Lecture notes: Motives and L-functions

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Abstract

These are lecture notes for the fall semester 2025-26 academic year.

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1 Lecture 1: Weil's Riemann-hypothesis for curves

Grothendieck sent a letter to Serre in 1964 detailing his idea for what a "motive" should be. An extract of this letter can be found in the annéxe of Serre's note on motives [Ser91]. Grothendieck's notion of a "motive" was motivated by proving the Weil conjectures. So what are the Weil conjectures? In 1949 Weil was interested in studying the number of solutions of equations over finite fields and he formulated the following conjecture:

Conjecture 1.1 (Weil Conjectures [Wei49])**.** *Let X be a smooth projective variety over $\mathbb{F}_p$ of dimension n such that $X \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ is irreducible and define the zeta function of X, $z(X,t)$ by*

$$\log z(X,t) := \sum_{m=1}^{\infty} |X(\mathbb{F}_{p^m})| \frac{t^m}{m}.$$

(1) ***Rationality and Riemann-Hypothesis:*** *Then there exists polynomials $P_1(t)$, $P_2(t)$, ..., $P_{2n-1}(t) \in \mathbb{Z}[t]$ where $P_i(t)$ factorizes as*

$$P_i(t) = (1 - a_{i1}t)(1 - a_{i2}t) \dots (1 - a_{ib_i}t)$$

where $|a_{ij}| = p^{i/2}$ such that

$$z(X, t) = \frac{P_1(t) \cdot \ldots \cdot P_{2n-1}(t)}{(1-t)P_2(t) \cdot \ldots \cdot P_{2n-2}(t)(1-p^n t)}.$$

(2) ***Betti numbers:*** *If X comes from reduction modulo p from some integral lift $\tilde{X}/\mathbb{Z}$, then the b_i are the Betti numbers of $\tilde{X}(\mathbb{C})$.*

Example 1.2. *(1) $X = *$, then $|X(\mathbb{F}_{p^m})| = 1$ and so $z(X, t) = \frac{1}{1-t}$.*

(2) $X = \mathbb{P}^1_{\mathbb{F}_p}$, then $|X(\mathbb{F}_{p^m})| = p^m + 1$ and so $z(X, t) = \frac{1}{(1-t)(1-pt)}$.

Remark 1.3. *There is an analogue of Weil's conjecture for Kähler manifolds given by Serre [Ser60]. The latter is a consequence of Hodge theory, while Weil's conjecture is about étale cohomology (and intersection theory as étale cohomology itself is not powerful enough).*

Weil proved these conjectures for the case of a curve a year earlier in [Wei48]. His proof relies on constructing a suitable object from X (which we now call a *pure* motive) and proving it has desirable properties. We now give Weil's proof[1] of the Riemann-Hypothesis following closely the exposition given by Sam Raskin [Ras07].

Proof. Let's relabel X by X_0 and now use X to denote the base change $X := X_0 \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$. Let $Y := X \times_{\overline{\mathbb{F}}_p} X$. **Spoiler: In the case of a curve, the motive attached to X will essentially capture the divisors of Y. So the remainder of the proof proceeds by studying divisors of Y.** Let $\Phi_{X_0} \colon X_0 \to X_0$ be the absolute Frobenius morphism[2] on X_0 and $\mathrm{Fr}_X = \Phi_{X_0} \times \mathrm{id}$ the Frobenius endomorphism of X.

A priori, there are two actions on $X_0(\overline{\mathbb{F}}_p) = X(\overline{\mathbb{F}}_p)$. One is given by the Frobenius endomorphism on X and the other is induced by the Galois action $\Phi_{\mathrm{Spec}(\overline{\mathbb{F}}_p)}$.

Lemma 1.4. *These two actions are the same.*

Proof. Exercise. □

Let $\Delta_X \colon X \to Y$ be the diagonal morphism and $\Gamma_{\mathrm{Fr}^n_X}$ the graph of the Frobenius endomorphism composed n times: Fr^n_X. Both Δ_X and $\Gamma_{\mathrm{Fr}^n_X}$ are closed immersions and cut out divisors in Y. We denote these divisors by $[\Delta_X]$ and $[\Gamma_{\mathrm{Fr}^n_X}]$, respectively.

Lemma 1.5. *We have $[\Gamma_{Fr^n_X}] = [(Fr_X \times \mathrm{id}_X)^*]^n[\Delta_X]$.*

Proof. First note that by functoriality of pullbacks $[(\mathrm{Fr}_X \times \mathrm{id}_X)^*]^n = (\mathrm{Fr}^n_X \times \mathrm{id}_X)^*$. Thus it suffices to show that for an arbitrary endomorphism $\psi \colon X \to X$, we have

$$[\Gamma_\psi] = (\psi \times \mathrm{id}_X)^*[\Delta_X] \tag{1.1}$$

where Γ_ψ is the graph of ψ in Y. We now work locally and assume $X = \mathrm{Spec}(A)$. Take a closed point $x \in X$ and a uniformizer $\pi \in \mathcal{O}_{X,x}$ and assume $\pi \in A$. By pulling back π along the two projections $Y \rightrightarrows X$ we get two global sections π_1, π_2 of Y. Then $\pi_1 - \pi_2$ generates $[\Delta_X]$. But then the LHS of (1.1) is generated[3] by $\psi^*(\pi_1) - \pi_2$.

□

[1] Weil's proof in [Wei48] is slightly different to what is presented here. In particular he relies on the Riemann-Roch theorem for surfaces.

[2] This is the morphism given by identity on the underlying topological space of X_0 and Frobenius on the ring of functions.

[3] To see the last statement, look at the graph morphism at the level of algebras $A \otimes_{\overline{\mathbb{F}}_p} A \to A$. This is given by $x \otimes y \mapsto \psi^*(x)y$ and one sees that the kernel is indeed generated by $\psi^*(\pi_1) - \pi_2$.

Lemma 1.6. *The cardinality of the set $X(\mathbb{F}_{p^n})$ is given by the intersection number*[4] $[\Gamma_{Fr^n_X}]\cdot[\Delta_X]$.

Proof. Before we begin the proof, let us recall what intersection numbers mean in the context of curves on surfaces.

Detour: Intersection numbers of closed curves on surfaces: Let C be a smooth closed curve on a smooth projective surface S and $D \in \mathrm{Div}(S)$. Then one definition of the intersection number is $C \cdot D = \deg(\mathcal{O}_S(D)|_C)$. Unravelling what this means $\mathcal{O}_S(D)|_C$ is a line-bundle on C and its degree is the degree of it's associated divisor.

Now let's go back to the proof of Lemma 1.6. First note that since we are in characteristic p, the differential of Fr^n_X vanishes. Thus if we look at the tangent spaces of Δ_X and $\Gamma_{\mathrm{Fr}^n_X}$, we see that their sum spans all of $T_xX \times T_xX = T_{(x,x)}Y$ at every point of intersection $(x,x) \in Y$. In the literature we say Δ_X and $\Gamma_{\mathrm{Fr}^n_X}$ meet *transversely.* The upshot of transversality is the following proposition[5].

Proposition 1.7. *In the setting of the previous **Detour**, suppose also that D is a closed smooth curve. If C and D intersect transversely then*

$$C \cdot D = |C \cap D|$$

Proof. Exercise. □

So Proposition 1.7 says that the intersection number $[\Gamma_{\mathrm{Fr}^n_X}] \cdot [\Delta_X]$ is just the number of (closed) points that Δ_X and $\Gamma_{\mathrm{Fr}^n_X}$ intersect at. Note that the points must indeed be closed as X is irreducible. On the other hand by Hilbert Nullstellensatz, the closed points of Y is just $X(\overline{\mathbb{F}}_p) \times X(\overline{\mathbb{F}}_p)$. The set of points which belong to the intersection of Δ_X and $\Gamma_{\mathrm{Fr}^n_X}$ is precisely $X(\mathbb{F}_{p^n})$ because the set of points fixed by Fr^n_X is the same as those fixed by $\Phi^n_{\mathrm{Spec}(\overline{\mathbb{F}}_p)}$ by Lemma 1.4.

□

We need one more ingredient to finish the proof: the idea of *numerical* equivalence of divisors.

Definition 1.8. We say that two divisors are *numerically* equivalent if their intersection numbers with any third divisor are equal[6]. We define $\mathrm{Num}(Y)$ to be the quotient of $\mathrm{Div}(Y)$ by numerical equivalence. In particular the intersection product descends to a non-degenerate symmetric bilinear form

$$\mathrm{Num}(Y) \times \mathrm{Num}(Y) \to \mathbb{Z}.$$

We will need the Hodge Index Theorem which describes the above linear form [Mum66, Lecture 18]:

Theorem 1.9 (**Hodge Index Theorem**). *Let S be a smooth projective surface over an algebraically closed field (of arbitrary characteristic). We have a direct sum decomposition*

$$Num(S) \otimes_{\mathbb{Z}} \mathbb{Q} = V \oplus V'$$

such that V has dimension 1 and the intersection form is positive definite on V and negative definite on V'.

[4] For the interested reader, Fulton's book [Ful84] develops intersection theory in rather great generality. In general one has to be careful outside of smooth/projective assumptions.

[5] It's so fundamental that I've labeled it a Proposition, even though we are inside a Lemma.

[6] Technically we haven't defined intersection numbers of divisors in general, but let's assume there is a reasonable definition for now.

Example 1.10. *Consider the quadric surface $S \subset \mathbb{P}^3$ given by the Segre embedding*

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\hookrightarrow \mathbb{P}^3 \\ [w : x] \times [y : z] &\mapsto [wy : wz : xy : xz] \end{aligned}$$

$S := V(x_0x_3 - x_1x_2)$. Show that $\mathrm{Div}(S) = Num(S) = \mathbb{Z} \oplus \mathbb{Z}$ and the intersection product is given by

$$\begin{aligned} \mathrm{Div}(S) \times \mathrm{Div}(S) &\to \mathbb{Z} \\ (a_1, b_1) \times (a_2, b_2) &\mapsto a_1b_2 + b_1a_2. \end{aligned}$$

Verify the Hodge Index theorem in this case.

Lemma 1.11. *We have $|X(\mathbb{F}_{p^n})| = p^n + O(p^{n/2})$.*

Proof. Let $[H]$ and $[V]$ be the divisors in $\mathrm{Div}(Y)$ corresponding to $X \times \{x_0\}$ and $\{x_0\} \times X$ for some closed point $x_0 \in X$, respectively. Since $[H] \cdot [V] = 1$ and $[H] \cdot [H] = 0$, these cannot be equal in $\mathrm{Num}(Y)$. Moreover $U := \mathbb{Q}[H] \oplus \mathbb{Q}[V]$ is a finite-dimensional subspace of $W := \mathrm{Num(Y)} \otimes_{\mathbb{Z}} \mathbb{Q}$, we can write

$$W = U \oplus U'$$

where U' is the orthogonal complement. We claim that the intersection form on U' is negative-definite: Indeed on matrix on U with respect to the basis $\{[H], [V]\}$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has one positive eigenvalue and so by the Hodge Index Theorem, the subspace on which the intersection form us positive-definite is contained in U.

