UNIVERSIDAD DE SANTIAGO DE CHILE

CIRAS

Hydrodynamics

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Introduction

Hydrodynamics, like thermodynamics, addresses systems with an enormous number of degrees-of-freedom because they are made of many particles whose interaction with each other manifests at large scale (e.g. through the concept of temperature which is a very simplified way to represent the underlying velocity distribution of particles). These systems are called complex systems and cannot be satisfyingly understood through the study of their individual components, either because of practical limitations (e.g. it is impossible to study the motion of 10^{23} particles) or because they are chaotic. In hydrodynamics, this transition to chaos will manifest through the notion of laminar and turbulent flows that we quantify with the Reynolds number (section 3.1.3). Complex systems exhibit specific properties such as non-linearity. It can be defined in different ways:

- Mathematically, an equation is linear if and only if any linear combination of two solutions is also a solution, that is to say:

$$\begin{aligned}&\text{Equation } (\mathcal{E}) \text{ is linear} \iff ...\\&... \left(f_1, f_2 \text{ solutions of } (\mathcal{E}) \implies \forall\, (\alpha_1, \alpha_2)\ \alpha_1 f_1 + \alpha_2 f_2 \text{ is solution of } (\mathcal{E}) \right)\end{aligned} \tag{1}$$

  In linear algebra for instance, you saw that vector spaces were a powerful tool to find the general expression of all solution of a linear equation using basis.

- From an epistemological point-of-view, the notion of linearity is illustrated by René Descartes' approach to solve problems: divide each difficulty into as many parts as is feasible and necessary to resolve it. Then, the combination of these partial solutions tells you about the global solution you never had to find directly. This method does not apply to complex systems and alternative strategies have been devised by philosophers of science over the past decades (e.g. E. Morin's Complex Thought).

Lane-Emden equation

When we studied the internal structure of stars, we saw that when the pressure could be directly deduced from the mass density through a power-law (i.e. when the flow behaves as a polytrope), the mass density profile obeys the Lane-Emden equation:

$$\frac{1}{\xi^2}\mathrm{d}_\xi \left(\xi^2 \mathrm{d}_\xi w\right) = -w^n \tag{2}$$

where n is the polytropic index, ξ is the dimensionless radius and w is the dimensionless mass density, which is a function of ξ. It is an ordinary differential equation (a.k.a. ODE) since in the assumption of a steady and spherically-symmetric star, variables depend only on one coordinate (either the distance r to the stellar center or the mass m encapsulated in a sphere of radius r). Then, the derivatives are exact ($\mathrm{d}_\xi w$) and not partial ($\partial_\xi w$). Otherwise, it would be a partial differential equation (a.k.a. PDE), like the Navier-Stokes equation in the general case. Is the Lane-Emden equation linear?

Therefore, in a complex system, it is generally not obvious whether the solution we find are the only ones and the relevant ones. As a matter of fact, the very existence of solutions to the central equation of hydrodynamics, the Navier-Stokes equation (section 3.1.3), is an unsolved problem whose resolution would make you madly rich.

XXX many qualitative statements, odm estimates

Contents

Chapter 1

Hydrostatics

1.1 Pressure

1.1.1 Definition

Experimentally, we see that the solid walls surrounding and/or immersed in any fluid experience a force perpendicular to the surface (e.g. an inflated balloon, Fig.XXX).

Surface and volume forces

In hydrodynamics, the forces which exert onto a fluid particle are twofold:

- Intrinsically volumetric forces, which exert onto the whole volume of the fluid particle such as the non-inertial pseudo-forces (centrifugal and Coriolis), the Lorentz force (if the fluid is electrically charged), or more simply, the weight per unit volume $\mathbf{f_{grav}} = \rho\,\mathbf{g}$ in a uniform gravitational field $\mathbf{g}$, with ρ the mass density.

  > Alternatively, we can express the weigth as a force per unit mass:
  >
  > $$\mathbf{f_{grav}} = \mathbf{g} \tag{1.1}$$

- Surface forces, which exert only on the surface of the fluid particle. For instance, friction is a surface force. For fluids, we usually call "pressure" the forces normal to the surface and "viscosity" those which are tangential to the surface (see section 3.1.2). Surface forces like pressure (section 1.1.1) and viscosity (section 3.1.3) can be reformulated as volume forces, provided we work with an infinitesimal volume.

Surface expression

Let $\mathrm{d}\,\mathbf{S}$ be an infinitesimal surface vector at the surface of an elementary volume immersed in a fluid (Fig. 1.1). Then, the elementary volume experiences a force $\mathrm{d}\,\mathbf{F}$ exerted by the surrounding fluid given by:

$$\mathrm{d}\,\mathbf{F} = -P\,\mathrm{d}\,\mathbf{S} \tag{1.2}$$

where P is the (isotropic) pressure at the center of the infinitesimal surface.

> Any volume is surrounded by a close surface and, by convention, the infinitesimal surface vector is always oriented outwards.

To compute the resultant of pressure forces onto a macroscopic surface, we decompose the infinitesimal surface vector $\mathrm{d}\,\mathbf{S}$ in its components and integrate over the surface (see exercise "The salad bowl" below).

> Pressure units are diverse: the Pascal (Pa, the MKSA unit), the bar (ba), the atmosphere (atm), the barye (Ba, the CGS unit)... They are linked by:
>
> $$1\mathrm{bar} \sim 1\mathrm{atm} \sim 10^5\mathrm{Pa} \sim 10^6\mathrm{Ba} \tag{1.3}$$

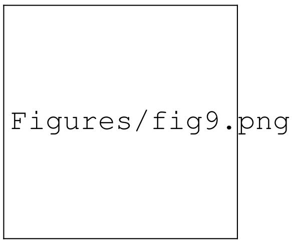

FIGURE 1.1: Infinitesimal surface element $\mathrm{d}S$ and its vector, normal to the surface. The pressure P in this point is the force applied per infinitesimal unit surface.

The salad bowl

Let it be a hollow half spherical shell of radius R and mass M on a horizontal plane (Fig. 1.2). At its top, we pour water (in blue) through a hole.

1. What is the direction of the total pressure force onto the sphere?
2. Express this force as a function of the amount of water in the sphere. When is it high enough to lift the half-sphere?

Volumetric expression

Let it be a infinitesimal volume $\mathrm{d}V$ in Cartesian coordinates (Fig.1.3). We can write the pressure forces applied to each of its 6 surfaces using equation (1.2). For instance, along the $\hat{\mathbf{x}}$ axis:

$$\begin{aligned} \mathrm{d}\,\mathbf{F}\cdot\hat{\mathbf{x}} &= P(x,y,z)\,\mathrm{d}y\,\mathrm{d}z - P(x+\mathrm{d}x\,,y,z)\,\mathrm{d}y\,\mathrm{d}z \\ \mathrm{d}\,\mathbf{F}\cdot\hat{\mathbf{x}} &= -\left[P(x+\mathrm{d}x\,,y,z) - P(x,y,z)\right]\mathrm{d}y\,\mathrm{d}z \end{aligned} \tag{1.4}$$

and, performing a Taylor expansion (section **??**) of P to 1^{st}-order along the $\hat{\mathbf{x}}$-axis only, we have:

$$P(x+\mathrm{d}x\,,y,z) = P(x,y,z) + \partial_x P\,\mathrm{d}x \tag{1.5}$$

so

$$\mathrm{d}\,\mathbf{F}\cdot\hat{\mathbf{x}} = -\partial_x P\,\underbrace{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}_{=\mathrm{d}V} \tag{1.6}$$

Performing the same computation along the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ axis, we can compute the full pressure force $\mathbf{f}$ applied per unit volume onto the volume by the surrounding fluid is given by:

$$\begin{aligned} \mathbf{f}_{\mathbf{press}} &= \frac{\mathrm{d}\,\mathbf{F}}{\mathrm{d}V} && (1.7)\\ &= \mathrm{d}\,\mathbf{F}\cdot\hat{\mathbf{x}} + \mathrm{d}\,\mathbf{F}\cdot\hat{\mathbf{y}} + \mathrm{d}\,\mathbf{F}\cdot\hat{\mathbf{z}} && (1.8)\\ &= -\partial_x P\,\hat{\mathbf{x}} - \partial_y P\,\hat{\mathbf{y}} - \partial_z P\,\hat{\mathbf{z}} && (1.9)\\ & && (1.10) \end{aligned}$$

$$\boxed{\mathbf{f}_{\mathbf{press}} = -\,\boldsymbol{\nabla} P} \tag{1.11}$$

Figures/fig14.png

FIGURE 1.2: Half-shell of radius R and mass M on a horizontal plane. It is initially hollow but we pour water (in blue) through an opening at the top.

Alternatively, we can express the pressure force as a force per unit mass:

$$\mathbf{f_{press}} = -\frac{1}{\rho}\,\boldsymbol{\nabla} P \tag{1.12}$$

Repeat the demonstration which led to equation (1.11) but in cylindrical and spherical coordinates.

1.1.2 Microscopic origin

Fundamentally, fluid pressure on a surface is the manifestation of a transfer of momentum from particles elastically[1] bouncing on this surface (Fig. **??**). In this picture, the only component of momentum which matters is the one normal to the surface[2]. There are two methods to derive an expression for the kinetic pressure, that is to say the pressure expressed as a function of the velocity and momentum of the particles.

[1] Meaning that the kinetic energy of the particle after the collision is the same as before.

[2] While to compute shear viscosity, we will look at the transport of the components which are tangent to the surface.

