a k -linear exact faithful tensor functor

$$\eta: \mathcal{C} \rightarrow \text{Mod}_R$$

that takes values in the subcategory Proj_R .

- (2) **Tannakian category:** \mathcal{C} is a Tannakian category over k if it admits a fibre functor with values in some nonzero k -algebra. It is a neutral Tannakian category if $R = k$.

The main property of Tannakian categories is that they are essentially representations of some group.

Theorem 5.8. Let \mathcal{C} be a Tannakian category over k . Then there exists a stack \mathcal{G} in groupoids and a canonical k -linear tensor functor

$$\mathcal{C} \rightarrow \text{Rep}_k(\mathcal{G})$$

which is an equivalence of categories²². If \mathcal{C} is a neutral Tannakian category then \mathcal{G} is represented by an affine group scheme.

Returning back to our false category of motives \dot{M}_k , it turns out for some technical issue, it cannot be Tannakian. To fix this one changes the commutativity constraint²³ as follows: Let

$$\dot{\psi}: M \otimes N \rightarrow N \otimes M \text{ where } \dot{\psi} = \oplus \dot{\psi}^{r,s} \text{ where } \dot{\psi}^{r,s} := M^r \otimes N^s \rightarrow N^s \otimes M^r$$

be the commutativity constraint. An explanation of the notation is in order: the grading $\dot{\psi}^{r,s}$ is coming from the grading induced on morphisms by

$$C_{\text{AH}}^{p+d}(X \times Y) \subset H_{\text{AH}}^{2d+2p}(X \times Y)(p+d) = \oplus_r \text{Hom}(H_{\text{AH}}^r(X), H_{\text{AH}}^{r+2p}(Y)(p)). \quad (5.1)$$

We now modify the commutativity constraint as

$$\psi: M \otimes N \rightarrow N \otimes M \text{ where } \psi = \oplus \psi^{r,s} \text{ where } \psi^{r,s} := (-1)^{rs} \dot{\psi}^{r,s} \quad (5.2)$$

²¹this essentially means that \mathcal{C} is equipped with a tensor product, has internal homs and duals.

²²Here $\text{Rep}_k(\mathcal{G})$ the category of cartesian functors $\mathcal{G} \rightarrow \text{Proj}$, where Proj is the stack such that $\text{Proj}(\text{Spec } R) := \text{Proj}_R$.

²³A commutativity constraint is part of the datum of a tensor category.

Definition 5.9 (True category of motives). We define the true category of motives M_k to be \dot{M}_k with the commutativity constraint ψ .

Proposition 5.10. *The category M_k is a semisimple Tannakian category over \mathbb{Q} .*

Proof. By similar ideas to Jannsen's proof that the category of Grothendieck's motives is an abelian semi-simple category (cf. Theorem 3.15), it suffices to show that $C_{AH}^{\dim(X)}(X \times X)$ is finite-dimensional semi-simple \mathbb{Q} -algebra.

Lemma 5.11. *For every $0 \leq r \leq 2d$, there exists $\psi_X \in C_{AH}^{2d-r}(X \times X)$ such that for every $\sigma: k \hookrightarrow \mathbb{C}$, the induced morphism (cf. (5.1))*

$$\psi^r: H_B^r((\sigma X)^{an}, \mathbb{R}) \times H_B^r((\sigma X)^{an}, \mathbb{R}) \rightarrow \mathbb{R}(-r)$$

*is a polarization of real Hodge structures*²⁴.

Proof. Recall the $*$ -operator appearing in Proposition 4.23

$$*: H_B^r((\sigma X)^{an}, \mathbb{R}) \rightarrow H_B^{2d-r}((\sigma X)^{an}, \mathbb{R})(2d-r).$$

Then we define ψ^r to be the composite (writing X for $(\sigma X)^{an}$ for the sake of brevity)

$$H_B^r(X, \mathbb{R}) \times H_B^r(X, \mathbb{R}) \xrightarrow{\text{id} \times *} H_B^r(X, \mathbb{R}) \times H_B^{2d-r}(X, \mathbb{R})(2d-r) \xrightarrow{\cup} H_B^{2d}(X)(d-r) \xrightarrow{\text{Tr}} \mathbb{R}(-r)$$

Exercise 5.12. *Show by unraveling the definitions that this is indeed an absolute Hodge cycle.*

It is positive-definite due to the Hodge-Riemann bilinear relations (i.e. the Hodge standard conjecture holds true in the complex setting). \square

Lemma 5.13. *Let ψ_X (and ψ_Y) be defined as in Lemma 5.11. For any $u \in C_{AH}^{\dim(Y)}(Y \times X)$, there exists $u' \in C_{AH}^{\dim(X)}(X \times Y)$ such that*

$$\psi_X(uy, x) = \psi_Y(y, u'x)$$

*for all $x \in H_B^r((\sigma X)^{an}, \mathbb{R})$ and $y \in H_B^r((\sigma Y)^{an}, \mathbb{R})$. Moreover*²⁵

$$\text{Tr}(u \circ u') = \text{Tr}(u' \circ u) \in \mathbb{Q} \text{ and } \text{Tr}(u \circ u') > 0 \text{ if } u \neq 0.$$

Proof. For the first part just take u' to be the adjoint of u . Such a u' exists because pairing is non-degenerate. The last part follows from formal properties of a polarization for a real Hodge structure (which I will skip). \square

The previous lemma, then implies that ψ_X is a *Weil* form on X and this implies that $C_{AH}^{\dim(X)}(X \times X)$ is a finite-dimensional semi-simple \mathbb{Q} -algebra (cf. [DM82, Definition 4.1 and Proposition 4.2]). \square

So at this point we know that M_k is a semisimple Tannakian category over \mathbb{Q} . Moreover for a fixed embedding $\sigma: k \hookrightarrow \mathbb{C}$ we have the Weil cohomology functor (given by rational Betti cohomology)

$$H_B: \text{SmProj}(k)^{\text{op}} \rightarrow \text{Vect}_{\mathbb{Q}} \\ X \mapsto \bigoplus_r H_B^r((\sigma X)^{an}, \mathbb{Q})$$

²⁴A polarization on a real Hodge structure V of weight n is a bilinear form $\phi: V \times V \rightarrow \mathbb{R}(-n)$ such that the real-valued form $(x, y) \mapsto (2\pi i)^n \phi(x, iy)$ is positive-definite and symmetric.

²⁵To make sense of this $u \circ u'$ is another correspondence or simply an element of $C_{AH}^{\dim(X)}(X \times X)$ and this acts on the graded space $\oplus_i H_{\mathbb{A}}^i(X)$ and the trace is taken there.

Essentially by formal reasons (cf. Proposition 3.14) this extends to a functor

$$\omega: M_k \rightarrow \text{Vect}_{\mathbb{Q}} \quad (5.3)$$

and it must be exact because every additive functor from a semi-simple abelian category is exact. It is faithful essentially by (5.1).

Definition 5.14 (Motivic Galois group of Deligne-Milne). The datum of ω in (5.3) together with the main theorem of Tannakian categories (cf. Theorem 5.8) gives rise to affine group scheme $G(\sigma)$ over \mathbb{Q} . This is the motivic Galois group of Deligne-Milne.

Proposition 5.15. *The group $G(\sigma)$ is a pro-reductive affine group scheme over \mathbb{Q} .*

Proof. Let $X \in \text{ob}(M_k)$ and let \mathcal{C}_X be the abelian tensor subcategory of M_k generated by X , X^\vee , \mathbb{L} and \mathbb{L}^\vee , where \mathbb{L} is the Lefschetz motive. Then $\omega|_{\mathcal{C}_X}$ is again a fibre functor and applying the main theorem of Tannakian categories gives $\mathcal{C}_X = \text{Rep}_k(G_X)$ for some affine group scheme over \mathbb{Q} . Moreover standard yoga involving Tannakian categories gives

$$G(\sigma) = \varprojlim_X G_X.$$

The fact that G_X are reductive follows from the following fact (cf. [DM82, 6.9]), which we won't prove:

Lemma 5.16. *Let G be a connected affine group scheme over k (a field of characteristic 0). Then G is reductive if and only if*

- (1) $\text{Rep}_k(G)$ has a tensor generator (in our case we can take $X \oplus \mathbb{L}$)
- (2) $\text{Rep}_k(G)$ has no non-trivial object X such that $\langle X \rangle$ is stable under \otimes (this is the reason we took \mathbb{L} in \mathcal{C}_X). Here $\langle X \rangle$ is the full subcategory of \mathcal{C}_X which is a subquotient of powers of X and X^\vee (cf. Definition 6.12 in loc.cit.)
- (3) $\text{Rep}_k(G)$ is semisimple (which is the case as M_k and hence \mathcal{C}_X is semisimple).

This shows that $G(\sigma)$ is pro-reductive. □

5.2 Structure of the motivic Galois group

To describe the motivic Galois group $G(\sigma)$ and compare it to the traditional Galois group $\text{Gal}(\bar{k}/k)$ we need the notion of *Artin motives*:

Definition 5.17 (Artin motives). Let $V_k^0 \subset M_k$ be the image of zero-dimensional varieties over k and let M_k^0 be the Tannakian subcategory of M_k generated by V_k^0 . The category M_k^0 is called the category of Artin motives.

It turns out that V_k^0 is already Tannakian:

Proposition 5.18. *We have that $M_k^0 = V_k^0$ and $M_k^0 \cong \text{Rep}_{\mathbb{Q}}(\text{Gal}(\bar{k}/k))$.*

Proof. Let X be a zero-dimensional variety over k . Then $X(\bar{k})$ is a finite set on which $\text{Gal}(\bar{k}/k)$ acts continuously (cf. [Sta18, Tag 03QR]). Thus $\mathbb{Q}^{X(\bar{k})}$ is a finite-dimensional \mathbb{Q} -representation of $\text{Gal}(\bar{k}/k)$. Let X and Y be zero-dimensional varieties over k . We compute

$$\begin{aligned} \text{Hom}_{M_k}(X, Y) &= C_{\text{AH}}^0(X \times Y) \\ &= (\mathbb{Q}^{X(\bar{k}) \times Y(\bar{k})})^{\text{Gal}(\bar{k}/k)} \\ &= \text{Hom}_{\text{Rep}_{\mathbb{Q}}(\text{Gal}(\bar{k}/k))}(\mathbb{Q}^{X(\bar{k})}, \mathbb{Q}^{Y(\bar{k})}) \end{aligned}$$

Thus

$$\begin{aligned} V_k^0 &\rightarrow \text{Rep}_{\mathbb{Q}}(\text{Gal}(\bar{k}/k)) \\ X &\mapsto \mathbb{Q}^{X(\bar{k})} \end{aligned}$$

is fully-faithful. It is essentially surjective by loc.cit. Therefore V_k^0 is abelian and thus Tannakian. \square

Now fix an embedding $\sigma: \bar{k} \hookrightarrow \mathbb{C}$. By standard Tannakian yoga the inclusion $M_k^0 \rightarrow M_k$ defines a homomorphism $\pi: G(\sigma) \rightarrow \text{Gal}(\bar{k}/k)$. Similarly the functor $M_k \rightarrow M_{\bar{k}}$ defines a homomorphism $\iota: G^0(\sigma) \rightarrow G(\sigma)$, where $G^0(\sigma)$ is the group such that $\text{Rep}_{\bar{k}}(G^0(\sigma)) \cong M_{\bar{k}}$.

Proposition 5.19. *The sequence*

$$1 \rightarrow G^0(\sigma) \xrightarrow{\iota} G(\sigma) \xrightarrow{\pi} \text{Gal}(\bar{k}/k) \rightarrow 1$$

is exact.

Proof. First as $M_k^0 \rightarrow M_k$ is fully faithful the morphism π is faithfully flat (this is also standard Tannakian yoga, but see [DM82, Corollary 5.2] for a proof). Similarly to prove ι is an injection by [DM82, Corollary 5.1] it suffices to prove the following: every motive (attached to) $X \in \text{SmProj}(\bar{k})$ is a subquotient of a motive $X' \otimes_k \bar{k}$ for some $X' \in \text{SmProj}(k)$. But X has a model X_0 over a finite extension k' of k and one can take $X' = \text{Res}_{k'/k} X_0$.

Exercise 5.20. *Try to show exactness at $G(\sigma)$.*

\square

Example 5.21. *Recall from the proof of Proposition 5.15 that $G(\sigma) = \varprojlim_X G_X$. Then one can compute G_X for various $X \in M_k$:*

(1) $G_{\mathbb{L}} = \mathbb{G}_m$.

(2) *If X is an elliptic curve and has no CM over \bar{k} (i.e. $\text{End}_{\bar{k}} X \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$, then $G_X = \text{GL}_2$).*

Remark 5.22. *One can think of the motivic Galois group $G(\sigma)$ as a generalization of [Sta18, Tag 03QR] to higher dimensions.*

6 Lecture 6: Various enriched realizations of motives

Recall that a Weil cohomology theory defined over a field F of characteristic 0 gives rise to a functor

$$H: \text{Mot}_{\text{rat}}(k) \rightarrow \text{GrVect}_F. \quad (6.1)$$

This lands in the category of graded finite-dimensional vector spaces GrVect_F . However several Weil cohomology theories carry additional structure (e.g. étale cohomology carries an action of Galois). It is often the case that there is some Tannakian category \mathcal{A} such that (6.1) actually lands in

$$H_{\mathcal{A}}: \text{Mot}_{\text{rat}}(k) \rightarrow \text{Gr-}\mathcal{A}$$

where $\text{Gr-}\mathcal{A}$ is a rigid tensor category of \mathcal{A} with a \mathbb{Z} -graduation²⁶. Similarly we get a functor

$$H_{\mathcal{A}}: \text{Mot}_{\text{hom}}(k) \rightarrow \text{Gr-}\mathcal{A} \quad (6.2)$$

It will be more convenient to drop the grading on both sides. For this we will need to suppose a weakened version of the standard conjecture C:

Conjecture 6.1 (Sign conjecture). *In the context of the Künneth conjecture, cf. §4.1, the Künneth sum of even-degree projections $\sum_i \pi_{2i}$ is algebraic.*

²⁶We refer the reader to [SR72, Chapitre IV] for a formal definition.

6.1 Detour on Tannakian categories

Let (C, \otimes) be a rigid tensor category. For each object $X \in C$ there is a canonical trace map:

$$\mathrm{Tr}_X: \mathrm{End}(X) \rightarrow \mathrm{End}(\mathbf{1})$$

defined by sending $f: X \rightarrow X$ to the composite

$$\mathbf{1} \rightarrow X \otimes X^\vee \xrightarrow{f \otimes X^\vee} X \otimes X^\vee \xrightarrow{\text{comm. constraint}} X^\vee \otimes X \rightarrow \mathbf{1}.$$

We let $\dim(X)$ denote the trace of id_X .

Theorem 6.2. *Assuming k is of characteristic 0, (C, \otimes) is Tannakian iff for all objects X , $\dim(X) \geq 0$.*

Exercise 6.3. *Assuming the sign conjecture, show that we get a tensor functor which still respects the grading*

$$H_{\mathcal{A}}: \mathrm{Mot}_{\mathrm{hom}}(k) \rightarrow \mathrm{Gr}\text{-}\mathcal{A}$$

where now the grading on both sides is a $\mathbb{Z}/2$ -grading (where one groups together all even graded parts and groups together all odd graded parts). Show that for X smooth projective, $\dim(X) = \chi(X)$. Moreover by modifying the commutativity constraint on $\mathrm{Mot}_{\mathrm{hom}}(k)$, show that we get an exact faithful tensor functor

$$H_{\mathcal{A}}: \mathrm{Mot}_{\mathrm{hom}}(k) \rightarrow \mathcal{A}.$$

6.2 Back to enriched realizations

From now on we assume the sign conjecture and assume we have a functor

$$H_{\mathcal{A}}: \mathrm{Mot}_{\mathrm{hom}}(k) \rightarrow \mathcal{A}.$$

We are interested in situations where $H_{\mathcal{A}}$ is full and whether the objects in the image are semi-simple²⁷. In fact these two properties are related:

Proposition 6.4. *Suppose $H_{\mathcal{A}}$ is full. TFAE:*

- (1) *The objects in the image of $H_{\mathcal{A}}$ are semi-simple.*
- (2) *Standard conjecture D holds and*

$$H_{\mathcal{A}}: \mathrm{Mot}_{\mathrm{hom}}(k) = \mathrm{Mot}_{\mathrm{num}}(k) \rightarrow \mathcal{A}$$

makes $\mathrm{Mot}_{\mathrm{num}}(k)$ a tannakian sub-category of \mathcal{A} .

Proof. (1) \implies (2): $\mathrm{Mot}_{\mathrm{hom}}(k)$ is therefore a semi-simple category. But a semi-simple sub-category of a Tannakian category is abelian and so by Theorem 3.15, Standard conjecture D holds. It is also Tannakian (by standard Tannakian yoga). (2) \implies (1): This again follows by Theorem 3.15. \square

We now consider four different Weil cohomology theories and their associated enrichments: Betti, étale, crystalline and de Rham.

