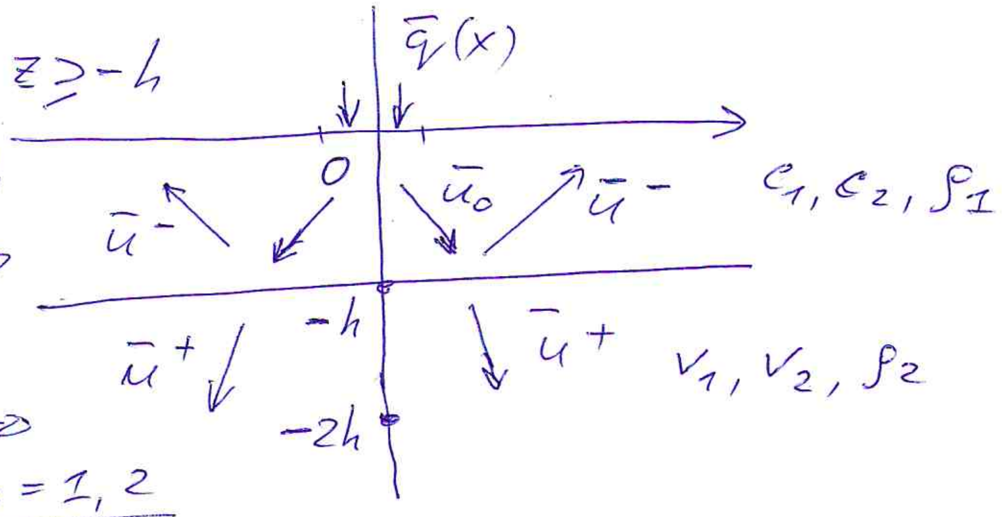
2022-10-30

Волна-Iго-2D   ФФП

$$\bar u = \begin{cases} \bar u_0 + \bar u^-, & z \geq -h \\ \bar u^+, & z < -h \end{cases}$$



$$\varkappa_n = \omega / c_n$$

$$k_n = \omega / v_n$$

$n = 1, 2$

1. Поле источника $\bar u_0$, $\bar X = (x, z)$

$$\bar u_0(x,z) = \frac{1}{2\pi}\int_\Gamma \bar{\mathcal{U}}_0(\alpha, z)\, e^{-i\alpha x}\, d\alpha = \bar u_1 + \bar u_2 = \bar u_p + \bar u_s$$

$$\bar{\mathcal{U}}_0 = K(\alpha, z)\, \bar Q(\alpha) = \cancel{\bar{\mathcal{U}}_{0,1}(\alpha)}) \sum_{n=1}^{2} \bar A_n(\alpha)\, e^{\sigma_n z}$$

K – м. Гр. для однородного полупр-ва со св-вами верхнего слоя (c_1, c_2, ρ_1)

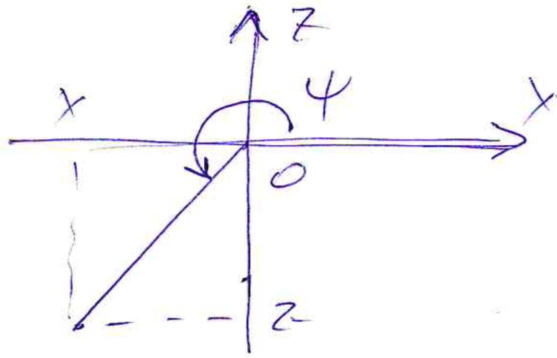
$\bar Q = \mathcal{F}_x[\bar q]$ – нагрузка (точечная).

$$\bar A_n = K_n \cdot \bar Q \text{ , где } K_n: \ K = \sum_{n=1}^{2} K_n(\alpha)\, e^{\sigma_n z}$$

$$\sigma_n = \sqrt{\alpha^2 - \varkappa_n^2} = \begin{cases} -i\sqrt{\varkappa_n^2 - \alpha^2} & \text{для } \alpha^2 < \varkappa_n^2 \\ \sqrt{\alpha^2 - \varkappa_n^2} > 0, & \text{для } \alpha^2 > \varkappa_n^2 \\ & \text{(вещ.)} \end{cases}$$

