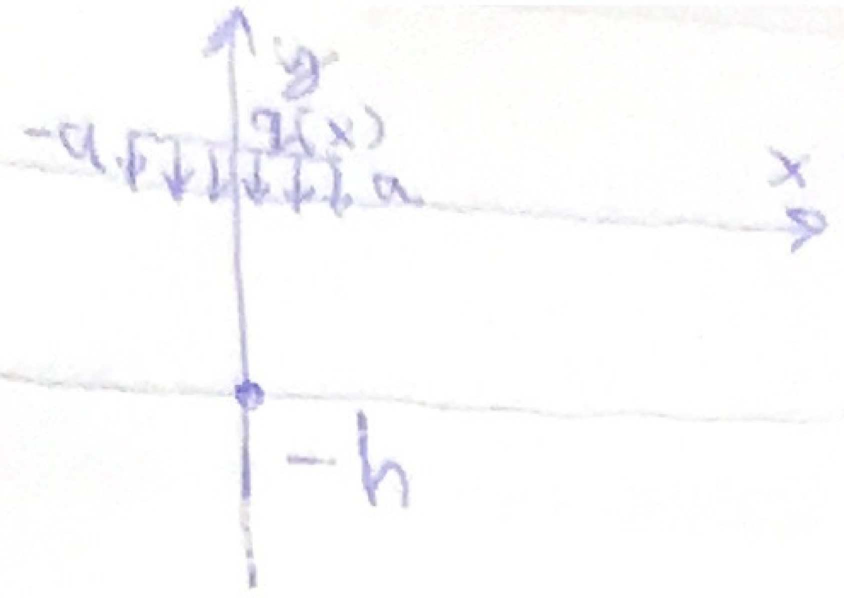

$$\begin{cases} c_{55}\dfrac{\partial^2 u}{\partial x^2}+c_{44}\dfrac{\partial^2 u}{\partial z^2}+e_{15}\dfrac{\partial^2 \varphi}{\partial x^2}+e_{24}\dfrac{\partial^2 \varphi}{\partial z^2}=\rho\ddot{u} \\ -\varepsilon_{11}\dfrac{\partial^2 \varphi}{\partial x^2}-\varepsilon_{22}\dfrac{\partial^2 \varphi}{\partial z^2}+e_{15}\dfrac{\partial^2 u}{\partial x^2}+e_{24}\dfrac{\partial^2 u}{\partial z^2}=0 \end{cases} \quad (1$$

$$c_{44}\left.\frac{\partial u}{\partial z}\right|_{z=0}+e_{24}\left.\frac{\partial \varphi}{\partial z}\right|_{z=0}=\tilde{q}(x), \qquad \tilde{q}(x)=\begin{cases} q(x), & x\in[-a;a]; \\ 0, & \text{иначе.}\end{cases}$$

$$c_{44}\left.\frac{\partial u}{\partial z}\right|_{z=-h}+e_{24}\left.\frac{\partial \varphi}{\partial z}\right|_{z=-h}=0$$

$$\frac{-\left.\dfrac{\partial\varphi}{\partial x}\right|_{z=0}}{-\varepsilon_{22}\left.\dfrac{\partial\varphi}{\partial z}\right|_{z=0}+e_{24}\left.\dfrac{\partial u}{\partial z}\right|_{z=0}}=\frac{-i}{\varepsilon_0} \qquad \frac{-\left.\dfrac{\partial\varphi}{\partial x}\right|_{z=-h}}{-\varepsilon_{22}\left.\dfrac{\partial\varphi}{\partial z}\right|_{z=-h}+e_{24}\left.\dfrac{\partial u}{\partial z}\right|_{z=-h}}=\frac{i}{\varepsilon_0}$$

Пусть $F_x(u(x,z))=\int\limits_{-\infty}^{+\infty}u(x,z)e^{i\alpha x}dx=U(\alpha,z)$; $F_x(\varphi(x,z))=\Phi(\alpha,z)$

$F_x^{-1}(U(\alpha,z))=\int\limits_{\Gamma}U(\alpha,z)e^{-i\alpha x}d\alpha=u(x,z)$; $F_x^{-1}(\Phi)=\varphi$