²⁷Recall that an object X in an abelian category \mathcal{C} is simple, if it only has (at most) two subobjects: itself and the zero object. An object is semi-simple if it is a sum of simple objects.

6.3 Hodge realisation (Betti cohomology)

In this setting we suppose $\sigma: k \hookrightarrow \mathbb{C}$. By Hodge theory, the Betti cohomology $H_B^i((\sigma X)^{\text{an}}, \mathbb{Q})$ is equipped with a rational Hodge structure:

Definition 6.5 (rational Hodge structure). A rational Hodge structure is a finite-dimensional vector space V over \mathbb{Q} together with a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{(p,q) \in \mathbb{Z}^2} V^{p,q} \quad (6.3)$$

such that $\overline{V^{p,q}} = V^{q,p}$. Morphisms are graded morphisms. We denote by $\text{HS}_{\mathbb{Q}}$ the category of rational Hodge structures.

The forgetful functor $\text{HS}_{\mathbb{Q}} \rightarrow \text{Vect}_{\mathbb{Q}}$ is a fibre functor and so $\text{HS}_{\mathbb{Q}}$ is a neutral Tannakian category. We get the corresponding *Hodge realisation*

$$H_{\text{Hodge}}: \text{Mot}_{\text{hom}}(k) \rightarrow \text{HS}_{\mathbb{Q}}.$$

Exercise 6.6 (Mumford-Tate group - an incarnation of the motivic galois group). *The bigraduation on $V \otimes_{\mathbb{Q}} \mathbb{C}$ given by (6.3) corresponds to a homomorphism $\mu: \mathbb{G}_m^2 \rightarrow \text{GL}(V \otimes_{\mathbb{Q}} \mathbb{C})$. For $V \in \text{HS}_{\mathbb{Q}}$, the corresponding Tannakian subcategory generated by V has an associated affine group scheme G_V (by Theorem 5.8). Show that G_V is the Mumford-Tate group of V , that is the smallest closed subgroup $MT(V) \subset \text{GL}(V)$ such that $MT(V)(\mathbb{C})$ contains the image of μ .*

Conjecture 6.7 (fullness of the Hodge realization). *Assume k is algebraically closed. The Hodge realization H_{Hodge} is a full functor.*

Remark 6.8 (Relationship with the Hodge conjecture). *Conjecture 6.7 is equivalent to the Hodge conjecture. Recall the Hodge conjecture states that all elements of bidegree $(0,0)$ in the Hodge decomposition of $\bigoplus_r H_B^{2r}((\sigma X)^{\text{an}}, \mathbb{Q})(r)$ are algebraic.*

Remark 6.9 (Polarizations and semi-simplicity). *All objects in the image of H_{Hodge} are semi-simple. This follows from the fact that all objects in the image are polarizable rational Hodge structures and polarizable rational Hodge structures are semi-simple.*

6.4 Tate realisation (étale cohomology)

In this setting k can be arbitrary and let \bar{k} be the separable closure of k . For $\ell \neq \text{char}(k)$, the étale cohomology groups $H_{\text{ét}}^i(X_{\bar{k}}, \mathbb{Q}_{\ell})$ are continuous representations of the absolute Galois group $\text{Gal}(\bar{k}/k)$. Let $\text{Rep}_{\mathbb{Q}_{\ell}} \text{Gal}(\bar{k}/k)$ be the category of continuous finite-dimensional \mathbb{Q}_{ℓ} -representations of $\text{Gal}(\bar{k}/k)$. The forgetful functor $\text{Rep}_{\mathbb{Q}_{\ell}} \text{Gal}(\bar{k}/k) \rightarrow \text{Vect}_{\mathbb{Q}_{\ell}}$ is a fibre functor and so $\text{Rep}_{\mathbb{Q}_{\ell}} \text{Gal}(\bar{k}/k)$ is a Tannakian category. We get the corresponding *Tate realisation*

$$H_{\text{Tate}}: \text{Mot}_{\text{hom}}(k) \rightarrow \text{Rep}_{\mathbb{Q}_{\ell}} \text{Gal}(\bar{k}/k)$$

Conjecture 6.10. *Suppose k is a finite extension of \mathbb{Q} or \mathbb{F}_p . Then*

- (1) *H_{Tate} is a full functor.*
- (2) *every object in the image of H_{Tate} is semi-simple.*

Remark 6.11 (Relationship with the Tate conjecture). *An element of $\bigoplus_r H_{\text{ét}}^{2r}(X_{\bar{k}}, \mathbb{Q}_{\ell})(r)$ which is $\text{Gal}(\bar{k}/k)$ -invariant is called an ℓ -adic Tate cycle. The Tate conjecture says that all ℓ -adic Tate cycles are a \mathbb{Q}_{ℓ} -linear combination of algebraic cycles. Conjecture 6.10(1) is equivalent to the Tate conjecture.*

6.5 Ogus realisation (crystalline cohomology)

Let $\Omega_{X/k}^\bullet$ be the complex of differential forms of X over k :

$$0 \rightarrow \mathcal{O}_X \xrightarrow{d} \Omega_{X/k}^1 \xrightarrow{d} \Omega_{X/k}^2 \xrightarrow{d} \dots$$

Then (algebraic) de Rham cohomology is defined to be the hypercohomology of this complex:

$$H_{\text{dR}}^i(X) := \mathbb{H}^i(\Omega_{X/k}^\bullet).$$

Now let k be a finite extension of \mathbb{Q} . Let v be an unramified prime of k where X has good reduction and let k_v be the completion of k at v . Then by Berthelot-Ogus's comparison theorem (cf. [BO83]) $H_{\text{dR}}^i(X) \otimes_k k_v$ is equipped with a Frobenius-semi-linear bijection:

Theorem 6.12 (Berthelot-Ogus). *Let A be a complete discrete valuation ring, with fraction field k and perfect residue field k_0 . We assume that k is of characteristic 0 and k_0 of characteristic $p > 0$. Let \mathfrak{X} be a projective and smooth scheme over A . Then, there is an isomorphism:*

$$H_{\text{dR}}^i(\mathfrak{X} \otimes_A k) \cong H_{\text{cris}}^i(\mathfrak{X} \otimes_A k_0) \otimes_{W(k_0)[\frac{1}{p}]} k.$$

Definition 6.13 (Ogus category). Let $\text{Og}(k)$ be the category whose objects are finite-dimensional vector spaces over k such that at almost all places v of k , the v -adic completion $V \otimes_k k_v$ is equipped with a Frobenius-semi-linear bijection. Morphisms are commutative diagrams in the obvious way.

It turns out that $\text{Og}(k)$ is actually a Tannakian category over \mathbb{Q} (one has to show that $\text{End}_{\text{Og}(k)}(\mathbf{1}) = \mathbb{Q}$). We get the corresponding *Ogus realisation*

$$H_{\text{Ogus}}: \text{Mot}_{\text{thom}}(k) \rightarrow \text{Og}(k).$$

Conjecture 6.14. *Suppose k is a finite extension of \mathbb{Q} . Then*

- (1) H_{Ogus} is a full functor.
- (2) every object in the image of H_{Ogus} is semi-simple.

Remark 6.15 (Relationship with the Ogus conjecture). *An element of $\oplus_r H_{\text{dR}}^{2r}(X)(r)$ which is Frobenius-invariant at almost all places unramified of k is called an Ogus cycle. The Ogus conjecture says that all Ogus cycles are algebraic cycles. Conjecture 6.14(1) is equivalent to the Ogus conjecture.*

6.6 Betti-de Rham realisation (The Grothendieck period conjecture)

In this setting we suppose $\sigma: k \hookrightarrow \mathbb{C}$. As for the Hodge realization, we are motivated by the de Rham-Betti comparison theorem (also known as the *period* isomorphism proven by Grothendieck in 1966 [Gro66])

$$H_{\text{dR}}^i(X) \otimes_k \mathbb{C} \cong H_B^i((\sigma X)^{\text{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}. \quad (6.4)$$

Definition 6.16 ($\text{Vect}_{k,\mathbb{Q}}$). Let $\text{Vect}_{k,\mathbb{Q}}$ be the category whose objects are triples (W, V, ω) where $W \in \text{Vect}_k$, $V \in \text{Vect}_{\mathbb{Q}}$ and

$$\omega: W \otimes_k \mathbb{C} \rightarrow V \otimes_{\mathbb{Q}} \mathbb{C}$$

is an isomorphism. Given two objects (W_1, V_1, ω_1) and (W_2, V_2, ω_2) . The group

$$\text{Hom}_{\text{Vect}_{k,\mathbb{Q}}}((W_1, V_1, \omega_1), (W_2, V_2, \omega_2))$$

is the subgroup $\text{Hom}_k(W_1, W_2) \oplus \text{Hom}_{\mathbb{Q}}(V_1, V_2)$ of pairs (ϕ_{dR}, ϕ_B) such that the following diagram is commutative:

$$\begin{array}{ccc} W_1 \otimes_k \mathbb{C} & \xrightarrow{\phi_{dR} \otimes_k \text{id}_{\mathbb{C}}} & W_2 \otimes_k \mathbb{C} \\ \downarrow \omega_1 & & \downarrow \omega_2 \\ V_1 \otimes_{\mathbb{Q}} \mathbb{C} & \xrightarrow{\phi_B \otimes_{\mathbb{Q}} \text{id}_{\mathbb{C}}} & V_2 \otimes_{\mathbb{Q}} \mathbb{C}. \end{array}$$

Exercise 6.17. Show that the forgetful functor

$$\begin{aligned} \text{Vect}_{k, \mathbb{Q}} &\rightarrow \text{Vect}_{\mathbb{Q}} \\ (W, V, \omega) &\mapsto V \end{aligned}$$

is a fibre functor and thus $\text{Vect}_{k, \mathbb{Q}}$ is a neutral Tannakian category over \mathbb{Q} .

Grothendieck's period isomorphism then gives us the de Rham-Betti realisation:

$$H_{dR-B}: \text{Mot}_{\text{hom}}(k) \rightarrow \text{Vect}_{k, \mathbb{Q}}.$$

Conjecture 6.18. Suppose k is a finite extension of \mathbb{Q} (or $k = \overline{\mathbb{Q}}$). Then

- (1) H_{dR-B} is a full functor.
- (2) every object in the image of H_{dR-B} is semi-simple.

Example 6.19. The isomorphism (6.4) is equivalent to giving a matrix of periods $\Omega_X \in \text{GL}_n(\mathbb{Q}) \backslash \text{GL}_n(\mathbb{C}) / \text{GL}_n(k)$.

Remark 6.20 (Relationship to the Grothendieck period conjecture). Assume again that k is a number field or $\overline{\mathbb{Q}}$. A de Rham-Betti cycle is an element of $\oplus_r H_B^{2r}((\sigma X)^{an}, \mathbb{Q})(r)$ which corresponds to an element of $\oplus_r H_{dR}^{2r}(X)(r)$ via the isomorphism (6.4). Then Conjecture 6.18(1) is equivalent to the following statement: All de Rham-Betti cycles are algebraic. When fixing a single r , we denote this conjecture by $\text{GPC}^r(X)$. In [Bos16], the authors refer to the latter statement as the Grothendieck period conjecture.

In [Gro66, note (10), p. 102], the original formulation of the Grothendieck period conjecture is stated differently. It is also related but not equivalent to the formulation in Remark 6.20. Bost shows that $\text{GPC}^1(X)$ is true in the case of an abelian variety [Bos13, Theorem 5.1]. We give a sketch of the proof:

Theorem 6.21. For X an abelian variety over $\overline{\mathbb{Q}}$, $\text{GPC}^1(X)$ is true.

Proof. We want to show that the realisation functor H_{dR-B}^1 restricted to the category of abelian varieties over $\overline{\mathbb{Q}}$ is fully faithful.

Step 1: identify morphisms between abelian varieties as morphisms of the associated universal vector extensions. For an abelian variety $A/\overline{\mathbb{Q}}$, let \mathbb{E}_A be the $\overline{\mathbb{Q}}$ -vector space $\Gamma(A, \Omega_{A/\overline{\mathbb{Q}}}^1)$. Then there is an extension

$$0 \rightarrow \mathbb{E}_{A^\vee} \rightarrow E(A) \rightarrow A \rightarrow 0$$

of commutative algebraic groups over $\overline{\mathbb{Q}}$. It is called the *universal vector extension* of A . The formation of $E(A)$ is functorial, that is for a morphism $A \rightarrow B$, we get a morphism $E(A) \rightarrow E(B)$.

Lemma 6.22. For any two abelian varieties A and B over $\overline{\mathbb{Q}}$, the morphism of \mathbb{Z} -modules $\text{Hom}(A, B) \rightarrow \text{Hom}(E(A), E(B))$ is an isomorphism.

Step 2: group of periods and Lie algebra of universal vector extension. The universal vector extension $E(A)$ has a lie algebra $\mathrm{Lie} E(A)$ and we can consider the exponential map of the complexified Lie algebra:

$$\exp_{E(A)_{\mathbb{C}}} : \mathrm{Lie} E(A)_{\mathbb{C}} \rightarrow E(A)_{\mathbb{C}}^{\mathrm{an}}$$

We denote by $\mathrm{Per}E(A)_{\mathbb{C}} := \ker \exp_{E(A)_{\mathbb{C}}}$, the group of periods. The exponential map is a covering of $E(A)$ by it's fundamental group. In this case $\mathrm{Per}E(A)_{\mathbb{C}} = \mathbb{Z}^{2 \dim A}$. Moreover by standard results on universal extensions $\mathrm{Lie} E(A)_{\mathbb{C}} = \mathbb{C}^{2 \dim A}$. Thus the inclusion $\mathrm{Per}E(A)_{\mathbb{C}} \hookrightarrow \mathrm{Lie} E(A)_{\mathbb{C}}$ extends to an isomorphism

$$c^{-1} : \mathrm{Per}E(A)_{\mathbb{C}} \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathrm{Lie} E(A)_{\mathbb{C}}$$

Therefore the triple $\mathrm{LiePer}E(A) := (\mathrm{Lie} E(A), \mathrm{Per}E(A)_{\mathbb{C}} \otimes_{\mathbb{Z}} \mathbb{Q}, c)$ gives rise to an object in $\mathrm{Vect}_{\overline{\mathbb{Q}}, \mathbb{Q}}$. Moreover the construction of $\mathrm{LiePer}E(A)$ is functorial in $E(A)$, that is for a morphism $E(A) \rightarrow E(B)$ we get a map on the corresponding LiePer pairs. As in Lemma 6.22, we have

Lemma 6.23. *For any two abelian varieties A and B over $\overline{\mathbb{Q}}$, the morphism of \mathbb{Z} -modules*

$$\mathrm{Hom}(E(A), E(B)) \rightarrow \mathrm{Hom}_{\mathrm{Vect}_{\overline{\mathbb{Q}}, \mathbb{Q}}}(\mathrm{LiePer}E(A), \mathrm{LiePer}E(B))$$

is an isomorphism.

Step 3: comparing de Rham-Betti cohomology to LiePer. Let

$$H_{\mathrm{dR-B}}^i(A) := (H_{\mathrm{dR}}^i(A), H_A^i((\sigma X)^{\mathrm{an}}, \mathbb{Q}), \omega)$$

where ω is the period isomorphism (6.4). Also we denote by $H_{i, \mathrm{dR-B}}(A) := H_{\mathrm{dR-B}}^i(A)^{\vee}$ the dual of $H_{\mathrm{dR-B}}^i(A)$ (in the Tannakian category $\mathrm{Vect}_{\overline{\mathbb{Q}}, \mathbb{Q}}$).

Lemma 6.24. *There is a canonical functorial isomorphism*

$$\mathrm{LiePer}E(A) \xrightarrow{\sim} H_{1, \mathrm{dR-B}}(A).$$

Combining Lemmas 6.22-6.24, shows that the realisation functor $H_{1, \mathrm{dR-B}}$ is fully-faithful on the category of abelian varieties over $\overline{\mathbb{Q}}$.

Step 4: Connection with the Néron-Severi group. Bost completes the proof as follows: From **Step 3**, we get an isomorphism

$$H_{1, \mathrm{dR-B}} : \mathrm{Hom}(A, A^{\vee}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Vect}_{\overline{\mathbb{Q}}, \mathbb{Q}}}(H_{1, \mathrm{dR-B}}(A), H_{1, \mathrm{dR-B}}(A^{\vee})) \quad (6.5)$$

On the other hand by the duality of homology-cohomology, there is a canonical isomorphism

$$H_{1, \mathrm{dR-B}}(A) \xrightarrow{\sim} H_{\mathrm{dR-B}}^1(A^{\vee}) \otimes \mathbb{Z}(1) \quad (6.6)$$

Substituting (6.6) into (6.5) gives an isomorphism

$$\mathrm{Hom}(A, A^{\vee}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Vect}_{\overline{\mathbb{Q}}, \mathbb{Q}}}(\mathbb{Z}(0), H_{\mathrm{dR-B}}^1(A) \otimes H_{\mathrm{dR-B}}^1(A) \otimes \mathbb{Z}(1)). \quad (6.7)$$

Recall that the Neron-Sévri group $N(A)$ is the group of divisors of A module algebraic equivalence. It can be shown that $N(A) \hookrightarrow \mathrm{Hom}(A, A^{\vee})$ and isomorphism (6.7) induces an isomorphism

$$N(A) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \text{de Rham-Betti cycles}$$

and this completes the sketch. \square

7 Lecture 7: Filtrations on the Chow ring

As usual let $X \in \text{SmProj}(k)$ where k is a field.

