Lecture notes: Motives and L-functions

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Abstract

These are lecture notes for the fall semester 2025-26 academic year.

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1 Lecture 1: Weil's Riemann-hypothesis for curves

Grothendieck sent a letter to Serre in 1964 detailing his idea for what a "motive" should be. An extract of this letter can be found in the annéxe of Serre's note on motives [Ser91]. Grothendieck's notion of a "motive" was motivated by proving the Weil conjectures. So what are the Weil conjectures? In 1949 Weil was interested in studying the number of solutions of equations over finite fields and he formulated the following conjecture:

Conjecture 1.1 (Weil Conjectures [Wei49])**.** *Let X be a smooth projective variety over $\mathbb{F}_p$ of dimension n such that $X \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ is irreducible and define the zeta function of X, $z(X,t)$ by*

$$\log z(X,t) := \sum_{m=1}^{\infty} |X(\mathbb{F}_{p^m})| \frac{t^m}{m}.$$

(1) ***Rationality and Riemann-Hypothesis:*** *Then there exists polynomials $P_1(t)$, $P_2(t)$, $\ldots$,$P_{2n-1}(t) \in \mathbb{Z}[t]$ where $P_i(t)$ factorizes as*

$$P_i(t) = (1 - a_{i1}t)(1 - a_{i2}t)\ldots(1 - a_{ib_i}t)$$

where $|a_{ij}| = p^{i/2}$ such that

$$z(X,t) = \frac{P_1(t) \cdot \ldots \cdot P_{2n-1}(t)}{(1-t)P_2(t) \cdot \ldots \cdot P_{2n-2}(t)(1-p^n t)}.$$

(2) **Betti numbers:** *If X comes from reduction modulo p from some integral lift $\tilde{X}/\mathbb{Z}$, then the b_i are the Betti numbers of $\tilde{X}(\mathbb{C})$.*

Example 1.2. (1) *$X = *$, then $|X(\mathbb{F}_{p^m})| = 1$ and so $z(X,t) = \frac{1}{1-t}$.*

(2) *$X = \mathbb{P}^1_{\mathbb{F}_p}$, then $|X(\mathbb{F}_{p^m})| = p^m + 1$ and so $z(X,t) = \frac{1}{(1-t)(1-pt)}$.*

Remark 1.3. *There is an analogue of Weil's conjecture for Kähler manifolds given by Serre [Ser60]. The latter is a consequence of Hodge theory, while Weil's conjecture is about étale cohomology (and intersection theory as étale cohomology itself is not powerful enough).*

Weil proved these conjectures for the case of a curve a year earlier in [Wei48]. His proof relies on constructing a suitable object from X (which we now call a *pure* motive) and proving it has desirable properties. We now give Weil's proof[1] of the Riemann-Hypothesis following closely the exposition given by Sam Raskin [Ras07].

Proof. Let's relabel X by X_0 and now use X to denote the base change $X := X_0 \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$. Let $Y := X \times_{\overline{\mathbb{F}}_p} X$. **Spoiler: In the case of a curve, the motive attached to X will essentially capture the divisors of Y. So the remainder of the proof proceeds by studying divisors of Y.** Let $\Phi_{X_0} \colon X_0 \to X_0$ be the absolute Frobenius morphism[2] on X_0 and $\mathrm{Fr}_X = \Phi_{X_0} \times \mathrm{id}$ the Frobenius endomorphism of X.

A priori, there are two actions on $X_0(\overline{\mathbb{F}}_p) = X(\overline{\mathbb{F}}_p)$. One is given by the Frobenius endomorphism on X and the other is induced by the Galois action $\Phi_{\mathrm{Spec}(\overline{\mathbb{F}}_p)}$.

Lemma 1.4. *These two actions are the same.*

Proof. Exercise. $\square$

Let $\Delta_X \colon X \to Y$ be the diagonal morphism and $\Gamma_{\mathrm{Fr}^n_X}$ the graph of the Frobenius endomorphism composed n times: Fr^n_X. Both Δ_X and $\Gamma_{\mathrm{Fr}^n_X}$ are closed immersions and cut out divisors in Y. We denote these divisors by $[\Delta_X]$ and $[\Gamma_{\mathrm{Fr}^n_X}]$, respectively.

Lemma 1.5. *We have $[\Gamma_{Fr^n_X}] = [(Fr_X \times \mathrm{id}_X)^*]^n[\Delta_X]$.*

Proof. First note that by functoriality of pullbacks $[(\mathrm{Fr}_X \times \mathrm{id}_X)^*]^n = (\mathrm{Fr}^n_X \times \mathrm{id}_X)^*$. Thus it suffices to show that for an arbitrary endomorphism $\psi \colon X \to X$, we have

$$[\Gamma_\psi] = (\psi \times \mathrm{id}_X)^*[\Delta_X] \tag{1.1}$$

where Γ_ψ is the graph of ψ in Y. We now work locally and assume $X = \mathrm{Spec}(A)$. Take a closed point $x \in X$ and a uniformizer $\pi \in \mathcal{O}_{X,x}$ and assume $\pi \in A$. By pulling back π along the two projections $Y \rightrightarrows X$ we get two global sections π_1, π_2 of Y. Then $\pi_1 - \pi_2$ generates $[\Delta_X]$. But then the LHS of (1.1) is generated[3] by $\psi^*(\pi_1) - \pi_2$.