по св-вам преобр. Фурье $F\left(\frac{\partial u}{\partial x}\right)=-i\alpha U$; $F\left(\frac{\partial \varphi}{\partial x}\right)=-i\alpha\Phi$;

$F\left(\frac{\partial^2 u}{\partial x^2}\right)=-\alpha^2 U$; $F\left(\frac{\partial^2 \varphi}{\partial x^2}\right)=-\alpha^2\Phi$; обозначим $F\left(\frac{\partial u}{\partial z}\right)=U'$;

$F\left(\frac{\partial \varphi}{\partial z}\right)=\Phi'$; $F\left(\frac{\partial^2 u}{\partial z^2}\right)=U''$; $F\left(\frac{\partial^2 \varphi}{\partial z^2}\right)=\Phi''$

Поскольку задача гармоническая $\rho\ddot{u}=-\omega^2\rho u$

Таким образом уравнения

$$c_{55}(-\alpha^2 U)+c_{44}U''+e_{15}(-\alpha^2\Phi)+e_{24}\Phi''=-\omega^2\rho U$$

$$-\varepsilon_{11}(-\alpha^2\Phi)-\varepsilon_{22}\Phi''+e_{15}(-\alpha^2 U)+e_{24}U''=0$$

гран. усл.

$$\frac{i\alpha\Phi|_{z=0}}{-\varepsilon_{22}\Phi'|_{z=0}+e_{24}U'|_{z=0}}=\frac{-i}{\varepsilon_0} \qquad \frac{i\alpha\Phi|_{z=-h}}{-\varepsilon_{22}\Phi'|_{z=-h}+e_{24}U'|_{z=-h}}=\frac{i}{\varepsilon_0}$$

$$c_{44}U'|_{z=0}+e_{24}\Phi'|_{z=0}=Q(\alpha); \qquad c_{44}U'|_{z=-h}+e_{24}\Phi'|_{z=-h}=0$$

$$(\omega^2\rho - c_{55}\alpha^2)U + (-\alpha^2 e_{15})\varphi = -c_{44}U'' - e_{24}\varphi'' \quad \Big| + \frac{e_{24}}{\varepsilon_{22}}\ \mathrm{II}$$

$$-\alpha^2 e_{15}\, U + \varepsilon_{11}\alpha^2 \varphi = -e_{24}U'' + \varepsilon_{22}\varphi'' \quad \Big| - \frac{e_{24}}{c_{44}}\ \mathrm{I}$$

(2

$$\left(\omega^2\rho - c_{55}\alpha^2 - \alpha^2 e_{15}\cdot\frac{e_{24}}{\varepsilon_{22}}\right)U + \left(-\alpha^2 e_{15} + \varepsilon_{11}\alpha^2\cdot\frac{e_{24}}{\varepsilon_{22}}\right)\varphi = \left(-c_{44} - e_{24}\cdot\frac{e_{24}}{\varepsilon_{22}}\right)U'' + \left(-e_{24} + \varepsilon_{22}\cdot\frac{e_{24}}{\varepsilon_{22}}\right)\varphi''$$

$$\left(-\alpha^2 e_{15} - \frac{e_{24}}{c_{44}}(\omega^2\rho - c_{55}\alpha^2)\right)U + \left(\varepsilon_{11}\alpha^2 + \frac{e_{24}}{c_{44}}\alpha^2 e_{15}\right)\varphi = \left(-e_{24} + \frac{e_{24}}{c_{44}}\cdot c_{44}\right)U'' + \left(\varepsilon_{22} + \frac{e_{24}}{c_{44}}\cdot e_{24}\right)\varphi''$$

$$\frac{\left(\omega^2\rho - c_{55}\alpha^2 - \frac{\alpha^2 e_{15} e_{24}}{\varepsilon_{22}}\right)U}{-c_{44} - \frac{e_{24}^2}{\varepsilon_{22}}} + \frac{-\alpha^2 e_{15} + \frac{\varepsilon_{11}\alpha^2 e_{24}}{\varepsilon_{22}}}{-c_{44} - \frac{e_{24}^2}{\varepsilon_{22}}}\varphi = U''$$

$$\frac{-\alpha^2 e_{15} - \frac{e_{24}}{c_{44}}(\omega^2\rho - c_{55}\alpha^2)}{\varepsilon_{22} + \frac{e_{24}^2}{c_{44}}}U + \frac{\varepsilon_{11}\alpha^2 + \frac{e_{24}}{c_{44}}\alpha^2 e_{15}}{\varepsilon_{22} + \frac{e_{24}^2}{c_{44}}}\varphi = \varphi''$$