$$\hat\sigma_n = \sqrt{\varkappa_n^2 - \alpha^2} > 0$$

1.2 Асимптотика $\bar{u}_0$



$$\begin{cases} x = R\cos\psi \\ z = R\sin\psi < 0 \end{cases}$$

$$R = |\bar{x}| = \sqrt{x^2+z^2}\,, \quad \pi < \psi < 2\pi$$

$$\text{I} \quad \bar{u}_0 = \bar{u}_1 + \bar{u}_2 = \bar{u}_p + \bar{u}_s$$

$$\text{II} \quad \bar{u}_n = \frac{\varkappa_n |\sin\psi|}{\sqrt{2\pi i}}\, \bar{a}_n(\psi)\, e^{i\varkappa_n R} \Big/ \sqrt{\varkappa_n R}\left[1 + O\left(\frac{1}{\varkappa_n R}\right)\right]$$

$$\varkappa_n R \gg 1\,, \quad n = 1, 2$$

$$\bar{a}_n = \bar{A}_n(\alpha_n) = K_n(\alpha_n)\, \bar{Q}(\alpha_n)$$

$$\alpha_n = -\varkappa_n \cos\psi\,, \quad \pi < \psi < 2\pi$$

2. <u>Отражённое поле $\bar{u}^-(x,z)$</u>. (+ прошедшее $\bar{u}^+$)

$$\Big\| \; \bar{u}^-(x,z) = \frac{1}{2\pi}\int_\Gamma \bar{U}^-(\alpha,z)\, e^{-i\alpha x}\, d\alpha, \quad \underline{z \geq -h}$$

$$\bar{U}^- = (U^-, W^-)^T$$

$$\left| \begin{aligned} U^- &= t_1(-i\alpha)\, e^{-\sigma_1(z+h)} + t_2 \sigma_2\, e^{-\sigma_2(z+h)} \\ W^- &= t_1(-\sigma_1)\, e^{-\sigma_1(z+h)} + t_2(-i\alpha)\, e^{-\sigma_2(z+h)} \end{aligned} \right.$$

$$\Big\| \; \bar{u}^+(x,z) = \frac{1}{2\pi}\int_\Gamma U^+(\alpha,z)\, e^{-i\alpha x}\, d\alpha, \quad z \leq -h$$

$$\left| \begin{aligned} U^+ &= t_3(-i\alpha)\, e^{s_1(z+h)} + t_4(-s_2)\, e^{s_2(z+h)} \\ W^+ &= t_3 s_1\, e^{s_1(z+h)} + t_4(-i\alpha)\, e^{s_2(z+h)} \end{aligned} \right.$$

$$\sigma_n = \sqrt{\alpha^2 - \varkappa_n^2}, \quad s_n = \sqrt{\alpha^2 - k_n^2}, \quad n = 1, 2$$

$\bar{t} = (t_1, t_2, t_3, t_4)^T$ определяется из системы

$$\boxed{\Big\| \; A\bar{t} = \bar{f},} \qquad \bar{f} = \bar{f}_1 e_1 + \bar{f}_2 e_2, \quad \underline{e_n \overset{def}{=} e^{-\sigma_n h}}$$

Матрица A и пр. часть $\bar{f}$ получаются из усл. непрерывности $\bar{u}$ и $\bar{\tau}$ на $\underline{z=-h}$:

$$\left|\begin{array}{l} \bar{u}_0+\bar{u}^-=\bar{u}^+ \\ \bar{\tau}_0+\bar{\tau}^-=\tau^+ \end{array}\right. \Rightarrow \left\{\begin{array}{l} \bar{u}^- - \bar{u}^+ = -\bar{u}_0 \\ \bar{\tau}^- - \bar{\tau}^+ = -\bar{\tau}_0 \end{array}\right., \ \underline{z=-h}$$

$$\bar{\tau} = T\bar{u} = (\tau_x, \sigma_z),$$

$$\left|\begin{array}{l} \tau_x = \mu\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right) \\ \sigma_z = \lambda\dfrac{\partial u}{\partial x} + \underbrace{(\lambda+2\mu)}_{Dflm}\dfrac{\partial w}{\partial z} \end{array}\right.$$