$\square$

Lemma 1.6. *The cardinality of the set $X(\mathbb{F}_{p^n})$ is given by the intersection number[4] $[\Gamma_{Fr^n_X}] \cdot [\Delta_X]$.*

[1] Weil's proof in [Wei48] is slightly different to what is presented here. In particular he relies on the Riemann-Roch theorem for surfaces.

[2] This is the morphism given by identity on the underlying topological space of X_0 and Frobenius on the ring of functions.

[3] To see the last statement, look at the graph morphism at the level of algebras $A \otimes_{\mathbb{F}_p} A \to A$. This is given by $x \otimes y \mapsto \psi^*(x)y$ and one sees that the kernel is indeed generated by $\psi^*(\pi_1) - \pi_2$.

[4] For the interested reader, Fulton's book [Ful84] develops intersection theory in rather great generality. In general one has to be careful outside of smooth/projective assumptions.

Proof. Before we begin the proof, let us recall what intersection numbers mean in the context of curves on surfaces.

Detour: Intersection numbers of closed curves on surfaces: Let C be a smooth closed curve on a smooth projective surface S and $D \in \mathrm{Div}(S)$. Then one definition of the intersection number is $C \cdot D = \deg(\mathcal{O}_S(D)|_C)$. Unravelling what this means $\mathcal{O}_S(D)|_C$ is a line-bundle on C and its degree is the degree of it's associated divisor.

Now let's go back to the proof of Lemma 1.6. First note that since we are in characteristic p, the differential of Fr_X^n vanishes. Thus if we look at the tangent spaces of Δ_X and $\Gamma_{\mathrm{Fr}_X^n}$, we see that their sum spans all of $T_xX \times T_xX = T_{(x,x)}Y$ at every point of intersection $(x,x) \in Y$. In the literature we say Δ_X and $\Gamma_{\mathrm{Fr}_X^n}$ meet *transversely.* The upshot of transversality is the following proposition[5].

Proposition 1.7. *In the setting of the previous* ***Detour****, suppose also that D is a closed smooth curve. If C and D intersect transversely then*

$$C \cdot D = |C \cap D|$$

Proof. Exercise. $\square$

So Proposition 1.7 says that the intersection number $[\Gamma_{\mathrm{Fr}_X^n}] \cdot [\Delta_X]$ is just the number of (closed) points that Δ_X and $\Gamma_{\mathrm{Fr}_X^n}$ intersect at. Note that the points must indeed be closed as X is irreducible. On the other hand by Hilbert Nullstellensatz, the closed points of Y is just $X(\overline{\mathbb{F}}_p) \times X(\overline{\mathbb{F}}_p)$. The set of points which belong to the intersection of Δ_X and $\Gamma_{\mathrm{Fr}_X^n}$ is precisely $X(\mathbb{F}_{p^n})$ because the set of points fixed by Fr_X^n is the same as those fixed by $\Phi^n_{\mathrm{Spec}(\overline{\mathbb{F}}_p)}$ by Lemma 1.4.

$\square$

We need one more ingredient to finish the proof: the idea of *numerical* equivalence of divisors.

Definition 1.8. We say that two divisors are *numerically* equivalent if their intersection numbers with any third divisor are equal[6]. We define $\mathrm{Num}(Y)$ to be the quotient of $\mathrm{Div}(Y)$ by numerical equivalence. In particular the intersection product descends to a non-degenerate symmetric bilinear form

$$\mathrm{Num}(Y) \times \mathrm{Num}(Y) \to \mathbb{Z}.$$

We will need the Hodge Index Theorem which describes the above linear form [Mum66, Lecture 18]:

Theorem 1.9 (**Hodge Index Theorem**)**.** *Let S be a smooth projective surface over an algebraically closed field (of arbitrary characteristic). We have a direct sum decomposition*

$$Num(S) \otimes_{\mathbb{Z}} \mathbb{Q} = V \oplus V'$$

such that V has dimension 1 and the intersection form is positive definite on V and negative definite on V'.

Example 1.10. *Consider the quadric surface $S \subset \mathbb{P}^3$ given by the Segre embedding*

$$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$
$$[w : x] \times [y : z] \mapsto [wy : wz : xy : xz]$$

[5]It's so fundamental that I've labeled it a Proposition, even though we are inside a Lemma.

[6]Technically we haven't defined intersection numbers of divisors in general, but let's assume there is a reasonable definition for now.

$S := V(x_0x_3 - x_1x_2)$. *Show that* $\mathrm{Div}(S) = \mathit{Num}(S) = \mathbb{Z} \oplus \mathbb{Z}$ *and the intersection product is given by*

$$\begin{aligned} \mathrm{Div}(S) \times \mathrm{Div}(S) &\to \mathbb{Z} \\ (a_1, b_1) \times (a_2, b_2) &\mapsto a_1b_2 + b_1a_2. \end{aligned}$$

Verify the Hodge Index theorem in this case.

Lemma 1.11. *We have* $|X(\mathbb{F}_{p^n})| = p^n + O(p^{n/2})$.