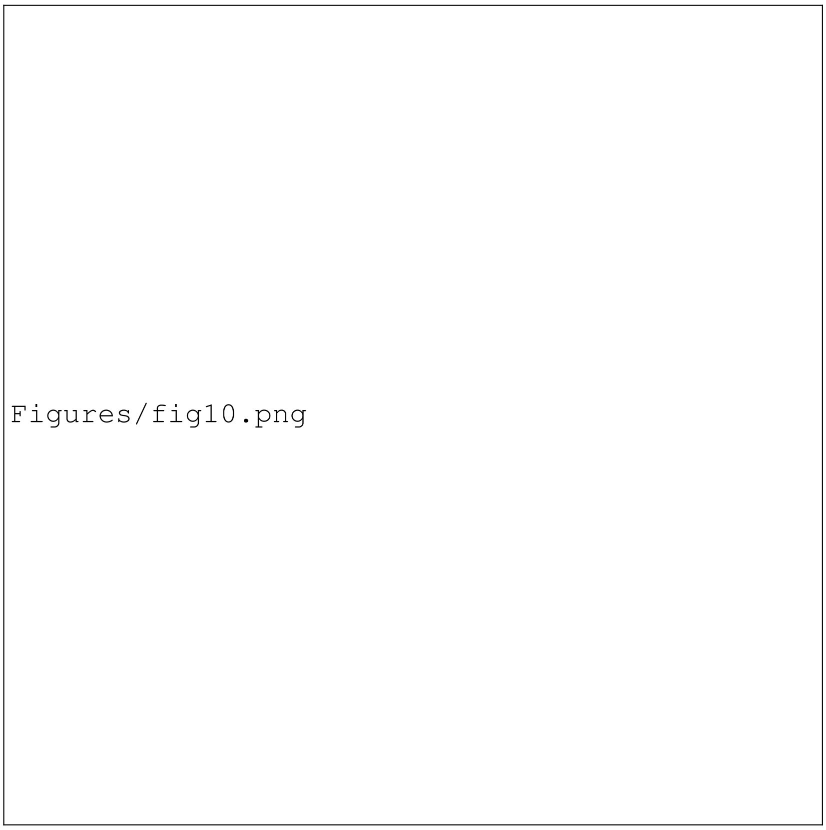

FIGURE 1.3: Infinitesimal volume $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in Cartesian coordinates, with pressure shown on two of the six surfaces (at x in red and at $x + \mathrm{d}x$ in blue).

> The elastic collision we look at is an intrinsically discontinuous event. We can define values x before and after the collision, along with differences Δx, but we cannot work with a time differential $\mathrm{d}t$.

Simplified reasoning

We can take a particle with a momentum $\mathbf{p}$ normal to the infinitesimal surface of area $\mathrm{d}S$. The difference between its momentum after and before elastic collision is $\Delta p = -p - (p) = -2p$. By conservation of momentum, it means that the surface was provided with a momentum $+2p$ during a duration Δt. According to Newton second law, the force applied to the surface by this particle is $F_1 = 2p/\Delta t$.

The total pressure is the total force per unit surface. To find it, we need to account for the contribution from all particles which impact the surface during Δt. How many particles will impact the surface during Δt? The first step to answer this question is to determine the number of particles in the volume where particles are susceptible to reach the surface within Δt. Its basis is given by the surface $\mathrm{d}S$ and its height is given by $v\Delta t$, and we write n the number density of particles in this volume.

However, this number density concerns all particles, including those which are not moving towards the surface and are not going to impact it within Δt. Which fraction of n matters in this calculation? We can considerably simplify the problem by assuming that the particles' momentum $\mathbf{p}$ is necessarily aligned with respect to one of the 3 spatial axis. The particles which will contribute to the pressure are only those which move along the axis normal to the surface and among those, we are only interested in those which move toward the surface.

Since the motion of particles is isotropic, only $1/3$ of particles move along the axis normal to the surface, and only half of them move towards the surface, so we are left with a factor $1/6$.

Finally, the kinetic pressure is given by the product of the force per unit surface for one particle (first factor) with the number of particles which will impact the surface within Δt:

$$P = \frac{2p}{\mathrm{d}S\,\Delta t} \cdot \frac{1}{6} n \cdot v \Delta t \,\mathrm{d}S \tag{1.13}$$

$$P = \frac{1}{3} nvp \tag{1.14}$$

As expected, the duration of the collision and the extent of the surface simplify and the kinetic pressure depends only on n, v and p.

Full reasoning

We relax the assumption that particles are moving along 3 axis only. Instead, we look at a particle with a fiducial momentum $\mathbf{p}$ such that it has an incident angle θ onto the surface (Fig. **??**). Within the duration Δt of the elastic collision, this particle will provide the surface with a momentum $2p\cos\theta$ (after projection onto the normal of the surface). This time, the volume where particles susceptible to reach the surface within Δt are located is $v\cos\theta\Delta t\,\mathrm{d}S$. The fraction of particles with a momentum $\mathbf{p}$ corresponding to the one we are looking at is:

$$\frac{\mathrm{d}\Omega}{4\pi} \tag{1.15}$$

where $\mathrm{d}\Omega = \mathrm{d}\theta \cdot \sin\theta\,\mathrm{d}\phi$ is the infinitesimal solid angle in the direction (θ, ϕ), and 4π steradian is the full solid angle. Since the motion of particles is isotropic, we can already integrate over ϕ from 0 to 2π and we get the fraction:

$$\frac{\sin\theta\,\mathrm{d}\theta}{2} \tag{1.16}$$

of particles with a momentum $\mathbf{p}$ corresponding to an incident angle θ. As a consequence, we deduce the kinetic pressure from a formula which should be compared to equation (1.13):

$$P = \int_0^{\pi/2} \frac{\sin\theta\,\mathrm{d}\theta}{2} n \cdot v\cos\theta\Delta t\,\mathrm{d}S \cdot \frac{2p\cos\theta}{\mathrm{d}S\,\Delta t} \tag{1.17}$$

$$P = nvp \underbrace{\int_0^{\pi/2} \cos^2\theta \sin\theta\,\mathrm{d}\theta}_{=1/3} \tag{1.18}$$

Luckily enough, we retrieve the same expression of the kinetic pressure as equation (1.14).

> For the sake of simplicity, we assumed that all particles have the same speed v and same momentum p. A more realistic approach would have required taking into account the underlying distributions and integrating over them.

1.1.3 Fundamental law of hydrostatics

Expression

For a fluid at rest in an inertial frame and in a gravitational field $\mathbf{g}$, gravity and pressure forces per unit volume counterbalance each other which yields the fundamental law of hydrostatics:

$$\mathbf{f_{grav}} + \mathbf{f_{press}} = \mathbf{0} \tag{1.19}$$

$$\boxed{\boldsymbol{\nabla} P = \rho\,\mathbf{g}} \tag{1.20}$$

Generally, in this course, we will work in the uniform gravitational field $\mathbf{g}$ at the Earth surface. We introduce the local Cartesian basis, with $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ horizontal and $\hat{\mathbf{z}}$ vertical and pointing upwards. Then, the projection of the fundamental law of hydrostatics onto the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ axis gives:

$$\begin{aligned}(1.20)\cdot\hat{\mathbf{x}} : \partial_x P = 0\\ (1.20)\cdot\hat{\mathbf{y}} : \partial_y P = 0\end{aligned} \tag{1.21}$$

and onto the $\hat{\mathbf{z}}$ axis:

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\rho g \tag{1.22}$$

where we used ordinary derivatives d rather than partial derivatives ∂ because P depends only on z.

The case of incompressible fluids

An incompressible fluid has a uniform density ρ. Therefore, we can immediately integrate equation (1.22) to determine the pressure profile $P(z)$, provided we give ourselves one boundary condition[3], for instance: $P(z=0) = P_0$. Then, we have, following the instructions for integration given in section **??**:

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\rho g \tag{1.23}$$

$$\mathrm{d}P = -\rho g\,\mathrm{d}z \tag{1.24}$$

$$\mathrm{d}P = \mathrm{d}\left(-\rho g z\right) \tag{1.25}$$

$$\int_{P_0}^{P(z)} \mathrm{d}P = \int_0^z \mathrm{d}\left(-\rho g z\right) \tag{1.26}$$

$$P(z) - P_0 = -\rho g z \tag{1.27}$$

$$P(z) = P_0 - \rho g z \tag{1.28}$$

where we could inject ρ in the differential element $\mathrm{d}z$ because in this specific case, the density is uniform so in particular, it does not depend on z. We retrieve the result that, if we dive into the sea (and take $z = 0$ to be the sea level and P_0 the air pressure at the sea level), the pressure increases (since $z < 0$). This increase is due to the term $-\rho g z$ in equation (1.28) which represents the weight of a cylinder-shaped fluid column of height $|z|$ and of basis unity (i.e. $1\mathrm{m}^2$ in MKSA, $1\mathrm{cm}^2$ in CGS). Physically, at a depth $-z$, it means that we feel the atmospheric pressure enhanced by the weight of the water above.

[3] Because it is a first order differential equation.

Repeat the integration of equation (1.22) but with the more general boundary condition $P(z = z_0) = P_0$.

Liquids VS gases

Generally, gases cannot be assumed to be incompressible: ρ significantly varies with z (see section 1.2.1 for instance). Yet, for the sake of this exercise, let us assume a gas of uniform mass density ρ_g and compare the pressure profile in the gas to the pressure profile in an incompressible liquid of mass density ρ_l. In which fluid does the pressure change faster? That is to say, for a given pressure density ΔP, which fluid requires a larger altitude variation Δz to produce the same ΔP?

> This is the reason why, hereafter, when we work with a vertical wall separating a gas from a liquid (e.g. the dams in section 1.2.3), we will neglect the variation of pressure in the gas along the height of the wall (but not in the liquid).

The Big Blue

1. At which depth do we feel a pressure twice as high as the atmospheric pressure P_0?
2. Located in the western Pacific Ocean, the Mariana Trench is approximately 10kms deep. Estimate the water pressure at the bottom of this trench.
3. The compressibility of a fluid determines the diminution of its volume or, equivalently, the increase of its density, when the pressure exerted on it increases (section **??** of the thermodynamic appendix in the course of Stellar Astrophysics). To determine the intrinsic compressibility of a fluid, we look at the relative change of density $\mathrm{d}\rho/\rho$. We define the isothermal compressibility χ_T as:

$$\chi_T = \frac{1}{\rho} \left.\frac{\partial \rho}{\partial p}\right|_T \tag{1.29}$$

   We measure $\chi_T \sim 5 \cdot 10^{-10}\mathrm{Pa}^{-1}$ for water at sea temperature (whose variation from top to bottom we neglect). In these conditions, what is approximately the relative change of mass density of water from the surface to the deepest point of the Mariana Trench? How fair was it to assume that sea water is an incompressible fluid?