умножим числитель и знам. в первом на $-\varepsilon_{22}$, во втором на c_{44}

$$\frac{c_{55}\alpha^2\varepsilon_{22} + \alpha^2 e_{15}e_{24} - \varepsilon_{22}\omega^2\rho}{\varepsilon_{22}c_{44} + e_{24}^2}U + \frac{\alpha^2 e_{15}\varepsilon_{22} - \varepsilon_{11}\alpha^2 e_{24}}{\varepsilon_{22}c_{44} + e_{24}^2}\varphi = U''$$

$$\frac{-\alpha^2 e_{15}c_{44} - e_{24}(\omega^2\rho - c_{55}\alpha^2)}{\varepsilon_{22}c_{44} + e_{24}^2}U + \frac{c_{44}\varepsilon_{11}\alpha^2 + e_{24}\alpha^2 e_{15}}{\varepsilon_{22}c_{44} + e_{24}^2}\varphi = \varphi''$$

Пусть $\bar{X} = \begin{pmatrix} U \\ \varphi \\ U' \\ \varphi' \end{pmatrix}$, тогда $A\bar{x} = \frac{\partial \bar{x}}{\partial z}$, где

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 0 & 0 \end{pmatrix}$$

$$a_1 = \frac{c_{55}\alpha^2\varepsilon_{22} + \alpha^2 e_{15}e_{24} - \varepsilon_{22}\omega^2\rho}{a_0}$$

$$a_2 = (\alpha^2 e_{15}\varepsilon_{22} - \varepsilon_{11}\alpha^2 e_{24})/a_0$$

$$a_3 = (c_{55}\alpha^2 e_{24} - e_{24}\omega^2\rho - \alpha^2 e_{15}c_{44})/a_0$$

$$a_4 = (\varepsilon_{11}c_{44}\alpha^2 + e_{24}\alpha^2 e_{15})/a_0$$

$$a_0 = \varepsilon_{22}c_{44} + e_{24}^2$$

Найдем собств. знач. A

$$\begin{vmatrix} -\gamma & 0 & 1 & 0 \\ 0 & -\gamma & 0 & 1 \\ a_1 & a_2 & -\gamma & 0 \\ a_3 & a_4 & 0 & -\gamma \end{vmatrix} = (-\gamma)\cdot(-1)^{1+1}\begin{vmatrix} -\gamma & 0 & 1 \\ a_2 & -\gamma & 0 \\ a_4 & 0 & -\gamma \end{vmatrix} + 1\cdot(-1)^{1+3}\begin{vmatrix} 0 & -\gamma & 1 \\ a_1 & a_2 & 0 \\ a_3 & a_4 & -\gamma \end{vmatrix} =$$

$$= -\gamma(-\gamma^3 + a_4\gamma) + a_1a_4 - a_2a_3 - a_1\gamma^2 =$$

$$= \gamma^4 - a_4\gamma^2 + a_1a_4 - a_2a_3 - a_1\gamma^2 = \gamma^4 - (a_1+a_4)\gamma^2 + a_1a_4 - a_2a_3$$

Т. о. $\sigma_1 = \sqrt{r_1}$; $-\sigma_1$; $\sigma_2 = \sqrt{r_2}$; $-\sigma_2$ – корни, где

$$r_1 = \frac{a_1+a_4-\sqrt{D}}{2}; \quad r_2 = \frac{a_1+a_4+\sqrt{D}}{2}; \quad D = (a_1+a_4)^2 - 4(a_1a_4 - a_2a_3)$$