Proof. Let $[H]$ and $[V]$ be the divisors in $\mathrm{Div}(Y)$ corresponding to $X \times \{x_0\}$ and $\{x_0\} \times X$ for some closed point $x_0 \in X$, respectively. Since $[H] \cdot [V] = 1$ and $[H] \cdot [H] = 0$, these cannot be equal in $\mathrm{Num}(Y)$. Moreover $U := \mathbb{Q}[H] \oplus \mathbb{Q}[V]$ is a finite-dimensional subspace of $W := \mathrm{Num(Y)} \otimes_{\mathbb{Z}} \mathbb{Q}$, we can write

$$W = U \oplus U'$$

where U' is the orthogonal complement. We claim that the intersection form on U' is negative-definite: Indeed on matrix on U with respect to the basis $\{[H], [V]\}$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has one positive eigenvalue and so by the Hodge Index Theorem, the subspace on which the intersection form us positive-definite is contained in U.

Let

$$\begin{aligned} T \colon W &\to W \\ D &\mapsto (\mathrm{Fr}_X \times \mathrm{id}_X)^* D \end{aligned}$$

Then $T([H]) = p[H]$ and $T([V]) = [V]$. We know by Lemma 1.5 that $T^n[\Delta_X] = [\Gamma_{\mathrm{Fr}_X^n}]$. For the following note that pullback/pushforward of divisors (up to linear equivalence) descends to numerical equivalence (this is essentially the content of the *moving lemma*). Moreover for any $D, E \in \mathrm{Num}(Y)$

$$\begin{aligned} (\mathrm{Fr}_X \times \mathrm{id}_X)^* D \cdot (\mathrm{Fr}_X \times \mathrm{id}_X)^* E &\overset{(1)}{=} D \cdot (\mathrm{Fr}_X \times \mathrm{id}_X)_* (\mathrm{Fr}_X \times \mathrm{id}_X)^* E \\ &\overset{(2)}{=} D \cdot pE \\ &\overset{(3)}{=} pD \cdot E \end{aligned}$$

where (1) follows from Proposition 1.12:

Proposition 1.12. *Given* $\varphi \colon Y \to Z$ *so that* $\varphi^* \colon \mathrm{Div}(Z) \to \mathrm{Div}(Y)$ *and* $\varphi_* \colon \mathrm{Div}(Y) \to \mathrm{Div}(Z)$ *are well-defined, we have*

$$C \cdot \varphi^*(D) = \varphi_*(C) \cdot D$$

for any $C \in \mathrm{Div}(Y)$ *and* $D \in \mathrm{Div}(Z)$.

Proof. Exercise. □

and (2) follows from Proposition

Proposition 1.13. *In the setting of Proposition 1.12* $\varphi_*\varphi^* \colon \mathrm{Div}(Z) \to \mathrm{Div}(Z)$ *is given by* $D \mapsto \deg(\varphi) D$.

Proof. Exercise □

and (3) follows by linearity of the intersection form.

Thus for all $v, w \in W$, we have $Tv \cdot Tw = p(v \cdot w)$. Since $[H] \cdot [\Delta_X] = [V] \cdot [\Delta_X] = 1$, we can write

$$[\Delta_X] = [H] + [V] + u'$$

for some $u' \in U'$. We then compute

$$\begin{aligned} |X(\mathbb{F}_{p^n})| &\overset{(a)}{=} [\Gamma_{\mathrm{Fr}_X^n}] \cdot [\Delta_X] \\ &\overset{(b)}{=} T^n[\Delta_X] \cdot [\Delta_X] \\ &\overset{(c)}{=} p^n + 1 + T^n u' \cdot u' \end{aligned}$$

where (a) follows from Lemma 1.6, (b) by Lemma 1.5 and (c) because $T([H]) = p[H]$ and $T([V]) = [V]$.

It's easy to check that $T^n u' \in U'$ and so we can apply the Cauchy-Schwarz inequality to get

$$|T^n u' \cdot u'| \leq \sqrt{|T^n u' \cdot T^n u'||u' \cdot u'|} = p^{n/2}|u' \cdot u'|$$

This completes the proof of Lemma 1.11.

□

A relatively straightforward analysis argument then concludes the proof of the Riemann-Hypothesis for curves. We won't include the details, as it's not what we are after conceptually.

□

2 Lecture 2: Algebraic cycles and adequate equivalence relations

For this lecture, I will be following the book by Murre-Nagel-Peters [MNP13, Chapter 1]. Another good reference is [And04, Chapitre 3]. Let k be an arbitrary field and X a k-variety.

Definition 2.1 (algebraic cycle)**.** An algebraic cycle on X is a formal finite integral linear combination $Z = \sum n_\alpha Z_\alpha$ of irreducible closed subvarieties Z_α of X. If all Z_α have the same dimension i, we say that Z is a dimension i cycle. We denote by $Z_i(X)$ the abelian group of dimension i cycles on X. When considering the codimension point of view we write $Z^{d-i}(X) := Z_i(X)$ if X is of dimension d. We write $Z(X) := \oplus_i Z^i(X)$ and consider it as a group with a graded structure.

Lemma 2.2. *Suppose X is smooth. Then two closed subvarieties V and W of X with codimensions i and j, respectively, have intersection*

$$V \cap W = \cup_\alpha Z_\alpha$$

where each Z_α is an irreducible subvariety of codimension at most i+j.