> Given their reduced vertical extension compared to the Earth radius, our atmosphere and oceans can both be treated in this plane-parallel approximation, neglecting the Earth's curvature (see section "Plane parallel atmospheres" of the Stellar Astrophysics course).

1.2 Applications

In the first section below, we consider a gas where the mass density varies, while in the other sections, we work with liquids which can be assimilated to incompressible and homogeneous fluids where the mass density is uniform.

1.2.1 Atmospheric models

In the exercises below, we solve the equation (1.22), that is to say we determine $P(z)$. In all of these exercises, the Earth's atmosphere is assumed to be an ideal gas whose mass density ρ, pressure P and temperature T are linked through the equation-of-state of ideal gases:

$$\frac{P}{\rho} = \frac{k_B T}{\mu m_p} \tag{1.30}$$

where μ is the mean molecular weight of the atmosphere, which is the average mass of a molecule in units of m_p.

Atmospheric mean molecular weight

The Earth's atmosphere is mostly composed of N_2 and O_2. N_2 (resp. O_2) represents $\alpha_N = 78\%$ (resp. $\alpha_O = 21\%$) in mass, and each atom of N_2 contains $A_N = 14$ nucleons (resp. each atom of O_2 contains $A_O = 16$ nucleons. Determine μ in the atmosphere, assuming that the composition of the atmosphere does not change with the altitude z.

Hereafter, we solve equation (1.22) with the following boundary condition: $P = P_0$ at $z = 0$, with the origin of the z-axis set at the sea level. We write ρ_0 the air mass density at $z = 0$.

Isothermal

1. Assuming that the temperature of the atmosphere is uniform ($T = T_0$), what are the profiles of mass density $\rho(z)$ and pressure $P(z)$?

2. Determine the characteristic length scale $H_\oplus$ of these profiles (a.k.a. the atmospheric scale height).

3. For $T = 300\text{K}$ and $g = |\mathbf{g}| = 10\text{m}\cdot\text{s}^{-2}$, evaluate $H_\oplus$ and validate a posteriori the estimate you made in question I.1.

Linear

In practice, the temperature of the atmosphere drops with the altitude. To represent this, we assume a linear temperature profile:

$$T(z) = T_0 - \alpha z \tag{1.31}$$

where α is a constant > 0 and $T_0 = \frac{\mu m_p}{k_B}\frac{P_0}{\rho_0}$ is the temperature at $z = 0$.

1. What are the new profiles of density $\rho(z)$ and pressure $P(z)$, expressed as a function of T_0 and α?

2. Is the scale height uniform? If not, what is its profile $H(z)$?

3. The temperature is $T_0 = 20^o\text{C}$ at sea level and $T_1 = -60^o\text{C}$ at the top of the troposphere located at $z_1 = 10\text{km}$. Determine the numerical value of α.

Square-root

Alternatively, we consider the following temperature profile:

$$T(z) = T_0\sqrt{1-\alpha z} \tag{1.32}$$

where α is a constant > 0 and T_0 is the temperature at $z = 0$.

1. What are the new profiles of density $\rho(z)$ and pressure $P(z)$?
2. Is the scale height uniform? If not, what is its profile $H(z)$?
3. Determine the new numerical value of α from the data of question I.3.c.iii.

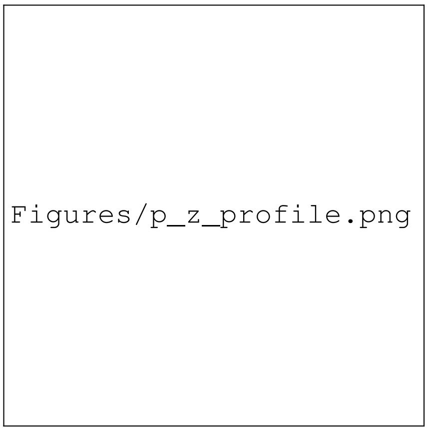

FIGURE 1.4: Altitude as a function of the pressure $z(P)$ in the Earth's troposphere.

Isentropic

If atmospheric thermodynamic processes are adiabatic (i.e. no heat exchange, $\delta Q = 0$) and mechanically reversible (i.e. no entropy created due to dissipative processes as shocks), then, we can assume that the entropy of the ideal gas remains constant:

$$P\rho^{-\gamma} = \text{cst} \quad \text{with } \gamma \text{ the adiabatic index} \tag{1.33}$$

1. Reformulate the equation (1.33) in terms of P and T first, and ρ and T then. Said otherwise, express the exponents a, b, c and d as a function of γ such that P^aT^b =cst and ρ^cT^d =cst.
2. Determine the profiles $\rho(z)$, $P(z)$ and $T(z)$.
3. Compare these profiles to those obtained above from a linear temperature profile.

Comparison to measures

The measured pressure profile in the Earth's troposphere is given in Fig. 1.4. Which of the 3 aforementioned models is the most accurate? You will use the free online tool WebPlotDigitizer to extract the data and fit it with the different profiles you found.

1.2.2 Pascal law

We consider an incompressible fluid (e.g. a liquid) in the device represented in Fig.XXX composed of a main part and of multiple thinner pipes. At the open top of the main part, we set a pump pear to increase air pressure above the fluid. We observe that the level of the air-liquid interface decreases in the main part, and increases in the thinner pipes (Fig.XXX). This observation enables us to deduce the following empirical statement which is known as Pascal law:

> A pressure change at any point in a confined incompressible fluid is transmitted throughout the fluid such that the same change occurs everywhere.

Hydrometer

Consider a perfectly vertical U-shaped tube with extremities in the open air (i.e. at air pressure P_0). Let us show that it can be used as a hydrometer, an instrument to measure the mass density of a liquid with respect to another. Let 1 and 2 be two non-miscible liquids at rest (Fig.XXX). At the interface between the two liquids, the pressure is continuous.

1. Let S be the U-shaped tube's transverse cross-section. We drop a volume V_2 of oil and a volume V_1 of water in the tube. The two liquids settle in the configuration given in Fig.XXX. What is the height h_2?

2. Apply Pascal law to deduce the link between the heights h_i and the mass densities ρ_i, with $i = 1, 2$.

3. How can you deduce the mass density ρ_2 of oil as a function of the mass density of water ρ_1 and the difference of heights $h_2 - h_1$?

Torricelli barometer

Consider the same U-shaped tube as in the previous exercise but now, one of the extremities is closed such as the volume above the liquid is void (Fig.XXX).

1. Use Pascal law to deduce the air pressure P_0 from the height h of the column.

2. Estimate the height h required to measure a typical air pressure if the liquid is water ($\rho \sim 1\text{g}\cdot\text{cm}^{-3}$) and if it is mercury ($\rho \sim 13.590\text{g}\cdot\text{cm}^{-3}$).

3. In practice, the pressure in the volume between the liquid and the closed extremity, P_1, is not totally zero, although $P_1 \ll P_0$. How does it alter the result? Was the air pressure we deduced previously under or overestimated?

4. Firefighters say that it is impossible to propel water up to a height of more than $h \sim 10\text{m}$ using a water pumping method based on void only. Comment.

   > In a building higher than 10m, the only way to provide water to the higher floors is by applying a pressure higher than the air pressure at the bottom.

Hydraulic press

In garages, cars have to be lifted for workers to be able to intervene on the chassis. Pascal law provides a convenient way to do so using a hydraulic press. The hydraulic press is composed of two cylinders filled with water and of different transverse sections S_1 and $S_2 < S_1$ connected by a thin pipe at their basis (Fig.XXX).

1. At equilibrium, only the air pressure applies to the water. What is the difference of altitude Δz between the two air-water interfaces?
2. How will the system react to any change of level $h_2 > 0$ in the cylinder 2?
3. If we exert a force F_2 onto the level 2, which force F_1 will it produce on the other side?

> Fun-fact: few time before inventing the hydraulic press, Joseph Bramah developed another convenient device, the flush toilet.

1.2.3 Dams

Simplified dam

We want to study a simplified dam modeled as a rectangular wall of height h and width L (Fig.XXX). The air pressure P_0 is uniform all over the dam and the water density is ρ_0.

1. Write the hydrostatic equation in water to determine the pressure profile $P(z)$ in water as a function of the altitude z.
2. Deduce the resultant of pressure forces exerted onto the dam.

Arch dam

An arch dam is so named because of its characteristic arched shape. The curved shape of these dams makes it possible to transfer the forces due to the push of the water to each side of the banks. Such a dam works on the same principle as vaults: for vaults, the load is concentrated on the pillars of the vaults, while for dams, the force is concentrated at the support points on the sides. This type of dam is therefore suitable for narrow valleys with very rigid slopes. Such a dam is modeled by a quarter cylinder of height H, radius R and opening angle α. The arc of the dam has a length L.

1. Determine the radius R as a function of L and α.
2. What are the two pressure forces which exert onto the dam?
3. Based on the symmetries of the problem (section **??**), determine the axis which carries the resultant of the pressure forces.
4. Determine the force exerted by the air on the dam.
5. Determine the force exerted by the water on the dam.
6. What is the resultant?

1.2.4 Archimedes buoyancy

Expression

We consider a rectangle-shaped solid body of density ρ_s. We fully immersed it in a fluid of density ρ_l and release it such as it is initially at rest (Fig.XXX). It has a height of h and a surface S at the top/bottom. We take its bottom as the origin $z = 0$ and the $\hat{\mathbf{z}}$-axis is oriented upwards. We compute all the forces applied to this body:

- At the top, it feels a downwards pressure force given by $-P(z = h)S\,\hat{\mathbf{z}}$.
- At the bottom, it feels an upwards pressure force given by $P(z = 0)S\,\hat{\mathbf{z}}$.
- On the sides, the situation is symmetric and the pressure forces cancel out.
- Finally, it feels its own weight $\rho_s hS\,\mathbf{g}$ (since its mass is given by $\rho_s hS$).