Найдем собств. векторы A

$$\begin{pmatrix} -\gamma & 0 & 1 & 0 \\ 0 & -\gamma & 0 & 1 \\ a_1 & a_2 & -\gamma & 0 \\ a_3 & a_4 & 0 & -\gamma \end{pmatrix} \begin{matrix} \\ \\ \gamma\,\mathrm{III} + a_1\,\mathrm{I} \\ \gamma\,\mathrm{III} + a_3\,\mathrm{I} \end{matrix} = \begin{pmatrix} -\gamma & 0 & 1 & 0 \\ 0 & -\gamma & 0 & 1 \\ 0 & \gamma a_2 & a_1-\gamma^2 & 0 \\ 0 & \gamma a_4 & a_3 & -\gamma^2 \end{pmatrix} \begin{matrix} \\ \\ +a_2\,\mathrm{II} \\ +a_4\,\mathrm{II} \end{matrix} =$$

$$= \begin{pmatrix} -\gamma & 0 & 1 & 0 \\ 0 & -\gamma & 0 & 1 \\ 0 & 0 & a_1-\gamma^2 & a_2 \\ 0 & 0 & a_3 & a_4-\gamma^2 \end{pmatrix} \begin{matrix} \\ \\ \\ \mathrm{IV}(a_1-\gamma^2) - a_3\,\mathrm{III} \end{matrix} = \begin{pmatrix} -\gamma & 0 & 1 & 0 \\ 0 & -\gamma & 0 & 1 \\ 0 & 0 & a_1-\gamma^2 & a_2 \\ 0 & 0 & 0 & 0^* \end{pmatrix}$$

$^* - (a_4-\gamma^2)(a_1-\gamma^2) - a_2a_3 = a_1a_4 - a_4\gamma^2 - \gamma^2a_1 + \gamma^4 - a_2a_3 = \gamma^4 - (a_1+a_4)\gamma^2 + a_1a_4 - a_2a_3$

Таким образом

$$\begin{cases} m_1 = \frac{1}{\gamma}m_3 \\ m_2 = \frac{1}{\gamma}m_4 \\ m_3 = \frac{a_2}{\gamma^2-a_1}m_4 \end{cases} \Rightarrow \begin{cases} m_1 = \frac{1}{\gamma}\frac{a_2}{\gamma^2-a_1}m_4 \\ m_2 = \frac{1}{\gamma}m_4 \\ m_3 = \frac{a_2}{\gamma^2-a_1}m_4 \end{cases}$$

Пусть $m_4 = \gamma\frac{\gamma^2-a_1}{a_2}$ $\Rightarrow \begin{cases} m_1 = 1 \\ m_2 = \frac{\gamma^2-a_1}{a_2} \\ m_3 = \gamma \end{cases}$

$$U = t_1e^{\sigma_1 z} + t_2e^{-\sigma_1 z} + t_3e^{\sigma_2 z} + t_4e^{-\sigma_2 z}$$

$$\varphi = t_1\frac{\sigma_1^2-a_1}{a_2}e^{\sigma_1 z} + t_2\frac{\sigma_1^2-a_1}{a_2}e^{-\sigma_1 z} + t_3\frac{\sigma_2^2-a_1}{a_2}e^{\sigma_2 z} + t_4\frac{\sigma_2^2-a_1}{a_2}e^{-\sigma_2 z}$$

$$U' = t_1\sigma_1e^{\sigma_1 z} + t_2(-\sigma_1)e^{-\sigma_1 z} + t_3\sigma_2e^{\sigma_2 z} + t_4(-\sigma_2)e^{-\sigma_2 z}$$

$$\varphi' = t_1p_1\sigma_1e^{\sigma_1 z} + t_2p_1(-\sigma_1)e^{-\sigma_1 z} + t_3p_2\sigma_2e^{\sigma_2 z} + t_4p_2(-\sigma_2)e^{-\sigma_2 z}$$