В Фурье символах:

$$\left\{\begin{array}{l} U^- - U^+ = -U_0 \\ W^- - W^+ = -W_0 \\ T_x^- - T_x^+ = -T_{x,0} \\ S_z^- - S_z^+ = -S_{z,0} \end{array}\right., \ \underline{\text{при } z=-h}$$

$$\left[\begin{array}{l} T_x^- = \mu_1\left[2i\alpha\sigma_1 t_1 - (\alpha^2+\sigma_2^2)t_2\right] \\ S_z^- = \mu_1(\alpha^2+\sigma_2^2)t_1 + \overset{2\mu_1 i\alpha\sigma_2}{\cancel{2i\alpha\sigma_2(\lambda_1+\mu_1)}}\, t_2, \end{array}\right.$$

$$\underline{z=-h}$$

$$\left[\begin{aligned} T_x^+ &= \mu_2\left[-2i\alpha s_1 t_3 - (\alpha^2+s_2^2)t_4\right] \\ s_z^+ &= \mu_2(\alpha^2+s_2^2)t_3 - \cancel{2i\alpha s_2(\lambda_2+\mu_2)} \; 2\mu_2 i\alpha s_2 \, t_4 \end{aligned}\right.$$

$\underline{z=-h}$

$$A = \begin{bmatrix} -i\alpha & \sigma_2 & i\alpha & s_2 \\ -\sigma_1 & -i\alpha & -s_1 & i\alpha \\ 2\mu_1 i\alpha\sigma_1 & -\mu_1(\alpha^2+\sigma_2^2) & 2\mu_2 i\alpha s_1 & \mu_2(\alpha^2+s_2^2) \\ \mu_1(\alpha^2+\sigma_2^2) & 2\mu_1 i\alpha\sigma_2 & -\mu_2(\alpha^2+s_2^2) & 2\mu_2 i\alpha s_2 \end{bmatrix}$$

$$\| \; \bar f = -\begin{pmatrix} \bar u_0 \\ \bar{T}_0 \end{pmatrix}, \text{ при } \underline{z=-h}$$

$$\bar u_0(-h) = \begin{pmatrix} u_1 \\ w_1 \end{pmatrix} e_1 + \begin{pmatrix} u_2 \\ w_2 \end{pmatrix} e_2 \; ; \; \underline{e_n = e^{-\sigma_n h}}$$

$$\bar V_n = \begin{pmatrix} u_n \\ w_n \end{pmatrix} = K_n(\alpha,-h)\,\bar Q, \quad n=1,2$$

$$\bar T_0 = T\left(-i\alpha, \frac{\partial}{\partial z}\right)\bar u_0 = \sum_{n=1}^{2} T_n \cdot \bar V_n \cdot e_n$$

$$T_n = \begin{bmatrix} \mu_1\sigma_n & -\mu_1 i\alpha \\ -\lambda_1 i\alpha & (\lambda_1+2\mu_1)\sigma_n \end{bmatrix}$$

Соответственно $\bar{f} = \bar{f}_1 e_1 + \bar{f}_2 e_2 \Rightarrow$

$$\bar{f}_n = \begin{pmatrix} -\bar{V}_n \\ \hline -T_n \bar{V}_n \end{pmatrix}, \quad n = 1, 2$$

Или

$$\bar{f}_1 = \left(-u_1, -W_1, -\mu_1(b_1 u_1 - i\alpha W_1), \lambda_1 i\alpha u_1 - lm_1 b_1 W_1\right)^T$$

$$\bar{f}_2 = \left(-u_2, -W_2, -\mu_1(b_2 u_2 - iW_2), \lambda_1 i\alpha u_2 - lm_1 b_2 W_2\right)^T$$

$$lm_1 \overset{Df}{=} \lambda_1 + 2\mu_1$$

В соотв. со структурой $\bar{f}$

$$\bar{t} = \bar{t}_1 e_1 + \bar{t}_2 e_2, \text{ где } A\bar{t}_n = \bar{f}_n, \quad n = 1, 2$$

т.е. каждая константа t_j расщепляется на два слагаемых:

$$t_j = t_1^{(j)} e_1 + t_2^{(j)} e_2, \text{ где } t_n^{(j)}:$$

$$\bar{t}_n = \left(t_n^{(1)}, t_n^{(2)}, t_n^{(3)}, t_n^{(4)}\right), \quad n = 1, 2$$