The net pressure force exerted onto the body is thus given by:

$$
\begin{aligned}
\mathbf{\Pi} &= -P(z = h)S\,\hat{\mathbf{z}} + P(z = 0)S\,\hat{\mathbf{z}} &(1.34)\\
&= -\left[P(z = h) - P(z = 0)\right] S\,\hat{\mathbf{z}} &(1.35)\\
&= \rho_l ghS\,\hat{\mathbf{z}} &(1.36)
\end{aligned}
$$

where we applied the law of hydrostatics (1.22) to obtain the last equality. Therefore, the resultant of the pressure forces applied to a solid body are equal to the opposite of the mass of the fluid which would occupy the same volume if the object were not present. It is called the Archimedes buoyancy (a.k.a. upthrust).

> It is the variation of pressure with altitude in the fluid which is at the origin of Archimedes' buoyancy, the pressure being greater under the object than above. Therefore, this result applies independently of the shape of the solid body.

Now, if we compute the total force exerted on the body, we have:

$$
\mathbf{\Pi} + \rho_s hS\,\mathbf{g} = (\rho_l - \rho_s)ghS\,\hat{\mathbf{z}} \tag{1.37}
$$

Therefore, for the body to stay still, it has to have the same mass density as the ambient liquid. If it is denser it sinks, it if it less dense it rises to the surface.

In a pool, a pound or the sea, why does blowing air out help to sink?

Scale

When you use a scale to determine your weight, the Archimedes buoyancy force from the ambient air applies.

1. What are the different forces exerted on you?
2. Define the apparent mass, the one measured by the scale.
3. By how much does it wrong the result?

> The higher the fluid density, the lower the apparent mass, hence the reason why the astronauts practice in pools to reproduce a lower gravity environment.

Hot air balloon

Tout-en-un XXX

> A final warning: when part of the surface of the immersed body is not in contact with the fluid (Fig.XXX), it is incorrect to write that the resultant of the pressure forces is equal to the Archimedes' buoyancy. In this case, the pressure force must be calculated by returning to the definition and integrating the force over the surface of the body in contact with the fluid.

Stability

In Physics, a system is stable if and only if it tends to restore itself to an equilibrium position after a small displacement. In this sentence, "small" means that in the general case, we will be able to perform Taylor expansions of the dynamic equation of the system around an equilibrium position. There are two types of stability to be considered when we study Archimedes buoyancy:

- From a point-mass point of view, we can study how is the Archimedes buoyancy force modified by a small vertical displacement: if it increases in the opposite direction of the displacement, it acts as a restoring force and the system is stable with respect to a vertical perturbation.
- If instead of seeing it as a point, we account for the spatial extent and the shape of the solid body, then, we can question its stability with respect to a slight rotation (Fig.XXX). In this case, it is important to remind that the weight applies to the center of mass (a.k.a. of gravity) to the solid body, while the Archimedes buoyancy applies to the center of mass of the displaced liquid, the one which would be in the volume occupied by the solid body if the solid body were not present. The two do not necessarily coincide. If the torques produced by the rotation point in the direction opposite to the rotational perturbation, the system is rotationally stable. Otherwise, it rocks.

Let us illustrate the latter aspect XXX

Iceberg

If a solid body is in equilibrium between two fluids, since there is continuity of pressure at the interface between the two fluids, the result obtained for the Archimedes buoyancy generalizes. In this case, we add the Archimedes buoyancy due to each of the two fluids. Let us apply this principle to an iceberg, that is to say a block of solid water of density ρ_s floating on liquid water ρ_l.

1. Which is the condition on ρ_s and ρ_l to explain that the iceberg floats? Does this relation between the mass density of water in solid and liquid states hold for other chemical species?
2. What are the forces applied to the iceberg?
3. If we neglect the Archimedes buoyancy due to air pressure, determine the equilibrium position of the iceberg and express the fraction of iceberg's volume immersed as a function of ρ_s and ρ_l.
4. The density of water increases with salinity. Is the fraction of iceberg's volume immersed larger or lower in a lake than in the sea?

5. Consider a glass of liquid with a floating ice cube made of the same chemical species. Once the ice cube melt, how does the liquid level change?

6. Before the aforementioned ice cube melts, we push it slightly downwards. According to the modifications induced in the Archimedes buoyancy force, does the system come back to equilibrium?

7. Now, let us investigate the rotational stability of the iceberg. Take a pyramid-shaped iceberg pointing up and another pointing down. Why is the former stable and the latter unstable?

Chapter 2

Kinematics

Kinematics is the branch of classical mechanics that describes the motion of a system without reference to the forces responsible for this motion. In point (resp. a solid) mechanics, this system is a point or a collection of points (resp. a solid with a certain size and orientation). In fluid mechanics, this system is a fluid whose motion will be described by a velocity field $\mathbf{v}(\mathbf{r}, t)$. For a solid, at a given time t, the motion of the system depends only on the velocity $\mathbf{v}(\mathbf{r_G}, t)$, which describes the motion of the center of mass G, and on the angular speed vector $\mathbf{\Omega}(t)$, which describes the orientation of the solid (Fig XXX). On the contrary, for a fluid, the velocity field $\mathbf{v}(\mathbf{r}, t)$ is a priori totally free. However, mass conservation sets a constrain to the velocity field in the form of a scalar equation which couples the mass density field $\rho(\mathbf{r}, t)$ (which is a scalar field) and the velocity field $\mathbf{v}(\mathbf{r}, t)$. Additional constrains may arise from simplifying assumptions: steady flow, incompressible fluid, incompressible flow, irrotational flow...

In this chapter, we emphasize on the physical meaning of the mathematical tools used to describe the motion of a fluid. Most of them were already introduced in electromagnetism (e.g. in the Maxwell equations) but the electric and magnetic fields are not as intuitive as the velocity field of a fluid. In particular, we will build upon the common intuition provided by the representation of flows (Fig.XXX) to enhance the reader's capacity to interpret the operators $\boldsymbol{\nabla} \cdot \bullet$ and $\boldsymbol{\nabla} \wedge \bullet$. Although more sophisticated than the ones you saw in point and solid mechanics, these operators have very concrete meanings.

2.1 Velocity field in a fluid

2.1.1 Lagrangian and Eulerian descriptions

Velocity of a fluid particle

In the fluid approximation, we treat a collection of particles as a continuous medium. We define a fluid particle as a volume element of mesoscopic size δ, that is to say:

- very small in comparison to the macroscopic scales L. Therefore, the scalar and velocity fields we work with behave as continuous functions.

- very large in comparison to the microscopic scales l. Therefore, fluid particles contain such a large number of physical particles (e.g. atoms, molecules, etc) that the relative standard deviation of quantities due to stochastic[1] motion is negligible (Fig XXX). For instance, even if the number N of particles in a given fluid particle vary with time because particles enter and leave in a random way due to their individual motion, N is large enough that the typical variation δN induced by stochastic motion is negligible compared to N (i.e. $\delta N \ll N$).

[1] In Physics, stochastic means that it cannot be predicted accurately, either because of intrinsic randomness (e.g. the wave function in quantum mechanics) or because the causes are out of the scope of the model. Here, we are in the later case since fluid mechanics does not address the particles' motion at the microscopic scale.

In this case, we can define the velocity vector $\mathbf{v}$ of a fluid particle as the average of the velocity vectors $\mathbf{v_i}$ of the N particles contained in the fluid particle (Fig XXX):

$$\mathbf{v} = \langle \mathbf{v} \rangle = \frac{\sum_{i=1}^{N} \mathbf{v_i}}{\sum_{i=1}^{N} 1} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{v_i} \tag{2.1}$$

The fluid motion in a given frame $\mathcal{R}$ is then given by all the fluid particles' velocity that we describe as a velocity field $\mathbf{v}(\mathbf{r}, t)$. We can subdivide the fluid in fluid particles in two different ways: either at the initial time (Lagrangian approach) or at a given time t (Eulerian approach).

Lagrangian approach

At an initial time t_0, we subdivide the fluid in a collection of fluid particles each centered on a point M_0 located in $\mathbf{r_0}$. We follow the motion in time of these fluid particles in a given frame $\mathcal{R}$. At a subsequent time t, we can then define the velocity vector[2] $\mathbf{v}^*(\mathbf{r_0}, t)$ of the fluid particle which was in M_0 at time t_0. We can then define the trajectory of the fluid particle as the successive positions of this fluid particle in time (section 2.1.3).

In this approach, the observer is bound to each fluid particle. As an illustration, one can imagine a river (Fig XXX): in the Lagrangian approach, we set at an initial time t_0 an infinite quantity of tiny buoys on water and we subsequently follow their motion. Each observer is bound to a given buoy and measure its velocity[3] $\mathbf{v}^*(\mathbf{r_0}, t)$ at each time t. Therefore, at a given location $\mathbf{r}$, the observers change in time.

In the Lagrangian approach, the observer follows a closed system, which enables us to directly apply the laws of classic mechanics. After all, when we apply Newton's second law to a falling apple for instance, we implicitly adopt a Lagrangian point of view since we follow the apple. However, this approach is very inconvenient to use the constrains set to the velocity or the pressure since these constrains are generally set at fixed points in space (e.g. the boundary conditions on the edges of a pipeline).

Eulerian approach

At each time t, we subdivide the fluid in a collection of fluid particles each centered on a point M located in $\mathbf{r}$. We define the velocity field $\mathbf{v}(\mathbf{r}, t)$ at a given time t as the velocity vector of the fluid particle which happens to be in $\mathbf{r}$ at time t. At a given time t, we define the streamlines as the field lines of the velocity field $\mathbf{v}(\mathbf{r}, t)$, that is to say the curves tangent in each of their points to the velocity field $\mathbf{v}(\mathbf{r}, t)$ (section 2.1.3).

In this approach, the observer is bound to a frame $\mathcal{R}$. As an illustration, one can imagine a river (Fig XXX): in the Eulerian approach, we set an infinite quantity of bridges over the river and an observer on each bridge. An observer located in $\mathbf{r}$ sees the buoys passing and measure their velocity $\mathbf{v}(\mathbf{r}, t)$ but at each time t, she sees a different buoy.

The drawback of this approach is that the velocities $\mathbf{v}(\mathbf{r}, t)$ do not correspond to the velocity of a given system. Instead, they describe the velocities of the different particles of fluid which happen to pass by the point $\mathbf{r}$ as they move. However, the advantage of this approach is that it is coherent with the thermodynamic description of the mass density and pressure fields $\rho(\mathbf{r}, t)$ and $P(\mathbf{r}, t)$. Furthermore, it suits well the concept of boundary conditions in fixed points.