7.1 Bloch-Beilinson Conjecture

In his Duke 1979 lectures [Blo10] Bloch and independently Beilinson [Bei87] both conjectured that there exists a descending filtration on the rational Chow groups $\text{CH}^i(X)_{\mathbb{Q}} := \text{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The underlying idea is that Grothendieck's theory of motives should not only be used as an *universal cohomology* theory, but also for studying the Chow groups of an algebraic variety.

The precise version of these conjectures was formulated by Jannsen in [Jan94, §2], which we will now review. For a fixed Weil cohomology theory \sim_{hom} , we also need the following subgroups:

$$\text{CH}_{\text{hom}}^i(X) := \{\alpha \in \text{CH}^i(X) \mid \text{cl}(\alpha) = 0\} \text{ and } \text{CH}_{\text{hom}}^i(X)_{\mathbb{Q}} := \text{CH}_{\text{hom}}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Conjecture 7.1 (Bloch-Beilinson filtration). *For each $i \geq 0$, there exists a descending filtration F_{BB}^{\bullet} on $\text{CH}^i(X)_{\mathbb{Q}}$ such that*

- (1) $F_{BB}^0 \text{CH}^i(X)_{\mathbb{Q}} = \text{CH}^i(X)_{\mathbb{Q}}$ and $F_{BB}^1 \text{CH}^i(X)_{\mathbb{Q}} = \text{CH}_{\text{hom}}^i(X)_{\mathbb{Q}}$.
- (2) $F_{BB}^r \text{CH}^i(X)_{\mathbb{Q}} \cdot F_{BB}^s \text{CH}^i(X)_{\mathbb{Q}} \subseteq F_{BB}^{r+s} \text{CH}^{i+j}(X)_{\mathbb{Q}}$.
- (3) For a morphism $f: X \rightarrow Y$, the induced morphisms f_* and f^* on the Chow ring $\bigoplus_i \text{CH}^i(X)_{\mathbb{Q}}$ respect the grading induced by F_{BB}^{\bullet} .
- (4) Assuming the Künneth Conjecture (Standard Conjecture C), the j th Künneth component π_j acts²⁸ on $\text{Gr}_{BB}^{\nu} \text{CH}^i(X)_{\mathbb{Q}}$ as $\delta_{j,2i-\nu} \cdot \text{id}$.
- (5) $F_{BB}^{i+1} \text{CH}^i(X)_{\mathbb{Q}} = 0$.

Remark 7.2. The way to make sense of (4) is as follows: Firstly by parts (2) and (3), as an action of a correspondence involves only operations of type (2) and (3), we get an induced action of Δ_i on the graded pieces. Each of the π_i are (honest) projections, in the sense that they either send an element to 0 or keep it the same (at least on the cohomological level). By part (1), the induced action of Δ_i are again projections on the cycle level and (4) is cutting out the contributions of each of these projectors.

Moreover, assuming the existence of an abelian category of *mixed motives* $\text{MM}(k)$ a more precise conjecture is formulated as follows

Conjecture 7.3 (A stronger version of Conjecture 7.1). *Keep (1), (2), (3) and (5) the same and replace (4) by:*

$$\text{Gr}_{BB}^{\nu} \text{CH}^i(X)_{\mathbb{Q}} = \text{Ext}_{\text{MM}(k)}^{\nu}(\mathbf{1}, (X, \Delta_{2i-\nu}, i)) \quad (7.1)$$

In Conjecture (7.3), formula (7.1) is called the *Beilinson formula*. Conjecture 7.1/7.3 is called the Bloch-Beilinson conjecture.

7.2 Conjectures of Murre

We now review 4 conjectures of Murre [Mur93] and then study their relation to the above Bloch-Beilinson conjecture. Assume X is pure of dimension d .

Conjecture 7.4 (Chow-Künneth Conjecture). *X has a Chow-Künneth decomposition over k defined below.*

²⁸or rather Δ_i where $\gamma_{X \times X}(\Delta_i) = \pi_i$.

Definition 7.5 (Chow-Künneth decomposition). We say that X admits a Chow-Künneth decomposition if there exists $p_i \in \mathrm{CH}^d(X \times X)_{\mathbb{Q}} = \mathrm{Corr}^0(X, X)$ for $0 \leq i \leq 2d$ such that

- (1) $\Delta_X = \sum_{i=0}^{2d} p_i$.
- (2) $p_i \circ p_j = 0$ if $i \neq j$ and each p_i is a projector.
- (3) $\gamma_{X \times X}(p_i) = \pi_i$, where π_i is the i th Künneth component of Δ_X (cf. Conjecture 4.1).

Assuming Conjecture 7.4, each of the projectors p_i clearly²⁹ operate on the Chow groups $\mathrm{CH}^i(X)_{\mathbb{Q}}$.

Conjecture 7.6 (Vanishing conjecture). *For every $0 \leq i \leq d$, the projectors $p_{2d}, p_{2d-1}, \dots, p_{2i+1}$ and p_0, p_1, \dots, p_{i-1} operate as zero on $\mathrm{CH}^i(X)_{\mathbb{Q}}$.*

Assuming Conjectures 7.4 and 7.6 we get the following descending filtration on $\mathrm{CH}^i(X)_{\mathbb{Q}}$:

$$\begin{aligned} F^0 \mathrm{CH}^i(X)_{\mathbb{Q}} &= \mathrm{CH}^i(X)_{\mathbb{Q}} \\ F^1 \mathrm{CH}^i(X)_{\mathbb{Q}} &= \ker(p_{2i}) \\ F^2 \mathrm{CH}^i(X)_{\mathbb{Q}} &= \ker(p_{2i}) \cap \ker(p_{2i-1}) \end{aligned}$$

and in general

$$F^\nu \mathrm{CH}^i(X)_{\mathbb{Q}} = \ker(p_{2i}) \cap \dots \cap \ker(p_{2i+1-\nu}). \quad (7.2)$$

Lemma 7.7 (Murre vs Bloch-Beilinson (1) and (5)). *Assuming Conjectures 7.4 and 7.6, we have*

- (1) $F^{i+1} \mathrm{CH}^i(X)_{\mathbb{Q}} = 0$.
- (2) $F^1 \mathrm{CH}^i(X)_{\mathbb{Q}} \subseteq \mathrm{CH}_{\mathrm{hom}}^i(X)_{\mathbb{Q}}$.

Proof. We have $F^{i+1} \mathrm{CH}^i(X)_{\mathbb{Q}} = \ker(p_{2i}) \cap \dots \cap \ker(p_i)$. On the other hand Conjecture 7.6 implies $\mathrm{id}_{\mathrm{CH}^i(X)_{\mathbb{Q}}} = p_{2i} + \dots + p_i$. This proves the first statement.

For the second statement we have the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^i(X)_{\mathbb{Q}} & \xrightarrow{p_{2i}} & \mathrm{CH}^i(X)_{\mathbb{Q}} \\ \downarrow \gamma_X & & \downarrow \gamma_X \\ H^{2i}(X) & \xrightarrow{\pi_{2i}} & H^{2i}(X). \end{array}$$

But by definition of the Künneth components the bottom arrow is identity and so $\ker p_{2i} \subseteq \ker \gamma_X = \mathrm{CH}_{\mathrm{hom}}^i(X)_{\mathbb{Q}}$. \square

We now come to the third conjecture of Murre.

Conjecture 7.8 (Third conjecture of Murre). *We have $F^1 \mathrm{CH}^i(X)_{\mathbb{Q}} = \mathrm{CH}_{\mathrm{hom}}^i(X)_{\mathbb{Q}}$.*

Finally the fourth conjecture of Murre:

Conjecture 7.9 (Fourth conjecture of Murre). *The filtration F^\bullet defined in (7.2) is independent of the choice of the projectors p_i .*

²⁹As they are degree 0 correspondences.

7.3 Bloch-Beilinson vs Murre

It turns out that the Bloch-Beilinson conjectures and the four conjectures of Murre are equivalent (cf. [Jan94, Theorem 5.2])

Theorem 7.10 (Jannsen). *Fix a field k . Then the Bloch-Beilinson conjectures (cf. Conjecture 7.1) are true for all smooth projective varieties over k iff the four conjectures of Murre (Conjectures 7.4-7.9) are true. Moreover, if these conjectures are true, then the Bloch-Beilinson filtration coincides with Murre's filtration.*

The crucial part of the proof of Theorem 7.10 is given by the following proposition cf. [Jan94, Proposition 5.8]:

Proposition 7.11. *Let X and Y be smooth projective varieties (of pure dimensions d and e , respectively). Suppose that X and Y have Chow-Künneth decompositions given by*

$$ch^i(X) = (X, p_i(X), 0) \text{ for } 0 \leq i \leq 2d \text{ and } ch^j(Y) = (Y, p_j(Y), 0) \text{ for } 0 \leq j \leq 2e.$$

Then the product variety $Z = X \times Y$ has a Chow-Künneth decomposition with projectors

$$p_m(Z) := \sum_{r+s=m} p_r(X) \times p_s(Y) \text{ for } 0 \leq m \leq 2d + 2e. \quad (7.3)$$

Moreover the following holds:

(1) *If, with the projectors (7.3), Conjecture 7.6 holds then*

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}(k)}(ch^i(X), ch^j(Y)) = 0 \text{ for } i < j.$$

(2) *If, with the projectors (7.3), Conjecture 7.8 holds then*

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}(k)}(ch^i(X), ch^i(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mot}_{\mathrm{hom}}(k)}(h^i(X), h^i(Y))$$

Proof. For the claim on the Chow-Künneth decomposition of Z , note that

$$\begin{aligned} \Delta_Z &= \Delta_X \times \Delta_Y \\ &= \left(\sum_{i=0}^{2d} p_i(X) \right) \times \left(\sum_{j=0}^{2e} p_j(Y) \right) \\ &= \sum_{m=0}^{2(d+e)} p_m(Z) \end{aligned}$$

It's easy to check that $p_{rs}(Z) := p_r(X) \times p_s(Y)$ are themselves projectors and mutually orthogonal. By Künneth decomposition, the relevant sums lift the Künneth projectors.

To simplify the proof of the next two statements, we assume in addition that the Chow-Künneth decomposition is *self-dual*³⁰ and for this we need the notion of the *transpose* of a correspondence:

Definition 7.12 (Transpose of a correspondence from X to Y). Given a correspondence C from X to Y , the transpose ${}^T C$ is the same cycle C , but considered as a subvariety of $Y \times X$.

³⁰It is conjecturally expected that there exists a Chow-Künneth decomposition which is self-dual

With Definition 7.12, we assume in addition that $p_{2d-i}(X) = {}^T p_i(X)$. Now by definition

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}(k)}(\mathrm{ch}^i(X), \mathrm{ch}^j(Y)) = \left\{ p_j(Y) \circ R \circ p_i(X) \mid R \in \mathrm{CH}^d(X \times Y)_{\mathbb{Q}} \right\}.$$

The action of $p_m(Z)$ on such a correspondence is given by

$$p_m(Z)_*(p_j(Y) \circ R \circ p_i(X)) = \sum_{r+s=m} (p_r(X) \times p_s(Y))_*(p_j(Y) \circ R \circ p_i(X)) \quad (7.4)$$

To simplify (7.4), we need Lieberman's Lemma (we stated a weakened version in Lemma 3.20 of Lecture 3, but didn't have enough time to prove it):

Exercise 7.13 (Lieberman's Lemma). *Let $f \in \mathrm{Corr}(X, Y)$, $\alpha \in \mathrm{Corr}(X, X')$, $\beta \in \mathrm{Corr}(Y, Y')$, then $(\alpha \times \beta)_*(f) = \beta \circ f \circ {}^T \alpha$.*

we get

$$\begin{aligned} \sum_{r+s=m} (p_r(X) \times p_s(Y))_*(p_j(Y) \circ R \circ p_i(X)) &= \sum_{r+s=m} p_s(Y) \circ p_j(Y) \circ R \circ p_i(X) \circ {}^T p_r(X) \\ &= p_j(Y) \circ R \circ p_i(X) \end{aligned}$$

because the only terms that survive are with $s = j$ and by the self-dual assumption $r = 2d - i$. Moreover $m = r + s = 2d - i + j$. Therefore if Conjecture 7.6 holds, then the above action is 0 for $m > 2d$ or equivalently $i < j$.

If Conjecture 7.8 holds, then $\ker p_{2d}(Z) = \mathrm{CH}_{\mathrm{hom}}^d(X \times Y)_{\mathbb{Q}}$. Now suppose $p_i(Y) \circ R \circ p_i(X)$ is homologically equivalent to 0. Then

$$0 = p_{2d}(Z)_*(p_i(Y) \circ R \circ p_i(X)).$$

But again by Lieberman's lemma $p_{2d}(Z)_*(p_i(Y) \circ R \circ p_i(X)) = p_i(Y) \circ R \circ p_i(X)$.

□

7.4 Some evidence for the conjectures in the case of a 3-fold

In this section we sketch the following result due to Murre in [Mur93].

Proposition 7.14. *Suppose $X = S \times C$ where S is a surface and C is a curve. Assume that S and C both admit k -rational points. Then X satisfies the four conjectures of Murre.*

Proof. We first show that X has a Chow-Künneth decomposition (i.e. Conjecture 7.4 is satisfied). By the first part of Proposition 7.11, it suffices to show that both C and S have a Chow-Künneth decomposition. The fact that C has a Chow-Künneth is trivial:

Exercise 7.15. *Consider the Chow-Künneth decomposition of C given by the decomposition $p_0(C) = \{e\} \times C$, $p_2(C) = C \times \{e\}$ for e a rational point of C and $p_1(C) = \Delta_C - p_0(C) - p_2(C)$. Show that $p_1(C)$ acts as identity on $\mathrm{CH}_{\mathrm{hom}}^1(C)_{\mathbb{Q}}$.*

For the surface S , the result is due to Murre [Mur90]. We won't go into his proof but let us mention that the Chow-Künneth decomposition he gives for S is self-dual in the sense of the proof of Proposition 7.11. We also restrict ourselves in studying $\mathrm{CH}^1(X)_{\mathbb{Q}}$. Thus it remains to check Conjectures 7.6 and 7.8 are satisfied for the seven projectors $p_0(X), p_1(X), \dots, p_6(X)$. Conjecture 7.9, then follows as the induced filtration on $\mathrm{CH}^1(X)_{\mathbb{Q}}$ coincides with $\mathrm{CH}^1(X)_{\mathbb{Q}} \supset \mathrm{CH}_{\mathrm{hom}}^1(X)_{\mathbb{Q}} \supset \{0\}$.

Lemma 7.16. *Conjectures 7.6 and 7.8 hold for $\mathrm{CH}^1(S \times C)_{\mathbb{Q}}$*

Proof. First we claim that if $D \in \mathrm{CH}_{\mathrm{hom}}^1(S \times C)_{\mathbb{Q}}$ then $D = D_1 \times C + S \times D_2$ with $D_1 \in \mathrm{CH}_{\mathrm{hom}}^1(S)_{\mathbb{Q}}$ and $D_2 \in \mathrm{CH}_{\mathrm{hom}}^1(C)_{\mathbb{Q}}$. By Matsusaka's Theorem, we can replace D by an integral multiple if necessary and assume $D \in \mathrm{CH}_{\mathrm{alg}}^1(S \times C) \subset \mathrm{CH}_{\mathrm{hom}}^1(S \times C)$ where

$$\mathrm{CH}_{\mathrm{alg}}^i(S \times C) := \{\alpha \in \mathrm{CH}^i(S \times C) \mid \alpha \sim_{\mathrm{alg}} 0\}.$$

The claim then follows from the following exercise:

Exercise 7.17. *Let $X, Y \in \mathrm{SmProj}(k)$ and D a divisor on $X \times Y$ such that $D \sim_{\mathrm{alg}} 0$. Then for some integer $m \neq 0$, we have $mD = D_1 \times Y + X \times D_2$ with D_1 (resp. D_2) a divisor on X (resp. Y).*

Now consider the projector

$$p_1(X) = p_1(S) \times p_0(C) + p_0(S) \times p_1(C)$$

Murre in [Mur90] also shows that $p_1(S)$ acts as identity on $\mathrm{CH}_{\mathrm{hom}}^1(S)_{\mathbb{Q}}$. By the previous claim together with Exercise 7.15, it follows that $p_1(X)$ acts as identity on $\mathrm{CH}_{\mathrm{hom}}^1(X)_{\mathbb{Q}}$.