Proof. We have that $V \cap W = \Delta^{-1}(V \times W)$ where $\Delta \colon X \to X \times X$ is the diagonal map. Since X is smooth, we can write $X \times X = \mathrm{Spec}(A)$ and $X = V(f_1, \ldots, f_c)$ where $f_1, \ldots, f_c$ is a regular sequence in A and $c = \dim(X)$. Then if $V \times W = \mathrm{Spec}(A/\mathfrak{p})$ then $V \cap W = \mathrm{Spec}(A/(\mathfrak{p} + (f_1, \ldots, f_c)))$. Then for $z \in Z_\alpha$ a closed point

$$\dim(V \times W) = \dim \mathcal{O}_{V\times W,z} \quad \text{and} \quad \dim(Z_\alpha) = \dim \mathcal{O}_{Z_\alpha,z} = \dim \mathcal{O}_{V\times W,z}/(f_1, \ldots, f_c).$$

From here one can compare the relevant dimensions by the fact that quotienting a local ring by an element in the maximal ideal, reduces the dimension by at most one. □

Definition 2.3 (proper intersection product of algebraic cycles)**.** In the setting of Lemma 2.2, we say that the intersection $V \cap W$ is *proper* (or V and W intersect *properly*) if the codimension of each Z_α is $i+j$. In this case the *intersection number* is defined by

$$i(V \cdot W; Z) := \sum_r (-1)^r \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathrm{Tor}_r^{\mathcal{O}_{X,Z}}(\mathcal{O}_{V,Z}, \mathcal{O}_{W,Z}))$$

where $A := \mathcal{O}_{?,Z}$ denotes the local ring of $?$ at the generic point of Z. We define the *intersection product*

$$V \cdot W := \sum_\alpha i(V \cdot W; Z_\alpha) Z_\alpha.$$

Definition 2.4 (proper pushforward)**.** Let $f\colon X \to Y$ be a proper morphism of k-varieties and $Z \subset X$ a k-dimensional closed irreducible subvariety. We define

$$f_*Z = \begin{cases} 0, & \text{if } \dim(f(Z)) < k \\ [R(Z)\colon R(f(Z))]f(Z), & \text{otherwise} \end{cases} \tag{2.1}$$

where $R(?)$ is the field of rational functions[7] on $?$. Extending by linearity induces a homomorphism

$$f_*\colon Z_k(X) \to Z_k(Y).$$

In general we say two algebraic cycles $\alpha, \beta \in Z(X)$ intersect properly if each components of α intersects each component of β properly.

Definition 2.5 (flat pullback)**.** Let $f\colon X \to Y$ be a flat morphism of k-varieties and $Z \subset Y$ a k-codimensional closed irreducible subvariety. We define

$$f^*Z = f^{-1}(Z)$$

Because f is flat, $f^{-1}(Z)$ turns out to be of codimension k (assuming it is non-empty). Extending by linearity induces a homomorphism

$$f^*\colon Z^k(Y) \to Z^k(X)$$

Definition 2.6. A *correspondence* from X to Y is a cycle in $X \times Y$. A correspondence $Z \in Z^t(X \times Y)$ acts on cycles on X as follows

$$\begin{aligned} Z\colon Z^i(X) &\to Z^{i+t-\dim(X)}(Y) \\ T &\mapsto pr_{Y*}(Z \cdot (T \times Y)) \end{aligned}$$

whenever defined. We call $t - \dim(X)$ the degree of the correspondence.

Example 2.7. *It turns out the notion of correspondences generalizes the notion of (proper) pushforward and (flat) pullback. Prove this.*

As we see correspondences (or intersection products) are not always defined. This is where the notion of *adequate equivalence* comes in. These are equivalences classes on the groups Z^i such that the intersection product is always defined.

[7]Note that $f|_Z\colon Z \to f(Z)$ is a dominant morphism, so the above degree is well-defined.

2.1 Adequate Equivalence

We now work in the category SmProj(k) of smooth projective varieties over k.

Definition 2.8 (Adequate Equivalence)**.** We say that an equivalence relation $\sim$ on $Z(X)$ is *adequate* if

(1) (**linearity**) $\sim$ is compatible with addition and graduation.

(2) (**moving lemma**) For all α, $\beta \in Z(X)$, $\exists \alpha' \sim \alpha$ such that α' and β intersect properly.

(3) (**correspondence**) $\sim$ is compatible with correspondences: In the setting of Definition 2.6 if $T \sim 0$ and Z intersects $T \times Y$ properly, then $Z(T) \sim 0$.

We write $Z_\sim(X) := Z(X)/\sim$ and for some field F, $Z_\sim(X)_F := Z(X) \otimes_{\mathbb{Z}} F/\sim$. We also write

$$Z^i_{\sim,0}(X) := \{Z \in Z^i(X) |\ Z \sim 0\}$$

The fact that intersection product is defined on the whole $Z_\sim(X)$ is a straightforward consequence of Definition 2.8 (cf. [Sam58, Proposition 6 and 7]).

Lemma 2.9. *For any adequate equivalence relation $\sim$ on $X \in SmProj(k)$, we have*

(1) $Z_\sim(X)$ is a graded ring with product induced by the intersection product of cycles.

(2) A correspondence Z from X to Y of degree r induces $Z_*\colon Z^i_\sim(X) \to Z^{i+r}_\sim(Y)$ and equivalent correspondences induce the same Z_*.

