Домашняя работа №2 Шамаев И.

2) $(\tilde{\partial}\bar{\partial}\partial y)(x)$ $y'''(x)$ $y(x)\in C^5$

$$(\tilde{\partial}\bar{\partial}\partial y)(x) = \frac{(\bar{\partial}\partial y)(x+h) - (\bar{\partial}\partial y)(x-h)}{2h} = \frac{1}{2h}\left[\frac{(\partial y)(x+h) - \partial(y)(x)}{h} - \frac{(\partial y)(x-h) - (\partial y)(x-2h)}{h}\right] = \frac{1}{2h^2}\left[\frac{y(x+2h)-y(x+h)}{h} - \frac{y(x+h)-y(x)}{h} - \left(\frac{y(x)-y(x-h)}{h} - \frac{y(x-h)-y(x-2h)}{h}\right)\right] =$$

$$= \frac{1}{2h^3}\left[y(x+2h) - y(x+h) - y(x+h) + \cancel{y(x)} - \cancel{y(x)} + y(x-h) + y(x-h) - y(x-2h)\right] = \frac{1}{2h^3}\left[y(x+2h) - 2y(x+h) + 2y(x-h) - y(x-2h)\right]$$

$$y(x\pm h) = y(x) \pm y'(x)h + y''(x)\frac{h^2}{2} \pm y'''(x)\frac{h^3}{6} + y^{IV}(x)\frac{h^4}{24} \pm y^{V}(\xi_\pm)\frac{h^5}{120}$$

$\xi_+ \in [x, x+h]$

$\xi_- \in [x-h, x]$ $\Rightarrow 2\left(y(x-h) - y(x+h)\right) = 2\left[-2y'(x)h - 2y'''(x)\frac{h^3}{6} - \frac{y^{V}(\xi_-) + y^{V}(\xi_+)}{120}h^5\right] = -4y'(x)h - 2y'''(x)\frac{h^3}{3} - \frac{y^{V}(\xi_-) + y^{V}(\xi_+)}{2}\cdot\frac{h^5}{30}$

По теореме о среднем $\exists\, \xi \in [\xi_-, \xi_+] \subset [x-h, x+h]: \frac{y^{V}(\xi_-) + y^{V}(\xi_+)}{2} = y^{V}(\xi)$

$$y(x\pm 2h) = y(x) \pm y'(x)\cdot 2h + y''(x)\frac{4h^2}{2} \pm y'''(x)\frac{8h^3}{6} + y^{IV}(x)\frac{16h^4}{24} \pm y^{V}(\xi_\pm)\cdot\frac{32h^5}{120}$$

$\xi_+ \in [x, x+2h]$, $\xi_- \in (x-2h, x]$ $\Rightarrow y(x+2h) - y(x-2h) = 2y'(x)\cdot 2h + 2y'''(x)\frac{8h^3}{6} + \frac{y^{V}(\xi_+) + y^{V}(\xi_-)}{2}\frac{32h^5}{60} = 4y'(x)h + \frac{8h^3}{3}y'''(x) +$

$$+\frac{y^V(\xi_+)+y^V(\xi_-)}{} \cdot \frac{8}{15}h^5$$

$$\exists\, \bar{\xi} \in [\xi_-, \xi_+] \subset [x-2h, x+2h] \Rightarrow \frac{y^V(\xi_+)+y^V(\xi_-)}{2} = y^V(\bar{\xi})$$

$$y'''(x) - (\tilde{\partial}\bar{\partial}\partial y)(x) = y'''(x) - \frac{1}{2h^3}\Big[4y'(x)h + \frac{8h^3}{3}y'''(x) + y^V(\bar{\xi})\cdot\frac{8}{15}h^5 -$$

$$-4y'(x)h - 2y'''(x)\frac{h^3}{3} - y^V(\xi)\cdot\frac{h^5}{30}\Big] = y'''(x) - y'''(x) - \frac{4}{15}h^2 y^V(\bar{\xi}) +$$

$$+y^V(\xi)\frac{h^2}{60} = \frac{h^2}{60}y^V(\xi) - \frac{4}{15}h^2 y^V(\bar{\xi}) = \frac{h^2}{15}\Big[\frac{y^V(\xi)}{4} - 4y^V(\bar{\xi})\Big] =$$

$$= O(h^2)$$

$$y(x) \in C^5 \Rightarrow |y^V| \le M_5 < \infty \Rightarrow |y'''(x) - \tilde{\partial}\bar{\partial}\partial y(x)| \le \frac{h^2}{15}\Big[\frac{M_5}{4} - 4M_5\Big]$$

