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Домашняя работа №3

1) $I=\int\limits_0^1 e^{-x^2}dx$, $\varepsilon=10^{-4}$

$$\int\limits_a^b f(x)dx \approx \sum_{i=0}^{n} \frac{f(x_{i-1})+f(x_i)}{2}(x_{i-1}-x_i) = \frac{h}{2}\sum_{i=0}^{n}(f(x_{i-1})+$$

$$+f(x_i)) = \frac{h}{2}\left[f(a)+f(b)+2\sum_{i=1}^{n-1} f(x_i)\right] = h\cdot\frac{f(a)-f(b)}{2} + h\sum_{i=1}^{n-1} f(x_i)$$

Получаем оценку:

$$R_n[f] = I(f) - S_n = (b-a)\frac{h^2}{12} f''(\xi)$$

$$|f''(\xi)| \le M_2 \quad , M_2 = \max_{x\in[a,b]} f''(x).$$

$$R_n[f] \le \frac{(b-a)^3}{12}\cdot\frac{1}{n^2}\cdot M_2 \le \varepsilon$$

$$(e^{-x^2})'' \quad 4x^2\cdot e^{-x^2} - 2e^{-x^2}$$

Найдем ~~xxxxx~~ max $\Rightarrow 2e^{-x^2}(2x^2-1)=0$

$$2x^2=1$$

$$x=\pm\sqrt{\tfrac{1}{2}}$$

$$\left.\begin{aligned}
&x=-\sqrt{\tfrac{1}{2}} \Rightarrow 2e^{-\frac{1}{2}}-2e^{-\frac{1}{2}}=0\\
&x=\sqrt{\tfrac{1}{2}} \Rightarrow 2e^{-\frac{1}{2}}-2e^{-\frac{1}{2}}=0\\
&x=0 \Rightarrow 0-2\cdot e^0=-2\\
&x=1 \Rightarrow 4e^{-1}-2e^{-1}=\frac{2}{e}
\end{aligned}\right\} \Rightarrow \max = \frac{2}{e} = M_2$$

$$\frac{(1-0)^3}{12}\cdot\frac{1}{n^2}\cdot\frac{2}{e} \le 10^{-4}$$

$$n^2 \ge \frac{10^4}{6e}$$

$$n \ge \frac{10^2}{\sqrt{6e}} \approx 24{,}7$$

Ответ: $n \ge 24{,}7$

2) $\int_0^2 (x+1) f(x)\, dx \approx C_1 f(0) + C_2 f(x_2) + C_3 f(2)$

$P_0 = 1;\; P_1 = x;\; P_2 = x^2$

1) $f(x) = 1;\; \int_0^2 (x+1) f(x)\, dx = \int_0^2 (x+1)\, dx = \left(\frac{x^2}{2} + x\right)\Big|_0^2 = 2 + 2 =$

$= 4 = C_1 + C_2 + C_3$

2) $f(x) = x;\; \int_0^2 x(x+1)\, dx = \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_0^2 = C_2 x_2 + 2C_3 +$

$+ C_1 f(0) \Rightarrow C_2 x_2 + 2C_3 = \frac{14}{3}$

3) $f(x) = x^2;\; \int_0^2 x^2(x+1)\, dx = C_1 \cdot 0 + C_2 \cdot x_2^2 + C_3 \cdot 4 = \left(\frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_0^2 =$

$= \frac{20}{3} = C_2 x_2 + 4C_3$

$$\left.\begin{array}{l} C_1 + C_2 + C_3 = 4 \\ C_2 x_2 + 2C_3 = \frac{14}{3} \\ C_2 x_2^2 + 4C_3 = \frac{20}{3} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} C_1 = 4 - C_2 - C_3 \\ C_3 = \frac{14 - C_2 x_2}{6} \\ C_2 x_2^2 + \frac{28 - 6x_2 C_2}{3} = \frac{20}{3} \end{array}\right.$$

$$C_2 = \frac{-8}{3x_2^2 - 6x_2} = \frac{-8}{3x_2(x_2 - 2)}$$

$$\cancel{C_3 = 14 + \frac{16}{x_2 - 2}} \quad C_3 = \frac{14 + \frac{16}{x^2 - 2}}{6} = \frac{14x_2 - 12}{6(x_2 - 2)} = \frac{7x_2 - 6}{3(x_2 - 2)}$$

$$C_1 = 4 + \frac{8}{3x_2(x_2 - 2)} - \frac{7x_2 - 6}{3(x_2 - 2)} = \frac{12x_2(x_2 - 2) + 8 - 7x_2^2 + 6x^2}{3x_2(x_2 - 2)} =$$

$$= \frac{5x_2^2 - 18x_2 + 8}{3x_2(x_2 - 2)}$$

Ответ: $\frac{5x_2^2-18x_2+8}{3x_2(x_2-2)} f(0) + \frac{-8}{3x_2(x_2+2)} f(x_2) + \frac{7x_2-6}{3(x_2-2)} f(2)$