Let

$$\begin{aligned} T \colon W &\to W \\ D &\mapsto (\mathrm{Fr}_X \times \mathrm{id}_X)^* D \end{aligned}$$

Then $T([H]) = p[H]$ and $T([V]) = [V]$. We know by Lemma 1.5 that $T^n[\Delta_X] = [\Gamma_{\mathrm{Fr}_X^n}]$. For the following note that pullback/pushforward of divisors (up to linear equivalence) descends to numerical equivalence (this is essentially the content of the *moving lemma*). Moreover for any $D, E \in \mathrm{Num}(Y)$

$$\begin{aligned} (\mathrm{Fr}_X \times \mathrm{id}_X)^* D \cdot (\mathrm{Fr}_X \times \mathrm{id}_X)^* E &\overset{(1)}{=} D \cdot (\mathrm{Fr}_X \times \mathrm{id}_X)_* (\mathrm{Fr}_X \times \mathrm{id}_X)^* E \\ &\overset{(2)}{=} D \cdot pE \\ &\overset{(3)}{=} pD \cdot E \end{aligned}$$

where (1) follows from Proposition 1.12:

Proposition 1.12. *Given $\varphi \colon Y \to Z$ so that $\varphi^* \colon \mathrm{Div}(Z) \to \mathrm{Div}(Y)$ and $\varphi_* \colon \mathrm{Div}(Y) \to \mathrm{Div}(Z)$ are well-defined, we have*

$$C \cdot \varphi^*(D) = \varphi_*(C) \cdot D$$

for any $C \in \mathrm{Div}(Y)$ and $D \in \mathrm{Div}(Z)$.

Proof. Exercise. □

and (2) follows from Proposition

Proposition 1.13. *In the setting of Proposition 1.12* $\varphi_*\varphi^*\colon \mathrm{Div}(Z) \to \mathrm{Div}(Z)$ *is given by* $D \mapsto \deg(\varphi)D$.

Proof. Exercise $\square$

and (3) follows by linearity of the intersection form.

Thus for all $v, w \in W$, we have $Tv \cdot Tw = p(v \cdot w)$. Since $[H] \cdot [\Delta_X] = [V] \cdot [\Delta_X] = 1$, we can write

$$[\Delta_X] = [H] + [V] + u' \tag{1.2}$$

for some $u' \in U'$. We then compute

$$\begin{aligned} |X(\mathbb{F}_{p^n})| &\overset{(a)}{=} [\Gamma_{\mathrm{Fr}_X^n}] \cdot [\Delta_X] \\ &\overset{(b)}{=} T^n[\Delta_X] \cdot [\Delta_X] \\ &\overset{(c)}{=} p^n + 1 + T^n u' \cdot u' \end{aligned}$$

where (a) follows from Lemma 1.6, (b) by Lemma 1.5 and (c) because $T([H]) = p[H]$ and $T([V]) = [V]$.

It's easy to check that $T^n u' \in U'$ and so we can apply the Cauchy-Schwarz inequality to get

$$|T^n u' \cdot u'| \leq \sqrt{|T^n u' \cdot T^n u'||u' \cdot u'|} = p^{n/2}|u' \cdot u'|$$

This completes the proof of Lemma 1.11.

$\square$

A relatively straightforward analysis argument then concludes the proof of the Riemann-Hypothesis for curves. We won't include the details, as it's not what we are after conceptually.

$\square$

2 Lecture 2: Algebraic cycles and adequate equivalence relations

For this lecture, I will be following the book by Murre-Nagel-Peters [MNP13, Chapter 1]. Another good reference is [And04, Chapitre 3]. Let k be an arbitrary field and X a k-variety.

Definition 2.1 (algebraic cycle)**.** An algebraic cycle on X is a formal finite integral linear combination $Z = \sum n_\alpha Z_\alpha$ of irreducible closed subvarieties Z_α of X. If all Z_α have the same dimension i, we say that Z is a dimension i cycle. We denote by $Z_i(X)$ the abelian group of dimension i cycles on X. When considering the codimension point of view we write $Z^{d-i}(X) := Z_i(X)$ if X is of dimension d. We write $Z(X) := \oplus_i Z^i(X)$ and consider it as a group with a graded structure.

Lemma 2.2. *Suppose X is smooth. Then two closed subvarieties V and W of X with codimensions i and j, respectively, have intersection*

$$V \cap W = \cup_\alpha Z_\alpha$$

where each Z_α is an irreducible subvariety of codimension at most i+j.

Proof. We have that $V \cap W = \Delta^{-1}(V \times W)$ where $\Delta\colon X \to X \times X$ is the diagonal map. Since X is smooth, we can write $X \times X = \mathrm{Spec}(A)$ and $X = V(f_1, \dots, f_c)$ where $f_1, \dots, f_c$ is a regular sequence in A and $c = \dim(X)$. Then if $V \times W = \mathrm{Spec}(A/\mathfrak{p})$ then $V \cap W = \mathrm{Spec}(A/(\mathfrak{p} + (f_1, \dots, f_c)))$. Then for $z \in Z_\alpha$ a closed point

$$\dim(V \times W) = \dim \mathcal{O}_{V \times W, z} \quad \text{and} \quad \dim(Z_\alpha) = \dim \mathcal{O}_{Z_\alpha, z} = \dim \mathcal{O}_{V \times W, z}/(f_1, \dots, f_c).$$

From here one can compare the relevant dimensions by the fact that quotienting a local ring by an element in the maximal ideal, reduces the dimension by at most one. □

Definition 2.3 (proper intersection product of algebraic cycles)**.** In the setting of Lemma 2.2, we say that the intersection $V \cap W$ is *proper* (or V and W intersect *properly*) if the codimension of each Z_α is $i+j$. In this case the *intersection number* is defined by

$$i(V \cdot W; Z) := \sum_r (-1)^r \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathrm{Tor}_r^{\mathcal{O}_{X,Z}}(\mathcal{O}_{V,Z}, \mathcal{O}_{W,Z}))$$

where $A := \mathcal{O}_{?,Z}$ denotes the local ring of ? at the generic point of Z. We define the *intersection product*

$$V \cdot W := \sum_\alpha i(V \cdot W; Z_\alpha) Z_\alpha.$$

Definition 2.4 (proper pushforward)**.** Let $f\colon X \to Y$ be a proper morphism of k-varieties and $Z \subset X$ a k-dimensional closed irreducible subvariety. We define

$$f_* Z = \begin{cases} 0, & \text{if } \dim(f(Z)) < k \\ [R(Z)\colon R(f(Z))] f(Z), & \text{otherwise} \end{cases} \tag{2.1}$$

where $R(?)$ is the field of rational functions[7] on ?. Extending by linearity induces a homomorphism

$$f_*\colon Z_k(X) \to Z_k(Y).$$

In general we say two algebraic cycles $\alpha, \beta \in Z(X)$ intersect properly if each components of α intersects each component of β properly.

Definition 2.5 (flat pullback)**.** Let $f\colon X \to Y$ be a flat morphism of k-varieties and $Z \subset Y$ a k-codimensional closed irreducible subvariety. We define

$$f^* Z = f^{-1}(Z)$$

Because f is flat, $f^{-1}(Z)$ turns out to be of codimension k (assuming it is non-empty). Extending by linearity induces a homomorphism

$$f^*\colon Z^k(Y) \to Z^k(X)$$

Exercise 2.6 (Projection formula)**.** *Prove* $f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta$

Definition 2.7. A *correspondence* from X to Y is a cycle in $X \times Y$. A correspondence $Z \in Z^t(X \times Y)$ acts on cycles on X as follows

$$\begin{aligned} Z\colon Z^i(X) &\to Z^{i+t-\dim(X)}(Y) \\ T &\mapsto pr_{Y*}(Z \cdot (T \times Y)) \end{aligned}$$

whenever defined. We call $t - \dim(X)$ the degree of the correspondence.

Example 2.8. *It turns out the notion of correspondences generalizes the notion of (proper) pushforward and (flat) pullback. Prove this.*

As we see correspondences (or intersection products) are not always defined. This is where the notion of *adequate equivalence* comes in. These are equivalences classes on the groups Z^i such that the intersection product is always defined.

[7]Note that $f|_Z\colon Z \to f(Z)$ is a dominant morphism, so the above degree is well-defined.

2.1 Adequate Equivalence

We now work in the category $\mathrm{SmProj}(k)$ of smooth projective varieties over k.

Definition 2.9 (Adequate Equivalence)**.** We say that an equivalence relation $\sim$ on $Z(X)$ is *adequate* if

(1) (**linearity**) $\sim$ is compatible with addition and graduation.

(2) (**moving lemma**) For all α, $\beta \in Z(X)$, $\exists \alpha' \sim \alpha$ such that α' and β intersect properly.

(3) (**correspondence**) $\sim$ is compatible with correspondences: In the setting of Definition 2.7 if $T \sim 0$ and Z intersects $T \times Y$ properly, then $Z(T) \sim 0$.

We write $Z_\sim(X) := Z(X)/\sim$ and for some field F, $Z_\sim(X)_F := Z(X) \otimes_{\mathbb{Z}} F/\sim$. We also write

$$Z^i_{\sim,0}(X) := \{Z \in Z^i(X) |\ Z \sim 0\}$$

The fact that intersection product is defined on the whole $Z_\sim(X)$ is a straightforward consequence of Definition 2.9 (cf. [Sam58, Proposition 6 and 7]).

Lemma 2.10. *For any adequate equivalence relation $\sim$ on $X \in SmProj(k)$, we have*

(1) $Z_\sim(X)$ is a graded ring with product induced by the intersection product of cycles.

(2) A correspondence Z from X to Y of degree r induces $Z_*\colon Z^i_\sim(X) \to Z^{i+r}_\sim(Y)$ and equivalent correspondences induce the same Z_*.

We now discuss the following adequate equivalence relations

- rational equivalence $\sim_{\mathrm{rat}}$
- algebraic equivalence $\sim_{\mathrm{alg}}$
- smash nilpotence equivalence $\sim_{\otimes\mathrm{nil}}$
- homological equivalence $\sim_{\mathrm{hom}}$
- numerical equivalence $\sim_{\mathrm{num}}$

2.1.1 Rational equivalence

Definition 2.11 (Rational equivalence)**.** A cycle $\alpha \in Z(X)$ is rationally equivalent to 0 ($\alpha \sim_{\mathrm{rat}} 0$) if there exists $\beta \in Z(X \times \mathbb{P}^1)$ such that $\beta(0)$ and $\beta(\infty)$ are well-defined and $\alpha = \beta(0) - \beta(\infty)$.

Lemma 2.12. *Rational equivalence corresponds to linear equivalence for codimension 1 cycles $Z^1(X)$.*

Proof. We first show $\mathrm{div}(f) \sim_{\mathrm{rat}} 0$ for $f \in R(X)$ a rational function. We can think of f as $f\colon U \to \mathbb{P}^1$ for some dense open $U \subset X$. Let $W \subset X \times \mathbb{P}^1$ be the closure of the graph of f. Then W gives a cycle $\beta \in Z(X \times \mathbb{P}^1)$ and essentially by definition $\mathrm{div}(f) = \beta(0) - \beta(\infty)$.