③ $$y'(x) \approx \frac{C_1 y(x) + C_2 y(x+2h) + C_3 y(x+3h)}{h} \qquad (1)$$

$$y''(x) \approx \frac{C_1 y(x-2h) + C_2 y(x-h) + C_3 y(x)}{h^2} \qquad (2)$$

Базис $\{1; x; x^2\}$

Наивысший порядок для полиномов, у которых ~~deg~~ $\deg \le 2$

$$y^{(k)}(x_0) = \sum_{i=0}^{n} C_i y(x_i) + R(y)$$

Получим, что $R(y)=0 \Rightarrow$ на основе базиса

$$\sum_{i=0}^{n} C_i x_i^m = \begin{cases} 0, & m = 0, 1, \dots, k-1; \\ \frac{m!}{(m-k)!} x_0^{k-m}, & m = k, k+1, \dots, n-1; \end{cases}$$

$$x_0 = 0 \Rightarrow y'(0) \approx \frac{C_1 y(0) + C_2 y(2h) + C_3 y(3h)}{h};$$

$$y''(0) \approx \frac{C_1 y(-2h) + C_2 y(-h) + C_3 y(0)}{h};$$

$y(x)=1 \qquad y'(x)=0 \qquad y'(0)=0$

$y(x)=x \qquad y'(x)=1 \qquad y'(0)=1$

$y(x)=x^2 \qquad y'(x)=2x \qquad y'(0)=0$

$$\left.\begin{aligned}
y'(0)&=\frac{C_1+C_2+C_3}{h}=0\\
y'(0)&=\frac{2hC_2+3hC_3}{h}=1\\
y'(0)&=\frac{4h^2C_2+9h^2C_3}{h}=0
\end{aligned}\right\} \Rightarrow \left.\begin{aligned}
C_1+C_2+C_3&=0\\
2C_2+3C_3&=1\\
4C_2+9C_3&=0
\end{aligned}\right] \Leftrightarrow \begin{cases} C_1=-\frac{5}{6}\\ C_2=\frac{3}{2}\\ C_3=-\frac{2}{3}\end{cases}$$

$$y'(0)=\frac{1}{6h}\left[-5y(0)+9y(2h)-4y(3h)\right];$$

Разложим $y(2h)$ и $y(3h)$ в ряд Тейлора в т. $x_0=0$

$y(2h)=y(0)+2hy'(0)+2h^2y''(0)+\frac{4}{3}h^3y'''(\xi_\pm)$

$y(3h)=y(0)+3hy'(0)+\frac{9}{2}h^2y''(0)+\frac{27}{6}h^3y'''(\xi_\pm)$

$R(y)=y'(0)-\frac{1}{6h}[-5y(0)+9y(0)\pm18hy'(0)+18h^2y''(0)\pm12h^3y'''(\xi_\pm)-$

$-4y(0)\mp12hy'(0)-18h^2y''(0)\mp\frac{54}{3}h^3y'''(\xi_+)]=y'(0)-\frac{1}{6}[6hy'(0)-6h^3y'''(\xi_\pm)]$

$\cancel{y'(0)}=y'(0)-y'(0)\mp h^2y'''(\xi\pm)=\pm h^2y'''(\xi_\pm)$

$R(y)=h^2[y'''(\xi_+)+y'''(\xi_-)] \Rightarrow$ второй порядок точности

Теперь для $y''(0)$

$$\left.\begin{aligned}
y(x)&=1\\
y(x)&=x\\
y(x)&=x^2
\end{aligned}\right\} \quad \left.\begin{aligned}
y'(x)&=0\\
y'(x)&=1\\
y'(x)&=2x
\end{aligned}\right\} \quad \begin{aligned}
y''(x)&=0\\
y''(x)&=0\\
y''(x)&=2
\end{aligned}$$

$$y''(0)=\frac{C_1+C_2+C_3}{h^2}=0$$
$$y''(0)=\frac{-2hC_1-C_2h}{h^2}=0 \quad \Rightarrow \quad \begin{cases} C_1+C_2+C_3=0\\ -2C_1-C_2=0\\ 4C_1+C_2=2\end{cases} \Leftrightarrow \begin{cases} C_1=1\\ C_2=-2\\ C_3=1\end{cases}$$
$$y''(0)=\frac{4h^2C_1+C_2h^2}{h^2}=2$$

$$y''(0)=\frac{1}{h^2}[y(-2h)-yy(-h)+y(0)]$$

$y(-2h)$ и $y(-h)$ в ряд Тейлора в точке $x_0=0$

$y(-h)=y(0)-hy'(0)+\frac{h}{2}y''(0)-\frac{h^3}{6}y'''(0)+\frac{h^4}{24}y^{IV}(\xi_-)$