[2] We note it with an asterisk superscript in order to differentiate the velocity of the fluid particle in the Lagrangian approach from the Eulerian velocity field we later introduce.

[3] Actually, it measures the velocity $-\mathbf{v}^*(\mathbf{r_0}, t)$ with respect to the riverside.

> The Eulerian approach is also the one used in electromagnetism to define the current densities and to write Maxwell equations. Therefore, the Eulerian approach is fundamentally linked to the notion of field theory.

Steady flow

A flow is steady if and only if its Eulerian fields $\mathbf{v}(\mathbf{r},t)$, $\rho(\mathbf{r},t)$ and $P(\mathbf{r},t)$ do not depend on time. In this case, the situation as seen by an observer located in a given point $\mathbf{r}$ does not change in time. Yet, the velocity of a given fluid particle can change in time since the fluid particle moves (Fig XXX).

> A steady flow should not be confused with a stationary flow, where the velocity field is uniformly null (i.e. the fluid is at rest).

Whether a flow is steady or not depends on the frame $\mathcal{R}$ chosen. Indeed, in a lake, the wake behind a duke is steady with respect to the duck, but it is not for an observer on the side.

2.1.2 Lagrangian derivative

Parachute XXX

In order to reconcile both approaches, we need a concept which connects them. For instance, we want to use the Eulerian notion of mass density field while being able to use the laws of classic mechanics, which apply to a closed system. Yet, the latter is only possible if we follow the fluid particles, that is to say if we adopt a Lagrangian point of view.

Incompressible flow

In a moving fluid, fluid particles conserve their mass during their motion since they are, by definition, closed systems[4]. Yet, in general, their volume varies, which can be quantified through the mass density field $\rho(\mathbf{r},t)$. If, at a given time t, the fluid particle located in $\mathbf{r}$ contains a mass $\mathrm{d}m$ and occupies a volume $\mathrm{d}V$, the mass density in $\mathbf{r}$ is given by:

$$\rho(\mathbf{r},t) = \frac{\mathrm{d}m}{\mathrm{d}V} \tag{2.2}$$

A flow is said to be incompressible if the volume of all fluid particles is conserved as they move. Since the mass of the fluid particles is conserved by definition, it is equivalent to say that a flow is incompressible if and only if the fluid particles conserve their mass density as they move.

> An incompressible flow should not be confused with an incompressible fluid (Fig XXX). In an incompressible fluid, mass density is the same everywhere, it does not depend on space (i.e. $\boldsymbol{\nabla}\rho = \mathbf{0}$). In steady state (i.e. $\partial_t = 0$), an incompressible flow has $\mathbf{v}\cdot\boldsymbol{\nabla}\rho = 0$ that is to say that the density is constant along a streamline. Yet, it can be different from a streamline to another, so it is not an incompressible fluid (see exercise "Matriochkas" hereafter). Liquids can typically be treated as incompressible fluids: for instance, from the surface to the bottom of the ocean, a few kilometers deep, the mass density of liquid water increases by less than a percent. Therefore, it is safe to assume that the mass density is uniform. On the other hand, gases are much more compressible and in general they cannot be treated as incompressible fluids. Yet, they can generally be treated as incompressible flows, provided they move at a speed much

[4] This is only possible provided the fluid approximation holds, that is to say the fluid particles are large compared to the microscopic scales, (i.e. $l \ll \delta$).

smaller than the local sound speed (Chap. 6).

Lagrangian derivative of mass density

Between time t and time $t + \mathrm{d}t$, when a fluid particle moves between the point $M(\mathbf{r})$ and the point $M'(\mathbf{r} + \mathrm{d}\,\mathbf{r})$, we must have, for an incompressible flow (Fig. XXX):

$$\rho(\mathbf{r} + \mathrm{d}\,\mathbf{r}\,, t + \mathrm{d}t) = \rho(\mathbf{r}, t) \tag{2.3}$$

In the more general case of a compressible flow, it is important to quantify the rate of change of the mass density between times t and $t + \mathrm{d}t$, while following the fluid particle located at $\mathbf{r}$ at time t and at $\mathbf{r} + \mathrm{d}\,\mathbf{r}$ at time $t + \mathrm{d}t$. We define the Lagrangian derivative of the mass density $\mathrm{D}\rho/\mathrm{D}t$ as the limit, when $\mathrm{d}t$ tends to zero, of the rate of change of the mass density measured while following the fluid particle located at $\mathbf{r}$ at time t and moving by $\mathrm{d}\,\mathbf{r}$ between times t and $t + \mathrm{d}t$:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \lim_{\mathrm{d}t \longrightarrow 0} \frac{\rho(\mathbf{r} + \mathrm{d}\,\mathbf{r}\,, t + \mathrm{d}t) - \rho(\mathbf{r}, t)}{\mathrm{d}t} \tag{2.4}$$

As an illustration, we will temporarily work in Cartesian coordinates such as:

$$\mathrm{d}\,\mathbf{r} = \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}} + \mathrm{d}z\,\hat{\mathbf{z}} \tag{2.5}$$

In this case, we can rewrite equation (2.4) as:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \lim_{\mathrm{d}t \longrightarrow 0} \frac{\rho(x + \mathrm{d}x\,, y + \mathrm{d}y\,, z + \mathrm{d}z\,, t + \mathrm{d}t) - \rho(x, y, z, t)}{\mathrm{d}t} \tag{2.6}$$

It can be interpreted as the derivative with respect to time of the function $\rho(x(t), y(t), z(t), t)$ which depends on time explicitly but also implicitly, through the dependence of the coordinates x, y and z on time. Therefore, we can use the chain rule and separate the 4 different variables (x, y, z and t) to obtain:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\rho}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial\rho}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial\rho}{\partial t} \tag{2.7}$$

where we used partial derivatives for ρ since it depends on multiple variables, and exact derivatives for x, y and z since the position of the fluid particle depends only on time t.

Let us introduce the Eulerian velocity field $\mathbf{v}(\mathbf{r}, t)$: by definition, it corresponds to the velocity of the fluid particle which passes by point $\mathbf{r}$ at time t. Therefore, between times t and $t + \mathrm{d}t$, this fluid particle moves by the infinitesimal displacement $\mathrm{d}\,\mathbf{r}$ given by:

$$\mathrm{d}\,\mathbf{r} = \mathbf{v}(\mathbf{r}, t)\,\mathrm{d}t = v_x\,\mathrm{d}t\,\hat{\mathbf{x}} + v_y\,\mathrm{d}t\,\hat{\mathbf{y}} + v_z\,\mathrm{d}t\,\hat{\mathbf{z}} \tag{2.8}$$

Thus, we have $\mathrm{d}i = v_i\,\mathrm{d}t$ (for $i = x, y, z$) and $\mathrm{d}i/\mathrm{d}t = v_i$. We can reinject this result in equation (2.7) to obtain:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = v_x\frac{\partial\rho}{\partial x} + v_y\frac{\partial\rho}{\partial y} + v_z\frac{\partial\rho}{\partial z} + \frac{\partial\rho}{\partial t} \tag{2.9}$$

which can be rewritten, using the operator $\mathbf{v}\cdot\boldsymbol{\nabla}$:

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\rho} \tag{2.10}$$

> When applied to a scalar field X, the operator $\mathbf{v}\cdot\boldsymbol{\nabla}$ can be splitted:
>
> $$(\mathbf{v}\cdot\boldsymbol{\nabla})X = \mathbf{v}\cdot(\boldsymbol{\nabla}X) \tag{2.11}$$
>
> where we recognize the gradient of $\boldsymbol{\nabla}X$ of the scalar field. Therefore, the specific expression in Cartesian, cylindrical and spherical coordinates of the operator $\mathbf{v}\cdot\boldsymbol{\nabla}$ applied to a scalar field can be deduced from the expression of the gradient operator $\boldsymbol{\nabla}$. The latter can be found on the Wikipedia page Del in cylindrical and spherical coordinates, at the entry "Gradient". Beware, when the operator $\mathbf{v}\cdot\boldsymbol{\nabla}$ is applied to a vector field, such a decomposition will not be possible (section 2.1.2).

The expression (2.10) does not depend on the system of coordinates chosen[5]. The right hand side contains two parts:

- The local derivative $\partial\rho/\partial t$, which describes the local change of mass density in time. It is null for a steady flow, when the fields do not depend on time.
- The convective derivative (a.k.a. the advective term) $\mathbf{v}\cdot\boldsymbol{\nabla}\rho$, which quantifies the influence of the fluid particle's displacement on the change of the mass density. It is null for a uniform flow ($\boldsymbol{\nabla}\rho = 0$), when the fields do not depend on space, or for a stationary flow ($\mathbf{v} = \mathbf{0}$).

Therefore, we can provide a quantitative definition of the notion of incompressible flow through:

$$\boxed{\text{Incompressible flow} \iff \frac{\mathrm{D}\rho}{\mathrm{D}t} = 0} \tag{2.12}$$

> Here, we see that an incompressible fluid (i.e. $\partial\rho/\partial t = 0$ and $\boldsymbol{\nabla}\rho = 0$) is a specific case of an incompressible flow (i.e. $\mathrm{D}\rho/\mathrm{D}t = 0$). However, in the general case, we can have $\mathrm{D}\rho/\mathrm{D}t = 0$ while $\partial\rho/\partial t \neq 0$ and $\boldsymbol{\nabla}\rho \neq 0$ (e) (e.g. exo astro estelar XXX).

Lagrangian derivative of velocity

In a similar way as what was done in the previous section, we define the Lagrangian derivative of the velocity $\mathrm{D}\,\mathbf{v}/\mathrm{D}t$ as the limit, when $\mathrm{d}t$ tends to zero, of the rate of change of the velocity $\mathbf{v}$ measured while following the fluid particle located at $\mathbf{r}$ at time t and moving by $\mathrm{d}\,\mathbf{r}$ between times t and $t+\mathrm{d}t$:

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \lim_{\mathrm{d}t\longrightarrow 0} \frac{\mathbf{v}(\mathbf{r}+\mathrm{d}\,\mathbf{r}\,, t+\mathrm{d}t) - \mathbf{v}(\mathbf{r}, t)}{\mathrm{d}t} \tag{2.13}$$

By construction, it is the acceleration vector of the fluid particle which passes by $\mathbf{r}$ at time t. Therefore, it will be this quantity which appears when we apply Newton's second law to fluids in the following chapters.