Now let $i \neq 1$ and $D \in \mathrm{CH}_{\mathrm{hom}}^1(X)_{\mathbb{Q}}$. Then

$$0 = (p_i(X) \circ p_1(X))_*(D) = p_i(X)_*(D). \quad (7.5)$$

Hence for $i \neq 1$ the projector $p_i(X)$ acts as zero on $\mathrm{CH}_{\mathrm{hom}}^1(X)_{\mathbb{Q}}$. In particular $\ker p_2(X) = \mathrm{CH}_{\mathrm{hom}}^1(X)_{\mathbb{Q}}$. Thus Conjecture 7.8 holds. To prove Conjecture 7.6, we have to show that for $i \neq 1, 2$, $p_i(X)$ acts as 0 on $\mathrm{CH}^1(X)_{\mathbb{Q}}$. Take $D \in \mathrm{CH}^1(X)_{\mathbb{Q}}$. Then

$$0 = (p_2(X) \circ p_i(X))_*(D) = p_2(X)_*(D_i)$$

where $D_i = p_i(X)_*(D)$. Thus $D_i \in \mathrm{CH}_{\mathrm{hom}}^1(X)_{\mathbb{Q}}$. Therefore

$$p_i(X)_*(D) = (p_i(X) \circ p_i(X))_*(D) = p_i(X)_*(D_i) = 0.$$

where the last equality follows from (7.5). Thus Conjecture 7.6 follows. □

□

Exercise 7.18. *Conjectures 7.6 and 7.8 hold for $\mathrm{CH}^3(S \times C)_{\mathbb{Q}}$*

8 Lecture 8: Voevodsky's derived category of mixed motives

So far we have attached motives to smooth projective varieties over a field. We now want to attach motives to arbitrary varieties giving a theory of *mixed* motives. No such theory exists, but there are various constructions of the derived category of mixed motives³¹. We shall study Voevodsky's construction in [Voe00]. We shall follow closely the lecture notes of Mazza-Voevodsky-Weibel³² [MVW06]. Crucially the construction no longer relies on the *moving lemma*, where smoothness and projectivity assumptions are important.

Remark 8.1. *Deligne was motivated by the yoga: pure motives \longrightarrow mixed motives, when he constructed a theory of mixed Hodge structures for non-smooth and non-proper varieties in [Del71] and [Del74b].*

³¹With the hope that there is a suitable t -structure whose heart would give the sought-after category.

³²The category of effective geometric motives is constructed in a different way to [Voe00].

8.1 Finite Correspondences

To solve the problem of partially defined intersections, Voevodsky introduced the notion of finite correspondences. Let Sm_k be the category of smooth separated schemes over k .

Definition 8.2 ($\mathrm{Corr}_{\mathrm{fin}}(X, Y)$). Let $X, Y \in \mathrm{Sm}_k$. The group $\mathrm{Corr}_{\mathrm{fin}}(X, Y)$ of finite correspondences from X to Y is the abelian subgroup of $Z(X \times_k Y)$ generated by integral³³ closed subschemes $W \subset X \times_k Y$ such that

- (1) the projection $p_1: W \rightarrow X$ is finite
- (2) the image $p_1(W) \subset X$ is an irreducible component of X .

Example 8.3. For $X, Y \in \mathrm{Sm}_k$, the graph Γ_f of a morphism $f: X \rightarrow Y$ is a finite correspondence from X to Y . Indeed $\Gamma_f \rightarrow X$ is an isomorphism and Γ_f is closed due to separated assumption.

Exercise 8.4 (Construction of composition of finite correspondences). Let $X, Y, Z \in \mathrm{Sm}_k$. In this exercise you will construct a composition law:

$$\circ: \mathrm{Corr}_{\mathrm{fin}}(Y, Z) \times \mathrm{Corr}_{\mathrm{fin}}(X, Y) \rightarrow \mathrm{Corr}_{\mathrm{fin}}(X, Z)$$

- (1) Given closed subsets $V \subset X \times Y$ and $W \subset Y \times Z$ which are finite and surjective over X and Y , respectively, show that $V \times Z$ and $X \times W$ intersect properly. This defines a cycle $[T]$ in $X \times Y \times Z$.
- (2) Show that³⁴ $p_*([T])$ where $p: X \times Y \times Z \rightarrow X \times Z$ is finite and surjective over X .

This allows us to define $W \circ V := p_*[(V \times Z) \cdot (X \times W)]$.

Definition 8.5 (Category of finite correspondences $\mathrm{Corr}_{\mathrm{fin}}(k)$). The category $\mathrm{Corr}_{\mathrm{fin}}(k)$ is the category with the same objects as Sm_k and morphisms

$$\mathrm{Hom}_{\mathrm{Corr}_{\mathrm{fin}}(k)}(X, Y) := \mathrm{Corr}_{\mathrm{fin}}(X, Y)$$

with composition given by Exercise 8.4.

Exercise 8.6. Show that $\Gamma_f \circ \Gamma_g = \Gamma_{f \circ g}$ and hence the functor

$$\begin{aligned} \mathrm{Sm}_k &\rightarrow \mathrm{Corr}_{\mathrm{fin}}(k) \\ X &\mapsto X \\ (f: X \rightarrow Y) &\mapsto \Gamma_f. \end{aligned}$$

is faithful.

Remark 8.7 (Category of finite correspondences over a general base). In [MVW06, Appendix 1A], for S a Noetherian scheme, a category $\mathrm{Corr}_{\mathrm{fin}}(S)$ is constructed. It's objects are schemes of finite type over S .

³³Recall a scheme is integral iff it is reduced and irreducible.

³⁴Strictly speaking the pushforward in Definition 2.4 is defined only for the proper case. However we can define it as the action of the graph of p viewed as a correspondence. In any case, what is important is that the components of $p_*([T])$ are just the supports of $p(T_i)$ for every irreducible component $T_i \subset T$.

8.2 The category of effective geometric motives

The category $\text{Corr}_{\text{fin}}(k)$ has hom-sets abelian groups and composition of morphisms is bilinear. Thus it is a preadditive category. Furthermore if $X = \coprod_i X_i$, then $\text{Corr}_{\text{fin}}(X, Y) = \oplus_i \text{Corr}_{\text{fin}}(X_i, Y)$. Thus it is an additive category (with finite disjoint union as a finitary (co)product).

Definition 8.8 ($\text{Corr}_{\text{fin}}(k)$ as a symmetric monoidal category). If X and Y are two objects in $\text{Corr}_{\text{fin}}(k)$, we define the tensor product

$$X \otimes Y = X \times_k Y$$

It follows that the bounded homotopy category $K^b(\text{Corr}_{\text{fin}}(k))$ is a triangulated tensor category. One defines triangles to be triangles that are isomorphic (in $K^b(\text{Corr}_{\text{fin}}(k))$) to the cone sequence

$$A \xrightarrow{f} B \rightarrow C(f) \rightarrow A[1]$$

Definition 8.9 (Construction of the triangulated tensor category $\text{DM}_{\text{gm}}^{\text{eff}}(k)$).

- (1) Localize $K^b(\text{Corr}_{\text{fin}}(k))$ with respect to the thick subcategory generated by complexes of the form
 - (a) **Homotopy:** $X \times \mathbb{A}^1 \rightarrow X$.
 - (b) **Mayer-Vietoris** $U \cap V \rightarrow U \oplus V \rightarrow X$, where U and V are Zariski open subsets of X such $X = U \cup V$. In other words invert the induced³⁵ map $C(U \cap V \rightarrow U \oplus V) \rightarrow X$.
- (2) Take the pseudo-abelian completion of the resulting quotient category³⁶.

The resulting category is the category of *effective geometric motives* $\text{DM}_{\text{gm}}^{\text{eff}}(k)$.

Definition 8.10 (Motive of a smooth scheme). We get a symmetric monoidal covariant³⁷ functor

$$M: \text{Sm}_k \rightarrow \text{DM}_{\text{gm}}^{\text{eff}}(k).$$

Example 8.11 (Immediate consequences from Definition 8.9).

- (1) **Homotopy-invariance:** $M(X \times \mathbb{A}^1) \cong M(X)$
- (2) **Mayer-Vietoris triangle:**

$$M(U \cap V) \rightarrow M(U) \oplus M(V) \rightarrow M(X) \rightarrow M(U \cap V)[1].$$

8.3 The category of geometric motives

Definition 8.12 (reduced motive \tilde{X}). For $X \in \text{Sm}_k$ and a rational point $e \in X(k)$ we can consider the triangle

$$M(\text{Spec } k) \xrightarrow{e} M(X) \rightarrow \tilde{X} \rightarrow \cdot$$

Similarly the structure morphism $p: X \rightarrow \text{Spec } k$, gives rise to a triangle

$$\tilde{Y} \rightarrow M(X) \xrightarrow{p} M(\text{Spec } k) \rightarrow \cdot$$

Exercise 8.13. Since $p \circ e = \text{id}_k$, show by using the axioms of triangulated categories that $\tilde{X} \cong \tilde{Y}$ and $M(X) \cong M(\text{Spec } k) \oplus \tilde{X}$ (i.e. the triangle splits).

³⁵Such an induced map comes from the universal property of the mapping cone: Given a morphism of chain of complexes $X \rightarrow Y \rightarrow Z$, if the composition is null-homotopic, then we get a map $C(X \rightarrow Y) \rightarrow Z$.

³⁶Voevodsky only does this step, in order to make comparisons with the category of pure motives.

³⁷As opposed to the associated functor for pure motives which is contravariant.

Definition 8.14 (Lefschetz motive). We set $\mathbb{Z}(1) := \widetilde{\mathbb{P}^1}[-2]$ and $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$ for $n \geq 0$.

Example 8.15. $M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$ and $\mathbb{Z} = M(\text{Spec } k)$.

Definition 8.16 (Construction of $\text{DM}_{\text{gm}}(k)$). We invert the Lefschetz motive. More formally: the objects are $M(r)$ for $M \in \text{DM}_{\text{gm}}^{\text{eff}}(k)$ and $r \in \mathbb{Z}$ and morphisms

$$\text{Hom}_{\text{DM}_{\text{gm}}(k)}(M(r), N(s)) := \varinjlim_n \text{Hom}_{\text{DM}_{\text{gm}}^{\text{eff}}(k)}(M \otimes \mathbb{Z}(n+r), N \otimes \mathbb{Z}(n+s)).$$

Remark 8.17. To compare $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ with $\text{DM}_{\text{gm}}(k)$ and to the usual category of (Chow) motives, we need to view $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ as a subcategory of motivic complexes $\text{DM}_{-}^{\text{eff}}(k)$.

8.4 Nisnevich sheaves with transfers

As mentioned in Remark 8.17, the geometric definition of $\text{DM}_{\text{gm}}^{\text{eff}}(k)$ (i.e. Definition 8.9) is not powerful enough (due to a lack of site). This section remedies this and provides a site-theoretic framework for mixed motives.

Definition 8.18 (Presheaf with transfers). A presheaf with transfers is a functor $F: \text{Corr}_{\text{fin}}(k)^{\text{op}} \rightarrow \mathbf{Ab}$, where \mathbf{Ab} is the category of abelian groups. Denote this category by $\text{PSh}(\text{Corr}_{\text{fin}}(k))$.

Such a presheaf is called *homotopy invariant* if the natural map

$$F(p_X): F(X) \rightarrow F(X \times \mathbb{A}^1)$$

is an isomorphism for all $X \in \text{Sm}_k$ where $p_X: X \times \mathbb{A}^1 \rightarrow X$ is the projection morphism.

Definition 8.19 ($\mathbb{Z}_{\text{tr}}(X)$). For $X \in \text{Sm}_k$, let $\mathbb{Z}_{\text{tr}}(X)$ denote the presheaf with transfers represented by X , so that $\mathbb{Z}_{\text{tr}}(X)(U) := \text{Corr}_{\text{fin}}(U, X)$. For a morphism $V \rightarrow U$, the map is defined $\mathbb{Z}_{\text{tr}}(X)(U) \rightarrow \mathbb{Z}_{\text{tr}}(X)(V)$ to be induced by composition of finite correspondences.

Remark 8.20. A presheaf with transfers attaches an abelian group $F(X)$ for every $X \in \text{Sm}_k$ and for every finite correspondence $Z \in \text{Corr}_{\text{fin}}(X, Y)$ a transfer map

$$\text{Tr}(Z): F(Y) \rightarrow F(X).$$

We need to work with a site that is finer than the Zariski topology but coarser than the étale topology. This is due to Nisnevich [Nis89].

Definition 8.21 (Nisnevich covering). A family of étale morphisms $\{p_i: U_i \rightarrow X\}_{i \in I}$ is said to be a Nisnevich covering of X if it has the Nisnevich lifting property: for all $x \in X$, there is an $i \in I$ and a $u \in U_i$ so that $p_i(u) = x$ and the induced map on residue fields $k(x) \rightarrow k(u)$ is an isomorphism.

Exercise 8.22. Show that Nisnevich coverings give a Grothendieck topology on Sm_k .

Example 8.23. If the characteristic of k is zero, then the two morphisms $j: \mathbb{A}^1 - \{a\} \hookrightarrow \mathbb{A}^1$ and $i: \mathbb{A}^1 - \{0\} \xrightarrow{z \mapsto z^2} \mathbb{A}^1$ is a Nisnevich covering of \mathbb{A}^1 iff $a \in (k^*)^2$. On the other hand it is an étale map for any non-zero a .

Recall the local rings in the étale topology are strict henselian rings, while for the Nisnevich topology they are just henselian rings.

Definition 8.24 (The category of Nisnevich sheaves with transfers: $\text{Nis}_{\text{tr}}(k)$). A presheaf with transfers is called a Nisnevich sheaf with transfers if its restriction to Sm_k^{op} is a sheaf for the Nisnevich topology.

We'll postpone the proof of the next proposition to the next lecture.

Proposition 8.25. *The presheaf with transfer $\mathbb{Z}_{tr}(X)$ is a Nisnevich sheaf with transfer.*

Theorem 8.26 ($\mathcal{N}is_{tr}(k)$ is a Grothendieck topos). *The embedding*

$$\mathcal{N}is_{tr}(k) \hookrightarrow PSh(\text{Corr}_{fin}(k))$$

has a left-adjoint which is left-exact.

Proof.

Lemma 8.27. *Let $p: U \rightarrow Y$ be a Nisnevich covering and $f: X \rightarrow Y$ a finite correspondence. Then there is a Nisnevich covering $p': V \rightarrow X$ and a finite correspondence $f': V \rightarrow U$ such that the following diagram commutes in $\text{Corr}_{fin}(k)$:*

$$\begin{array}{ccc} V & \xrightarrow{f'} & U \\ \downarrow p' & & \downarrow p \\ X & \xrightarrow{f} & Y. \end{array} \quad (8.1)$$

Proof. We can assume that f is defined by a closed irreducible $Z \subset X \times Y$ which surjects onto X . Let $Z_U := Z \times_Y U \subset X \times U$. Since the projection $Z_U \rightarrow Z$ is a Nisnevich cover and $Z \rightarrow X$ is finite, we claim that $Z_U \rightarrow Z$ splits Nisnevich locally³⁸ on X . Indeed we can assume³⁹ $X = \text{Spec}(R)$ where R is henselian. Then by [Sta18, Tag 04GH], Z is a disjoint union of henselian schemes and so the required map has a section. But then $s(V \times_X Z) \subset V \times U$ is finite over V (as $Z \rightarrow X$ is finite) and $V \times_X Z \rightarrow V$ is a surjection (again since this is true for $Z \rightarrow X$). \square

Lemma 8.28. *Let \mathcal{F} be a presheaf with transfers. Denote by $\mathcal{F}_{\mathcal{N}is}$ the sheafification of $\mathcal{F}|_{Sm_k}$ for the Nisnevich topology. Then there exists a unique Nisnevich sheaf with transfers \mathcal{G} such that $\mathcal{G}|_{Sm_k} = \mathcal{F}_{\mathcal{N}is}$ equipped with a morphism of presheaves with transfers $\mathcal{F} \rightarrow \mathcal{G}$.*

Proof. We first show uniqueness. Suppose there are two Nisnevich sheaves with transfers \mathcal{G}_1 and \mathcal{G}_2 such that $\mathcal{G}_1|_{Sm_k} = \mathcal{G}_2|_{Sm_k} = \mathcal{F}_{\mathcal{N}is}$. Given a morphism $f: X \rightarrow Y \in \text{Corr}_{fin}(k)$ we need to check that $\mathcal{G}_1(f) = \mathcal{G}_2(f)$ as morphisms $\mathcal{G}_1(Y) \rightarrow \mathcal{G}_1(X)$. Let $y \in \mathcal{G}_1(Y) = \mathcal{G}_2(Y)$. Choose a Nisnevich cover $p: U \rightarrow Y$ such that $y|_U \in \mathcal{G}_1(U)$ is the image of some $u \in \mathcal{F}(U)$. Then applying Lemma 8.27, we get diagram (8.1). We now compute

$$\begin{aligned} \mathcal{G}_1(p')\mathcal{G}_1(f)(y) &= \mathcal{G}_1(f')\mathcal{G}_1(p)(y) \\ &= \mathcal{G}_1(f')(y|_U) \\ &= \mathcal{G}_2(f')(y|_U) \\ &= \mathcal{G}_2(p')\mathcal{G}_2(f)(y) \\ &= \mathcal{G}_1(p')\mathcal{G}_2(f)(y) \end{aligned}$$

This implies $\mathcal{G}_1(f) = \mathcal{G}_2(f)$ as p' is a covering. This shows uniqueness.