решение $A\bar{t}_n = \bar{f}_n$

Или в полярных коорд. (R, φ):

$$\begin{cases} \theta_{11} = \hat{\sigma}_1 \sin\varphi - \alpha\cos\varphi \\ \theta_{12} = \hat{\sigma}_1 \sin\varphi + (\hat{\sigma}_2 - \hat{\sigma}_1)b - \alpha\cos\varphi \\ \theta_{21} = \hat{\sigma}_2 \sin\varphi + (\hat{\sigma}_1 - \hat{\sigma}_2)b - \alpha\cos\varphi \\ \theta_{22} = \hat{\sigma}_2 \sin\varphi - \alpha\cos\varphi \end{cases}$$

здесь $\underline{b = h/R}$

Стац. точки α_{nm} определяются из у-й

$$\| \theta'_{nm}(\alpha) = 0$$

$$\begin{cases} \theta'_{11} = -\dfrac{\alpha}{\hat{\sigma}_1}\sin\varphi - \cos\varphi = 0\ ; \quad \underline{\alpha_{11} = -x_1\cos\varphi} \\ \theta'_{12} = -\dfrac{\alpha}{\hat{\sigma}_1}\sin\varphi + \left(\dfrac{\alpha}{\hat{\sigma}_1} - \dfrac{\alpha}{\hat{\sigma}_2}\right)b - \cos\varphi = 0, \ - \underline{\text{числ.}} \\ \theta'_{21} = -\dfrac{\alpha}{\hat{\sigma}_2}\sin\varphi - \alpha b\left(\dfrac{1}{\hat{\sigma}_1} - \dfrac{1}{\hat{\sigma}_2}\right) - \cos\varphi = 0, \ ^{-\text{числ.}} \\ \theta'_{22} = -\dfrac{\alpha}{\hat{\sigma}_2}\sin\varphi - \cos\varphi = 0\ ; \quad \underline{\alpha_{22} = -x_2\cos\varphi} \end{cases}$$

Радикалы в т. $\alpha = \alpha_{nm}$

1) $n = m$ $\quad \hat{\sigma}_1(\alpha_{11}) = \varkappa_1 \sin\psi > 0$;

$\hat{\sigma}_2(\alpha_{22}) = \varkappa_2 \sin\psi > 0$

$$\Theta^{0}_{11} = \Theta_{11}(\alpha_{11}) = \varkappa_1$$

$$\Theta^{0}_{22} = \Theta_{22}(\alpha_{22}) = \varkappa_2$$

$$\Theta''_{nn} = -\sin\psi \left[\frac{1}{\hat{\sigma}_n}\left(1 - \frac{\alpha^2}{\hat{\sigma}_n^{2}}\right)\right], \quad \left|\; \begin{aligned} \alpha_n &= -\varkappa_n \cos\psi \\ \hat{\sigma}_n &= \varkappa_n \sin\psi \end{aligned}\right.$$

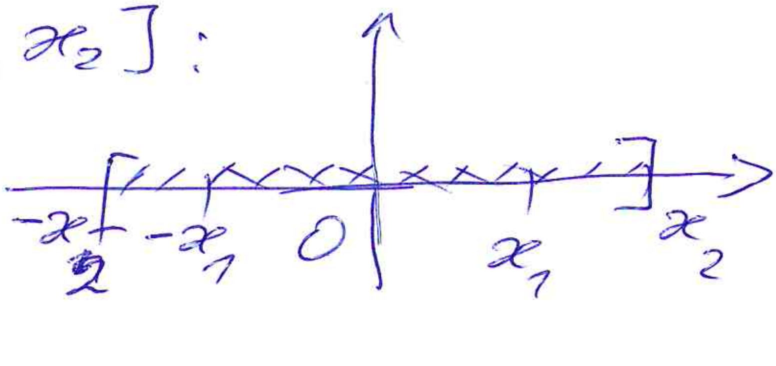
$$\Theta''_{nn}(\alpha_{nn}) = -\frac{1}{\varkappa_n \sin^2\psi} < 0 \Rightarrow \underline{\operatorname{sgn} \Theta''_{nn} = -1}$$

$$|\det \Theta''_{nn}|^{-1/2} = \sqrt{\varkappa_n}\, \sin\psi$$

2) $n \neq m$ (домножено на $\hat{\sigma}_1 \hat{\sigma}_2$):

$$\begin{cases} \Theta'_{12}: \ \alpha \hat{\sigma}_2 \sin\psi + \alpha b(\hat{\sigma}_1 - \hat{\sigma}_2) + \hat{\sigma}_1 \hat{\sigma}_2 \cos\psi = 0 \\ \Theta'_{21}: \ \alpha \hat{\sigma}_1 \sin\psi - \alpha b(\hat{\sigma}_1 - \hat{\sigma}_2) + \hat{\sigma}_1 \hat{\sigma}_2 \cos\psi = 0 \end{cases}$$