We can use the decomposition of the vector field in Cartesian coordinates:

$$\mathbf{v} = v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}} \tag{2.14}$$

in order to decompose the Lagrangian derivative in:

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \frac{\mathrm{D}v_x}{\mathrm{D}t}\,\hat{\mathbf{x}} + v_x\underbrace{\frac{\mathrm{D}\hat{\mathbf{x}}}{\mathrm{D}t}}_{=0} + \frac{\mathrm{D}v_y}{\mathrm{D}t}\,\hat{\mathbf{y}} + v_y\underbrace{\frac{\mathrm{D}\hat{\mathbf{y}}}{\mathrm{D}t}}_{=0} + \frac{\mathrm{D}v_z}{\mathrm{D}t}\,\hat{\mathbf{z}} + v_z\underbrace{\frac{\mathrm{D}\hat{\mathbf{z}}}{\mathrm{D}t}}_{=0} \tag{2.15}$$

[5] Such a coordinate-independent expression is called "intrinsic".

where the derivatives of the basis' vectors are null since they are fixed in space and time. Now, we can use the expression of the Lagrangian derivative of a scalar field (2.10) to obtain:

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \left(\frac{\partial v_x}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} v_x\right)\hat{\mathbf{x}} + \left(\frac{\partial v_y}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} v_y\right)\hat{\mathbf{y}} + \left(\frac{\partial v_z}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} v_z\right)\hat{\mathbf{z}} \tag{2.16}$$

We can gather the local derivatives on one side, and the advective terms on the other side to obtain:

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \frac{\partial(v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}})}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla} v_x)\,\hat{\mathbf{x}} + (\mathbf{v}\cdot\boldsymbol{\nabla} v_y)\,\hat{\mathbf{y}} + (\mathbf{v}\cdot\boldsymbol{\nabla} v_z)\,\hat{\mathbf{z}} \tag{2.17}$$

which becomes, in intrinsic form:

$$\boxed{\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}} \tag{2.18}$$

Once again, the right hand side of equation (2.18) contains two parts:

- The local derivative $\partial\mathbf{v}/\partial t$, which describes the local change of velocity in time. It is null for a steady flow, when the fields do not depend on time.
- The convective derivative (a.k.a. the advective term) $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$, which quantifies the influence of the fluid particle's displacement on the change of the velocity.

It is important to understand that even for a steady flow, a fluid particle can be accelerated if the advective term is not null, for instance if the velocity field $\mathbf{v}$ depends on space.

Alternative expression

The operator $\mathbf{v}\cdot\boldsymbol{\nabla}$ applied to the velocity vector field $\mathbf{v}$ can be re-expressed using operators you are more familiar with. In Cartesian coordinates, show that we have:

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = \boldsymbol{\nabla}\left(\frac{\mathbf{v}^2}{2}\right) + (\boldsymbol{\nabla}\wedge\mathbf{v})\wedge\mathbf{v} \tag{2.19}$$

> The specific expression in Cartesian, cylindrical and spherical coordinates of the operator $\mathbf{v}\cdot\boldsymbol{\nabla}$ applied to a vector field can be found on the Wikipedia page Del in cylindrical and spherical coordinates, at the entry "Directional derivative".

2.1.3 Description of flow motion

Trajectories

The most intuitive way to describe the motion of a fluid is to focus on a specific fluid particle and determine its trajectory, that is to say the locus of its location as a function of time. If we write $\mathbf{r}(t)$ the location of the fluid particle at a given time, its trajectory is defined as:

$$\forall t \quad \{\text{all points } M(\mathbf{r}(t))\} \tag{2.20}$$

Therefore, to plot a trajectory, we need to follow a given fluid particle in time. From a visualization point-of-view, it means that we want to inject a tracer particle (e.g. a dead leaf or a yellow rubber duck for a fluid particle at the surface) at the initial location of the particle and shoot a picture with a long exposure time to follow the tracer (Fig.XXX). Since we focus on a specific fluid particle and its time-dependent location $\mathbf{r}(t)$, the concept of trajectory is intrinsically Lagrangian.

> Fluid particles' trajectories are also called pathlines.

Streamlines

The streamlines are the field lines of the velocity field (section **??**). Therefore, to determine the streamlines, we need to instantaneously capture the infinitesimal motion $\mathrm{d}\boldsymbol{l}$ of all fluid particles. From a visualization point-of-view, it means that we want to shoot a picture with an infinitely short exposure time $\mathrm{d}t$. Since we look simultaneously at all the flow at a given time, the concept of streamline is intrinsically Eulerian.

F

or each of the velocity fields given in section 2.1.4, determine the streamlines.

Comparison

In general, the streamlines and the trajectories do not correspond (Fig XXX). They only do when the flow is steady (see exercise "Gravity waves" in 2.4). In practice, to visualize the streamlines, we inject tracer particles uniformly and simultane- ously in the whole fluid at a given time, and we shoot a picture with a very short exposure time. Doing so, we can assume that within this short exposure time, the flow is steady so the particles' trajectories correspond to the streamlines. The particles draw tiny segments whose reunion corresponds to the streamlines (Fig.XXX). More generally, there are many different flow visualization techniques.

2.1.4 Physical meaning of $\nabla \cdot \mathbf{v}$ and $\nabla \wedge \mathbf{v}$

Using the first order spatial derivatives of the Eulerian velocity field, we can construct a scalar and a vector field, respectively the divergence $\boldsymbol{\nabla} \cdot \mathbf{v}$ and the curl $\boldsymbol{\nabla} \wedge \mathbf{v}$. To understand their physical meaning, we will study 3 fiducial velocity fields and examine how they deform a fluid particle. For the sake of simplicity, we will focus on purely two-dimensional Cartesian velocity fields, that is to say $v_z = 0$, $\partial_z v_x = 0$ and $\partial_z v_y = 0$. Therefore, at time t, the fluid particle is initially a square of side L and volume $V(t) = L^3$ defined by its vertices $O(0,0)$, $A(L,0)$, $B(L,L)$ and $C(0,L)$ (Fig.XXX).

First example: variation of volume without rotation

We consider the velocity field:

$$\mathbf{v} = \underbrace{ax}_{\hat{=} v_x} \hat{\mathbf{x}} + \underbrace{by}_{\hat{=} v_y} \hat{\mathbf{y}} \tag{2.21}$$

with a and b two constants.

What is the unit of a and b?

1. First, let us compute the divergence and the curl of this velocity field:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = a + b \tag{2.22}$$

$$\boldsymbol{\nabla} \wedge \mathbf{v} = \begin{pmatrix} \partial_x \\ \partial_y \\ 0 \end{pmatrix} \wedge \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}} = \mathbf{0} \tag{2.23}$$

2. Second, let us examine the deformation of the fluid particle $OABC$. Between times t and $t + \mathrm{d}t$, the vertices move to O', A', B' and C':

$$\mathbf{OO'} = \mathbf{v}(O)\,\mathrm{d}t = \mathbf{0} \tag{2.24}$$
$$\mathbf{AA'} = \mathbf{v}(A)\,\mathrm{d}t = aL\,\mathrm{d}t\,\hat{\mathbf{x}} \tag{2.25}$$
$$\mathbf{BB'} = \mathbf{v}(B)\,\mathrm{d}t = aL\,\mathrm{d}t\,\hat{\mathbf{x}} + bL\,\mathrm{d}t\,\hat{\mathbf{y}} \tag{2.26}$$
$$\mathbf{CC'} = \mathbf{v}(C)\,\mathrm{d}t = bL\,\mathrm{d}t\,\hat{\mathbf{y}} \tag{2.27}$$

which enables us to plot the new shape of the fluid particles after an infinitesimal amount of time $\mathrm{d}t$ (Fig.XXX).

3. Third, we compute the new volume $V(t + \mathrm{d}t)$ of the fluid particle after an infinitesimal amount of time $\mathrm{d}t$:

$$V(t + \mathrm{d}t) \sim L_x L_y L_z = (L + aL\,\mathrm{d}t)(L + bL\,\mathrm{d}t)L \tag{2.28}$$
$$\sim L^3 (1 + a\,\mathrm{d}t)(1 + b\,\mathrm{d}t) \tag{2.29}$$
$$\sim L^3 \left[1 + (a + b)\,\mathrm{d}t\right] \tag{2.30}$$

where the last equality was obtained by neglecting the second order terms in $(\mathrm{d}t)^2$. The relative[6] variation of volume of the fluid particle is thus given by:

$$\frac{\mathrm{d}V}{V} = \frac{V(t + \mathrm{d}t) - V(t)}{V(t)} = (a + b)\,\mathrm{d}t = (\boldsymbol{\nabla} \cdot \mathbf{v})\,\mathrm{d}t \tag{2.31}$$

so:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{V} \frac{\mathrm{d}V}{\mathrm{d}t} \tag{2.32}$$

It shows that the divergence can be interpreted as a relative rate of change of fluid particles' volume (a result we will generalize in section 2.2.5). On the other hand, we notice that $\boldsymbol{\nabla} \wedge \mathbf{v} = \mathbf{0}$ and that the fluid particle did not rotate.

Second example: deformation at constant volume and without rotation

We consider the velocity field:

$$\mathbf{v} = \underbrace{ay}_{\hat{=}v_x}\,\hat{\mathbf{x}} + \underbrace{ax}_{\hat{=}v_y}\,\hat{\mathbf{y}} \tag{2.33}$$

with a a constant.

[6] Which means that we compare the variation of volume to the volume itself by dividing by V.