We now prove existence. We need to define a morphism $\mathcal{F}_{\mathcal{N}is}(Y) \rightarrow \mathcal{F}_{\mathcal{N}is}(X)$ for each $f: X \rightarrow Y \in \text{Corr}_{fin}(X, Y)$ \square

Lemma 8.29. *Let $X \in Sm_k$ and $U \rightarrow X$ a Nisnevich covering of X . The Čech complex*

$$\dots \mathbb{Z}_{tr}(U \times_X U) \rightarrow \mathbb{Z}_{tr}(U) \rightarrow \mathbb{Z}_{tr}(X) \rightarrow 0 \quad (8.2)$$

is exact as a complex of Nisnevich sheaves.

³⁸That is there is an étale cover $V \rightarrow X$ so that $V \times_X Z_U \rightarrow V \times_X Z$ has a section s .

³⁹by taking the limit across all Nisnevich neighborhoods of some point $x \in X$.

Proof. By [Nis89, Corollary 1.17], the Nisnevich topology on Sm_k has enough points given by henselian localizations. This means that to prove exactness of (8.2), it suffices to prove exactness of

$$\dots \varinjlim_i \mathbb{Z}_{\mathrm{tr}}(U \times_X U)(S_i) \rightarrow \varinjlim_i \mathbb{Z}_{\mathrm{tr}}(U)(S_i) \rightarrow \varinjlim_i \mathbb{Z}_{\mathrm{tr}}(X)(S_i) \rightarrow 0 \quad (8.3)$$

where each $S_i \in \mathrm{Sm}_k$ is affine and $S = \varprojlim_i S_i$ is a henselian scheme. For a closed subscheme $Z \subset X \times S$, which is finite over S , denote by $L(Z/S)$ the free abelian group generated by irreducible components of Z which are finite and surjective over S . Then (8.3) is the colimit of complexes of the form

$$\dots L(Z_U \times_Z Z_U/S) \rightarrow L(Z_U/S) \rightarrow L(Z/S) \rightarrow 0 \quad (8.4)$$

where $Z_U = Z \times_X U$ and the colimit is taken over all Z closed subschemes of $X \times S$ which are finite and surjective over S . Thus it suffices to show exactness of (8.4). Since S is henselian and $Z \rightarrow S$ is finite, Z is a finite product of henselian rings and so $Z_U \rightarrow Z$ splits. Let $s_1: Z \rightarrow Z_U$ be a splitting. Put $(Z_U)_Z^k = Z_U \times_Z \dots \times_Z Z_U$ and set $s_k: (Z_U)_Z^k \rightarrow (Z_U)_Z^{k+1}$ to be $s_1 \times_Z \mathrm{id}_{(Z_U)_Z^k}$. The s_k induce homomorphisms of abelian groups

$$\sigma_k: L((Z_U)_Z^k/S) \rightarrow L((Z_U)_Z^{k+1}/S).$$

Exercise 8.30. Show that the σ_k provide a chain homotopy from the identity morphism of (8.4) to the zero morphism of (8.4). □

We now produce a morphism

$$\mathcal{F}_{\mathrm{Nis}}(Y) \rightarrow \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Sm}_k)}(\mathbb{Z}_{\mathrm{tr}}(Y), \mathcal{F}_{\mathrm{Nis}})$$

where the Hom is in the category of Nisnevich sheaves on Sm_k .

For $y \in \mathcal{F}_{\mathrm{Nis}}(Y)$, choose a Nisnevich cover $p: U \rightarrow Y$ such that $y|_{U \in \mathcal{F}_{\mathrm{Nis}}(U)}$ is the image of some $u \in \mathcal{F}(U)$. By Yoneda's lemma ($\mathcal{F}(U) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Corr}_{\mathrm{fin}}(k))}(\mathbb{Z}_{\mathrm{tr}}(U), \mathcal{F})$) and so u determines a morphism $\mathbb{Z}_{\mathrm{tr}}(U) \rightarrow \mathcal{F}$ of presheaves with transfer. By shrinking U if necessary, we can assume that the difference map $d: \mathcal{F}(U) \rightarrow \mathcal{F}(U \times_Y U)$ sends u to 0. Consider now the commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow & \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Sm}_k)}(\mathbb{Z}_{\mathrm{tr}}(Y), \mathcal{F}_{\mathrm{Nis}}) & \longrightarrow & \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Sm}_k)}(\mathbb{Z}_{\mathrm{tr}}(U), \mathcal{F}_{\mathrm{Nis}}) & \longrightarrow & \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Sm}_k)}(\mathbb{Z}_{\mathrm{tr}}(U_Y^2), \mathcal{F}_{\mathrm{Nis}}) \\ & & & \uparrow & & \uparrow \\ & & & \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Corr}_{\mathrm{fin}}(k))}(\mathbb{Z}_{\mathrm{tr}}(U), \mathcal{F}) & \longrightarrow & \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Corr}_{\mathrm{fin}}(k))}(\mathbb{Z}_{\mathrm{tr}}(U_Y^2), \mathcal{F}) \end{array}$$

The top row is exact by Lemma 8.29 and so we get $[y] \in \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Sm}_k)}(\mathbb{Z}_{\mathrm{tr}}(Y), \mathcal{F}_{\mathrm{Nis}})$

Exercise 8.31. Check that $[y]$ is independent of the choice of U and u .

Finally to define a morphism $\mathcal{F}_{\mathrm{Nis}}(Y) \rightarrow \mathcal{F}_{\mathrm{Nis}}(X)$ for each finite correspondence f , it suffices to define a pairing

$$\mathrm{Corr}_{\mathrm{fin}}(X, Y) \times \mathcal{F}_{\mathrm{Nis}}(Y) \rightarrow \mathcal{F}_{\mathrm{Nis}}(X).$$

Indeed given $f \in \mathrm{Corr}_{\mathrm{fin}}(X, Y) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathrm{Corr}_{\mathrm{fin}}(k))}(\mathbb{Z}_{\mathrm{tr}}(X), \mathbb{Z}_{\mathrm{tr}}(Y))$ and $y \in \mathcal{F}_{\mathrm{Nis}}(Y)$, the previous paragraph gives $[y]: \mathbb{Z}_{\mathrm{tr}}(Y) \rightarrow \mathcal{F}_{\mathrm{Nis}}$. Thus we get the composition

$$\mathbb{Z}_{\mathrm{tr}}(X)(X) \rightarrow \mathbb{Z}_{\mathrm{tr}}(Y)(X) \rightarrow \mathcal{F}_{\mathrm{Nis}}(X)$$

Exercise 8.32. Show that the image of the identity map gives the required pairing (i.e. the constructed morphisms are natural).

Exactness follows because sheafification is an exact procedure. \square

Definition 8.33 (The category of effective motivic complexes: $\mathrm{DM}_{-}^{\mathrm{eff}}(k)$). By Theorem 8.26, we can consider the derived category of bounded above complexes of Nisnevich sheaves with transfers: $D^{-}(\mathrm{Nis}_{\mathrm{tr}}(k))$. The category $\mathrm{DM}_{-}^{\mathrm{eff}}(k)$ is the full subcategory of $D^{-}(\mathrm{Nis}_{\mathrm{tr}}(k))$ whose cohomology sheaves are homotopy invariant.

9 Lecture 9: The localization theorem of $\mathrm{DM}_{-}^{\mathrm{eff}}(k)$

The following is the postponed proof of Proposition 8.25 from the previous lecture.

Proposition 9.1. *The presheaf with transfer $\mathbb{Z}_{\mathrm{tr}}(X)$ is a Nisnevich sheaf with transfer.*

Proof. We need to check the following two properties

- (1) For every surjective Nisnevich morphism of smooth separated schemes $U \rightarrow Y$, the sequence

$$0 \rightarrow \mathbb{Z}_{\mathrm{tr}}(X)(Y) \rightarrow \mathbb{Z}_{\mathrm{tr}}(X)(U) \rightarrow \mathbb{Z}_{\mathrm{tr}}(X)(U \times_Y U)$$

is exact

- (2) $\mathbb{Z}_{\mathrm{tr}}(X)(U \amalg V) = \mathbb{Z}_{\mathrm{tr}}(X)(U) \oplus \mathbb{Z}_{\mathrm{tr}}(X)(V)$

For (2) note that $\mathbb{Z}_{\mathrm{tr}}(X)(U \amalg V) = \mathrm{Corr}_{\mathrm{fin}}(U \amalg V, X) = \mathrm{Corr}_{\mathrm{fin}}(U, X) \oplus \mathrm{Corr}_{\mathrm{fin}}(V, X)$.

For (1) we can assume Y is connected and therefore irreducible (as it is smooth). But then a finite correspondence in $\mathrm{Corr}_{\mathrm{fin}}(Y, X)$ is dominant and determined by the fiber at the generic point⁴⁰ of Y . Thus we get exactness at $\mathbb{Z}_{\mathrm{tr}}(X)(Y)$, as the map $\mathbb{Z}_{\mathrm{tr}}(X)(Y) \rightarrow \mathbb{Z}_{\mathrm{tr}}(X)(U)$ is just pullback of cycles. It remains to show exactness at $\mathbb{Z}_{\mathrm{tr}}(X)(U)$. Take $Z_U \in \mathrm{Corr}_{\mathrm{fin}}(U, X)$ whose images in $\mathrm{Corr}_{\mathrm{fin}}(U \times_Y U, X)$ coincide along the two projections.

Exercise 9.2. *Show that there is a Zariski open $V \subset Y$ and $Z_V \in \mathrm{Corr}_{\mathrm{fin}}(V, X)$ agreeing with Z_U in $\mathrm{Corr}_{\mathrm{fin}}(U \times_Y V, X)$.*

Exercise 9.3. *Given an étale surjection $U \rightarrow Y$ with Y irreducible, show that there exists an irreducible components $U_1 \subset U$ such that $U_1 \rightarrow Y$ is still a surjection.*

By Exercise 9.3 we can assume U is irreducible, and we can write $Z_V = \sum n_i Z_i$ and $Z_U = \sum n_i Z'_i$ so that Z_i and Z'_i agree in $\mathrm{Corr}_{\mathrm{fin}}(U \times_Y V, X)$ (since U and V are irreducible, the same trick as in the first paragraph works to see that one gets the claimed presentation for Z_V and Z_U). Thus we can assume $Z_V = Z_i$ and $Z_U = Z'_i$.

Let Z be the closure of Z_V in $Y \times X$. Since $Z \times_Y V = Z_V$ is irreducible and dominant over V , it means Z is irreducible and dominant over Y .

Exercise 9.4. *Prove that Z maps to Z_U along the pullback of $U \times X \rightarrow Y \times X$.*

This means that the components of $Z \times_Y U$ are finite over U . By faithfully flat descent, this means Z is finite over Y . \square

We want to construct a functor $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \rightarrow \mathrm{DM}_{-}^{\mathrm{eff}}(k)$. For this we need to view $\mathrm{DM}_{-}^{\mathrm{eff}}(k)$ as a localization of $D^{-}(\mathrm{Nis}_{\mathrm{tr}}(k))$, just like $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ was defined as a certain localization.

Definition 9.5. Define the cosimplicial scheme Δ^{\bullet} over k which is defined by

$$\Delta^n = \mathrm{Spec} k[x_0, \dots, x_n] / \left(\sum_{i=0}^n x_i = 1 \right)$$

The i th face map $\delta_i: \Delta^n \rightarrow \Delta^{n+1}$ is given by $x_i = 0$.

⁴⁰Think of what is happening along $U \times X \rightarrow Y \times X \rightarrow Y$.

Definition 9.6 (Suslin complex). Let \mathcal{F} be a presheaf with transfers. Define the presheaf with transfers $C_n(\mathcal{F})$ by

$$C_n(\mathcal{F})(X) := \mathcal{F}(X \times \Delta^n)$$

The *Suslin complex* $C_*(\mathcal{F})$ is the complex with differential

$$d_n := \sum_i (-1)^i \delta_i^* : C_n(\mathcal{F}) \rightarrow C_{n-1}(\mathcal{F}).$$

Proposition 9.7. *The Suslin complex in Definition 9.6 defines a functor*

$$C_* : \text{Nis}_{\text{tr}}(k) \rightarrow DM_{-}^{\text{eff}}(k)$$

Proof. A priori $C_* : \text{Nis}_{\text{tr}}(k) \rightarrow D^-(\text{Nis}_{\text{tr}}(k))$. We first show that the cohomology presheaves of $C_*(\mathcal{F})$ are homotopy-invariant.

Lemma 9.8. *For a rational point $\alpha \in \mathbb{A}^1(k)$, let $i_\alpha : X \hookrightarrow X \times \mathbb{A}^1$ be the inclusion $x \mapsto (x, \alpha)$. Then \mathcal{F} is homotopy invariant iff*

$$\mathcal{F}(i_0) = \mathcal{F}(i_1) : \mathcal{F}(X \times \mathbb{A}^1) \rightarrow \mathcal{F}(X)$$

for all $X \in \text{Sm}_k$.

Proof. Denote by $p_X : X \times \mathbb{A}^1 \rightarrow X$, the projection. Then $p_X \circ i_\alpha = \text{id}_X$ and so if \mathcal{F} is homotopy invariant (i.e. $\mathcal{F}(p_X)$ is an isomorphism), then $\mathcal{F}(i_0) = \mathcal{F}(i_1)$. For the converse suppose $\mathcal{F}(i_0) = \mathcal{F}(i_1)$. Applying \mathcal{F} to the multiplication map $m : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ gives the diagram

$$\begin{array}{ccccc} & & \mathcal{F}(X \times \mathbb{A}^1) & \xrightarrow{\mathcal{F}(i_0)} & \mathcal{F}(X) \\ & \swarrow \mathcal{F}(\text{id}_{X \times \mathbb{A}^1}) & \downarrow \mathcal{F}(\text{id}_X \times m) & & \downarrow \mathcal{F}(p_X) \\ \mathcal{F}(X \times \mathbb{A}^1) & \xleftarrow{\mathcal{F}(i_1 \times \text{id}_{\mathbb{A}^1})} & \mathcal{F}(X \times \mathbb{A}^1 \times \mathbb{A}^1) & \xrightarrow{\mathcal{F}(i_0 \times \text{id}_{\mathbb{A}^1})} & \mathcal{F}(X \times \mathbb{A}^1). \end{array}$$

Thus $\mathcal{F}(p_X) \circ \mathcal{F}(i_0) = \mathcal{F}(\text{id}_{X \times \mathbb{A}^1})$. Thus it follows that $\mathcal{F}(p_X)$ is onto. But we know it is injective as it has a section. Thus it is an isomorphism and \mathcal{F} is homotopy invariant. \square

Lemma 9.9. *Let \mathcal{F} be a presheaf with transfers. The induced chain maps $i_0^*, i_1^* : C_*(\mathcal{F})(X \times \mathbb{A}^1) \rightarrow C_*(\mathcal{F})(X)$ are chain homotopic.*

Proof. For each $i = 0, \dots, n$ define $\theta_i : \Delta^{n+1} \rightarrow \Delta^n \times \mathbb{A}^1$ to be the map that sends the vertex v_j defined by $x_j = 1$ and $x_{j'} = 0$ for $j' \neq j$ to $v_j \times \{0\}$ for $j \leq i$ and to $v_{j-1} \times \{1\}$, otherwise. These θ_i induces maps

$$h_i : \mathcal{F}(\text{id}_X \times \theta_i) : C_n(\mathcal{F})(X \times \mathbb{A}^1) \rightarrow C_{n+1}(\mathcal{F})(X).$$

Exercise 9.10. *Check that $i_1^* = \delta_0 \circ h_0$ and $i_0^* = \delta_{n+1} \circ h_n$ and that the alternating sum $\sum (-1)^i h_i$ is a chain homotopy from i_1^* to i_0^* .* \square

By Lemmas 9.8 and 9.9, it follows that the cohomology presheaves of $C_*(\mathcal{F})$ are homotopy-invariant. The Proposition now follows from Theorem 9.11, which we will not prove. \square

Theorem 9.11. *Let \mathcal{F} be a homotopy invariant presheaf with transfers. Then the Zariski sheaf⁴¹ given by the Zariski sheafification on Sm_k is homotopy invariant. Moreover $\mathcal{F}_{\text{Zar}} = \mathcal{F}_{\text{Nis}}|_{\text{Sm}_k}$ where \mathcal{F}_{Nis} is given by Theorem 8.26.*

Before we extend the functor C_* to the full triangulated category $D^-(\text{Nis}_{\text{tr}}(k))$, we need to put a tensor structure on it.

⁴¹viewed as an honest sheaf on Sm_k , without transfers.