We now discuss the following adequate equivalence relations

- rational equivalence $\sim_{\text{rat}}$
- algebraic equivalence $\sim_{\text{alg}}$
- smash nilpotence equivalence $\sim_{\otimes\text{nil}}$
- homological equivalence $\sim_{\text{hom}}$
- numerical equivalence $\sim_{\text{num}}$

2.1.1 Rational equivalence

Definition 2.10 (Rational equivalence)**.** A cycle $\alpha \in Z(X)$ is rationally equivalent to 0 ($\alpha \sim_{\text{rat}} 0$) if there exists $\beta \in Z(X \times \mathbb{P}^1)$ such that $\beta(0)$ and $\beta(\infty)$ are well-defined and $\alpha = \beta(0) - \beta(\infty)$.

Lemma 2.11. *Rational equivalence corresponds to linear equivalence for codimension 1 cycles $Z^1(X)$.*

Proof. We first show $\operatorname{div}(f) \sim_{\text{rat}} 0$ for $f \in R(X)$ a rational function. We can think of f as $f\colon U \to \mathbb{P}^1$ for some dense open $U \subset X$. Let $W \subset X \times \mathbb{P}^1$ be the closure of the graph of f. Then W gives a cycle $\beta \in Z(X \times \mathbb{P}^1)$ and essentially by definition $\operatorname{div}(f) = \beta(0) - \beta(\infty)$.

For the converse suppose $\alpha \in Z^1(X)$ and $\alpha \sim_{\text{rat}} 0$. Take a component $Z' \subset X \times \mathbb{P}^1$ of β (with β part of Definition 2.10). Then Z' dominates $\mathbb{P}^1$. Let $Z \subset X$ be the image of Z' under the projection to X. Then $Z \subset X$ is closed (as projection is proper) and $Z' \to Z$ is proper and dominant with fibers of dimension 0 or 1.

There are two cases as to whether $\dim(Z) < \dim(Z')$ or $\dim(Z) = \dim(Z')$.

If $\dim(Z) < \dim(Z')$, then $Z' = Z \times \mathbb{P}^1$ and $[Z'_0] - [Z'_\infty] = [Z] - [Z] = 0$.

If $\dim(Z) = \dim(Z')$, then $Z' \to Z$ is generically finite (i.e. inverse image of generic point is finite). Then I leave it as an exercise[8] to show that $[Z'_0] - [Z'_\infty] = \operatorname{div}(\operatorname{Nm}(f))$ where $f\colon Z' \to \mathbb{P}^1$ viewed as a rational function on Z'. $\square$

[8] The main idea is essentially in the proof of [Ful84, Proposition 1.4(b)]

The technical difficulty in proving that rational equivalence is indeed an adequate equivalence relation lies in proving the *moving lemma*. The proof is roughly as follows: We embed $X \hookrightarrow \mathbb{P}^N$ and given $V, W \subset X$, we need to move V so that it intersects W properly. There are two cases to consider as to whether $X = \mathbb{P}^N$ or not. In the former, there is some general linear transformation that makes V and W intersect properly. In the later case, one considers a linear subspace $L \subset \mathbb{P}^N$ and the cone $C(L, V)$. For the details we refer to [Ful84, Example 11.4.1].

Example 2.12. *What goes wrong with the moving lemma for rational equivalence if we relax the smoothness assumption? What if we keep smoothness and relax the projectivity assumption?*

Definition 2.13 (Chow ring)**.** The corresponding graded ring $CH(X) := Z_{\text{rat}}(X)$ is called the Chow ring. We will also denote by $\text{Corr}(X, Y) := CH(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ the *correspondences* from X to Y.

Lemma 2.14. *Among all adequate equivalence relations, rational equivalence is the finest.*

Proof. Let $\sim$ be an adequate equivalence relation. It suffices to prove $[0] \sim [\infty]$ as then by using a correspondence, we get the definition of rational equivalence. Since the point $1 \in \mathbb{P}^1$ does not intersect itself properly, by the moving lemma there exists $\sum_i n_i[x_i] \in Z^1(\mathbb{P}^1) \sim 1$ with $x_i \in \mathbb{P}^1$ such that x_i intersects 1 properly. In other words $x_i \neq 1$. Consider now the correspondence $Z \in Z^1(\mathbb{P}^1 \times \mathbb{P}^1)$ given by the graph of the polynomial

$$1 - \prod_i \left(\frac{x - x_i}{1 - x_i}\right)^{m_i} \tag{2.2}$$

for a collection of $m_i > 0$ and $T = \sum_i n_i[x_i] - 1$. Then $Z(T)$ is just the pushforward of T by (2.2). The pushforward of T is just $mn[1] - m[0]$ where $m = \sum_i m_i$ and $n = \sum_i n_i$. Since this holds for arbitrary m_i, we get $n[1] \sim [0]$. Applying the condition of correspondence to the automorphism $x \mapsto \frac{1}{x}$, we get $n[1] \sim [\infty]$, from which we can conclude. $\square$

2.1.2 Algebraic equivalence

Definition 2.15 (Algebraic Equivalence)**.** This is the same definition as rational equivalence but with $\mathbb{P}^1$ replaced by any smooth projective irreducible curve and the two points 0 and ∞ by any two k-rational points on the curve. In other words $\alpha \in Z(X)$ is $\sim_{\text{alg}} 0$ if there exists a smooth irreducible projective curve C and $\beta \in Z(X \times C)$ and two points $a, b \in C(k)$ such that $\beta(a) = 0$ and $\beta(b) = \alpha$.