1. First, let us compute the divergence and the curl of this velocity field:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{2.34}$$

$$\boldsymbol{\nabla} \wedge \mathbf{v} = \begin{pmatrix} \partial_x \\ \partial_y \\ 0 \end{pmatrix} \wedge \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}} = (a - a)\, \hat{\mathbf{z}} = \mathbf{0} \tag{2.35}$$

2. Second, let us examine the deformation of the fluid particle $OABC$. Between times t and $t + \mathrm{d}t$, the vertices move to O', A', B' and C':

$$\mathbf{OO'} = \mathbf{v}(O)\, \mathrm{d}t = \mathbf{0} \tag{2.36}$$
$$\mathbf{AA'} = \mathbf{v}(A)\, \mathrm{d}t = aL\, \mathrm{d}t\, \hat{\mathbf{y}} \tag{2.37}$$
$$\mathbf{BB'} = \mathbf{v}(B)\, \mathrm{d}t = aL\, \mathrm{d}t\, \hat{\mathbf{x}} + aL\, \mathrm{d}t\, \hat{\mathbf{y}} \tag{2.38}$$
$$\mathbf{CC'} = \mathbf{v}(C)\, \mathrm{d}t = aL\, \mathrm{d}t\, \hat{\mathbf{x}} \tag{2.39}$$

which enables us to plot the new shape of the fluid particles after an infinitesimal amount of time $\mathrm{d}t$ (Fig.XXX).

3. Third, we notice that the volume of the fluid particle did not change since the area of the triangles OCC' and OAA' on one hand, and $BB'C$ and $BB'A$ on the other hand compensate each other. Therefore, we have (i) $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ and the volume of the fluid particle did not change, and (ii) $\boldsymbol{\nabla} \wedge \mathbf{v} = \mathbf{0}$ and the fluid particle did not rotate.

Third example: rotation without variation of volume nor deformation

We consider the velocity field:

$$\mathbf{v} = \underbrace{-ay}_{\hat{=} v_x} \hat{\mathbf{x}} + \underbrace{ax}_{\hat{=} v_y} \hat{\mathbf{y}} \tag{2.40}$$

with a a constant.

1. First, let us compute the divergence and the curl of this velocity field:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{2.41}$$

$$\boldsymbol{\nabla} \wedge \mathbf{v} = \begin{pmatrix} \partial_x \\ \partial_y \\ 0 \end{pmatrix} \wedge \begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}} = 2a\, \hat{\mathbf{z}} \tag{2.42}$$

2. Second, let us examine the deformation of the fluid particle $OABC$. Between times t and $t + \mathrm{d}t$, the vertices move to O', A', B' and C':

$$\mathbf{OO'} = \mathbf{v}(O)\, \mathrm{d}t = \mathbf{0} \tag{2.43}$$
$$\mathbf{AA'} = \mathbf{v}(A)\, \mathrm{d}t = aL\, \mathrm{d}t\, \hat{\mathbf{y}} \tag{2.44}$$
$$\mathbf{BB'} = \mathbf{v}(B)\, \mathrm{d}t = -aL\, \mathrm{d}t\, \hat{\mathbf{x}} + aL\, \mathrm{d}t\, \hat{\mathbf{y}} \tag{2.45}$$
$$\mathbf{CC'} = \mathbf{v}(C)\, \mathrm{d}t = -aL\, \mathrm{d}t\, \hat{\mathbf{x}} \tag{2.46}$$

which enables us to plot the new shape of the fluid particles after an infinitesimal amount of time $\mathrm{d}t$ (Fig.XXX).

3. The volume of the fluid particle did not change but the new fluid particle is obtained by rotation of infinitesimal angle $\mathrm{d}\theta$ around the $\hat{\mathbf{z}}$-axis given by:

$$\mathrm{d}\theta = \frac{AA'}{OA} = \frac{aL\,\mathrm{d}t}{L} = a\,\mathrm{d}t \quad \text{so} \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = a \tag{2.47}$$

In a similar way as we do for a solid, we can define the angular speed vector $\mathbf{\Omega}$ as:

$$\mathbf{\Omega} = \dot{\theta}\,\hat{\mathbf{z}} = a\,\hat{\mathbf{z}} \tag{2.48}$$

and we have, using the expression (2.42) of the curl:

$$\mathbf{\Omega} = \frac{1}{2}\boldsymbol{\nabla} \wedge \mathbf{v} \tag{2.49}$$

General case

Near a point $O(0,0)$, one can always perform a first order Taylor expansion of the velocity field using equation (**??**) to obtain the approximate expression of the velocity field in a point $M(x,y)$ infinitely close:

$$\mathbf{v}(M) \sim \mathbf{v}(O) + \begin{pmatrix} \alpha x + \beta y \\ \delta x + \gamma y \end{pmatrix} \tag{2.50}$$

where we did not represent the z-component, and where α, β, δ and γ stand for the partial derivatives evaluated in O. We can re-write this expression as:

$$\mathbf{v}(M) \sim \mathbf{v}(O) + \begin{pmatrix} \alpha x \\ \gamma y \end{pmatrix} + \frac{1}{2}\begin{pmatrix} (\beta+\gamma)y \\ (\beta+\gamma)x \end{pmatrix} + \frac{1}{2}\begin{pmatrix} (\beta-\gamma)y \\ -(\beta-\gamma)x \end{pmatrix} = \mathbf{v}(O) + \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3} \tag{2.51}$$

where $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ are the velocity fields of the 3 examples above. Therefore, we see that any velocity field in a fluid can be decomposed in 4 components:

- A uniform part $\mathbf{v}(O)$ which describes the bulk motion of the fluid particle.
- A component $\mathbf{v_1}$ which describes a variation of volume without rotation.
- A component $\mathbf{v_2}$ which describes a deformation at constant volume and without rotation.
- A component $\mathbf{v_3}$ which describes a rotation without variation of volume nor deformation.

> For a solid, there is no variation of volume (i.e. $\mathbf{v_1} = \mathbf{0}$) nor deformation (i.e. $\mathbf{v_2} = \mathbf{0}$) possible.

Physical meaning of $\boldsymbol{\nabla} \cdot \mathbf{v}$

In section 6.6.2, we will generalize the result (2.32) to show that the divergence of the velocity field in a point M at a time t is the rate of relative variation of the volume of the fluid particle which happens to pass by M at time t:

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{V}\frac{\mathrm{D}\rho}{\mathrm{D}t}} \tag{2.52}$$

Since we focus on the fluid particle, we must use the Lagrangian derivative since we study the volume variation as the fluid particle flows through the point M.

> We will see in section 2.2.5 that, due to mass conservation, an incompressible flow has a velocity field which is necessarily divergence-free (i.e. $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ everywhere). In this case, the fluid particles' volume remains constant and the streamlines of the velocity field present topological properties similar to the magnetic field lines in electromagnetism, since Maxwell-Thomson equation states that the magnetic field is always divergence-free (i.e. $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$).

Physical meaning of $\boldsymbol{\nabla} \wedge \mathbf{v}$

We define the vorticity $\boldsymbol{\omega}$ as:

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{v} \tag{2.53}$$

and because the result (2.49) can be generalized, the vorticity describes the local rotation of fluid particles. We define an irrotational flow as:

$$\boxed{\text{Irrotational flow} \iff \boldsymbol{\nabla} \wedge \mathbf{v} = \mathbf{0}} \tag{2.54}$$

In this case, we say that the velocity is a curl-free vector field. Also, due to the cancellation rule "$\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} f) = \mathbf{0}$ for any scalar field f", an irrotational flow is necessarily associated to a scalar field Φ such as:

$$\mathbf{v} = \boldsymbol{\nabla}\Phi \tag{2.55}$$

and the velocity field is said to be potential. This is the reason why an irrotational flow is also called a potential flow.

> Like for a conservative force in classic mechanics, which can be written as the (opposite of) gradient of a potential energy, this approach considerably simplifies the physical interpretation of the problem since it replaces a vector field ($\mathbf{v}$) by a scalar field (Φ).

Irrotationality and streamlines

Beware, the fact that a flow is irrotational or not does not mean anything about its streamlines being straight or not. For instance, the streamlines of a flow with a velocity field $\mathbf{v} = ax\,\hat{\mathbf{y}}$ are straight (Fig.XXX) and yet, the flow is not irrotational (because $\boldsymbol{\nabla} \wedge \mathbf{v} \neq \mathbf{0}$). On the reverse, the streamlines of a flow with a velocity field $\mathbf{v} = \Gamma/(2\pi r)\,\hat{\boldsymbol{\theta}}$ in cylindrical coordinates are circular (Fig.XXX) and yet, the flow is irrotational. Indeed, using the tabulated expression of the curl operator in cylindrical coordinates, one can show that $\boldsymbol{\nabla} \wedge \mathbf{v} = \mathbf{0}$ (exercise: do it). In this case, the fact that the flow is irrotational manifests through the fact that the fluid particles are in circular translation (like the capsules in a ferris wheel): for instance, in a sink that empties, the orientation of a match at the surface does not change because it is in circular translation (see these animations).

Finally, it is important to keep in mind that the fact that a flow is irrotational or not depends on the frame $\mathcal{R}$ we work in. Indeed, for a fluid at rest in a cylindrical pipe rotating around its main axis $\hat{\mathbf{z}}$ at uniform angular speed $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$, we have $\boldsymbol{\nabla} \wedge \mathbf{v} = \mathbf{0}$ in the co-rotating frame (Fig. XXX). Yet, in the lab frame, the fluid's velocity field is $\mathbf{v} = r\omega\,\hat{\boldsymbol{\theta}}$ and we have $\boldsymbol{\nabla} \wedge \mathbf{v} = 2\,\boldsymbol{\omega} \neq \mathbf{0}$.