For the converse suppose $\alpha \in Z^1(X)$ and $\alpha \sim_{\mathrm{rat}} 0$. Take a component $Z' \subset X \times \mathbb{P}^1$ of β (with β part of Definition 2.11). Then Z' dominates $\mathbb{P}^1$. Let $Z \subset X$ be the image of Z' under the projection to X. Then $Z \subset X$ is closed (as projection is proper) and $Z' \to Z$ is proper and dominant with fibers of dimension 0 or 1.

There are two cases as to whether $\dim(Z) < \dim(Z')$ or $\dim(Z) = \dim(Z')$.

If $\dim(Z) < \dim(Z')$, then $Z' = Z \times \mathbb{P}^1$ and $[Z'_0] - [Z'_\infty] = [Z] - [Z] = 0$.

If $\dim(Z) = \dim(Z')$, then $Z' \to Z$ is generically finite (i.e. inverse image of generic point is finite). Then I leave it as an exercise[8] to show that $[Z'_0] - [Z'_\infty] = \mathrm{div}(\mathrm{Nm}(f))$ where $f\colon Z' \to \mathbb{P}^1$ viewed as a rational function on Z'. □

[8] The main idea is essentially in the proof of [Ful84, Proposition 1.4(b)]

The technical difficulty in proving that rational equivalence is indeed an adequate equivalence relation lies in proving the *moving lemma*. The proof is roughly as follows: We embed $X \hookrightarrow \mathbb{P}^N$ and given $V, W \subset X$, we need to move V so that it intersects W properly. There are two cases to consider as to whether $X = \mathbb{P}^N$ or not. In the former, there is some general linear transformation that makes V and W intersect properly. In the later case, one considers a linear subspace $L \subset \mathbb{P}^N$ and the cone $C(L, V)$. For the details we refer to [Ful84, Example 11.4.1].

Exercise 2.13. *What goes wrong with the moving lemma for rational equivalence if we relax the smoothness assumption? What if we keep smoothness and relax the projectivity assumption?*

Definition 2.14 (Chow ring)**.** The corresponding graded ring $CH(X) := Z_{\text{rat}}(X)$ is called the Chow ring. We will also denote by $\text{Corr}(X, Y) := CH(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ the *correspondences* from X to Y.

Lemma 2.15. *Among all adequate equivalence relations, rational equivalence is the finest.*

Proof. Let $\sim$ be an adequate equivalence relation. It suffices to prove $[0] \sim [\infty]$ as then by using a correspondence, we get the definition of rational equivalence. Since the point $1 \in \mathbb{P}^1$ does not intersect itself properly, by the moving lemma there exists $\sum_i n_i[x_i] \in Z^1(\mathbb{P}^1) \sim 1$ with $x_i \in \mathbb{P}^1$ such that x_i intersects 1 properly. In other words $x_i \neq 1$. Consider now the correspondence $Z \in Z^1(\mathbb{P}^1 \times \mathbb{P}^1)$ given by the graph of the polynomial

$$1 - \prod_i \left(\frac{x - x_i}{1 - x_i}\right)^{m_i} \tag{2.2}$$

for a collection of $m_i > 0$ and $T = \sum_i n_i[x_i] - 1$. Then $Z(T)$ is just the pushforward of T by (2.2). The pushforward of T is just $mn[1] - m[0]$ where $m = \sum_i m_i$ and $n = \sum_i n_i$. Since this holds for arbitrary m_i, we get $n[1] \sim [0]$. Applying the condition of correspondence to the automorphism $x \mapsto \frac{1}{x}$, we get $n[1] \sim [\infty]$, from which we can conclude. $\square$

2.1.2 Algebraic equivalence

Definition 2.16 (Algebraic Equivalence)**.** This is the same definition as rational equivalence but with $\mathbb{P}^1$ replaced by any smooth projective irreducible curve and the two points 0 and ∞ by any two k-rational points on the curve. In other words $\alpha \in Z(X)$ is $\sim_{\text{alg}} 0$ if there exists a smooth irreducible projective curve C and $\beta \in Z(X \times C)$ and two points $a, b \in C(k)$ such that $\beta(a) = 0$ and $\beta(b) = \alpha$.

Example 2.17 (algebraic equivalence is coarser than rational equivalence)**.** *Take an elliptic curve E over $\mathbb{C}$ and two distinct points $a, b \in |E|$. Then $a - b$ is not a divisor of any rational function. This is because we can make an identification $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ and we see by Cauchy's residue theorem that any rational function has it's sum of residues equal to 0. So any rational function cannot have a simple pole. In summary $a - b \not\sim_{rat} 0$.*

On the other hand E is equipped with a degree 2 cover over $\mathbb{P}^1$ with 4 ramification points (by Hurwitz's theorem). If we take the graph $Z \subset E \times \mathbb{P}^1$ of this cover, then we get that these 4 ramification points must be algebraically equivalent[9]*.*

2.2 Smash Nilpotent equivalence

Definition 2.18 (Smash Nilpotent equivalence)**.** For $Z \in Z(X)$ we say $Z \sim_{\otimes} 0$ iff for some positive integer n, $Z^n \sim_{\text{rat}} 0$ where we view $Z^n \in Z(X^n)$.

Theorem 2.19 ($\sim_{\otimes}$ vs $\sim_{\text{alg}}$)**.** *We have $Z^i_{alg,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \subset Z^i_{\otimes,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

[9] We can just take the graph of the identity map to conclude that any two rational points on any smooth projective curve are algebraically equivalent.

Theorem 2.19 is due independently to Voevodsky [Voe95] and Voisin [Voi96].

Proof. We proceed in several steps as in [MNP13, Appendix B].
Step 0: Reduce to $k = \overline{k}$.

Exercise 2.20. *For any (adequate) equivalence relation $\sim$, there is a natural map*

$$Z_\sim(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to (Z_\sim(X_{\overline{k}}) \otimes_{\mathbb{Z}} \mathbb{Q})^{\mathrm{Gal}(\overline{k}/k)}$$

given by restricting a closed subvariety over k to one over $\overline{k}$. Prove that this map is a bijection.

Exercise 2.20 allows us to assume k is algebraically closed.
Step 1: Reduce to the case of a smooth projective curve. Take $Z \sim_{\mathrm{alg}} 0$. Then by definition $\exists\, \Gamma \in \mathrm{Corr}(C, X)$ and two points $a, b \in C(k)$ such that $Z = \Gamma_*(a-b)$. Thus taking products gives $Z^n = (\Gamma^n)_*(a-b)^n$ and so it suffices to show $(a-b)^n \sim_{\mathrm{alg}} 0$ on C^n. In fact we shall show

$$(a-b)^n \sim_{\mathrm{alg}} 0 \quad \text{for} \quad n > g,$$

where g is the genus of the curve C.
Step 2: Reducing $(a-b)^n$ as a divisor on the n-fold symmetric product of C. A priori $(a-b)^n \in Z(C^n)$. However the symmetric group S_n induces an action on C^n and clearly $(a-b)^n \in (Z(C^n) \otimes_{\mathbb{Z}} \mathbb{Q})^{S_n}$.

Exercise 2.21. *Show that $(Z(C^n) \otimes_{\mathbb{Z}} \mathbb{Q})^{S_n} \cong Z(C^n/S^n) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Exercise 2.21 allows us to view $(a-b)^n$ in the quotient variety C^n/S^n (the n-fold symmetric product of C).
Step 3: Comparison of C/S^n with the Jacobian[10] $J(C)$. Fix a base point $e \in C(k)$. Denote by

$$\begin{aligned} \pi_n \colon C^n &\to C^n/S^n \\ (x_1, \ldots, x_n) &\mapsto [x_1, \ldots, x_n] \end{aligned}$$

the natural surjection and

$$\begin{aligned} \varphi_n \colon C^n/S^n &\to J(C) \\ [x_1, \ldots, x_n] &\mapsto \sum_i (x_i - e). \end{aligned}$$

Lemma 2.22. *The induced map $(\varphi_n)_* \colon CH_0(C^n/S^n) \to CH_0(J(C))$ is an isomorphism for all $n \geq g$.*

Proof. If $n = g$, then we claim that φ_n is a birational morphism. Indeed by Riemann-Roch

$$\ell(x_1 + \cdots + x_g) = g + 1 - g + \ell(K - x_1 - \cdots - x_g)$$

if none of the x_i are base points of the canonical divisor K, then since $\ell(K) = g$, we get[11] $\ell(K - x_1 - \cdots - x_g) = 0$ and so $\ell(x_1 + \cdots + x_g) = 1$. This means that φ_n is an isomorphism[12] outside of a finite set of points. So it is a birational morphism. By [Ful84, Example 16.1.11], the group CH_0 is invariant for birational morphisms.

[10]The Jacobian variety $J(C)$ is the variety which represents the functor $T \mapsto \{\text{invertible sheaves of degree 0 on } X \times T\}$.

[11]This is related to [Har77, Chapter IV, Proposition 3.1]

[12]Because the fiber of φ_n is just the set of points $[x_1, \ldots, x_n]$ such that $\sum_i x_i$ form a complete linear system. This also means fibers are projective.

Suppose $n > g$ and consider the natural embedding

$$\begin{aligned} \iota \colon C^g/S^g &\to C^n/S^n \\ [x_1, \dots, x_g] &\mapsto [x_1, \dots, x_g, \underbrace{e, \dots, e}_{n-g}] \end{aligned}$$

Then $(\varphi_n)_* \iota_* = (\varphi_g)_*$. Since $(\varphi_g)_*$ is an isomorphism, it follows that ι_* is an injection. It remains to show that it is a surjection. So take $y \in C^n/S^n$ and consider the image $z \in J(C)$ and some point $x \in C^g/S^g$ which maps to z under equivalence. Then $\iota(x)$ and y belong to the fiber $\varphi_n^{-1}(z)$. But the fibers[13] of φ_n are projective and any two points are rationally equivalent. This proves the lemma. □

Step 4: Application of Bloch's theorem. We have that $(\varphi_{g+1})_*((a-b)^{g+1}) = (\varphi_1(a) - \varphi_1(b))^{*(g+1)}$ and this vanishes by [Blo76].

□

Exercise 2.23. *Use ideas from the proof of Theorem 2.19 to show that the cartesian product of two non-zero Chow cycles can be zero.*

2.3 Homological equivalence

To define *homological equivalence* we need to define a *Weil cohomology theory*. Let F be a field of characteristic 0. We denote $\mathrm{GrVect}_F^{\geq 0}$ be the category of finite dimensional graded F-vector spaces, where the grading is concentrated in non-negative degrees.

Definition 2.24. A Weil cohomology theory is a functor

$$H \colon \mathrm{SmProj}(k)^{\mathrm{opp}} \to \mathrm{GrVect}_F^{\geq 0}$$

which satisfies the following axioms:

(1) A one-dimensional F-vector space $F(1)$, which gives rise to *Tate* twists.

(2) $\exists$ a graded cup product $\cup \colon H(X) \times H(X) \to H(X)$ such that if $a \in H^i(X)$, $b \in H^j(X)$, then $a \cup b = (-1)^{ij} b \cup a$.