Example 2.16 (algebraic equivalence is coarser than rational equivalence)**.** *Take an elliptic curve E over $\mathbb{C}$ and two distinct points $a, b \in |E|$. Then $a - b$ is not a divisor of any rational function. This is because we can make an identification $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ and we see by Cauchy's residue theorem that any rational function has it's sum of residues equal to 0. So any rational function cannot have a simple pole. In summary $a - b \not\sim_{alg} 0$.*

On the other hand E is equipped with a degree 2 cover over $\mathbb{P}^1$ with 4 ramification points (by Hurwitz's theorem). If we take the graph $Z \subset E \times \mathbb{P}^1$ of this cover, then we get that these 4 ramification points must be algebraically equivalent[9]*.*

2.2 Smash Nilpotent equivalence

Definition 2.17 (Smash Nilpotent equivalence)**.** For $Z \in Z(X)$ we say $Z \sim_{\otimes} 0$ iff for some positive integer n, $Z^n \sim_{\text{rat}} 0$ where we view $Z^n \in Z(X^n)$.

Theorem 2.18 ($\sim_{\otimes}$ vs $\sim_{\text{alg}}$)**.** *We have $Z^i_{alg,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \subset Z^i_{\otimes,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

[9] We can just take the graph of the identity map to conclude that any two rational points on any smooth projective curve are algebraically equivalent.

Theorem 2.18 is due independently to Voevodsky [Voe95] and Voisin [Voi96].

Proof. We proceed in several steps as in [MNP13, Appendix B].
Step 0: Reduce to $k = \overline{k}$.

Exercise 2.19. *For any (adequate) equivalence relation $\sim$, there is a natural map*

$$Z_\sim(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to (Z_\sim(X_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{Q})^{\mathrm{Gal}(\overline{k}/kk)}$$

given by restricting a closed subvariety over k to one over $\overline{k}$. Prove that this map is a bijection.

Exercise 2.19 allows us to assume k is algebraically closed.
Step 1: Reduce to the case of a smooth projective curve. Take $Z \sim_{\mathrm{alg}} 0$. Then by definition $\exists\, \Gamma \in \mathrm{Corr}(C, X)$ and two points $a, b \in C(k)$ such that $Z = \Gamma_*(a-b)$. Thus taking products gives $Z^n = (\Gamma^n)_*(a-b)^n$ and so it suffices to show $(a-b)^n \sim_{\mathrm{alg}} 0$ on C^n. In fact we shall show

$$(a-b)^n \sim_{\mathrm{alg}} 0 \quad \text{for} \quad n > g,$$

where g is the genus of the curve C.
Step 2: Reducing $(a-b)^n$ as a divisor on the n-fold symmetric product of C. A priori $(a-b)^n \in Z(C^n)$. However the symmetric group S_n induces an action on C^n and clearly $(a-b)^n \in (Z(C^n) \otimes_{\mathbb{Z}} \mathbb{Q})^{S_n}$.

Exercise 2.20. *Show that $(Z(C^n) \otimes_{\mathbb{Z}} \mathbb{Q})^{S_n} \cong Z(C^n/S^n) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Exercise 2.20 allows us to view $(a-b)^n$ in the quotient variety C^n/S^n (the n-fold symmetric product of C).
Step 3: Comparison of C/S^n with the Jacobian[10] $J(C)$. Fix a base point $e \in C(k)$. Denote by

$$\begin{aligned} \pi_n \colon C^n &\to C^n/S^n \\ (x_1, \dots, x_n) &\mapsto [x_1, \dots, x_n] \end{aligned}$$

the natural surjection and

$$\begin{aligned} \varphi_n \colon C^n/S^n &\to J(C) \\ [x_1, \dots, x_n] &\mapsto \sum_i (x_i - e). \end{aligned}$$

Lemma 2.21. *The induced map $(\varphi_n)_* \colon CH_0(C^n/S^n) \to CH_0(J(C))$ is an isomorphism for all $n \geq g$.*

Proof. If $n = g$, then we claim that φ_n is a birational morphism. Indeed by Riemann-Roch

$$\ell(x_1 + \cdots + x_g) = g + 1 - g + \ell(K - x_1 - \cdots - x_g)$$

if none of the x_i are base points of the canonical divisor K, then since $\ell(K) = g$, we get $\ell(K - x_1 - \cdots - x_g) = 0$ and so $\ell(x_1 + \cdots + x_g) = 1$. This means that φ_n is an isomorphism[11] outside of $[e, e, \dots, e]$. So it is a birational morphism. By [Ful84, Example 16.1.11], the group CH_0 is invariant for birational morphisms.

[10] The Jacobian variety $J(C)$ is the variety which represents the functor $T \mapsto \{\text{invertible sheaves of degree 0 on } X \times T\}$.

[11] Because the fiber of φ_n is just the set of points $[x_1, \dots, x_n]$ such that $\sum_i x_i$ form a complete linear system. This also means fibers are projective.