Корни ищем численно (Hamin) на интервале $[-\varkappa_2, \varkappa_2]$:



Если искомый корень лежит в интервалах $\varkappa_1 < |\alpha| < \varkappa_2$ (т.е. вне $|\alpha| < \varkappa_1$), то $\hat{\sigma}_1$ становится чисто мнимой $\Rightarrow$

$e^{i\hat{\sigma}_1 R} \to 0$ <u>экспоненциально</u> $\Rightarrow$

$\Rightarrow e^{i\Theta_{nm} R} \to 0 \Rightarrow$ такая стац. точка вклада не даёт

<u>Вывод:</u> для $n \neq m$ стац. точку ищем только на интервале $\underline{|\alpha| < \varkappa_1}$

$$\left[\begin{aligned} \Theta''_{12} &= -\sin\psi \frac{\hat{\sigma}_1^2 - \alpha^2}{\hat{\sigma}_1^3} + b\left[\frac{\hat{\sigma}_1^2 - \alpha^2}{\hat{\sigma}_1^3} - \frac{\hat{\sigma}_2^2 - \alpha^2}{\hat{\sigma}_2^3}\right] \\ \Theta''_{21} &= \underbrace{-\sin\psi \frac{\hat{\sigma}_2^2 - \alpha^2}{\hat{\sigma}_2^3}}_{(\mathrm{I})} - b\underbrace{\left[\ldots\ldots\right]}_{(\mathrm{II})} \end{aligned}\right.$$

$$\hat{\sigma}_n^2 - \alpha^2 = \varkappa_n^2$$

$$[\ldots] = \frac{\varkappa_1^2}{\hat{\sigma}_1^3} - \frac{\varkappa_2^2}{\hat{\sigma}_2^3}$$

При $b = h/R \to 0$ $\Theta''_{nm} \to \Theta''_{nn} \Rightarrow$

как правило $\operatorname{sgn}\Theta''_{21} = \operatorname{sgn}\Theta''_{12} = \underline{-1}$, но знак может поменяться: второе слаг. станет по модулю больше первого.

Асимптотика

1) $\underline{n=m}$, $\bar{u}_{11}=\bar{u}_{pp}$, $\bar{u}_{22}=\bar{u}_{ss}$

$$\bar{u}_{nn}=\frac{\varkappa_n \sin\varphi}{\sqrt{2\pi i}}\,\bar{a}_{nn}(\varphi)\,e^{i\varkappa_n R}/\sqrt{\varkappa_n R}\,\times$$

$$\times\left[1+O(1/\varkappa_n R)\right]$$

$$\varkappa_n R \gg 1,\quad n=1,2,\quad \underline{0<\varphi<\pi}$$

$$\bar{a}_{nn}=\bar{A}_{nn}(\alpha_{nn}),\quad \alpha_{nn}=-\varkappa_n \cos\varphi$$

$$\bar{A}_{nn}=\bar{V}_n(\alpha)\,t_n^{(n)}(\alpha)$$

2) $\underline{n\neq m}$, $\bar{u}_{12}=\bar{u}_{ps}$, $\bar{u}_{21}=\bar{u}_{sp}$

$$\bar{u}_{nm}=\frac{\sqrt{\varkappa_n}}{\sqrt{2\pi i\left(-\theta''_{nm}(\alpha_{nm})\right)}}\cdot\bar{a}_{nm}(\varphi)\,e^{i\varkappa_{nm}R}\times$$

$$\varkappa_{nm}=\theta_{nm}(\alpha_{nm}),\qquad \frac{1}{\sqrt{\varkappa_n R}}\left[1+O(1/\varkappa_n R)\right]$$

$$\bar{a}_{nm}=\begin{cases}\bar{A}_{nm}(\alpha_{nm}), & \text{где } \alpha_{nm}\text{ — численно},\ \alpha_{nm}\in[-\varkappa_1,\varkappa_1]\\ 0, & |\alpha_{nm}|>\varkappa_1\end{cases}$$