(3) one has Poincaré duality (assume X has pure dimension d): $\exists$ a trace isomorphism

$$\mathrm{Tr} \colon H^{2d}(X)(d) \xrightarrow{\sim} F$$

such that

$$H^i(X) \times H^{2d-i}(X)(d) \xrightarrow{\cup} H^{2d}(X)(d) \xrightarrow{\mathrm{Tr}} F$$

is a perfect pairing.

(4) A Künneth map

$$H(X) \otimes H(Y) \xrightarrow{(pr_X)^* \otimes (pr_Y)^*} H(X \times Y)$$

which is a (graded) isomorphism.

(5) there are cycle class maps

$$\gamma_X \colon \mathrm{CH}^i(X) \to H^{2i}(X)(i)$$

which satisfy various compatibilities[14].

[13] fiber above a point is a complete linear system

[14] I will state them explicitly when we need them.

(6) If X is pure of dimension d and $\iota\colon Y \hookrightarrow X$ is a smooth hyperplane, then *weak Lefschetz* holds:

$$H^i(X) \xrightarrow{\iota^*} H^i(Y)$$

is an isomorphism if $i < d-1$ and an injection for $i = d-1$.

(7) With the setting as in (6), the Lefschetz operator $L(\alpha) := \alpha \cup \gamma_X(Y)$ induces isomorphisms

$$L^i\colon H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X)$$

for $0 \le i \le d$. This is known as *Hard Lefschetz*.

Example 2.25. *For $k = \mathbb{C}$, we get many examples of Weil cohomology theories:*

(1) singular cohomology groups: $H^i(X_{an})$ where X_{an} is the complex manifold attached to X.

(2) classical de Rham cohomology: $H^i_{dR}(X_{an}, \mathbb{C})$.

(3) algebraic de Rham cohomology: $H^i_{dR} := \mathbb{H}(X, \Omega^\bullet_{X/\mathbb{C}})$.

The fact that these are indeed Weil cohomology groups follows from classical reasons together with comparison isomorphisms. On the other hand the fact that $H^i_{ét}(X, \mathbb{Q}_\ell)$ is a Weil cohomology theory (in particular satisfies Hard Lefschetz) is deep work of Deligne [Del80].

Definition 2.26 (Homological equivalence)**.** Fix a Weil cohomology theory. Then for $Z \in Z(X)$ we say $Z \sim_{\text{hom}} 0$ if $\gamma_X(Z) = 0$.

We can compare homological equivalence to algebraic and smash nilpotent equivalence.

Lemma 2.27 ($\sim_\otimes$ and $\sim_{\text{alg}}$ vs $\sim_{\text{hom}}$)**.** *(1) $Z^i_{alg,0}(X) \subset Z^i_{hom,0}(X)$.*

(2) $Z^i_{\otimes,0}(X) \subset Z^i_{hom,0}(X)$.

Proof. For (1), note that $\alpha \sim_{\text{alg}} 0$ means that for some smooth projective curve C, $\alpha = pr_{X*}pr_C^*([a] - [b])$ for two rational points $a, b \in C$. Now cycle map is compatible with push-forward and pullbacks (one of the conditions I didn't state in Definition 2.24(4)). So we can reduce to the case of a curve. We then conclude by Matsusaka's theorem:

Theorem 2.28 (Matsusaka's Theorem)**.**

$$Z^1_{hom,0}(X) = \{D \in Z^1(X) | \;\; nD \sim_{alg} 0 \textit{ for some } n \in \mathbb{Z}\}$$

For part (2), note that $\alpha \sim_\otimes 0$ means $\alpha^n \sim_{\text{rat}} 0$ for some $n > 0$. Then

$$\gamma_{X^n}(\alpha^n) = \underbrace{\gamma_X(\alpha) \otimes \ldots \otimes \gamma_X(\alpha)}_{n}$$

is zero. So each of $\gamma_X(\alpha) = 0$. □

Exercise 2.29. *Find an alternative proof of Lemma 2.27(1) using part (2) and Voevodsky-Voisin Theorem 2.19.*

2.4 Numerical equivalence

Definition 2.30 (Numerical Equivalence)**.** Let X be of pure dimension d. For $Z \in Z^i(X)$, we say $Z \sim_{\text{num}} 0$ if for every $W \in Z^{d-i}(X)$ such that $Z \cdot W$ is defined, we have $\deg(Z \cdot W) = 0$.

We can compare homological equivalence and numerical equivalence.

Lemma 2.31 ($\sim_{\text{hom}}$ vs $\sim_{\text{num}}$)**.** *$Z^i_{hom,0}(X) \subset Z^i_{num,0}(X)$.*

Proof. We will need to use that γ_X (the cycle class map) is compatible with intersection products:

$$\gamma_X(\alpha \cdot \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta) \tag{2.3}$$

and compatible with points P:

$$\mathrm{Tr} \circ \gamma_X = \deg \qquad \text{on } CH^d(X). \tag{2.4}$$

Conditions (2.3) and (2.4) are the remaining conditions I didn't state in Definition 2.24(4)).

By property (2.4), we see that the result holds for $i = d$ (i.e. zero cycles). Suppose now $i < d$ and $Z \in Z^i_{\text{hom},0}(X)$ and $W \in Z^{d-i}(X)$. such that $Z \cdot W$ is defined. Then

$$\begin{aligned} 0 &\overset{(i)}{=} \mathrm{Tr}(\gamma_X(Z) \cup \gamma_X(W)) \\ &\overset{(ii)}{=} \mathrm{Tr}(\gamma_X(Z \cdot W)) \\ &\overset{(iii)}{=} \deg(Z \cdot W) \end{aligned}$$

where (i) holds because $\gamma_X(Z) = 0$, (ii) holds because of (2.3) and (iii) holds because of (2.4). $\square$

Exercise 2.32. *Show that by realizing the degree map as a correspondence, that $Z^i_{\sim,0}(X) \subset Z^i_{num,0}(X)$ for any non-trivial adequate equivalence relation.*

Summarizing Lemmas 2.15, 2.27(1) and 2.31 have shown the following chain

$$Z^i_{\text{rat},0}(X) \subset Z^i_{\text{alg},0}(X) \subset Z^i_{\text{hom},0}(X) \subset Z^i_{\text{num},0}(X)$$

As part of the standard conjectures:

Conjecture 2.33 (Standard Conjecture D)**.** *$Z^i_{hom,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = Z^i_{num,0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

3 Lecture 3: Grothendieck's pure motives

For this lecture, I will be following the book by Murre-Nagel-Peters [MNP13, Chapter 2]. Another good reference is [And04, Chapitre 4]. Let k be an arbitrary field and X and Y smooth projective k-varieties.

The next definition is along the same lines as Definitions 2.7 and 2.14.

Definition 3.1 (correspondences and degree r correspondence)**.** For an adequate equivalence relation $\sim$, we denote the graded vector spaces of correspondences:

$$\mathrm{Corr}_\sim(X, Y) := Z_\sim(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and if X is pure of dimension d, we also consider the degree r correspondences by

$$\mathrm{Corr}^r_\sim(X, Y) := Z^{d+r}_\sim(X \times Y) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We also need to know how to compose correspondences:

Definition 3.2 (composition of correspondences)**.** We define composition

$$\mathrm{Corr}_{\sim}(X,Y)\times\mathrm{Corr}_{\sim}(Y,Z)\to\mathrm{Corr}_{\sim}(X,Z)$$
$$(f,g)\mapsto g\circ f$$

by $g\circ f := \mathrm{pr}_{XZ*}\{(f\times Z)\cdot(X\times g)\}$.

Exercise 3.3. *Check that Definition 3.2 restricts to give a composition of degree 0 correspondences. In general composition respects the grading.*

Definition 3.4 (projectors)**.** A projector for X is an element $p\in\mathrm{Corr}_{\sim}(X,X)$ for which $p\circ p=p$.

Exercise 3.5. *Show that the diagonal Δ_X is an example of a projector.*

We now proceed to give the construction of (pure) motives in several steps. For the following fix an adequate equivalence relation $\sim$.
Construction of (pure) motives:
Step 1: Consider the category $Z_{\sim}\mathrm{SmProj}(k)$ with

(1) objects: same as $\mathrm{SmProj}(k)^{\mathrm{op}}$

(2) morphisms: degree 0 correspondences. More precisely $\mathrm{Hom}(X,Y):=\mathrm{Corr}^0_{\sim}(X,Y)$.

We are hoping that the category we construct is abelian and it's formal nonsense[15] to see that one should keep track of idempotent morphisms (i.e. projectors). This leads to
Step 2: Consider the category of *effective* motives $\mathrm{Mot}^{\mathrm{eff}}_{\sim}(k)$ with

(1) objects: pairs (X,p) with $X\in\mathrm{SmProj}(k)$ and p a projector.

(2) morphisms: $\mathrm{Hom}((X,p),(Y,q)):=q\circ\mathrm{Corr}^0_{\sim}(X,Y)\circ p$.

Exercise 3.6. *Show that the mapping $X\mapsto(X,\Delta_X)$ realizes $Z_{\sim}$SmProj(k) as a full subcategory of Mot$^{\mathrm{eff}}_{\sim}(k)$.*

Finally we want to include duals (i.e. Tate twists):
Step 3: The category of pure motives $\mathrm{Mot}_{\sim}(k)$ with

(1) objects: triples (X,p,m) with (X,p) an object of $\mathrm{Mot}^{\mathrm{eff}}_{\sim}(k)$ and $m\in\mathbb{Z}$.

(2) morphisms: $\mathrm{Hom}((X,p,m),(Y,q,n)):=q\circ\mathrm{Corr}^{n-m}_{\sim}(X,Y)\circ p$

Exercise 3.7. *Show that the mapping $(X,p)\mapsto(X,p,0)$ realizes Mot$^{\mathrm{eff}}_{\sim}(k)$ as a full subcategory of Mot$_{\sim}(k)$.*

The category $\mathrm{Mot}_{\sim}(k)$ has a natural structure of a symmetric monoidal category with duals. We touch on this in the next example.