3) $I[f] = \int_{-1}^{0} f(x)\,dx$

$S_3[f] = C\sum_{i=1}^{3} f(x_i)$

$C = \frac{b-a}{n} = \frac{1}{3}$

$I = \frac{1}{3}(f(x_1) + f(x_2) + f(x_3))$

$P_0 = 1;\ P_2 = x;\ P_3 = x^2;\ P_4 = x^3$

$$\left.\begin{array}{ll} f(x)=1: & 1=1 \\ f(x)=x: & \frac{1}{3}(x_1+x_2+x_3) = -\frac{1}{2} \\ f(x)=x^2: & \frac{1}{3}(x_1^2+x_2^2+x_3^2) = \frac{1}{3} \\ f(x)=x^3: & \frac{1}{3}(x_1^3+x_2^3+x_3^3) = -\frac{1}{4} \end{array}\right\} \quad \begin{array}{l} \int_{-1}^{0} x\,dx = -\frac{1}{2} \\ \int_{-1}^{0} x^2\,dx = \frac{1}{3} \\ \int_{-1}^{0} x^3\,dx = -\frac{1}{4} \end{array}$$

$$\left\{\begin{array}{l} x_1+x_2+x_3 = -\frac{3}{2} \\ x_1^2+x_2^2+x_3^2 = 1 \\ x_1^3+x_2^3+x_3^3 = -\frac{3}{4} \end{array}\right.$$

Пусть $x_1 = -\frac{1}{2}$

$$\left\{\begin{array}{l} x_2 = -x_3 - 1 \\ 2x_3^2 + 2x_3 = -\frac{1}{4} \\ x_1^3 + x_2^3 + x_3^3 = -\frac{3}{4} \end{array}\right\} \Rightarrow$$

$\Rightarrow x_3^2 + x_3 + \frac{1}{8} = 0$

$D = 1 - \frac{1}{2} = \frac{1}{2}$

$x_3 = \frac{-1 \pm \sqrt{\frac{1}{2}}}{2}$

Пусть $x_3 = \frac{-1+\frac{1}{\sqrt{2}}}{2} = -\frac{1}{2} + \frac{1}{2\sqrt{2}} \Rightarrow$

$\Rightarrow x_2 = \frac{1}{2} - \frac{1}{2\sqrt{2}} - 1 = -\frac{1}{2} - \frac{1}{2\sqrt{2}}$

Ответ: $I[f] \approx \frac{1}{3}\left[f\left(-\frac{1}{2}\right) + f\left(-\frac{1}{2}+\frac{1}{2\sqrt{2}}\right) + f\left(-\frac{1}{2}-\frac{1}{2\sqrt{2}}\right)\right]$

4) $I[f] = \int_0^1 \sqrt{x} f(x)\,dx \approx S[f] = C_1 f(x_1) + C_2 f(x_2)$

~~[illegible]~~ $I_p[f] = \int_a^b p(x) f(x)\,dx,\ p(x) > 0$ –

~~[illegible]~~ – весовая функция : $\int_a^b p(x)\,dx < \infty$

$(f, g) = \int_a^b p(x) f(x) \cdot g(x)\,dx = \int_0^1 \sqrt{x} f(x) g x\,dx$

$P_0(x) = 1;\ P_1(x) = x:\ (q_2, P_0) = 0 \Rightarrow \int_0^1 \sqrt{x}(x^2 + ax + b)\,dx = 0$

$(q_2, P_1) = 0 \Rightarrow \int_0^1 \sqrt{x}(x^2 + ax + b)x\,dx = 0$

$\int_0^1 (x^{\frac{5}{2}} + ax^{\frac{3}{2}} + bx^{\frac{1}{2}})\,dx = \left(\frac{2}{7}x^{\frac{7}{2}} + \frac{2}{5}ax^{\frac{5}{2}} + \frac{2}{3}bx^{\frac{3}{2}}\right)\Big|_0^1 =$

$= \frac{2}{7} + \frac{2}{5}a + \frac{2}{3}b = \frac{70b + 42a + 30}{105} = 0$

$70b + 42a + 30 = 0$

$35b + 21a + 15 = 0$

$\int_0^1 (x^{\frac{7}{2}} + ax^{\frac{5}{2}} + bx^{\frac{3}{2}})\,dx = \left(\frac{2}{9}x^{\frac{9}{2}} + \frac{2}{7}x^{\frac{7}{2}}a + \frac{2}{5}bx^{\frac{5}{2}}\right)\Big|_0^1 =$