9.1 Tensor structure on $D^-(\mathbf{Nis}_{\mathrm{tr}}(k))$

First we define a tensor product on $\mathbf{Nis}_{\mathrm{tr}}(k)$. We set $\mathbb{Z}_{\mathrm{tr}}(X) \otimes \mathbb{Z}_{\mathrm{tr}}(Y) := \mathbb{Z}_{\mathrm{tr}}(X \times Y)$. For general \mathcal{F} , we have a canonical surjection

$$\bigoplus_{(X, s \in \mathcal{F}(X))} \mathbb{Z}_{\mathrm{tr}}(X) \rightarrow \mathcal{F}$$

Iterating this construction we get a canonical left resolution $\mathcal{L}(\mathcal{F})$ of \mathcal{F} which consists of direct sums of presheaves of the form $\mathbb{Z}_{\mathrm{tr}}(X)$ for $X \in \mathbf{Sm}_k$. We then set

$$\mathcal{F} \otimes \mathcal{G} := H_0^{\mathrm{Nis}}(\mathcal{L}(\mathcal{F}) \otimes \mathcal{L}(\mathcal{G})).$$

This induces a tensor structure on $D^-(\mathbf{Nis}_{\mathrm{tr}}(k))$. The unit $\mathbf{1}$ is $\mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec} k)$.

For presheaves with transfers \mathcal{F}, \mathcal{G} , denote by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ the presheaf with transfers given by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(X) = \mathrm{Hom}(\mathcal{F} \otimes \mathbb{Z}_{\mathrm{tr}}(X), \mathcal{G})$$

Exercise 9.12. Show that $\mathcal{H}om(-, -)$ is the internal Hom-object with respect to the tensor product, that is there is a canonical isomorphism

$$\mathrm{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})) \rightarrow \mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}).$$

9.2 Localization Theorem

Proposition 9.13 (Localization Theorem). Assume k is perfect. The functor C_* extends to a functor

$$RC_*: D^-(\mathbf{Nis}_{\mathrm{tr}}(k)) \rightarrow DM_-^{\mathrm{eff}}(k).$$

which is left adjoint to the natural embedding. The functor RC_* identifies $DM_-^{\mathrm{eff}}(k)$ with the localization of $D^-(\mathbf{Nis}_{\mathrm{tr}}(k))$ with respect to the thick subcategory generated by complexes of the form

$$\mathbb{Z}_{\mathrm{tr}}(X \times \mathbb{A}^1) \xrightarrow{\mathbb{Z}_{\mathrm{tr}}(p_X)} \mathbb{Z}_{\mathrm{tr}}(X)$$

for all $X \in \mathbf{Sm}_k$.

Proof. Denote by \mathcal{A} the class of objects in $D^-(\mathbf{Nis}_{\mathrm{tr}}(k))$ of the given form $\mathbb{Z}_{\mathrm{tr}}(X \times \mathbb{A}^1) \rightarrow \mathbb{Z}_{\mathrm{tr}}(X)$ and let \mathcal{A} be the minimal triangulated subcategory in $D^-(\mathbf{Nis}_{\mathrm{tr}}(k))$ which contains \mathcal{A} which is closed under direct summands and direct sums⁴². Consider the localization $D^-(\mathbf{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$ with respect to the class of morphisms whose cone are in \mathcal{A} .

To prove the proposition, we need to prove the following two statements:

- (1) For any $\mathcal{F} \in \mathbf{Nis}_{\mathrm{tr}}(k)$, the canonical morphism $\mathcal{F} \rightarrow C_*(\mathcal{F})$ is an isomorphism in $D^-(\mathbf{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$.
- (2) For any object T of $DM_-^{\mathrm{eff}}(k)$ and any object B of \mathcal{A} one has $\mathrm{Hom}(B, T) = 0$.

Indeed (1) implies $DM_-^{\mathrm{eff}}(k) \rightarrow D^-(\mathbf{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$ is surjective on isomorphism classes and (2) implies $DM_-^{\mathrm{eff}}(k) \rightarrow D^-(\mathbf{Nis}_{\mathrm{tr}}(k))/\mathcal{A}$ is fully faithful.

proof of (2): We can assume⁴³ $B := \mathbb{Z}_{\mathrm{tr}}(X \times \mathbb{A}^1) \rightarrow \mathbb{Z}_{\mathrm{tr}}(X)$. To get a grip on $\mathrm{Hom}(B, T)$, we need some preparation:

Lemma 9.14. For any $X \in \mathbf{Sm}_k$, $i \in \mathbb{Z}$, $\mathcal{F} \in \mathbf{Nis}_{\mathrm{tr}}(k)$, there is a canonical isomorphism

$$\mathrm{Ext}_{\mathbf{Nis}_{\mathrm{tr}}(k)}^i(\mathbb{Z}_{\mathrm{tr}}(X), \mathcal{F}) = H_{\mathrm{Nis}}^i(X, \mathcal{F}).$$

⁴²A thick subcategory.

⁴³Since \mathcal{A} is formed by taking shifts, cones and summands of objects of \mathcal{A} .

Proof. By Theorem 8.26, $\mathrm{Nis}_{\mathrm{tr}}(k)$ is a Grothendieck topos and so it has enough injective objects. The result is true for $i = 0$ and so by taking an injective resolution of \mathcal{F} , we have to show that $H_{\mathrm{Nis}}^i(X, \mathcal{I}) = 0$ for any injective Nisnevich sheaf with transfers \mathcal{I} and $i > 0$. It suffices to show higher Čech cohomology vanishes. Let $U \rightarrow X$ be a Nisnevich cover and $\alpha \in \check{H}_{\mathrm{Nis}}^i(U/X, \mathcal{I})$. Then α is given by a section a of $\mathcal{I}(U_X^i)$ or equivalently a morphism $\mathbb{Z}_{\mathrm{tr}}(U_X^i) \rightarrow \mathcal{I}$. Since a is a cocycle: $a \in \ker(\mathcal{I}(U_X^i) \rightarrow \mathcal{I}(U_X^{i+1}))$, this means that the section a is equivalent to a morphism $\mathrm{coker}(\mathbb{Z}_{\mathrm{tr}}(U_X^{i+1}) \rightarrow \mathbb{Z}_{\mathrm{tr}}(U_X^i)) \rightarrow \mathcal{I}$. But by Lemma 8.29, $\mathrm{Im}(\mathbb{Z}_{\mathrm{tr}}(U_X^{i+1}) \rightarrow \mathbb{Z}_{\mathrm{tr}}(U_X^i)) = \ker(\mathbb{Z}_{\mathrm{tr}}(U_X^i) \rightarrow \mathbb{Z}_{\mathrm{tr}}(U_X^{i-1}))$ and so $a \in \mathrm{Im}(\mathcal{I}(U_X^{i-1}) \rightarrow \mathcal{I}(U_X^i))$ and thus higher Čech cohomology vanishes. \square

Lemma 9.15. *For any $X \in \mathrm{Sm}_k$, $i \in \mathbb{Z}$ and $K \in D^-(\mathrm{Nis}_{\mathrm{tr}}(k))$ there is a canonical isomorphism*

$$\mathrm{Hom}_{D^-(\mathrm{Nis}_{\mathrm{tr}}(k))}(\mathbb{Z}_{\mathrm{tr}}(X), K[i]) = \mathbb{H}_{\mathrm{Nis}}^i(X, K)$$

where the groups on the RHS are the hypercohomology of K in the Grothendieck over-category Sm_k/X .

Proof. This is just a decorated version of Lemma 9.14. For K concentrated in a single degree, this is precisely Lemma 9.14. We can then extend to bounded complexes K by taking triangles and induction. To extend to bounded above complexes, we need that Nisnevich topology has bounded cohomological dimension by [Nis89, Theorem 1.32]. Essentially by bounded cohomological dimension, we can take injective resolutions by [Sta18, Tag 07K7]. \square

Returning to the **proof of (2)**, by Lemma 9.15, it suffices to show that the induced morphism

$$\mathbb{H}^*(X, T) \rightarrow \mathbb{H}^*(X \times \mathbb{A}^1, T) \quad (9.1)$$

is an isomorphism. This is the *derived* version of Theorem 9.11 and relies on k being perfect and similarly we won't prove this.

proof of (1): We need the following lemma:

Lemma 9.16. *The category \mathcal{A} is a \otimes -ideal: $A \in \mathcal{A}$, $B \in D^-(\mathrm{Nis}_{\mathrm{tr}}(k))$ implies $A \otimes B \in \mathcal{A}$.*

Proof. We can assume $A = \mathbb{Z}_{\mathrm{tr}}(X \times \mathbb{A}^1) \rightarrow \mathbb{Z}_{\mathrm{tr}}(X)$ and by the existence of the canonical left resolution by direct sums of $\mathbb{Z}_{\mathrm{tr}}(Y)$, we can assume $B = \mathbb{Z}_{\mathrm{tr}}(Y)$. But then $A \otimes B = \mathbb{Z}_{\mathrm{tr}}(X \times Y \times \mathbb{A}^1) \rightarrow \mathbb{Z}_{\mathrm{tr}}(X \times Y)$, which is in \mathcal{A} . \square

and also the notion of \mathbb{A}^1 -homotopy equivalence:

Definition 9.17 (\mathbb{A}^1 -homotopy equivalence). (1) We say that two morphism $f, g: \mathcal{F} \rightarrow \mathcal{G}$ of presheaves with transfer are \mathbb{A}^1 -homotopic if there is a map

$$h: \mathcal{F} \otimes \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1) \rightarrow \mathcal{G}$$

with $f = h \circ (\mathrm{id}_{\mathcal{F}} \otimes i_0)$ and $g = h \circ (\mathrm{id}_{\mathcal{F}} \otimes i_1)$, where $i_0, i_1: \mathbf{1} \rightarrow \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1)$ induced by the inclusions $i_0, i_1: \mathrm{Spec}(k) \rightarrow \mathbb{A}^1$.

- (2) A morphism of presheaves with transfer $f: \mathcal{F} \rightarrow \mathcal{G}$ is an \mathbb{A}^1 -homotopy equivalence if there is a morphism $g: \mathcal{G} \rightarrow \mathcal{F}$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity morphisms of \mathcal{G} and \mathcal{F} , respectively.

Lemma 9.18. *Let $f: \mathcal{F} \rightarrow \mathcal{G}$ be a \mathbb{A}^1 -homotopy equivalence of presheaves with transfer. Then the cone of f belongs to \mathcal{A} .*

Proof. The cone of f belonging to \mathcal{A} is equivalent to saying that f becomes an isomorphism in the localized category $D^-(\text{Nis}_{\text{tr}}(k))/\mathcal{A}$. It is enough to show that an endomorphism of a presheaf with transfers which is homotopic to the identity equals the identity morphism in $D^-(\text{Nis}_{\text{tr}}(k))/\mathcal{A}$. Unraveling the definitions it suffices to show that the two morphisms

$$\text{id}_{\mathcal{F}} \otimes i_0: \mathcal{F} \rightarrow \mathcal{F} \otimes \mathbb{Z}_{\text{tr}}(\mathbb{A}^1) \text{ and } \text{id}_{\mathcal{F}} \otimes i_1: \mathcal{F} \rightarrow \mathcal{F} \otimes \mathbb{Z}_{\text{tr}}(\mathbb{A}^1)$$

are equal in $D^-(\text{Nis}_{\text{tr}}(k))/\mathcal{A}$. Note that $i_0 - i_1: \mathbf{1} \rightarrow \mathbb{Z}_{\text{tr}}(\mathbb{A}^1)$ goes to 0 after composition with $\mathbb{Z}_{\text{tr}}(\mathbb{A}^1) \rightarrow \mathbf{1}$. Thus it factors through a map⁴⁴ $\phi: \mathbf{1} \rightarrow (\mathbb{Z}_{\text{tr}}(\mathbb{A}^1) \rightarrow \mathbf{1})$. This $\text{id}_{\mathcal{F}} \otimes i_0 - \text{id}_{\mathcal{F}} \otimes i_1$ factors through $\text{id} \otimes \phi: \mathcal{F} \rightarrow \mathcal{F} \otimes (\mathbb{Z}_{\text{tr}}(\mathbb{A}^1) \rightarrow \mathbf{1})$. Now by Lemma 9.16, $\mathcal{F} \otimes (\mathbb{Z}_{\text{tr}}(\mathbb{A}^1) \rightarrow \mathbf{1}) \in \mathcal{A}$ and so $\text{id} \otimes \phi$ is the zero map in $D^-(\text{Nis}_{\text{tr}}(k))/\mathcal{A}$. \square

Coming back to the **proof of (1)**, let $C_{\geq 1}(\mathcal{F})$ be the cone of our map $\mathcal{F} \rightarrow C_*(\mathcal{F})$. We need to show that $C_{\geq 1}(\mathcal{F}) \in \mathcal{A}$. To understand what $C_{\geq 1}(\mathcal{F})$ looks like, consider the homomorphisms

$$\eta_n: \mathcal{F} \rightarrow C_n(\mathcal{F})$$

which on sections X is induced by the projection $X \times \Delta^n \rightarrow X$.

Lemma 9.19. *The morphisms η_n are \mathbb{A}^1 -homotopy equivalences*

Proof. Since Δ^n is isomorphic to \mathbb{A}^n , we have

$$C_n(\mathcal{F}) = C_1(C_{n-1}(\mathcal{F}))$$

and so it suffices to show that η_1 is a homotopy equivalence. Let $\alpha: C_1(\mathcal{F}) \rightarrow \mathcal{F}$ be the morphism which on sections is induced by $X \times \{0\} \rightarrow X \times \mathbb{A}^1$. Then $\alpha \circ \eta_1 = \text{id}_{\mathcal{F}}$ and it suffices to show there exists a morphism $h: C_1(\mathcal{F}) \otimes \mathbb{Z}_{\text{tr}}(\mathbb{A}^1) \rightarrow C_1(\mathcal{F})$ such that

$$h \circ (\text{id} \otimes i_0) = \eta_1 \circ \alpha \text{ and } h \circ (\text{id} \otimes i_1) = \text{id} \quad (9.2)$$

To construct h , first note that by Exercise 9.12

$$\text{Hom}(C_1(\mathcal{F}), \mathcal{H}\text{om}(\mathbb{Z}_{\text{tr}}(\mathbb{A}^1), C_1(\mathcal{F}))) \xrightarrow{\sim} \text{Hom}(C_1(\mathcal{F}) \otimes \mathbb{Z}_{\text{tr}}(\mathbb{A}^1), C_1(\mathcal{F})).$$

Now

$$\begin{aligned} \mathcal{H}\text{om}(\mathbb{Z}_{\text{tr}}(\mathbb{A}^1), C_1(\mathcal{F}))(X) &= \text{Hom}(\mathbb{Z}_{\text{tr}}(\mathbb{A}^1) \otimes \mathbb{Z}_{\text{tr}}(X), C_1(\mathcal{F})) \\ &= \text{Hom}(\mathbb{Z}_{\text{tr}}(\mathbb{A}^1 \times X), C_1(\mathcal{F})) \\ &= C_1(\mathcal{F})(\mathbb{A}^1 \times X) \\ &= C_2(\mathcal{F})(X) \end{aligned}$$

Thus it suffices to find a suitable homomorphism $C_1(\mathcal{F}) \rightarrow C_2(\mathcal{F})$. On sections this corresponds to a morphism $\mathcal{F}(X \times \mathbb{A}^1) \rightarrow \mathcal{F}(X \times \mathbb{A}^2)$ and we define this to be induced by multiplication $\mathbb{A}^2 \rightarrow \mathbb{A}^1$.

Exercise 9.20. *Show that the induced morphism h satisfies (9.2).*

\square

Now let $\tilde{C}_n(\mathcal{F}) = \text{coker}(\eta_n)$. Now the differential in $C_n(\mathcal{F})$ takes $\text{Im}(\eta_n)$ to $\text{Im}(\eta_{n-1})$ and so the $\tilde{C}_n(\mathcal{F})$ fit into a complex.

Exercise 9.21. *Show that $\tilde{C}_n(\mathcal{F})$ is quasi-isomorphic to $C_n(\mathcal{F})$.*

⁴⁴To see this consider the triangle $\mathbb{Z}_{\text{tr}}(\mathbb{A}^1) \rightarrow \mathbf{1} \rightarrow (\mathbb{Z}_{\text{tr}}(\mathbb{A}^1) \rightarrow \mathbf{1}) \rightarrow \cdot$ and consider the induced triangle after taking $\text{Hom}(\mathbf{1}, -)$.

But by Lemma 9.19 and 9.18, the complex $\tilde{C}_n(\mathcal{F})$ is in \mathcal{A} and so we are done. \square

Definition 9.22 (Tensor structure on $DM_-^{\text{eff}}(k)$). Via the embedding $DM_-^{\text{eff}}(k) \hookrightarrow D^-(\text{Nis}_{\text{tr}}(k))$, for objects $M, N \in DM_-^{\text{eff}}(k)$ we set

$$M \otimes N := RC_*(M \otimes N)$$

Remark 9.23. The inclusion $DM_-^{\text{eff}}(k) \hookrightarrow D^-(\text{Nis}_{\text{tr}}(k))$ does not preserve tensor structures.

10 Lecture 10: The embedding theorem of $DM_-^{\text{eff}}(k)$

We begin by showing that $DM_-^{\text{eff}}(k)$ is a pseudo-abelian triangulated category.