Example 3.8. *By Exercises 3.6 and 3.7, we get that*

$$\mathrm{End}_{Mot_{\sim}(k)}((\mathbb{P}^1,\Delta_{\mathbb{P}^1}))=Corr^0(\mathbb{P}^1,\mathbb{P}^1)=\mathbb{Q}\oplus\mathbb{Q}.$$

Moreover $\Delta_{\mathbb{P}^1}=e_0\oplus e_1$ with $e_0=\{0\}\times\mathbb{P}^1$ and $e_1=\mathbb{P}^1\times\{0\}$ and this allows us to write

$$(\mathbb{P}^1,\Delta_{\mathbb{P}^1}):=\mathbf{1}\oplus\mathbb{L}\tag{3.1}$$

[15] The method of passing from **Step 1** to **Step 2** is an instance of a more general idea of taking a pseudo-abelian completion of an additive category.

where $\mathbf{1} = (\mathrm{Spec}(k), \mathrm{id})$ *corresponds to the motive of a point and* $\mathbb{L} = (\mathbb{P}^1, \mathbb{P}^1 \times \{0\})$ *the Lefschetz motive. Note that* (3.1) *is a definition that follows from pseudo-abelian completion. It's then an exercise*[16] *to show that* $\mathbb{L} \cong (\mathrm{Spec}(k), \mathrm{id}, -1) =: \mathbf{1}(-1)$*. The dual* $\mathbf{1}(1) := (\mathrm{Spec}(k), \mathrm{id}, 1)$ *is called the Tate motive. In general the definition of the dual of* (X, Δ_X) *is* $(X, \Delta_X) \otimes \mathbb{L}^{-d}$*, where* d *is the dimension of* X*.*

Definition 3.9 (symmetric monoidal structure)**.** $(X, p, m) \otimes (Y, q, n) := (X \times Y, p \times q, m + n)$

Exercise 3.10. *Show that* $Z^r_\sim(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}(\mathbb{L}^r, (X, \Delta_X))$*.*

Under the isomorphism given by Exercise 3.10, for $\alpha \in Z^r_\sim(X)$, we write $\alpha_* \in \mathrm{Hom}(\mathbb{L}^r, (X, \Delta_X))$ for the corresponding morphism. The corresponding dual morphism $\alpha^* \colon (X, \Delta_X) \otimes \mathbb{L}^r \to \mathbb{L}^{\dim(X)}$.

Definition 3.11 (Chow motives and Grothendieck's (numerical) motives)**.** We denote by $\mathrm{CHM}(k) := M_{\mathrm{rat}}(k)$ the category of Chow motives and $\mathrm{NM}(k) := M_{\mathrm{num}}(k)$ the category of Grothendieck motives (or numerical motives).

The next result is due to Scholl [Sch94, Corollary 3.5].

Proposition 3.12. *Assume that* k *is not contained in the algebraic closure of a finite field. Then the category of Chow motives CHM*(k) *is not an abelian category.*

Proof. Given the conditions on k, there exists an elliptic curve E/k of positive rank[17]. Let $P \in E(k)$ be a point of infinite order. Then by writing the Δ_E as in 1.2, we obtain a decomposition (again by definition of pseudo-abelian completion)

$$(E, \Delta_E) = \mathbf{1} \oplus \mathbb{L} \oplus h^1 E.$$

Then the divisor $[P] - [0]$ is a point on the Jacobian $J(E)$ and determines a non-zero morphism $\eta_* \colon \mathbb{L} \to h^1(E)$ by

Lemma 3.13. *We have an isomorphism* $\mathrm{Hom}(\mathbb{L}, h^1(E)) \cong J(E)(k) \otimes \mathbb{Q}$*.*

Proof. By Exercise 3.10, we have $\mathrm{Hom}(\mathbb{L}, (E, \Delta_E)) \cong Z^1_{\mathrm{rat}}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. On the other hand

$$\mathrm{Hom}(\mathbb{L}, \mathbf{1}) = \mathrm{Hom}((\mathrm{Spec}(k), \mathrm{id}, -1), (\mathrm{Spec}(k), \mathrm{id}, 0)) \subset \mathrm{Corr}^1_{\mathrm{rat}}(k, k) = 0,$$

and

$$\mathrm{Hom}(\mathbb{L}, \mathbb{L}) = \mathrm{Hom}(\mathbf{1}, \mathbf{1}) = \mathbb{Q}.$$

The projection morphism $Z^1_{\mathrm{rat}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ is just the degree map. Thus $\mathrm{Hom}(\mathbb{L}, h^1(E))$ identifies as the kernel of this map which gives the result. ☐

The composite $\eta_* \circ \eta^* \colon h^1(E) \otimes \mathbb{L} \to h^1(E)$. Note that

$$\mathrm{Hom}(h^1(E) \otimes \mathbb{L}, h^1(E)) \subset \mathrm{Hom}((E, \Delta_E, -1), (E, \Delta_E, 0)) \subset \mathrm{Corr}^1(E, E) = Z^2_{\mathrm{rat}}(E \times E).$$

and it is a check to see that $\eta_* \circ \eta^*$ corresponds to zero-cycle $c = (P, P) + (0, 0) - (P, 0) - (0, P)$. Assume $P = 2Q$ for $Q \in E(k)$. Then in $Z^2_{\mathrm{rat}}(E \times E)$ we can write

$$c = [(P, P) + (0, 0) - 2(Q, Q)] + [2(Q, Q) - (P, 0) - (0, P)]$$

and this is rationally equivalent to zero. Thus $\eta_* \circ \eta^* = 0$. This means η_* is not a monomorphism. If $\mathrm{CHM}(k)$ were abelian, then $\ker(\eta_*)$ would be a proper subobject of $\mathbb{L}$. Tensoring by the Tate motive would give a proper suboject of $\mathbf{1}$. But the unit object in an abelian category with a symmetric monoidal structure is completely decomposable and this gives a contradiction since $\mathrm{End}(\mathbf{1}) = \mathbb{Q}$ (the only idempotents of $\mathbb{Q}$ are 0 and 1). ☐

[16] Take a look at [Sta18, Tage 0FGD].

[17] I can't find a reference for this, unless k is a number field.

Proposition 3.14. *A Weil cohomology theory*

$$H\colon \mathit{SmProj}(k)^{opp} \to \mathit{GrVect}_F^{\geq 0}$$

factorizes as

$$\begin{aligned} H\colon \mathit{SmProj}(k)^{opp} \to \mathit{Mot}_{rat}(k) &\xrightarrow{G} \mathit{GrVect}_F^{\geq 0} \\ X \mapsto (X, \Delta_X, 0) \end{aligned}$$

Furthermore G precisely corresponds to the datum of H iff $G(\mathbf{1}(1))$ is non-zero only in degree -2.

The next Theorem is an important result due to Jannsen [Jan92] and arguably the most important result concerning pure motives:

Theorem 3.15. *Assume $k = \overline{k}$ and let $\sim$ be any adequate equivalence relation. TFAE*

(1) $\mathit{Mot}_\sim(k)$ is an abelian semi-simple category.

(2) $\sim$ is numerical equivalence

(3) For all $X \in \mathit{SmProj}(k)$ of pure dimension, $\mathit{Corr}^0(X, X)$ is a finite-dimensional semi-simple $\mathbb{Q}$-algebra.

Proof. (1) $\implies$ (2): Suppose for the sake of contradiction that $\mathrm{Mot}_\sim(k)$ is abelian and semi-simple but $Z_{\sim,0}(X) \neq Z_{\mathrm{num},0}(X)$. On the other hand we know that by Exercise 2.32: $Z_{\sim,0}(X) \subset Z_{\mathrm{num},0}(X)$. So take $Z \in Z^i_{\mathrm{num},0}(X)$ but $Z \notin Z^i_{\sim,0}(X)$. This Z gives a non-zero morphism

$$f\colon \mathbf{1} = (\mathrm{Spec}(k), \mathrm{id}, 0) \to (X, \mathrm{id}, i)$$

in $\mathrm{Mot}_\sim(k)$. Since $\mathrm{Mot}_\sim(k)$ is abelian and semi-simple, there is a morphism

$$g\colon (X, \mathrm{id}, i) \to \mathbf{1}$$

such that $g \circ f = \mathrm{id}_{\mathbf{1}}$. Such a g is given by $W \in Z^{d-i}_\sim(X)$. Then by the definition of composition of correspondences

$$\begin{aligned} g \circ f &= \mathrm{pr}_{\mathrm{Spec}(k)\times\mathrm{Spec}(k),*}((\mathrm{Spec}(k) \times Z \times \mathrm{Spec}(k)) \cdot (\mathrm{Spec}(k) \times W \times \mathrm{Spec}(k)) \\ &= \deg(Z \cdot W)\,\mathrm{Spec}(k) \times \mathrm{Spec}(k) \end{aligned}$$

where the second equality is by defintion of degree as pushforward onto a point. But $g \circ f = \mathrm{id}_{\mathbf{1}}$ and so $\deg(Z \cdot W) = 1$. But this contradicts $Z \in Z^i_{\mathrm{num},0}(X)$.

(2) $\implies$ (3): Fix a Weil cohomology theory (in this case we take étale cohomology with coefficients $\mathbb{Q}_\ell$ where $\ell \neq \mathrm{char}(k)$) and recall the cycle map (ignoring Tate twist)

$$\gamma_X\colon \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{2i}(X)$$

and define $A^i(X) := \mathrm{im}(\gamma_X) \subset H^{2i}(X)$ and set $B^i(X) := Z^i_{\mathrm{num}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. By Lemma 2.31 or Exercise 2.32, we get a surjection

$$A^i(X) \twoheadrightarrow B^i(X).$$

Let $d = \dim(X)$. We need to show $B^d(X \times X)$ is a finite-dimensional semi-simple $\mathbb{Q}$-algebra. By the above surjectivity statement, since $A^d(X \times X)$ is finite-dimensional, so is $B^d(X \times X)$. It remains to show it is semi-simple.

Lemma 3.16. *$B^d(X \times X)$ is a semi-simple $\mathbb{Q}$-algebra.*

Proof. By standard results of non-commutative algebra, it suffices to show $J(B^d(X \times X)) = 0$ where $J(R)$ is the Jacobson radical[18]. Since $J(B^d(X \times X)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = J(B^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$, we reduce to showing

$$J(B^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) = 0.$$

So put $A = A^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, $B = B^d(X \times X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ and $J_A = J(A)$ and $J_B = J(B)$. We have a surjection

$$\Phi \colon A \twoheadrightarrow B$$

and we need to show $J_B = 0$. By formal arguments one shows $\Phi(J_A) = J_B$. So take $f_B \in J_B$, which lifts to $f_A \in J_A$ and so f_A is nilpotent in A (as J_A is nilpotent ideal). Then for any $g \in A$ the Lefschetz trace formula[19] gives

$$\operatorname{Tr}(f_A \cup g) = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}_{H^i_{\text{ét}}(X, \mathbb{Q}_\ell)}(f_A \circ g) \tag{3.2}$$

Since the Jacobson radical is a two-sided ideal, we get $f_A \circ g \in J_A$. So the RHS of (3.2) vanishes. But $\operatorname{Tr}(f_A \cup g) = \deg(f_A \cdot g)$. So this means f_A is numerically 0 and so $f_B = 0$, as desired. ☐

(3) $\implies$ (1). We won't prove this. ☐

3.1 Manin's identity principle and Lieberman's lemma

We now change gears and ask ourselves how to tell whether a correspondence is trivial or not. The next example shows something funny can happen.

Example 3.17 (Detection of trivial correspondences)**.** *Consider an elliptic curve E/k and four different points $a, b, c, d \in E(k)$. Then consider $p = \{a - b\} \times \{c - d\} \in CH^2(E \times E)$ is not zero. Viewing $p \in Corr^1_{rat}(E, E)$, we get an induced map*

$$\begin{aligned} p_* \colon CH^i(E) &\to CH^{i+1}(E) \\ T &\mapsto pr_{E*}(p \cdot (T \times E)) \end{aligned}$$

for every $i \geq 0$. Clearly $p_* = 0$ (recall that any two points are algebraically equivalent so $a \cdot T = b \cdot T$).