$y(-2h)=y(0)-2hy'(0)+2h^2y''(0)-\frac{4}{3}h^3y'''(0)+\frac{2}{3}h^4y^{IV}(\xi_-)$

$R(y)=y''(0)-\frac{1}{h^2}[\cancel{y(0)}+\cancel{y(0)}-\cancel{2hy'(0)}+2h^2y''(0)-\frac{4}{3}h^3y'''(0)+$

$+\frac{2}{3}h^4y^{IV}(\xi_-)-\cancel{2y(0)}+\cancel{2hy'(0)}-h^2y''(0)+\frac{h^3}{3}y'''(0)-\frac{h^4}{12}y^{IV}(\xi)]=$

$=y''(0)-\frac{1}{h^2}[h^2y''(0)-h^3y'''(0)+\frac{7}{12}y^{IV}(\xi)h^4]=\cancel{y''(0)}-\cancel{y''(0)}+hy'''(0)-$

$-\frac{7}{12}h^4y^{IV}(\xi)=hy'''(0)-\frac{7}{12}h^2y^{IV}(\xi)\Rightarrow$ второй порядок

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$|y^{(k)}(x)|\le M_k \qquad \max|y(x)-\varphi(x)|\le\varepsilon$

$|y'''(x)|\le M_3$

для (1): $|y'(0)-\frac{1}{6h}[5y(0)+3y(2h)-4y(3h)]|+|\frac{1}{6h}[5y(0)+$

$+3y(2h)-4y(3h)]-\frac{1}{6h}[5\varphi(0)+3\varphi(2h)-4\varphi(3h)]|\le h^2|y'''(\xi)|+$

$+\frac{5}{6h}|y(0)-\varphi(0)|+\frac{1}{2h}|3y(2h)-3\varphi(2h)|+\frac{2}{3h}|-4y(3h)+4\varphi(3h)|\le$

$4\varphi(3h)|\le M_3h^2+\frac{(5+3+4)\varepsilon}{6h}$

$$\left(M_3h^2+\frac{2\varepsilon}{h}\right)'_h=2M_3h-\frac{2\varepsilon}{h^2}=0\Rightarrow h_{min}=\sqrt[3]{\frac{\varepsilon}{M_3}}$$

Для (2): $\left|y''(0) - \frac{1}{h^2}[\varphi(-2h) - 2\varphi(-h) + \varphi(0)]\right| \le$

$\le \left|y''(0) - \frac{1}{h^2}[y(-2h) - 2y(-h) + y(0)]\right| + \left|\frac{y(-2h) - 2y(-h) + y(0)}{h^2} -\right.$

$\left.- \frac{\varphi(-2h) - 2\varphi(-h) + \varphi(0)}{h^2}\right| \le \frac{7}{12} h^2 |y^{IV}(\xi)| + \frac{1}{h^2}|y(-2h) - \varphi(-2h)| +$

$+ \frac{2}{h^2}|y(-h) + \varphi(-h)| + \frac{1}{h^2}|y(0) + \varphi(0)| \le \frac{7h^3M_3}{12} + \frac{4\varepsilon}{h^2}$

$\cancel{\frac{7hM_3}{12}} \le \frac{7h^2M_3}{12} + \frac{4\varepsilon}{h^2}$

$$\left(\frac{7}{12}h^2M_3 + \frac{4\varepsilon}{h^2}\right)'_h = \frac{7hM_3}{6} - \frac{8\varepsilon}{h^3} = 0$$

$$\frac{7M_3}{6}h^4 - 8\varepsilon = 0$$

$$h = \sqrt[4]{\frac{8\varepsilon \cdot 6}{7M_3}} \Rightarrow h = \sqrt[4]{\frac{48\varepsilon}{7M_3}}$$

Ответ: $h_{1\,min} = \sqrt[3]{\frac{\varepsilon}{M_3}}$, $h_{2\,min} = \sqrt[4]{\frac{48\varepsilon}{7M_3}}$;