Proposition 10.1. *Let k be a perfect field. Then the full subcategory $HI(k)$ of the category $\text{Nis}_{\text{tr}}(k)$ consisting of homotopy invariant sheaves is abelian and the inclusion functor $HI(k) \rightarrow \text{Nis}_{\text{tr}}(k)$ is exact.*

Proof. By (9.1), a morphism $f: \mathcal{F} \rightarrow \mathcal{G}$ between homotopy-invariant sheaves with transfer has ker and coker also homotopy-invariant⁴⁵. \square

Corollary 10.2. *The $DM_-^{\text{eff}}(k)$ is a pseudo-abelian triangulated category.*

Proof. Only pseudo-abelian property is not clear. Since $DM_-^{\text{eff}}(k)$ is closed under countable direct sums, it follows by [Sta18, Tag 05QW] that it is pseudo-abelian. \square

We have previously constructed a functor (The Yoneda embedding): $\mathbb{Z}_{\text{tr}}: \text{Corr}_{\text{fin}}(k) \rightarrow \text{Nis}_{\text{tr}}(k)$ and by [Sta18, Tag 014X] we get an induced functor $\mathbb{Z}_{\text{tr}}: K^b(\text{Corr}_{\text{fin}}(k)) \rightarrow D^-(\text{Nis}_{\text{tr}}(k))$.

Theorem 10.3 (Embedding Theorem). *Let k be a perfect field. Then there exists a full embedding $i: DM_{\text{gm}}^{\text{eff}}(k) \rightarrow DM_-^{\text{eff}}(k)$ with dense image and a commutative diagram of tensor triangulated functors of the form*

$$\begin{array}{ccc} K^b(\text{Corr}_{\text{fin}}(k)) & \xrightarrow{\mathbb{Z}_{\text{tr}}} & D^-(\text{Nis}_{\text{tr}}(k)) \\ \downarrow M & & \downarrow RC_* \\ DM_{\text{gm}}^{\text{eff}}(k) & \xrightarrow{i} & DM_-^{\text{eff}}(k). \end{array}$$

Moreover for $X \in \text{Sm}_k$, there is a canonical isomorphism $i(M(X)) = RC_*(\mathbb{Z}_{\text{tr}}(X)) \cong C_*(\mathbb{Z}_{\text{tr}}(X))$.

Proof. Note first that by the proof of the localization theorem, $RC_*(\mathbb{Z}_{\text{tr}}(X)) \cong C_*(\mathbb{Z}_{\text{tr}}(X))$. We now show that i exists. Since $DM_-^{\text{eff}}(k)$ is pseudo-abelian by Corollary 10.2, so we can replace $DM_{\text{gm}}^{\text{eff}}(k)$ by the localization of $K^b(\text{Corr}_{\text{fin}}(k))$ with respect to the thick subcategory generated by complexes of the form

- (1) **Homotopy:** $X \times \mathbb{A}^1 \rightarrow X$.
- (2) **Mayer-Vietoris** $U \cap V \rightarrow U \oplus V \rightarrow X$, where U and V are Zariski open subsets of X such $X = U \cup V$.

Recall by the localization theorem that $DM_-^{\text{eff}}(k) = D^-(\text{Nis}_{\text{tr}}(k))/\mathcal{A}$ and so it suffices to show that the two types of morphisms (1) and (2) above, are already inverted in $D^-(\text{Nis}_{\text{tr}}(k))/\mathcal{A}$.

⁴⁵The cone of f encodes the kernel and cokernel and the cohomology of the cone coincides with the cohomology of f .

- (1) **Homotopy:** The complex $X \times \mathbb{A}^1 \rightarrow X$ gets sent to $\mathbb{Z}_{\text{tr}}(X \times \mathbb{A}^1) \rightarrow \mathbb{Z}_{\text{tr}}(X)$ which is a generator in \mathcal{A} .
- (2) **Mayer-Vietoris:** The complex $U \cap V \rightarrow U \oplus V \rightarrow X$ gets sent to $\mathbb{Z}_{\text{tr}}(U \cap V) \rightarrow \mathbb{Z}_{\text{tr}}(U) \oplus \mathbb{Z}_{\text{tr}}(V) \rightarrow \mathbb{Z}_{\text{tr}}(X)$ (via \mathbb{Z}_{tr}). Thus it suffices to show

$$0 \rightarrow \mathbb{Z}_{\text{tr}}(U \cap V) \rightarrow \mathbb{Z}_{\text{tr}}(U) \oplus \mathbb{Z}_{\text{tr}}(V) \rightarrow \mathbb{Z}_{\text{tr}}(X) \rightarrow 0 \quad (10.1)$$

is exact (as RC_* is an exact functor). It is exact by Lemma 8.29.

Next we show i is a full embedding. First note that \mathbb{Z}_{tr} is the composition

$$K^b(\text{Corr}_{\text{fin}}(k)) \rightarrow D^-(\text{PSh}(\text{Corr}_{\text{fin}}(k))) \rightarrow D^-(\text{Nis}_{\text{tr}}(k)).$$

The first functor is clearly full (as it comes from the Yoneda embedding which is fully faithful). Thus it suffices to show that if \mathcal{F} is a presheaf with transfers such that $\mathcal{F}_{\text{Nis}} = 0$, then it comes from the localizing subcategory generated by morphisms of type (1) and (2). Let \mathcal{T} be the smallest triangulated subcategory of $D^-(\text{PSh}(\text{Corr}_{\text{fin}}(k)))$ generated by the image of morphisms of type (1) and (2) which is closed under direct summands and direct sums and consider the family of functors $H^i: \text{Sm}_k \rightarrow \text{Ab}$

$$H^i(X) = \text{Hom}_{D^-(\text{PSh}(\text{Corr}_{\text{fin}}(k)))/\mathcal{T}}(\mathbb{Z}_{\text{tr}}(X), \mathcal{F}[i]).$$

It suffices to show⁴⁶ that $H^i(X) = 0$ for all X and $i \in \mathbb{Z}$. The functor H^i is a presheaf with transfer which satisfies Mayer-Vietoris and so by [BG73, Theorem 1'], it suffices to show the Zariski sheafification of H^i is zero. But the H^i are also homotopy-invariant and so by Theorem 9.11, the Nisnevich sheafification coincides with the Zariski sheafification and so it suffices to prove $H^i_{\text{Nis}} = 0$. By definition of morphisms in derived categories, the morphism $\mathbb{Z}_{\text{tr}}(X) \rightarrow \mathcal{F}[i]$ in $D^-(\text{PSh}(\text{Corr}_{\text{fin}}(k)))/\mathcal{T}$ is represented by a *roof*: $\mathbb{Z}_{\text{tr}}(X) \xrightarrow{f} K \xleftarrow{g} \mathcal{F}[i]$ where the cone of g belongs to \mathcal{T} . Moreover, the Suslin complex defines a functor

$$C_*: D^-(\text{PSh}(\text{Corr}_{\text{fin}}(k))) \rightarrow D^-(\text{PSh}(\text{Corr}_{\text{fin}}(k)))$$

because it takes exact sequences of presheaves with transfer to exact complexes of presheaves with transfer and as was shown in the proof of the localization theorem, the canonical morphism

$$K \rightarrow C_*(K)$$

is an isomorphism in $D^-(\text{PSh}(\text{Corr}_{\text{fin}}(k)))/\mathcal{T}$. Thus it suffices to show that there is a Nisnevich cover $U \rightarrow X$ such that composition

$$\mathbb{Z}_{\text{tr}}(U) \rightarrow \mathbb{Z}_{\text{tr}}(X) \xrightarrow{f} K \rightarrow C_*(K)$$

is zero. Take an injective resolution of $K \rightarrow I^\bullet$. Since $K_{\text{Nis}} = 0$, $I_{\text{Nis}} = 0$ and so locally the sections of $C_*(K)$ vanish. \square

Definition 10.4 (Motivic complex of X). In view of Theorem 10.3 we define the motivic complex of $X \in \text{Sm}_k$ by $C_*(X) := C_*(\mathbb{Z}_{\text{tr}}(X))$.

⁴⁶This is because the representable presheaves form a spanning class.

10.1 Consequences of the localization and embedding theorems

In this section we discuss consequences of Proposition 9.13 and Theorem 10.3.

Lemma 10.5 (Projective bundle formula). *Let $X \in Sm_k$ and \mathcal{E} a vector bundle over X . Let $\pi: \mathbb{P}_X(\mathcal{E}) \rightarrow X$ be the associated projective bundle. Then there is a canonical isomorphism*

$$M(\mathbb{P}_X(\mathcal{E})) = \bigoplus_{n=0}^{\dim(\mathcal{E})-1} M(X)(n)[2n].$$

Proof. By taking a Zariski trivialization of \mathcal{E} and using the Mayer-Vietoris sequence, we can assume \mathcal{E} is trivial. Then $\mathbb{P}_X(\mathcal{E}) = \mathbb{P}_k^{\dim(\mathcal{E})} \times X$ and so it suffices to assume $X = \text{Spec}(k)$ and so we need to show

$$M(\mathbb{P}^n) = \bigoplus_{i=0}^n \mathbb{Z}(i)[2i].$$

Consider the cover $\mathbb{P}^n = \mathbb{A}^n \cup (\mathbb{P}^n \setminus \{0\})$. Then by the homotopy-invariance property $M(\mathbb{A}^n) = M(\text{Spec}(k)) = \mathbb{Z}$. Now $\mathbb{P}^n \setminus \{0\}$ is an \mathbb{A}^1 -bundle over⁴⁷ \mathbb{P}^{n-1} . Thus by induction we get

$$M(\mathbb{P}^n \setminus \{0\}) = \bigoplus_{i=0}^{n-1} \mathbb{Z}(i)[2i].$$

We also need the motive of $\mathbb{A}^n \cap (\mathbb{P}^n \setminus \{0\}) = \mathbb{A}^n \setminus \{0\}$.

Lemma 10.6. *There is a canonical isomorphism $M(\mathbb{A}^n \setminus \{0\}) = \mathbb{Z}(n)[2n-1] \oplus \mathbb{Z}$.*

Proof. By definition we have $M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$. Mayer-Vietoris then gives

$$M(\mathbb{A}^1 \setminus \{0\}) \rightarrow M(\mathbb{A}^1) \oplus M(\mathbb{A}^1) \rightarrow M(\mathbb{P}^1) \rightarrow M(\mathbb{A}^1 \setminus \{0\})[1].$$

Thus $M(\mathbb{A}^1 \setminus \{0\})$ is the cone of $\mathbb{Z}[-1] \rightarrow \mathbb{Z}(1)[1]$, which is $\mathbb{Z} \oplus \mathbb{Z}(1)[1]$. For general n , consider the cover $\mathbb{A}^n \setminus \{0\} = \mathbb{A}^n \setminus \mathbb{A}^{n-1} \cup \mathbb{A}^n \setminus \mathbb{A}^1$. Then $\mathbb{A}^n \setminus \mathbb{A}^{n-1} = \{x_1 \neq 0\}$ is an affine bundle over $\mathbb{A}^1 \setminus \{0\}$ and so $M(\mathbb{A}^n \setminus \mathbb{A}^{n-1}) = \mathbb{Z} \oplus \mathbb{Z}(1)[1]$ and $\mathbb{A}^n \setminus \mathbb{A}^1 = \{x_2 \cdots x_n \neq 0\}$ is an affine bundle over $\mathbb{A}^{n-1} \setminus \{0\}$ and so by induction $M(\mathbb{A}^n \setminus \mathbb{A}^1) = \mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z}$. Moreover Mayer-Vietoris gives the distinguished triangle:

$$(\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \otimes (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z}) \rightarrow (\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \oplus (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z}) \rightarrow M(\mathbb{A}^n \setminus \{0\}) \rightarrow \cdot$$

Note that we got $M(\mathbb{A}^n \setminus \mathbb{A}^{n-1} \cap \mathbb{A}^n \setminus \mathbb{A}^1) = M(\mathbb{A}^n \setminus \mathbb{A}^{n-1} \times_k \mathbb{A}^n \setminus \mathbb{A}^1)$ as $\mathbb{A}^n \setminus \mathbb{A}^{n-1} \cap \mathbb{A}^n \setminus \mathbb{A}^1 = \mathbb{A}^n \setminus \mathbb{A}^{n-1} \times_{\mathbb{A}^n} \mathbb{A}^n \setminus \mathbb{A}^1$ is an affine bundle over $\mathbb{A}^n \setminus \mathbb{A}^{n-1} \times_k \mathbb{A}^n \setminus \mathbb{A}^1$. This then gives the result. \square

Returning to the proof of Lemma 10.5, Mayer-Vietoris for the cover $\mathbb{P}^n = \mathbb{A}^n \cup (\mathbb{P}^n \setminus \{0\})$ gives

$$M(\mathbb{A}^n \setminus \{0\}) \rightarrow M(\mathbb{A}^n) \oplus M(\mathbb{P}^n \setminus \{0\}) \rightarrow M(\mathbb{P}^n) \rightarrow \cdot$$

which yields the result. \square

Proposition 10.7 (Blow up triangle). *Every blow-up diagram⁴⁸*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{i}} & Y = \text{Bl}_Z X \\ \downarrow \sigma|_E & & \downarrow \sigma \\ Z & \xrightarrow{i} & X. \end{array}$$

⁴⁷This means we can construct a surjective morphism $p: \mathbb{P}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$ whose fibers are \mathbb{A}^1 .

⁴⁸This means the square is a pullback square, $Z \rightarrow X$ is closed immersion and there is an induced isomorphism $X \setminus Z \cong Y \setminus E$.

induces a distinguished triangle

$$M(E) \xrightarrow{(\sigma|_E)_* + \tilde{i}_*} M(Z) \oplus M(Y) \xrightarrow{i_* - \sigma_*} M(X) \rightarrow \cdot$$

in $DM_{gm}^{\text{eff}}(k)$ (assuming k is perfect).

Proof. Consider the complex $C := E \rightarrow Z \oplus Y \rightarrow X$ in $\text{Corr}_{\text{fin}}(k)$ (it's a complex by definition of the morphisms). This complex sits in a distinguished triangle:

$$C \rightarrow \text{Cone}(E \rightarrow Z \oplus Y) \rightarrow X \rightarrow \cdot$$

and so it suffices to show that the image of $E \rightarrow Z \oplus Y \rightarrow X$ in $DM_{gm}^{\text{eff}}(k)$ is 0. Consider the complex Φ of sheaves with transfers of the form

$$\mathbb{Z}_{\text{tr}}(E) \rightarrow \mathbb{Z}_{\text{tr}}(Z) \oplus \mathbb{Z}_{\text{tr}}(Y) \rightarrow \mathbb{Z}_{\text{tr}}(X).$$

By the embedding theorem, it suffices to show that for any homotopy invariant sheaf with transfers \mathcal{F} and any $i \in \mathbb{Z}$, we have

$$\text{Hom}_{D^-(\text{Nis}_{\text{tr}}(k))}(\Phi, \mathcal{F}[i]) = 0.$$

At this point, just as we have the *freely generated* Nisnevich sheaf with transfers $\mathbb{Z}_{\text{tr}}(-)$, we need to consider the analogue (without transfers):

Definition 10.8 ($\mathbb{Z}_{\text{Nis}}(X)$ for $X \in \text{Sm}_k$). Consider the presheaf $h_X: \text{Sm}_k \rightarrow \text{Ab}$ given by sending U to $\mathbb{Z}[\text{Hom}_{\text{Sm}_k}(U, X)]$. We write $\mathbb{Z}_{\text{Nis}}(X)$ for the sheafification of h_X with respect to the Nisnevich topology.

Remark 10.9. By definition $\mathbb{Z}_{\text{Nis}}(X)$ is a subsheaf of $\mathbb{Z}_{\text{tr}}(X)|_{\text{Sm}_k}$.

Let Φ_0 be the complex of Nisnevich sheaves

$$\mathbb{Z}_{\text{Nis}}(E) \rightarrow \mathbb{Z}_{\text{Nis}}(Z) \oplus \mathbb{Z}_{\text{Nis}}(Y) \rightarrow \mathbb{Z}_{\text{Nis}}(X).$$

In the same spirit as Lemmas 9.14/9.15 (but much more formally):

Exercise 10.10. For any $X \in \text{Sm}_k$, $i \in \mathbb{Z}$ and $K \in D^-(\text{Nis}(k))$ there is a canonical isomorphism

$$\text{Hom}_{D^-(\text{Nis}(k))}(\mathbb{Z}_{\text{tr}}(X), K[i]) = \mathbb{H}_{\text{Nis}}^i(X, K)$$

where $\text{Nis}(k)$ is the category of Nisnevich sheaves on Sm_k .