Manin's identity principle [Man68, pg. 450] gives some characterization of detecting non-trivial correspondences. To state it, we need to think of a correspondence as a functor of points (just like schemes). For the following assume we are working with the rational adequate equivalence (for simplicity):

Definition 3.18 (correspondence as a functor of points)**.** Given $T \in \operatorname{SmProj}(k)$, we put $X(T) := \operatorname{Corr}(T, X)$. Then for $f \in \operatorname{Corr}(X, Y)$, we get the induced morphism

$$\begin{aligned} f_T \colon X(T) &\to Y(T) \\ \alpha &\mapsto f \circ \alpha \end{aligned}$$

Theorem 3.19 (Manin's identity principle)**.** *Let $f, g \in Corr(X, Y)$. TFAE*

(1) $f = g$

(2) $f_T = g_T$ *for all* $T \in SmProj(k)$

(3) $f_X = g_X$

[18] The Jacobson radical of a ring R, $J(R) := \{r \in R| \quad rM = 0 \text{ for all } M \text{ simple}\}$.

[19] Such a formula is a formal consequence of the Weil cohomology theory axioms.

Proof. The only non-trivial direction is (3) $\implies$ (1). But one can check that $f = f \circ \Delta_X$, which gives the resut. $\square$

In practice we need Lieberman's lemma to actually make use of Manin's identity principle.

Lemma 3.20 (Lieberman's lemma)**.** *In the setting of Definition 3.18 $f \circ \alpha = (\Delta_T \times f)_*(\alpha)$.*

Proof. By definition of action of correspondences:

$$(\Delta_T \times f)_*(\alpha) = p^{TTXY}_{TY*}((\Delta_T \times f) \cdot (p^{TTXY*}_{TX}(\alpha))) \tag{3.3}$$

where

$$p^{TTXY}_{TY} : T \times \underline{T} \times X \times \underline{Y} \to \underline{T} \times \underline{Y} \text{ and } p^{TTXY}_{TX} : \underline{T} \times T \times \underline{X} \times Y \to \underline{T} \times \underline{X}.$$

We can rewrite (3.3) as

$$\begin{aligned} p^{TTXY}_{TY*}((\Delta_T \times f) \cdot (p^{TTXY*}_{TX}(\alpha))) &= p^{TTXY}_{TY*}((\Delta_T \times f) \cdot (T \times \alpha \times Y)) \\ &= p^{TTXY}_{TY*}((\Delta_T \times X \times Y) \cdot (T \times T \times f) \cdot (T \times \alpha \times Y)) \end{aligned}$$

Now writing p^{TTXY}_{TY} as the composition:

$$p^{TTXY}_{TY} : T \times T \times X \times Y \xrightarrow{p} T \times X \times Y \xrightarrow{q} T \times Y$$

we get

$$\begin{aligned} p^{TTXY}_{TY*}((\Delta_T \times X \times Y) \cdot (T \times T \times f) \cdot (T \times \alpha \times Y)) &= q_* p_*((\Delta_T \times X \times Y) \cdot p^*(T \times f) \cdot (T \times \alpha \times Y)) \\ &= q_*(p_*((\Delta_T \times X \times Y) \cdot (T \times \alpha \times Y))) \cdot (T \times f)) \\ &= q_*(((\alpha \circ \Delta_T) \times Y) \cdot (T \times f)) \\ &= f \circ \alpha \circ \Delta_T \\ &= f \circ \alpha. \end{aligned}$$

where the second isomorphism follows from the projection formula (cf. Exercise 2.6) and the third/fourth follow from defnition of composition of correspondences. The final equality is just that Δ_T acts as identity when composing. $\square$

Corollary 3.21. *In the context of Manin's identity principle (cf. Theorem 3.19), we get $f = g$ iff*

$$(\mathrm{id}_T \times f)_* = (\mathrm{id}_T \times g)_*$$

considered as maps on the Chow groups

$$CH(T \times X) \to CH(T \times Y) \quad \forall\, T$$

As an application of Manin's identity principle, we sketch the proof of the following:

Lemma 3.22. *Let $\mathscr{E}$ be a locally free sheaf of rank $(m+1)$ on $S \in SmProj(k)$ and let $\pi\colon \mathbb{P}_S(\mathscr{E}) \to S$ be the associated projective bundle. Then there is an isomorphism of motives in $Mot_{rat}(k)$*

$$(\mathbb{P}_S(\mathscr{E}), \Delta_{\mathbb{P}_S(E)}, 0) \xrightarrow{\sim} \bigoplus_{i=0}^{m} (S, \Delta_S, -i)$$

Proof. Let $\xi = \mathcal{O}(1)$ be the tautological line bundle on $\mathbb{P}_S(\mathscr{E})$. Then there is a *projective space bundle formula* [Sta18, Tag 0ERV]:

$$\lambda\colon CH(\mathbb{P}_S(\mathscr{E})) \xrightarrow{\sim} \bigoplus_{i=0}^{m} CH(S)[\xi^i].$$

Moreover the isomorphism λ (and it's inverse μ) are induced by correspondences. Also the morphism λ and μ are compatible with base change $T \to \mathrm{Spec}(k)$.

So this means that $(\mathrm{id}_T \times \lambda) \circ (\mathrm{id}_T \times \mu) = \mathrm{id}$ for all T. The result then follows by Corollary 3.21. $\square$

3.2 $M_{\mathbf{rat}}(k)$ vs category of abelian varieties up to isogeny

We prove that the category of Chow motives contains as a full subcategory the category of abelian varieties up to isogeny.

Recall from the proof of Proposition 3.12 for any curve $X \in \mathrm{SmProj}(k)$, we can write

$$(X, \Delta_X) = \mathbf{1} \oplus \mathbb{L} \oplus h^1 X. \tag{3.4}$$

Then in the spirit of Lemma 3.13 we have

Proposition 3.23. *Given two curves $X, X' \in SmProj(k)$ we have*

$$\mathrm{Hom}(h^1 X, h^1 X') = \mathrm{Hom}_{AV}(J(X), J(X')) \otimes_{\mathbb{Z}} \mathbb{Q}$$

.

Proof. By Weil's theorem [Wei71, Theorem 22, Chapitre VI]

$$Z^1_{\mathrm{rat}}(X \times X') \otimes_{\mathbb{Z}} \mathbb{Q} = (Z^1_{\mathrm{rat}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (Z^1_{\mathrm{rat}}(X') \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus \mathrm{Hom}_{\mathrm{AV}}(J(X), J(X')) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Note that $Z^1_{\mathrm{rat}}(X \times X') \otimes_{\mathbb{Z}} \mathbb{Q} = Hom((X, \Delta_X), (X', \Delta_{X'}))$. By using the decomposition (3.4), it's a check to get the result. □

To get the result one needs the Poincaré reducibility theorem [Mum74, Chapter IV, §19, Theorem 1]:

$$\{\text{category of AV}\}/\text{isogeny} = \text{pseudo-abelian completion of } \{J(C) | C \text{ curve}\}.$$

4 Lecture 4: Grothendieck's standard conjectures

Up to this point we have defined motives. Motives are expected to have good properties, but it turns out that these are still open. In this lecture, we will discuss the so-called *standard conjectures* concerning motives. These were originally formulated by Grothendieck in [Gro69]. In this lecture we will discuss some results in [Kle68] and [Kle94]. We have already seen standard conjecture D in (cf. Conjecture 2.33). In this lecture we will take a look at the remaining standard conjectures:

(1) Standard Conjectures C (Künneth Conjecture)

(2) Standard Conjectures A and B (Conjectures of Lefschetz type)

(3) Standard Conjecture H (Conjecture of Hodge type)

Let $X \in \mathrm{SmProj}(k)$. We fix a Weil cohomology $H(X)$ over a characteristic 0 field and recall we have

$$\gamma_X \colon \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{2i}(X)$$

and define $A^i(X) := \mathrm{im}(\gamma_X) \subset H^{2i}(X)$. We call the elements of $A^i(X)$ the *algebraic* classes.

4.1 Künneth conjecture (Standard conjecture C)

Assume X is pure of dimension d. Let $\Delta_X \in \mathrm{CH}^d(X \times X)$ be the diagonal and consider its class

$$\gamma_{X \times X}(\Delta_X) \in H^{2d}(X \times X) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X)$$

where the equality is the Künneth decomposition (cf. axiom (4) in Definition 2.24). So we can write

$$\gamma_{X \times X}(\Delta_X) = \pi_0 + \pi_1 + \ldots + \pi_i + \ldots + \pi_{2d}$$

with $\pi_i \in H^{2d-i}(X) \otimes H^i(X)$.

Conjecture 4.1 (Künneth conjecture)**.** *The Künneth components π_i are algebraic: $\exists$ cycle classes $\Delta_i \in CH^d(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\gamma_{X\times X}(\Delta_i) = \pi_i$.*

Exercise 4.2. *Let X be a scheme with a* cellular decomposition*: that is there exists a filtration*

$$X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \emptyset$$

by closed subschemes with each $X_i \backslash X_{i-1}$ a disjoint uniond of schemes U_{ij} isomorphic to affine spaces $\mathbb{A}^{n_{ij}}$. Then $Z^k(X)$ is finitely generated by $\{[V_{ij}]\}$, where V_{ij} is the closure of U_{ij} in X. Show in this case one has

$$CH(X \times X) \cong CH(X) \otimes_{\mathbb{Z}} CH(X).$$

Exercise 4.3. *Show that any $X \in SmProj(k)$ which satisifes the* Chow-Künneth *decomposition:*

$$CH(X \times X) \cong CH(X) \otimes_{\mathbb{Z}} CH(X)$$

implies that γ_X is in fact an isomorphism. Use this to show that for such X, the Künneth conjecture (trivially) holds.

Remark 4.4. *Projective space $\mathbb{P}^n$ satisfies the condition of Exercise 4.2. In general if X is a* linear *scheme, then it satisfies the conditions of Exercise 4.3 (cf. [Tot14, Proposition 1]).*

The next proposition is less trivial and is due to Katz-Messing [KM74, Theorem 2 part 1)].