$= \frac{2}{9} + \frac{2}{7}a + \frac{2}{5}b = \frac{70 + 90a + 126b}{315} = 0$

$35 + 45a + 63b = 0$

$\begin{cases} 21a + 35b + 15 = 0 \\ 45a + 63b + 35 = 0 \end{cases} \Big| \cdot \frac{45}{21}$

$\begin{cases} 21a + 36b = -15 \\ -12b = -\frac{20}{7} \end{cases} \Rightarrow \begin{cases} a = -\frac{7}{63} \\ b = \frac{5}{21} \end{cases} \Rightarrow \begin{cases} a = -\frac{10}{9} \\ b = \frac{5}{21} \end{cases}$

$$q_2(x) = x^2 - \frac{10}{9}x + \frac{5}{21}$$

$$D = \frac{100}{81} - \frac{20}{21} = \frac{160}{567}$$

$$x_{1,2} = \frac{\frac{10}{9} \pm \sqrt{\frac{160}{567}}}{2} = \frac{5}{9} \pm \sqrt{\frac{40}{567}} = \frac{5}{9} \pm \frac{1}{9}\sqrt{\frac{40}{7}} = \frac{5}{9} \pm \frac{2}{9}\sqrt{\frac{10}{7}}$$

Веса квадратуры:

$$\left.\begin{array}{l} C_1 + C_2 = \int_0^1 \sqrt{x}\,dx = \frac{2}{3} \\ C_1 x_1 + C_2 x_2 = \int_0^1 x\sqrt{x}\,dx = \frac{2}{5} \end{array}\right\} \Leftrightarrow \begin{cases} C_1 = \frac{2}{3} - C_2 \\ \left(\frac{2}{3} - C_2\right)\left(\frac{5}{9} - \frac{2}{9}\sqrt{\frac{10}{7}}\right) + C_2\left(\frac{5}{9} + \frac{2}{9}\sqrt{\frac{10}{7}}\right) = \end{cases}$$

$$= \frac{2}{5}$$

$$\frac{10}{27} - \cancel{\frac{5}{9}C_2} - \frac{4}{27}\sqrt{\frac{10}{7}} + \frac{2}{9}C_2\sqrt{\frac{10}{7}} + \frac{5}{9}C_2 + \frac{2}{9}C_2\sqrt{\frac{10}{7}} = \frac{2}{5}$$

$$\frac{4}{9}C_2\sqrt{\frac{10}{7}} - \frac{4}{27}\sqrt{\frac{10}{7}} = \frac{4}{135}$$

$$C_2 - \frac{1}{3} = \frac{9}{135}\cdot\sqrt{\frac{7}{10}} \Rightarrow \underline{C_2 = \frac{\sqrt{7}}{15\sqrt{10}} + \frac{1}{3}}$$

$$C_1 = \frac{2}{3} - \frac{\sqrt{7}}{15\sqrt{10}} - \frac{1}{3} \Rightarrow \underline{C_1 = \frac{1}{3} - \frac{\sqrt{7}}{15\sqrt{10}}}$$

Ответ: $S[f] = \left(\frac{1}{3} - \frac{\sqrt{7}}{15\sqrt{10}}\right) f\left(\frac{5}{9} - \frac{2}{9}\sqrt{\frac{10}{7}}\right) + \left(\frac{\sqrt{7}}{15\sqrt{10}} + \frac{1}{3}\right) f\left(\frac{5}{9} + \frac{2}{9}\sqrt{\frac{10}{7}}\right)$

5) $I[f] = \int_{-1}^{1} f(x)\,dx \approx \frac{1}{4}\left[f(-1) + 3f\left(-\frac{1}{3}\right) + 3f\left(\frac{1}{3}\right) + f(1)\right] = S_4'[f]$

$f(x) \in C^4[-1;1]$

$$f(x) = f(x_0) + f'(x)(x-x_0) + \frac{f''(x)(x-x_0)^2}{2} + \frac{f'''(x)(x-x_0)^3}{6} + \frac{f^{IV}(x)(x-x_0)^4}{24} + \frac{f^{V}(x)(x-x_0)^5}{120} + O(h^6)$$

$$R[h] = \frac{b-a}{2880} \max_{x\in[-1;1]} |f^{IV}(x)| \cdot h^4 = \Big/ h^4 = \frac{(b-a)^4}{N^4} \Big/ = \frac{(b-a)^5}{2880 \cdot n^4} \cdot$$

$$\cdot \max_{x\in[-1;1]} |f^{IV}(x)|$$