Suppose $n > g$ and consider the natural embedding

$$\iota\colon C^g/S^g \to C^n/S^n$$
$$[x_1, \dots, x_g] \mapsto [x_1, \dots, x_g, \underbrace{e, \dots, e}_{n-g}]$$

Then $(\varphi_n)_* \iota_* = (\varphi_g)_*$. Since $(\varphi_g)_*$ is an isomorphism, it follows that ι_* is an injection. It remains to show that it is a surjection. So take $y \in C^n/S^n$ and consider the image $z \in J(C)$ and some point $x \in C^g/S^g$ which maps to z under equivalence. Then $\iota(x)$ and y belong to the fiber $\varphi_n^{-1}(z)$. But the fibers[12] of φ_n are projective and any two points are rationally equivalent. This proves the lemma. $\square$

Step 4: Application of Bloch's theorem. We have that $(\varphi_{g+1})_*((a-b)^{g+1}) = (\varphi_1(a) - \varphi_1(b))^{*(g+1)}$ and this vanishes by [Blo76].

$\square$

Exercise 2.22. *Use ideas from the proof of Theorem 2.18 to show that for an elliptic curve $E/\mathbb{C}$, $Z^1_{alg,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \not\subset Z^1_{\otimes,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

2.3 Homological equivalence

To define *homological equivalence* we need to define a *Weil cohomology theory.* Let F be a field of characteristic 0. We denote GrVect_F be the category of finite dimensional graded F-vector spaces.

Definition 2.23. A Weil cohomology theory is a functor

$$H\colon \mathrm{SmProj}(k)^{\mathrm{opp}} \to \mathrm{GrVect}_F$$

which satisfies the following axioms:

(1) $\exists$ a graded cup product $\cup\colon H(X) \times H(X) \to H(X)$ such that if $a \in H^i(X)$, $b \in H^j(X)$, then $a \cup b = (-1)^{ij} b \cup a$.

(2) one has Poincaré duality (assume X has pure dimension d): $\exists$ a trace isomorphism

$$\mathrm{Tr}\colon H^{2d}(X) \xrightarrow{\sim} \mathbb{Q}$$

such that

$$H^i(X) \times H^{2d-i} \xrightarrow{\cup} H^{2d}(X) \xrightarrow{\mathrm{Tr}} \mathbb{Q}$$

is a perfect pairing.

(3) A Künneth map

$$H(X) \otimes H(Y) \xrightarrow{(pr_X)^* \otimes (pr_Y)^*} H(X \times Y)$$

which is a (graded) isomorphism.

(4) there are cycle class maps

$$\gamma_X\colon \mathrm{CH}^i(X) \to H^{2i}(X)$$

which satisfy various compatibilities[13].

[12] fiber above a point is a complete linear system
[13] I will state them explicitly when we need them.

(5) If X is pure of dimension d and $\iota \colon Y \hookrightarrow X$ is a smooth hyperplane, then *weak Lefschetz* holds:

$$H^i(X) \xrightarrow{\iota^*} H^i(Y)$$

is an isomorphism if $i < d-1$ and an injection for $i = d-1$.

(6) With the setting as in (5), the Lefschetz operator $L(\alpha) := \alpha \cup \gamma_X(Y)$ induces isomorphisms

$$L^{d-i} \colon H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X)$$

for $0 \le i \le d$. This is known as *Hard Lefschetz*.

Example 2.24. *For $k = \mathbb{C}$, we get many examples of Weil cohomology theories:*

(1) singular cohomology groups: $H^i(X_{an})$ where X_{an} is the complex manifold attached to X.

(2) classical de Rham cohomology: $H^i_{dR}(X_{an}, \mathbb{C})$.

(3) algebraic de Rham cohomology: $H^i_{dR} := \mathbb{H}(X, \Omega^\bullet_{X/\mathbb{C}})$.

The fact that these are indeed Weil cohomology groups follows from classical reasons together with comparison isomorphisms. On the other hand the fact that $H^i_{\acute{e}t}(X, \mathbb{Q}_\ell)$ is a Weil cohomology theory (in particular satisfies Hard Lefschetz) is deep work of Deligne [Del80].

Definition 2.25 (Homological equivalence)**.** Fix a Weil cohomology theory. Then for $Z \in Z(X)$ we say $Z \sim_{\mathrm{hom}} 0$ if $\gamma_X(Z) = 0$.

We can compare homological equivalence to algebraic and smash nilpotent equivalence.

Lemma 2.26 ($\sim_\otimes$ and $\sim_{\mathrm{alg}}$ vs $\sim_{\mathrm{hom}}$)**.** *(1) $Z^i_{alg,0}(X) \subset Z^i_{hom,0}(X)$.*

(2) $Z^i_{\otimes,0}(X) \subset Z^i_{hom,0}(X)$.