Proposition 4.5. *Suppose $k = \mathbb{F}_q$ and $X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ is irreducible. Then the Künneth conjecture holds for X.*

Proof. Fix a prime $\ell \neq p = \mathrm{char}(\mathbb{F}_q)$ and let Fr be the relative Frobenius morphism of X over $\mathbb{F}_q$. Deligne has proved (cf. [Del74, Théorème I.6]), as part of his proof of the Weil conjectures that the polynomial in T

$$\det(1 - TFr \mid H^i_{\text{ét}}(X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Q}_\ell)) \tag{4.1}$$

lies in $\mathbb{Z}[T]$ and its reciprocal zeros all have complex absolute value $q^{i/2}$ for every $i \geq 0$. As a first step Katz-Messing (cf. [KM74, Theorem 1]) show that the term (4.1) is independent of the Weil cohomology theory, that is:

Lemma 4.6. *We have* $\det(1 - TFr \mid H^i_{\acute{e}t}(X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Q}_\ell)) = \det(1 - TFr \mid H^i(X))$ *where $H^i(X)$ is our chosen Weil cohomology theory from the start of this lecture.*

Proof. We won't prove this, but let me mention that it relies on Poincaré duality and the weak Lefschetz axiom. □

It follows that the polynomials $G_i(T) = \det(1 - TFr \mid H^i(X))$ are pairwise relatively prime in $\mathbb{Q}[T]$ because their roots have different absolute value. Let $\Pi_i(T) \in \mathbb{Q}[T]$ be a polynomial such that

$$G_j(T) | \Pi_i(T) \text{ for all } j \neq i \text{ and } \Pi_i(T) = 1 \bmod G_i(T).$$

Such a polynomial exists by the Chinese remainder theorem. By the Cayley-Hamilton theorem, it follows that the operator

$$\Pi_i(\mathrm{Fr})\colon \bigoplus_{j=0}^{2d} H^j(X) \to H^i(X) \tag{4.2}$$

is exactly the projection operator and these are algebraic. But note that by Poincaré duality we can rewrite the Künneth formula as

$$H^{2d}(X \times X) = \mathrm{Hom}_{\mathrm{GrVect}_F^{\geq 0}}(H(X), H(X)).$$

This means that by (4.2) $\gamma_{X\times X}(\Delta_X) = p_1 + \ldots + p_{2d}$, where $p_i \in \mathrm{Corr}^0(X)$ corresponding to (the graph of) $\Pi_i(\mathrm{Fr})$. □

Exercise 4.7. *Using decomposition* (3.4), *show that the Künneth conjecture holds for curves.*

4.2 Conjectures of Lefschetz type (Standard conjectures A and B)

Assume X is pure of dimension d and let $Y \hookrightarrow X$ be a smooth hyperplane section. Recall the Lefschetz operator

$$\begin{aligned} L\colon H^i(X) &\to H^{i+2}(X) \\ \alpha &\mapsto \alpha \cup \gamma_X(Y). \end{aligned}$$

Recall for H a Weil cohomology theory, we assume hard Lefschetz (cf. Definition 2.24(7))

$$L^i\colon H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X).$$

Lemma 4.8. *The Lefschetz operator L is algebraic. More precisely it is represented by the algebraic cycle $\Delta_*(Y) \in A^{d+1}(X \times X)$ and $\Delta\colon X \to X \times X$ is the diagonal map.*

Proof. For $u = \Delta_*(Y)$, it suffices to show

$$\gamma_X(p_{2*}(p_1^*\alpha \cdot u)) = \alpha \cup \gamma_X(Y).$$

Indeed we have

$$\begin{aligned} \gamma_X(p_{2*}(p_1^*\alpha \cdot u)) &= \gamma_X(p_{2*}(p_1^*\alpha \cdot \Delta_*Y)) \\ &= p_{2*}(p_1^*\alpha \cup \Delta_*Y) \\ &= \alpha \cup p_{2*}\Delta_*Y \end{aligned}$$

where the second equality follows by compatibility of γ_X and pushforwards, the third equality by a version of the projection formula (cf. Exercise 2.6). But note that $p_2 \circ \Delta = \mathrm{id}_X$, so we are done. ☐

Using Hard Lefschetz we can define a unique linear map $\Lambda\colon H^i(X) \to H^{i-2}(X)$ for each $2 \le i \le 2d$ as follows:

1) For $2 \le i \le d$ which makes the following diagram commutative:

$$\begin{tikzcd} H^i(X) \arrow[r, "L^{d-i}"] \arrow[d, dotted, "\Lambda"'] & H^{2d-i}(X) \arrow[d, "L"] \\ H^{i-2}(X) \arrow[r, "L^{d-i+2}"] & H^{2d-i+2}(X). \end{tikzcd}$$

2) For $i = d+1$, $\Lambda := L^{-1}$ where $L\colon H^{d-1}(X) \xrightarrow{\sim} H^{d+1}(X)$.

3) For $d+2 \le i \le 2d$ which makes the following diagram commutative:

$$\begin{tikzcd} H^{2d-i}(X) \arrow[r, "L^{i-d}"] \arrow[d, "L"'] & H^i(X) \arrow[d, dotted, "\Lambda"] \\ H^{2d-i+2}(X) \arrow[r, "L^{i-d-2}"] & H^{i-2}(X). \end{tikzcd}$$

By Poincaré duality and Künneth formula, we have

$$\Lambda \in \mathrm{Hom}(H^i(X), H^{i-2}(X)) = H^{2d-i}(X) \otimes H^{i-2}(X) \subset H^{2d-2}(X \times X).$$

So we can view Λ canonically as an element in $H^{2d-2}(X \times X)$.

Conjecture 4.9 (Standard conjecture B)**.** *The operator Λ is algebraic: $\exists$ cycle $Z \in CH^{d-1}(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\gamma_{X\times X}(Z) = \Lambda$.*

Lemma 4.10 (Conjecture B implies the Künneth conjecture)**.** *Standard conjecture B implies the Künneth conjecture*

Proof.

Exercise 4.11. *Verify the formula*

$$\pi_i = \Lambda^{d-i}\left(1 - \sum_{j>2d-i} \pi_j\right) L^{d-i}\left(1 - \sum_{j<i} \pi_j\right)$$

where $\Lambda^{d-i} \colon H^{2d-i}(X) \xrightarrow{\sim} H^i(X)$ *(an inverse to* L^{d-i}*).*

We can then proceed by induction as π_0 and π_{2d} are algebraic. □

It turns out that Standard conjecture B is independent of the choice of hyperplane section which defines L (and hence Λ), cf. [Kle94, Theorem 4.1(2)].

Proposition 4.12. *If standard conjecture B holds for one choice of* L*, then it holds for all choices.*

Proof. It suffices to show that standard conjecture B is equivalent to the following statement (which is independent of L):

For each $i \leq d$, there exists an algebraic correspondence

$$\nu_i \colon H^{2d-i}(X) \xrightarrow{\sim} H^i(X). \tag{4.3}$$

Indeed if standard conjecture B is true, then Λ is algebraic and thus by taking a sufficiently large composition Λ^i is also algebraic and induces the above isomorphism.

For the converse suppose (4.3) holds. Then $u := \nu_i \circ L^{d-i}$ is algebraic.

Exercise 4.13. *By looking at the characteristic polynomial of* u*, show that* u^{-1} *is algebraic.*

By exercise 4.13, it follows that $\theta_i := u^{-1} \circ v_i$ is an algebraic inverse of L^{d-i}. The result then follows from the following exercise:

Exercise 4.14. *Show that*

$$\Lambda := \sum_{i \leq d} (\pi_{i-1}\theta_i L^{d-i+1}\pi_i + \pi_{2d-i}L^{d-i+1}\theta_{i+2}\pi_{2d-i+2}).$$

□

Remark 4.15. *Standard conjecture B holds true for abelian varities. This result is due to Lieberman-Kleiman [Kle68, Theorem 2A11].*

To state standard conjecture A, note that we have a commutative diagram (for $d \geq 2i$)

$$\begin{tikzcd} A^i(X) \arrow[r] \arrow[d] & A^{d-i}(X) \arrow[d, dotted] \\ H^{2i}(X) \arrow[r, "L^{d-2i}"] & H^{2d-2i}(X). \end{tikzcd} \tag{4.4}$$

The top arrow $A^i(X) \to A^{d-i}(X)$ exists because L is algebraic (cf. Lemma 4.8). In fact is injective.

Conjecture 4.16 (Standard Conjecture A)**.** *Hard Lefschetz is true on cycles. That is the top arrow* $A^i(X) \hookrightarrow A^{d-i}(X)$ *in diagram* (4.4) *is an isomorphism.*

It turns out that Standard conjectures A and B are equivalent, cf. [Kle94, Corollary 4.2]

Proposition 4.17. *Standard conjecture A holds iff standard conjecture B holds.*

Proof. (B) $\implies$ (A): Indeed if (B) is true then we get an algebraic inverse to L^{d-2i} (given by Λ^{d-2i}) and so we get an inverse map $A^{d-i}(X) \to A^i(X)$. Thus (A) is true.

(A) $\implies$ (B): We won't prove this in detail but highlight some steps.

Exercise 4.18. *Show that each $x \in H^i(X)$ has a unique decomposition*

$$x = \sum_{j \geq max(i-d,0)} L^j x_j$$

where $x_j \in \ker(L|H^{i-2j}(X))$.

Given Exercise 4.18 we can define an operator ${}^c\Lambda$ given by:

$${}^c\Lambda(x) := \sum_{j \geq max(i-d,1)} j(n-i+j+1)L^{j-1}x_j$$

Exercise 4.19. *Show that Λ is algebraic iff ${}^c\Lambda$ is algebraic.*

Since we have assumed conjecture A is true, in particular it is true for $X \times X$ equipped with Lefschetz operator $1 \otimes L + L \otimes 1$. Then [Kle68, Proposition 1.4.6(ii) and Proposition 2.1] implies that $1 \otimes^c \Lambda +^c \Lambda \otimes 1$ carries algebraic classes to algebraic classes (a priori it is only defined at the level of cohomology). Moreover Proposition 1.3.4 in loc.cit. shows that it carries Δ_X to $2^c\Lambda$. Therefore ${}^c\Lambda$ is algebraic and we are done by Exercise 4.19.

□

4.3 Hodge standard conjecture

In general by hard Lefschetz

$$L^{d-i} \colon H^i(X) \xrightarrow{\sim} H^{2d-i}(X)$$

is an isomorphism. However in the spirit of Exercise 4.18:

Definition 4.20 (primitive cohomology)**.** We define primitive cohomology as

$$P^i(X) := \ker(L^{d-i+1} \colon H^i(X) \to H^{2d-i+2}(X)).$$

Definition 4.21 (primitive algebraic classes)**.** We define primitive algebraic classes as

$$A^i_{\text{prim}}(X) := A^i(X) \cap P^{2i}(X)$$

For $i \leq d/2$, the cup product gives a pairing

$$\begin{aligned} A^i_{\text{prim}}(X) \times A^i_{\text{prim}}(X) &\to \mathbb{Q} \\ (x,y) &\mapsto (-1)^i \mathrm{Tr} \circ (L^{d-2i}(x) \cup y) \end{aligned}$$

Conjecture 4.22 (Hodge standard conjecture)**.** *The pairing defined above is positive definite.*

The next result can be found in [Kle94, Proposition 5.1].

Proposition 4.23. *Given the Hodge standard conjecture, standard conjecture A is equivalent to standard conjecture D.*

Proof. We need to define a version of the Hodge star operator (appearing in Hodge theory):

$$\begin{aligned} *\colon H^i(X) &\to H^{2d-i}(X) \\ x &\mapsto \sum_{j \geq max(i-d,0)} (-1)^{(i-2j)(i-2j+1)/2} L^{d-i+j} x_j \end{aligned}$$

where the $x_j \in H^{i-2j}(X)$ are those appearing in Exercise 4.18.

Now suppose the Hodge conjecture is true.

Exercise 4.24. *By using Exercise 4.18 for x, show that the pairing*

$$\begin{aligned} A^i(X) \times A^i(X) &\to \mathbb{Q} \\ (x,y) &\mapsto Tr(x \cup *y) \end{aligned}$$

is positive-definite.