Матрица гран. условий

$$\frac{\alpha\varphi|_{z=0}}{-\varepsilon_{22}\varphi'|_{z=0}+e_{24}U'|_{z=0}}=\frac{-1}{\varepsilon_0};\quad \varepsilon_{22}\varphi'|_{z=0}-e_{24}U'|_{z=0}=\varepsilon_0\alpha\varphi|_{z=0}$$

$$\varepsilon_{22}\varphi'|_{z=0}-e_{24}U'|_{z=0}-\varepsilon_0\alpha\varphi|_{z=0}=0$$

$$\frac{\alpha\varphi|_{z=-h}}{-\varepsilon_{22}\varphi'|_{z=-h}+e_{24}U'|_{z=-h}}=\frac{1}{\varepsilon_0};\quad -\varepsilon_{22}\varphi'|_{z=-h}+e_{24}U'|_{z=-h}=\varepsilon_0\alpha\varphi|_{z=-h}$$

$$\varepsilon_{22}\varphi'|_{z=-h}-e_{24}U'|_{z=-h}+\varepsilon_0\alpha\varphi|_{z=-h}=0$$

$$t_1(c_{44}\sigma_1+e_{24}p_1\sigma_1)+t_2(c_{44}(-\sigma_1)+e_{24}p_1(-\sigma_1))+t_3(c_{44}\sigma_2+e_{24}p_2\sigma_2)+t_4(c_{44}(-\sigma_2)+e_{24}p_2(-\sigma_2$$

Т.о.,
$$t_1b_1+t_2(-b_1)+t_3b_2+t_4(-b_2)=0$$
$$t_1b_1e^{-\sigma_1h}+t_2(-b_1)e^{\sigma_1h}+t_3b_2e^{-\sigma_2h}+t_4(-b_2)e^{\sigma_2h}=0$$

$$b_1=c_{44}\sigma_1+e_{24}p_1\sigma_1;\quad b_2=c_{44}\sigma_2+e_{24}p_2\sigma_2$$

$$t_1(\varepsilon_{22}p_1\sigma_1-e_{24}\sigma_1-\varepsilon_0\alpha p_1)+t_2(\varepsilon_{22}p_1(-\sigma_1)-e_{24}(-\sigma_1)-\varepsilon_0\alpha p_1)+$$
$$+t_3(\varepsilon_{22}p_2\sigma_2-e_{24}\sigma_2-\varepsilon_0\alpha p_2)+t_4(\varepsilon_{22}p_2(-\sigma_2)-e_{24}(-\sigma_2)-\varepsilon_0\alpha p_2)=0$$

$$A=\begin{pmatrix} b_1 & -b_1 & b_2 & -b_2\\ b_1e^{-\sigma_1h} & -b_1e^{\sigma_1h} & b_2e^{-\sigma_2h} & -b_2e^{\sigma_2h}\\ c_1 & c_2 & c_3 & c_4\\ d_1e^{-\sigma_1h} & d_2e^{\sigma_1h} & d_3e^{-\sigma_2h} & d_4e^{\sigma_2h}\end{pmatrix}$$

$$c_1=\varepsilon_{22}p_1\sigma_1-e_{24}\sigma_1-\varepsilon_0\alpha p_1;\quad c_2=\varepsilon_{22}p_1(-\sigma_1)-e_{24}(-\sigma_1)-\varepsilon_0\alpha p_1;$$
$$c_3=\varepsilon_{22}p_2\sigma_2-e_{24}\sigma_2-\varepsilon_0\alpha p_2;\quad c_4=\varepsilon_{22}p_2(-\sigma_2)-e_{24}(-\sigma_2)-\varepsilon_0\alpha p_2;$$
$$d_1=\varepsilon_{22}p_1\sigma_1-e_{24}\sigma_1+\varepsilon_0\alpha p_1;\quad d_2=\varepsilon_{22}p_1(-\sigma_1)-e_{24}(-\sigma_1)+\varepsilon_0\alpha p_1;$$
$$d_3=\varepsilon_{22}p_2\sigma_2-e_{24}\sigma_2+\varepsilon_0\alpha p_2;\quad d_4=\varepsilon_{22}p_2(-\sigma_2)-e_{24}(-\sigma_2)+\varepsilon_0\alpha p_2.$$