Proof. For (1), note that $\alpha \sim_{\mathrm{alg}} 0$ means that for some smooth projective curve C, $\alpha = pr_{X*}pr_C^*([a] - [b])$ for two rational points $a, b \in C$. Now cycle map is compatible with push-forward and pullbacks (one of the conditions I didn't state in Definition 2.23(4)). So we can reduce to the case of a curve. We then conclude by Matsusaka's theorem:

Theorem 2.27 (Matsusaka's Theorem)**.**

$$Z^1_{hom,0}(X) = \{D \in Z^1(X) |\ \ nD \sim_{alg} 0 \textit{ for some } n \in \mathbb{Z}\}$$

For part (2), note that $\alpha \sim_\otimes 0$ means $\alpha^n \sim_{\mathrm{rat}} 0$ for some $n > 0$. Then

$$\gamma_{X^n}(\alpha^n) = \underbrace{\gamma_X(\alpha) \otimes \ldots \otimes \gamma_X(\alpha)}_{n}$$

is zero. So each of $\gamma_X(\alpha) = 0$. $\square$

Exercise 2.28. *Find an alternative proof of Lemma 2.26(1) using part (2) and Voevodsky-Voisin Theorem 2.18.*

2.4 Numerical equivalence

Definition 2.29 (Numerical Equivalence)**.** Let X be of pure dimension d. For $Z \in Z^i(X)$, we say $Z \sim_{\text{num}} 0$ if for every $W \in Z^{d-i}(X)$ such that $Z \cdot W$ is defined, we have $\deg(Z \cdot W) = 0$.

We can compare homological equivalence and numerical equivalence.

Lemma 2.30 ($\sim_{\text{hom}}$ vs $\sim_{\text{num}}$)**.** *$Z^i_{hom,0}(X) \subset Z^i_{num,0}(X)$.*

Proof. We will need to use that γ_X (the cycle class map) is compatible with intersection products:

$$\gamma_X(\alpha \cdot \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta) \tag{2.3}$$

and compatible with points P:

$$\mathrm{Tr} \circ \gamma_X = \deg \qquad \text{on } CH^d(X). \tag{2.4}$$

Conditions (2.3) and (2.4) are the remaining conditions I didn't state in Definition 2.23(4)).

By property (2.4), we see that the result holds for $i = d$ (i.e. zero cycles). Suppose now $i < d$ and $Z \in Z^i_{\text{hom},0}(X)$ and $W \in Z^{d-i}(X)$. such that $Z \cdot W$ is defined. Then

$$\begin{aligned} 0 &\overset{(i)}{=} \mathrm{Tr}(\gamma_X(Z) \cup \gamma_X(W)) \\ &\overset{(ii)}{=} \mathrm{Tr}(\gamma_X(Z \cdot W)) \\ &\overset{(iii)}{=} \deg(Z \cdot W) \end{aligned}$$

where (i) holds because $\gamma_X(Z) = 0$, (ii) holds because of (2.3) and (iii) holds because of (2.4). □

Summarizing Lemmas 2.14, 2.26(1) and 2.30 have shown the following chain

$$Z^i_{\text{rat},0}(X) \subset Z^i_{\text{alg},0}(X) \subset Z^i_{\text{hom},0}(X) \subset Z^i_{\text{num},0}(X)$$

As part of the standard conjectures:

Conjecture 2.31 (Standard Conjecture D)**.** *$Z^i_{hom,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = Z^i_{num,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

3 Lecture 3: Grothendieck's pure motives

4 Lecture 4: Voevodsky's mixed motives (approach by transfers)

5 Lecture 5: Homotopy category of Morel-Voevodsky

6 Lecture 6: Motives over rigid-analytic varities - part I (after Ayoub)

7 Lecture 7: Motives over rigid-analytic varieties - part II (after Ayoub)

8 Lecture 8: 6-functor formalism of motives over rigid-analytic varieties

The work of Ayoub-Gallauer-Vezzani [AGV22] produced a 6-functor formalism for rigid-analytic varieties.

This in turn relied on the work of Ayoub [Ayo15], where for a quite general adic space S, he constructed a category of (étale) rigid analytic motives over S with rational coefficients $\mathbf{RigDA}_{\text{ét}}(S, \mathbb{Q})$.

In [Ayo15], he extended the work of the theory of motives of over an algebraic variety. Given a scheme S there are two known approaches to constructing a theory of motives over S:

(1) the homotopic approach of Morel-Voevodsky leading to the homotopic category $\mathbf{H}(S)$ (cf. [MV99]) and its stable version $\mathbf{SH}(S)$ (cf. [Jar00]).

(2) the "approach by transfers" by [VSF00].

9 Appendix

9.1 Solutions to exercises

Solution 9.1 (To Example 1.10)**.** *The Segre embedding shows that S is a closed subspace of projective space. Thus it is also projective. It's also smooth (being the product of two smooth varieties), so* $\operatorname{Div}(S)$ *is well-defined. Next we show* $\operatorname{Div}(S) = \mathbb{Z} \oplus \mathbb{Z}$.

To show $\operatorname{Div}(S) \cong \mathbb{Z} \oplus \mathbb{Z}$ *we refer to [Har77, Example 6.6.1]. We can take as generators divisors (up to linear equivalence) $p := 0 \times \mathbb{P}^1$ and $q := \mathbb{P}^1 \times 0$. Then $p \cdot q = 1$ (as they meet transversely and they intersect at a single point) and $p \cdot p = 0$ because we can move p to another parallel line with no intersection. Similarly for q. This determines the intersection product claimed formula by [Har77, Theorem 1.1].*

It is easy to see that $\operatorname{Div}(S) = Num(S)$*, since have basically described* $\operatorname{Div}(S)$ *and it's intersection product above.*

The claimed signature of the intersection form then follows.

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