(A) $\implies$ (D): Now standard conjecture A implies the canonical pairing (given by cup product) $A^i(X) \times A^{d-i}(X) \to \mathbb{Q}$ is perfect. Thus if $x \in Z^i_{\text{num},0}(X)$, then $x \in Z^i_{\text{hom},0}(X)$.

(D) $\implies$ (A): In this case we use $A^i(X) \hookrightarrow A^{d-i}(X)$ and again the positive-definiteness of Exercise 4.24.

□

Remark 4.25. *If k is of characteristic zero, then the Hodge standard conjecture is true and is a consequence of Hodge theory.*

5 Lecture 5: Motivic Galois groups I

Up to now we have mainly focused on the geometric aspects of motives. In this lecture we start to look at the arithmetic aspects. Grothendieck wanted to build some kind of Galois group coming from a fiber functor[20]

$$\text{Mot}_{\text{num}}(k) \to \{\text{category of finite vector spaces over } k\}.$$

Such a fiber functor can come from a Weil cohomology theory, but the issue is that we don't have standard conjecture D and so a priori one one gets a fiber functor from $\text{Mot}_{\text{hom}}(k)$. Now the issue is that $\text{Mot}_{\text{hom}}(k)$ is no longer abelian and so not Tannakian.

There are several approaches of circumventing conjecture D and modifying the source category $\text{Mot}_{\text{num}}(k)$ just enough so that one has a fiber functor and the category remains abelian. We will study the approach of Deligne-Milne in [DM82].

5.1 Absolute Hodge cycles

Let k be an algebraically closed field of finite transcendence degree over $\mathbb{Q}$ and $X \in \text{SmProj}(k)$. We set

$$H^n_{\text{ét}}(X) := \varprojlim_r H^n_{\text{ét}}(X, \mathbb{Z}/r\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \text{ and } H^n_{\text{ét}}(X)(m) := H^n_{\text{ét}}(X) \otimes_{\mathbb{A}_f} (\varprojlim_r \mu_r \otimes_{\mathbb{Z}} \mathbb{Q})$$

and

$$H^n_{\text{dR}}(X)(m) := H^n_{\text{dR}}(X).$$

Finally we set

$$H^n_{\mathbb{A}}(X)(m) := H^n_{\text{dR}}(X)(m) \times H^n_{\text{ét}}(X)(m).$$

[20]This is the reason that Grothendieck (and his student Saavedra Rivano) introduced the notion of Tannakian category.

Given an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$, there are canonical isomorphisms:

$$\sigma^*_{dR}\colon H^n_{\mathrm{dR}}(X)(m) \otimes_{k,\sigma} \mathbb{C} \xrightarrow{\sim} H_{\mathrm{dR}}(\sigma X)(m) \text{ and } \sigma^*_{\text{ét}}\colon H^n_{\text{ét}}(X)(m) \xrightarrow{\sim} H^n_{\text{ét}}(\sigma X)(m)$$

where $\sigma X := X \times_k \mathbb{C}$. We pute $\sigma^* := \sigma^*_{\mathrm{dR}} \times \sigma^*_{\text{ét}}$. We now put

$$H^n_B(X)(m) := H^n_B((\sigma X)^{\mathrm{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} (2\pi i)^m \mathbb{Q}$$

so that the standard comparison isomorphisms give

$$H^n_B(X)(m) \otimes_{\mathbb{Q}} (\mathbb{C} \times \mathbb{A}_f) \xrightarrow{\sim} H^n_{\mathrm{dR}}(\sigma X)(m) \times H^n_{\text{ét}}(\sigma X)(m)$$

Definition 5.1 (Hodge cycle relative to σ)**.** An element $t \in H^{2p}_{\mathbb{A}}(X)(p)$ is a Hodge cycle relative to σ if

(1) t is rational relative to σ: $\sigma^*(t)$ lies in the rational subspace $H^{2p}_B(X)(p)$ of $H^{2p}_{\mathrm{dR}}(\sigma X)(m) \times H^{2p}_{\text{ét}}(\sigma X)(m)$.

(2) it is of bidegree (p,p).

Definition 5.2 (Absolute Hodge cycle)**.** An element $t \in H^{2p}_{\mathbb{A}}(X)(p)$ is an absolute Hodge cycle if it is a relative Hodge cycle for every embedding $\sigma\colon k \hookrightarrow \mathbb{C}$. We denote by $C^p_{\mathrm{AH}}(X)$ the $\mathbb{Q}$-vector space of absolute Hodge cycles $t \in H^{2p}_{\mathbb{A}}(X)(p)$.

Example 5.3. *The cycle class maps*

$$\gamma_{dR}\colon CH^p(X) \to H^{2p}_{dR}(X)(p) \text{ and } \gamma_{\acute{e}t}\colon CH^p(X) \to H^{2p}_{\acute{e}t}(X)(p)$$

and we claim that $t := (\gamma_{dR}(Z), \gamma_{\acute{e}t}(Z))$ is an absolute Hodge cycle. Indeed for any $\sigma\colon k \hookrightarrow \mathbb{C}$, $\sigma^*(t) = \gamma_B(\sigma Z)$. This is because the cycle class maps are all compatible via the comparison isomorphisms cf. [Del71, 2.2.5.1]. In addition it is of degree (p,p) by a calculation. The Hodge conjecture predicts that there are no other absolute Hodge cycles.

Definition 5.4 (False category of motives)**.** By repeating the procedure of taking the pseudo-abelian completion with morphisms given by $\mathrm{C}^p_{\mathrm{AH}}(X \times Y)$, we get the false category of motives $\dot{M}_k$. More precisely this is the category given by

(1) **objects:** triples (X,p,m) with $X \in \mathrm{SmProj}(k)$, $p \in \mathrm{C}^d_{\mathrm{AH}}(X \times X)$ a projector ($d = \dim(X)$) and $m \in \mathbb{Z}$.

(2) **morphisms:** $\mathrm{Hom}((X,p,m),(Y,q,n)) := q \circ \mathrm{C}^{n-m+d}_{\mathrm{AH}}(X \times Y) \circ p$ with composition given by cup product.

To state the main theorem regarding $\dot{M}_k$, let us recall the notion of a Tannakian category. For a more comprehensive treatment of Tannakian categories, cf. [SR72].

Definition 5.5. Let $\mathcal{C}$ be a (rigid) abelian[21] tensor category with $\mathrm{End}(\mathbf{1}) = k$.

(1) **fibre functor:** A fibre functor on $\mathcal{C}$ with values in a k-algebra R is a k-linear exact faithful tensor functor

$$\eta\colon \mathcal{C} \to \mathrm{Mod}_R$$

that takes values in the subcategory Proj_R.

(2) **Tannakian category:** $\mathcal{C}$ is a Tannakian category if it admits a fibre functor with values in some nonzero k-algebra.

[21] this essentially means that $\mathcal{C}$ has a tensor product with duals and a $\mathbf{1}$.

The main property of Tannakian categories is that they are essentially representations of some group.

Theorem 5.6. *Let $\mathcal{C}$ be a Tannakian category over k. Then there exists a stack $\mathcal{G}$ in groupoids and a canonical k-linear tensor functor*

$$\mathcal{C} \to \mathrm{Rep}_k(\mathcal{G})$$

which is an equivalence of categories[22].

Returning back to our false category of motives $\dot{M}_k$, it turns out for some technical issue, it cannot be Tannakian. To fix this one changes the commutativity constraint[23] as follows: Let

$$\dot{\psi}\colon M \otimes N \to N \otimes M \text{ where } \dot{\psi} = \oplus \dot{\psi}^{r,s} \text{ where } \dot{\psi}^{r,s} := M^r \otimes N^s \to N^s \otimes M^r$$

be the commutativity constraint. An explanation of the notation is in order: the grading $\dot{\psi}^{r,s}$ is coming from the grading induced on morphisms by

$$\mathrm{C}^{p+d}_{\mathrm{AH}}(X \times Y) \subset H^{2d+2p}(X \times Y)(p+n) = \oplus_r \mathrm{Hom}(H^r(X), H^{r+2p}(Y)(p)).$$

We now modify the commutativity constraint as

$$\psi\colon M \otimes N \to N \otimes M \text{ where } \psi = \oplus \psi^{r,s} \text{ where } \psi^{r,s} := (-1)^{rs} \dot{\psi}^{r,s}$$

Definition 5.7 (True category of motives)**.** We define the true category of motives M_k to be $\dot{M}_k$ with the commutativity constraint ψ.

Proposition 5.8. *The category M_k is a semisimple Tannakian category over $\mathbb{Q}$.*

6 Lecture 6: Motivic Galois groups II

7 Lecture 7: Voevodsky's mixed motives (approach by transfers)

8 Lecture 8: Homotopy category of Morel-Voevodsky

9 Lecture 9: Motives over rigid-analytic varities - part I (after Ayoub)

10 Lecture 10: Motives over rigid-analytic varieties - part II (after Ayoub)

11 Lecture 11: 6-functor formalism of motives over rigid-analytic varieties

The work of Ayoub-Gallauer-Vezzani [AGV22] produced a 6-functor formalism for rigid-analytic varieties.

This in turn relied on the work of Ayoub [Ayo15], where for a quite general adic space S, he constructed a category of (étale) rigid analytic motives over S with rational coefficients $\mathbf{RigDA}_{\text{ét}}(S, \mathbb{Q})$.

In [Ayo15], he extended the work of the theory of motives of over an algebraic variety. Given a scheme S there are two known approaches to constructing a theory of motives over S:

[22] Here $\mathrm{Rep}_k(\mathcal{G})$ the category of cartesian functors $\mathcal{G} \to \mathrm{Proj}$, where Proj is the stack such that $\mathrm{Proj}(\mathrm{Spec}\, R) := \mathrm{Proj}_R$.

[23] A commutativity constraint is part of the datum of a tensor category.

(1) the homotopic approach of Morel-Voevodsky leading to the homotopic category $\mathbf{H}(S)$ (cf. [MV99]) and its stable version $\mathbf{SH}(S)$ (cf. [Jar00]).

(2) the "approach by transfers" by [VSF00].

12 Appendix

12.1 Solutions to exercises

Solution 12.1 (To Example 1.10)**.** *The Segre embedding shows that S is a closed subspace of projective space. Thus it is also projective. It's also smooth (being the product of two smooth varieties), so* $\mathrm{Div}(S)$ *is well-defined. Next we show* $\mathrm{Div}(S) = \mathbb{Z} \oplus \mathbb{Z}$*.*

To show $\mathrm{Div}(S) \cong \mathbb{Z} \oplus \mathbb{Z}$ *we refer to [Har77, Example 6.6.1]. We can take as generators divisors (up to linear equivalence) $p := 0 \times \mathbb{P}^1$ and $q := \mathbb{P}^1 \times 0$. Then $p \cdot q = 1$ (as they meet transversely and they intersect at a single point) and $p \cdot p = 0$ because we can move p to another parallel line with no intersection. Similarly for q. This determines the intersection product claimed formula by [Har77, Theorem 1.1].*

It is easy to see that $\mathrm{Div}(S) = Num(S)$*, since have basically described* $\mathrm{Div}(S)$ *and it's intersection product above.*

The claimed signature of the intersection form then follows.

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