ФДП 2023-11-14

$$U(\alpha, z) = t_1 e^{\sigma_1 z} + t_2 e^{-\sigma_1 z} + t_3 e^{\sigma_2 z} + t_4 e^{-\sigma_2 z}$$

$$\varphi(\alpha, z) = t_1 p_1 e^{\sigma_1 z} + t_2 p_1 e^{-\sigma_1 z} + t_3 p_2 e^{\sigma_2 z} + t_4 p_2 e^{-\sigma_2 z}$$

$$U'(\alpha, z) = t_1 \sigma_1 e^{\sigma_1 z} + t_2(-\sigma_1) e^{-\sigma_1 z} + t_3 \sigma_2 e^{\sigma_2 z} + t_4(-\sigma_2) e^{-\sigma_2 z}$$

$$\varphi'(\alpha, z) = t_1 p_1 \sigma_1 e^{\sigma_1 z} + t_2 p_1(-\sigma_1) e^{-\sigma_1 z} + t_3 p_2 \sigma_2 e^{\sigma_2 z} + t_4 p_2(-\sigma_2) e^{-\sigma_2 z}$$

где $(t_1, t_2, t_3, t_4)^T = \bar{t}$ – решение системы, $A\bar{t} = \bar{Q}$, где

$\bar{Q} = (0; 0; 0; Q(\alpha))^T$ $\quad Q(\alpha) =$

$$A = \begin{pmatrix} b_1 & -b_1 & b_2 & -b_2 \\ b_1 e^{-\sigma_1 h} & -b_1 e^{\sigma_1 h} & b_2 e^{-\sigma_2 h} & -b_2 e^{\sigma_2 h} \\ c_1 & c_2 & c_3 & c_4 \\ d_1 e^{-\sigma_1 h} & d_2 e^{\sigma_1 h} & d_3 e^{-\sigma_2 h} & d_4 e^{\sigma_2 h} \end{pmatrix}$$

$b_1 = c_{44} \sigma_1 + e_{24} p_1 \sigma_1$; $\quad b_2 = c_{44} \sigma_2 + e_{24} p_2 \sigma_2$

$c_1 = \varepsilon_{22} p_1 \sigma_1 - e_{24} \sigma_1 - \varepsilon_0 \alpha p_1$; $\quad c_2 = \varepsilon_{22} p_1(-\sigma_1) - e_{24}(-\sigma_1) - \varepsilon_0 \alpha p_1$;

$c_3 = \varepsilon_{22} p_2 \sigma_2 - e_{24} \sigma_2 - \varepsilon_0 \alpha p_2$; $\quad c_4 = \varepsilon_{22} p_2(-\sigma_2) - e_{24}(-\sigma_2) - \varepsilon_0 \alpha p_2$;

$d_1 = \varepsilon_{22} p_1 \sigma_1 - e_{24} \sigma_1 + \varepsilon_0 \alpha p_1$; $\quad d_2 = \varepsilon_{22} p_1(-\sigma_1) - e_{24}(-\sigma_1) + \varepsilon_0 \alpha p_1$;

$d_3 = \varepsilon_{22} p_2 \sigma_2 - e_{24} \sigma_2 + \varepsilon_0 \alpha p_2$; $\quad d_4 = \varepsilon_{22} p_2(-\sigma_2) - e_{24}(-\sigma_2) + \varepsilon_0 \alpha p_2$.

$p_1 = \dfrac{\sigma_1^2 - a_1}{a_2}$; $\quad p_2 = \dfrac{\sigma_2^2 - a_1}{a_2}$

$\sigma_1 = \sqrt{r_1}$; $\sigma_2 = \sqrt{r_2}$; $r_1 = \dfrac{a_1 + a_4 - \sqrt{D}}{2}$; $r_2 = \dfrac{a_1 + a_4 + \sqrt{D}}{2}$; $D = (a_1 + a_4)^2 - 4(a_1 a_4 - a_2 a_3)$

$a_1 = (c_{55} \alpha^2 \varepsilon_{22} + \alpha^2 e_{15} e_{24} - \varepsilon_{22} \omega^2 \rho)/a_0$; $\quad a_2 = (\alpha^2 e_{15} \varepsilon_{22} - \varepsilon_{11} \alpha^2 e_{24})/a_0$;

$a_3 = (c_{55} \alpha^2 e_{24} - e_{24} \omega^2 \rho - \alpha^2 e_{15} c_{44})/a_0$; $\quad a_4 = (\varepsilon_{11} c_{44} \alpha^2 + e_{24} \alpha^2 e_{15})/a_0$

$a_0 = \varepsilon_{22} c_{44} + e_{24}^2$