By Lemmas 9.14/9.15 and Exercise 10.10, the induced morphism

$$\text{Hom}_{D^-(\text{Nis}_{\text{tr}}(k))}(\Phi, \mathcal{F}[i]) \rightarrow \text{Hom}_{D^-(\text{Nis}(k))}(\Phi_0, \mathcal{F}[i])$$

is an isomorphism. The complex Φ_0 is left-exact because the blow-up square is a pullback square. Moreover

$$\text{coker}(\mathbb{Z}_{\text{Nis}}(Z) \oplus \mathbb{Z}_{\text{Nis}}(Y) \rightarrow \mathbb{Z}_{\text{Nis}}(X)) = \text{coker}(\mathbb{Z}_{\text{Nis}}(Y) \xrightarrow{\mathbb{Z}_{\text{Nis}}(\sigma)} \mathbb{Z}_{\text{Nis}}(X)) \quad (10.2)$$

since $\sigma|_E: E \rightarrow Z$ is a projective space bundle (i.e. locally on the base for the Zariski topology it is projection $T \times \mathbb{P}^d \rightarrow T$) and so $\mathbb{Z}_{\text{Nis}}(E) \rightarrow \mathbb{Z}_{\text{Nis}}(Z)$ is surjective. Let us explain this last point: It is certainly surjective in the case $E \rightarrow Z$ is the projection $T \times \mathbb{P}^d \rightarrow T$. Otherwise use

Exercise 10.11. Show that both cokernels in (10.2) are unchanged by replacing Z by a Zariski neighbourhood of Z .

Therefore we have canonical isomorphisms

$$\mathrm{Hom}_{D^-(\mathrm{Nis}(k))}(\Phi_0, \mathcal{F}[i]) = \mathrm{Ext}_{\mathrm{Nis}(k)}^i(\mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(\sigma)), \mathcal{F})$$

It is then a calculation to show the LHS is zero (cf. [VSF00, Chapter 3, Propostion 5.21]). \square

Proposition 10.12. *In the situation of Proposition 10.7, suppose in addition that Z is of (pure) codimension c in X . Then there is a canonical isomorphism*

$$M(Y) = M(X) \oplus (\oplus_{n=1}^{c-1} M(Z)(n)[2n])$$

Proof. Lemma 10.5 and Propostion 10.7 imply that we have a distinguished triangle

$$\oplus_{n=1}^{c-1} M(Z)(n)[2n] \rightarrow M(Y) \rightarrow M(X) \rightarrow .$$

So it suffices to show that the morphism $M(Y) \rightarrow M(X)$ has a canonical splitting. Denote by $q: Y_{Z \times \{0\}} := \mathrm{Bl}_{Z \times \{0\}}(X \times \mathbb{A}^1) \rightarrow X \times \mathbb{A}^1$ and $E_{Z \times \{0\}}$ be the corresponding fiber product in the blow up square and consider the diagram

$$\begin{array}{ccc} M(E) & \xrightarrow{1} & M(E_{Z \times \{0\}}) \\ \downarrow 3 & & \downarrow 4 \\ M(Z) \oplus M(Y) & \xrightarrow{2} & M(Z \times \{0\}) \oplus M(Y_{Z \times \{0\}}) \\ \downarrow 5 & & \downarrow 6 \\ M(X) & \xrightarrow{M(\mathrm{id} \times \{0\})} & M(X \times \mathbb{A}^1) \\ \downarrow 7 & & \downarrow 8 \\ M(E)[1] & \xrightarrow{1[1]} & M(E_{Z \times \{0\}})[1] \end{array} \quad (10.3)$$

By homotopy invariance $M(\mathrm{id} \times \{0\})$ is an isomorphism equal to the isomorphism $M(\mathrm{id} \times \{1\})$. But by the universal property of blow-ups (cf. [Har77, Proposition II.7.14]), this means the morphism $\mathrm{id} \times \{1\}: X \rightarrow X \times \mathbb{A}^1$ lifts to $X \rightarrow Y_{Z \times \{0\}}$. This splits the morphism 6.

We now study morphism 1. First note by [Har77, Theorem II.8.2.4], that $E \rightarrow Z$ corresponds to the projective space bundle $\mathbb{P}(N(Z)) \rightarrow Z$ where $N(Z)$ is the conormal sheaf of $Z \hookrightarrow X$. Similarly $E_{Z \times \{0\}} \rightarrow Z \times \{0\}$ corresponds to the projective space bundle $\mathbb{P}(N(Z) \oplus \mathcal{O}_Z) \rightarrow Z$ and so 1 is induced by the canonical embedding $\mathbb{P}(N(Z)) \rightarrow \mathbb{P}(N(Z) \oplus \mathcal{O}_Z)$. Thus by Lemma 10.5, the morphism 1 has a canonical splitting. Thus the composition of 2 together with the splittings of 4 and 1 give us a canonical splitting of 3. This gives a splitting of 5. \square

Corollary 10.13. *In the setting of Proposition 10.12, there exists a canonical morphism $g_Z: M(X) \rightarrow M(Z)(c)[2c]$.*

Proof. There are two morphism $M(X) \rightarrow M(Z \times \{0\}) \oplus M(Y_{Z \times \{0\}})$ in diagram (10.3). The first is the lifting of $M(\mathrm{id} \times \{1\})$ as we constructed in the proof of Proposition 10.12. The second is the composition of the canonical splitting of 5 with the morphism 2. Let f be their difference⁴⁹. Compstion of f with the morphism 6 is the zero morphism. Thus the morphism f lifts to a morphism⁵⁰ $M(X) \rightarrow M(E_{Z \times \{0\}})$. But again note $E_{Z \times \{0\}} = \mathbb{P}(N(Z) \oplus \mathcal{O}_Z)$ and $N(Z) \oplus \mathcal{O}_Z$ is a vector bundle of dimension $c + 1$ and so by Lemma 10.5

$$M(E_{Z \times \{0\}}) = \bigoplus_{n=0}^c M(Z)(n)[2n].$$

Thus we get our desired morphism $g_Z: M(X) \rightarrow M(Z)(c)[2c]$ by projection. \square

⁴⁹The underlying category is additive.

⁵⁰This is by the axioms of triangulated category. In general this morphism is not canoncial, but it is in our case as we can work in the honest category of chain complexes as all our morphisms are defined there.

11 Lecture 12: Gysin distinguished triangle and duality

Theorem 11.1 (Gysin distinguished triangle). *Let k be a perfect field. Let $i: Z \hookrightarrow X$ be a codimension n closed immersion in Sm_k and open complement $j: U \hookrightarrow X$. Then there is a canonical distinguished triangle in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$:*

$$M(U) \xrightarrow{j_*} M(X) \rightarrow M(Z)(n)[2n] \rightarrow \cdot$$

Proof. Denote by $M(X/U)$ the object in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ which corresponds to the image of the complex $U \rightarrow X$ (with X in degree 0) in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$ via $M: K^b(\mathrm{Corr}_{\mathrm{fin}}(k)) \rightarrow \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$. Thus we have a distinguished triangle:

$$M(U) \xrightarrow{j_*} M(X) \rightarrow M(X/U) \rightarrow \cdot$$

and it suffices to construct an isomorphism $M(X/U) \rightarrow M(Z)(n)[2n]$. □

Remark 11.2. *Let us mention without proof that $\mathrm{DM}_{\mathrm{gm}}(k)$ remains a triangulated tensor category and the canonical embedding $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k) \rightarrow \mathrm{DM}_{\mathrm{gm}}(k)$ is fully faithful in the case k is a perfect field (Voevodsky’s cancellation theorem). However the reasons for this are less formal than the case for pure (effective) motives.*

11.1 Motivic Cohomology

Recall that by Exercise 3.10 $Z_{\sim}^r(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}(\mathbf{1}(-r), (X, \Delta_X))$. In a *similar* spirit motivic cohomology is defined as

Definition 11.3 (Motivic cohomology). For $X \in \mathrm{Sm}_k$ we put

$$H^p(X, \mathbb{Z}(q)) := \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(k)}(M(X), \mathbb{Z}(q)[p])$$

11.2 Comparison to Chow Motives

12 Lecture 12: Homotopy category of Morel-Voevodsky

13 Lecture 13: Motives over rigid-analytic varieties - part I (after Ayoub)

14 Lecture 14: Motives over rigid-analytic varieties - part II (after Ayoub)

15 Lecture 15: 6-functor formalism of motives over rigid-analytic varieties

The work of Ayoub-Gallauer-Vezzani [AGV22] produced a 6-functor formalism for rigid-analytic varieties.

This in turn relied on the work of Ayoub [Ayo15], where for a quite general adic space S , he constructed a category of (étale) rigid analytic motives over S with rational coefficients $\mathrm{RigDA}_{\mathrm{\acute{e}t}}(S, \mathbb{Q})$.

In [Ayo15], he extended the work of the theory of motives of over an algebraic variety. Given a scheme S there are two known approaches to constructing a theory of motives over S :

- (1) the homotopic approach of Morel-Voevodsky leading to the homotopic category $\mathbf{H}(S)$ (cf. [MV99]) and its stable version $\mathbf{SH}(S)$ (cf. [Jar00]).
- (2) the “approach by transfers” by [VSF00].

16 Appendix

16.1 Solutions to exercises

Solution 16.1 (To Example 1.10). *The Segre embedding shows that S is a closed subspace of projective space. Thus it is also projective. It's also smooth (being the product of two smooth varieties), so $\text{Div}(S)$ is well-defined. Next we show $\text{Div}(S) = \mathbb{Z} \oplus \mathbb{Z}$.*

To show $\text{Div}(S) \cong \mathbb{Z} \oplus \mathbb{Z}$ we refer to [Har77, Example 6.6.1]. We can take as generators divisors (up to linear equivalence) $p := 0 \times \mathbb{P}^1$ and $q := \mathbb{P}^1 \times 0$. Then $p \cdot q = 1$ (as they meet transversely and they intersect at a single point) and $p \cdot p = 0$ because we can move p to another parallel line with no intersection. Similarly for q . This determines the intersection product claimed formula by [Har77, Theorem 1.1].

It is easy to see that $\text{Div}(S) = \text{Num}(S)$, since we have basically described $\text{Div}(S)$ and its intersection product above.

The claimed signature of the intersection form then follows.

References

- [AGV22] J. Ayoub, M. Gallauer, and A. Vezzani. The six-functor formalism for rigid analytic motives. *Forum of Mathematics, Sigma*, 10:e61, 2022.
- [And04] Y. André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses*. Société Mathématique de France, 2004.
- [Ayo15] J. Ayoub. Motifs des variétés analytiques rigides. *Société Mathématique de France. Mémoires*, 1, 01 2015.
- [Bei87] A. A. Beilinson. Height pairing between algebraic cycles. volume 1289, pages 1–26. Springer-Verlag, 1987.
- [BG73] K. S. Brown and S. M. Gersten. Algebraic k-theory as generalized sheaf cohomology. In H. Bass, editor, *Higher K-Theories*, pages 266–292, Berlin, Heidelberg, 1973. Springer Berlin Heidelberg.
- [Blo76] Some elementary theorem about algebraic cycles on abelian varieties. *Inventiones mathematicae*, 34:215–228, 1976.
- [Blo10] S. Bloch. *Lectures on Algebraic Cycles*. New Mathematical Monographs. Cambridge University Press, 2 edition, 2010.
- [BO83] P. Berthelot and A. Ogus. F-isocrystals and de rham cohomology. i. *Invent. Math.*, 72:159–199, 1983.
- [Bos13] J.-B. Bost. Algebraization, transcendence, and d-group schemes. *Notre Dame Journal of Formal Logic*, 54(3–4), January 2013.
- [Bos16] Bost, Jean-Benoît and Charles, François. Some remarks concerning the Grothendieck Period Conjecture. *Journal für die reine und angewandte Mathematik*, 2016(714):175–208, 2016.
- [Del71] P. Deligne. Théorie de Hodge : II. *Publications Mathématiques de l’IHÉS*, 40:5–57, 1971.
- [Del74a] P. Deligne. La conjecture de Weil, I. *Publications Mathématiques de l’IHÉS*, 43:273–308, 1974.

- [Del74b] P. Deligne. Théorie de Hodge: III. *Publications Mathématiques de l'IHÉS.*, 44:5–77, 1974.
- [Del80] P. Deligne. La conjecture de Weil : II. *Publications Mathématiques de l'IHÉS.*, 52:137–252, 1980.
- [DM82] P. Deligne and J. Milne. Tannakian categories. In *Hodge Cycles, Motives, and Shimura Varieties*, volume 900 of *Lecture Notes in Mathematics*, pages 101–228. Springer Berlin Heidelberg, 1982.
- [Ful84] W. Fulton. *Intersection Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, 1 edition, 1984.
- [Gro66] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Publications Mathématiques de l'IHÉS.*, 29:95–103, 1966.
- [Gro69] A. Grothendieck. Standard conjectures on algebraic cycles. In *Algebraic Geometry*, pages 193–199. Oxford University Press, London, 1969.
- [Har77] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, 1977.
- [Jan92] U. Jannsen. Motives, numerical equivalence, and semi-simplicity. *Inventiones mathematicae*, 107(3):447–452, 1992.
- [Jan94] U. Jannsen. Motivic sheaves and filtrations on chow groups. In *Motives*, volume 55 of *Proceedings of Symposia in Pure Mathematics*, pages 245–302. American Mathematical Society, Providence, RI, 1994.
- [Jar00] R. Jardine. Motivic symmetric spectra. *Doc. Math.*, 5, 01 2000.
- [Kle68] S. L. Kleiman. Algebraic cycles and the weil conjectures. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 359–386. North-Holland, Amsterdam, 1968.
- [Kle94] S. L. Kleiman. The standard conjectures. In *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1994.
- [KM74] N. M. Katz and W. Messing. Some consequences of the riemann hypothesis for varieties over finite fields. *Inventiones mathematicae*, 23:73–78, 1974.
- [Man68] J. I. Manin. Correspondences, motifs and monoidal transformations. *Mat. Sb. (N.S.)*, 77 (119):475–507, 1968.
- [MNP13] J. Murre, J. Nagel, and C. Peters. *Lectures on the Theory Of Pure Motives*. 04 2013.
- [Mum66] D. Mumford. *Lectures on Curves on an Algebraic Surface*, volume 59 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, New Jersey, 1966.
- [Mum74] D. Mumford. *Abelian Varieties*. Oxford University Press, 2nd edition, 1974.
- [Mur90] J. Murre. On the motive of an algebraic surface. *Journal für die reine und angewandte Mathematik*, 409:190–204, 1990.
- [Mur93] J. Murre. On a conjectural filtration on the chow groups of an algebraic variety: Part I and II. *Indagationes Mathematicae*, 4(2):177–201, 1993.

- [MV99] F. Morel and V. Voevodsky. A1-homotopy theory of schemes. *Publications Mathématiques de l'IHÉS*, 90(1):45–143, 1999.
- [MVW06] C. Mazza, V. Voevodsky, and C. A. Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2006.
- [Nis89] Y. Nisnevich. The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K -theory. In *Algebraic K-theory: connections with geometry and topology*, volume 279 of *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.*, pages 241–342. Kluwer Acad. Publ., Dordrecht, 1989.
- [Ras07] S. Raskin. The Weil conjectures for curves. <https://math.uchicago.edu/~may/VIGRE/VIGRE2007/REUPapers/FINALFULL/Raskin.pdf>, 2007. Lecture notes.
- [Sam58] P. Samuel. Relations d'équivalence en géométrie algébrique. In *Proceedings of the International Congress of Mathematicians 1958*, pages 157–160. Cambridge University Press, 1958.
- [Sch94] A. J. Scholl. Classical motives. In *Motives*, volume 55 of *Proc. Sympos. Pure Math.*, pages 163–187. Amer. Math. Soc., Providence, RI, 1994. Seattle, WA, 1991.
- [Ser60] J.-P. Serre. Analogues kähleriens de certaines conjectures de weil. *Annals of Mathematics*, 71(2):392–394, 1960.
- [Ser91] J.-P. Serre. Motifs. In *Astérisque*, number 198-199-200, pages 333–349. Société Mathématique de France, Paris, 1991. Journées Arithmétiques de Luminy, 17-21 Juillet 1989.
- [SR72] N. Saavedra Rivano. *Catégories Tannakiennes*. Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin-New York, 1972.
- [Sta18] T. Stacks Project Authors. *Stacks Project*. <https://stacks.math.columbia.edu>, 2018.
- [Tot14] B. Totaro. Chow groups, chow cohomology, and linear varieties. *Forum of Mathematics, Sigma*, 2:e17, 2014.
- [Voe95] V. Voevodsky. A nilpotence theorem for cycles algebraically equivalent to zero. *Internat. Math. Res. Notices*, (4):187–198, 1995.
- [Voe00] V. Voevodsky. Triangulated categories of motives over a field. *Cycles, transfers, and motivic homology theories*, pages 188–238, 2000.
- [Voi96] C. Voisin. Remarks on zero-cycles of self-products of varieties. In *Moduli of Vector Bundles*, page 21. Chapman and Hall/CRC, 1996.
- [VSF00] V. Voevodsky, A. Suslin, and E. M. Friedlander. *Cycles, Transfers, and Motivic Homology Theories. (AM-143)*. Princeton University Press, 2000.
- [Wei48] A. Weil. *Sur les courbes algébriques et les variétés qui s'en déduisent*. Hermann, 1948.
- [Wei49] A. Weil. Numbers of solutions of equations in finite fields. *Bulletin of the American Mathematical Society*, 55(5):497–508, 1949.
- [Wei71] A. Weil. *Variétés abéliennes et courbes algébriques*. Hermann, Paris, 1971.