2.2 Local mass conservation equation

2.2.1 Flow rates

Volumetric flow rate

Let (S) be an infinitesimal oriented surface of normal vector $\mathrm{d}\,\mathbf{S}$. Then, we define the volumetric flow rate D_V as the amount of volume crossing the surface (S) per unit time, and counted positively (resp. negatively) in the direction of $\mathrm{d}\,\mathbf{S}$ (resp. in the direction opposite to $\mathrm{d}\,\mathbf{S}$):

$$D_V = \frac{\delta V}{\mathrm{d}t} \tag{2.56}$$

In order to provide an expression which relates D_V to the velocity field, we want to determine the expression of δV, the infinitesimal amount of volume which crosses the surface (S) within an infinitesimal amount of time $\mathrm{d}t$. To do so, we consider the infinitesimal surface element $\mathrm{d}\,\mathbf{S}$ centered in a point located in $\mathbf{r}$ (Fig. XXX). The velocity of fluid particles near this point is $\mathbf{v}(\mathbf{r}, t)$ such as between times t and $t + \mathrm{d}t$, fluid particles move by an infinitesimal displacement vector:

$$\mathrm{d}\,\mathbf{r} = \mathbf{v}(\mathbf{r}, t)\,\mathrm{d}t \tag{2.57}$$

Notice that, like in Fig. XXX, $\mathbf{v}(\mathbf{r}, t)$ is not necessarily colinear with $\mathrm{d}\,\mathbf{S}$. Therefore, the fluid which crosses the infinitesimal surface element $\mathrm{d}\,\mathbf{S}$ between times t and $t + \mathrm{d}t$ is contained in the oblique cylinder of base $\mathrm{d}\,\mathbf{S}$ and of generatrix $\mathrm{d}\,\mathbf{r}$. Its infinitesimal volume[7] $\delta^2 V$ is:

$$\delta^2 V = \mathrm{d}\,\mathbf{r} \cdot \mathrm{d}\,\mathbf{S} = \mathbf{v}(\mathbf{r}, t) \cdot \mathrm{d}\,\mathbf{S}\,\mathrm{d}t \tag{2.58}$$

We now integrate over the whole surface (S) to obtain the infinitesimal volume δV crossing (S) within $\mathrm{d}t$:

$$\delta V = \iint_{(S)} \mathbf{v}(\mathbf{r}, t) \cdot \mathrm{d}\,\mathbf{S}\,\mathrm{d}t = \mathrm{d}t \iint_{(S)} \mathbf{v}(\mathbf{r}, t) \cdot \mathrm{d}\,\mathbf{S} \tag{2.59}$$

We conclude that the volumetric flow rate D_V defined in equation (2.56) can be written as[8]:

$$\boxed{D_V = \iint_{(S)} \mathbf{v}(\mathbf{r}, t) \cdot \mathrm{d}\,\mathbf{S}} \tag{2.60}$$

The scalar product which enters this expression means that if the velocity of the fluid particles is parallel to the surface (S) in any point of the surface (i.e. $\mathbf{v}(\mathbf{r}, t) \perp \mathrm{d}\,\mathbf{S}$), then, there is no flow across this surface.

> When $\mathrm{d}t \longrightarrow 0$, the ratio $\delta V/\,\mathrm{d}t$ is not a derivative since δV is not the infinitesimal variation of a function of time. It explains why, rigorously speaking, we must write δV rather than $\mathrm{d}V$.

Mass flow rate

Let (S) be an infinitesimal oriented surface of normal vector $\mathrm{d}\,\mathbf{S}$. Then, we define the mass flow rate D_m as the amount of mass crossing the surface (S) per unit time, and counted

[7] The exponent 2 in $\delta^2 V$ indicates that it is a 2^{nd}-order infinitesimal because it is the amount of volume crossing an infinitesimal surface within an infinitesimal amount of time.

[8] This is an algebric volumetric flow rate since its sign can be positive or negative whether $\mathbf{v}$ is in the direction of, or in the opposite direction of $\mathrm{d}\,\mathbf{S}$ respectively.

positively (resp. negatively) in the direction of $\mathrm{d}\,\mathbf{S}$ (resp. in the direction opposite to $\mathrm{d}\,\mathbf{S}$):

$$D_m = \frac{\delta m}{\mathrm{d}t} \tag{2.61}$$

Like previously, we evaluate the infinitesimal amount of mass $\delta^2 m$ crossing an infinitesimal surface element $\mathrm{d}\,\mathbf{S}$ within an infinitesimal amount of time $\mathrm{d}t$. It is related to the infinitesimal amount of volume derived in equation (2.58) by the definition of the mass density ρ:

$$\delta^2 m = \rho(\mathbf{r}, t)\delta^2 V = \rho(\mathbf{r}, t)\, \mathbf{v}(\mathbf{r}, t) \cdot \mathrm{d}\,\mathbf{S}\,\mathrm{d}t \tag{2.62}$$

We now integrate over the whole surface (S) to obtain the infinitesimal mass δm crossing (S) within $\mathrm{d}t$:

$$\delta m = \iint\limits_{(S)} \rho(\mathbf{r}, t)\, \mathbf{v}(\mathbf{r}, t) \cdot \mathrm{d}\,\mathbf{S}\,\mathrm{d}t = \mathrm{d}t \iint\limits_{(S)} \rho(\mathbf{r}, t)\, \mathbf{v}(\mathbf{r}, t) \cdot \mathrm{d}\,\mathbf{S} \tag{2.63}$$

We conclude that the mass flow rate D_m defined in equation (2.61) can be written as:

$$D_m = \iint\limits_{(S)} \rho(\mathbf{r}, t)\, \mathbf{v}(\mathbf{r}, t) \cdot \mathrm{d}\,\mathbf{S} \tag{2.64}$$

We introduce the mass flow density vector $\mathbf{j}$ defined as:

$$\mathbf{j} = \rho\,\mathbf{v} \tag{2.65}$$

such as:

$$\boxed{D_m = \iint\limits_{(S)} \mathbf{j}(\mathbf{r}, t) \cdot \mathrm{d}\,\mathbf{S}} \tag{2.66}$$

2.2.2 Consequences for boundary conditions

Fixed obstacle

By definition, fluid particles cannot flow through a fixed, impenetrable and indeformable obstacle. Let us write the surface element vector $\mathbf{dS} = \mathrm{d}S\,\mathbf{n}$ with $\mathbf{n}$ the unit vector normal to the surface and of same orientation as $\mathbf{dS}$ (i.e. $\mathrm{d}S > 0$). It means that for any point at surface of a fixed obstacle:

$$\mathbf{v} \cdot \mathbf{n} = 0 \tag{2.67}$$

Therefore, in any point of the surface of a fixed obstacle, the normal component of the velocity is always zero.

Deformable obstacle

Let us consider an impenetrable obstacle which can however be deformed (e.g. an inflatable balloon). Locally, it means that the obstacle is not fixed in the frame of study $\mathcal{R}$. We consider the point P on the obstacle, we write $\mathbf{v_S}$ the velocity of this point and we define the frame $\mathcal{R}'$ moving with this point. In $\mathcal{R}'$, the situation is analogous to the aforementioned case of a fixed obstacle. The Galilean transformation of fluid velocity from frame $\mathcal{R}$ to frame $\mathcal{R}'$ gives the fluid velocity $\mathbf{v}' = \mathbf{v} - \mathbf{v_S}$ in the frame $\mathcal{R}'$ and we use equation (2.67) to get, for any point at the surface of a deformable obstacle:

$$\mathbf{v} \cdot \mathbf{n} = \mathbf{v_S} \cdot \mathbf{n} \tag{2.68}$$

Therefore, in any point of the surface of a deformable obstacle, the normal component of the fluid velocity is the same as the normal component of the velocity of the point on the obstacle.

Interface between two non-miscible fluids

Let us consider the interface between two non-miscible fluids indexed 1 and 2. It behaves as a deformable surface that the two fluids cannot cross such as we have, for any point at the interface between the two fluids:

$$\mathbf{v_1} \cdot \mathbf{n} = \mathbf{v_2} \cdot \mathbf{n} \tag{2.69}$$

Generalization

More generally, the normal component of the fluid velocity must be continuous through the boundaries of the fluid.

> We will see in chapter 3 that viscosity enforces the continuity of the tangential component of the velocity at the boundaries. For instance, the velocity must cancel at the surface surface of a fixed and impenetrable obstacle.

2.2.3 Local mass conservation equation

Let (V) be a fixed volume of mass m centered in a point $M(\mathbf{r})$ and surrounded by the closed surface (Σ). We follow the thermodynamic convention by counting positively (resp. negatively) the mass which enters into this volume (resp. which leaves this volume). Between times t and $t + \mathrm{d}t$, the infinitesimal amount of mass δm which enters into this volume is given by the definition of the mass flow rate (2.66):

$$\delta m = -\,\mathrm{d}t \oiint_{(\Sigma)} \mathbf{j} \cdot \mathbf{d\,S} \tag{2.70}$$

where $\mathrm{d}\,\mathbf{S}$ is the surface element which is always, for a closed surface, oriented outwards, hence the minus sign. On the other hand, the intrinsic variation of mass m of the volume (V) between times t and $t + dt$ is given by:

$$\mathrm{d}m = m(t + \mathrm{d}t) - m(t) \tag{2.71}$$

$$= \iiint_{(V)} \rho(\mathbf{r}, t + \mathrm{d}t)\,\mathrm{d}V - \iiint_{(V)} \rho(\mathbf{r}, t)\,\mathrm{d}V \tag{2.72}$$

$$= \iiint_{(V)} \left[\rho(\mathbf{r}, t + \mathrm{d}t) - \rho(\mathbf{r}, t)\right] \mathrm{d}V \tag{2.73}$$

> Here, we evaluate ρ at the same locations $\mathbf{r}$ both at times t and $t + \mathrm{d}t$ because we adopt an Eulerian viewpoint since the volume (V) is fixed and does not follow the fluid motion.

We perform a Taylor expansion of the term within brackets in equation (2.71) up to 1^{st}-order in $\mathrm{d}t$ to get:

$$\mathrm{d}m = \iiint_{(V)} \frac{\partial \rho}{\partial t}\, \mathrm{d}t\, \mathrm{d}V \tag{2.74}$$

$$= \mathrm{d}t \iiint_{(V)} \frac{\partial \rho}{\partial t}\, \mathrm{d}V \tag{2.75}$$

Since the mass is locally conserved[9], we have:

$$\mathrm{d}m = \delta m \tag{2.76}$$

In order to bring together the two integrals (2.70) and (2.75), we apply the Green-Ostrogradski divergence theorem (section **??**) to re-express δm:

$$\delta m = -\,\mathrm{d}t \iiint_{(V)} (\boldsymbol{\nabla} \cdot \mathbf{j})\, \mathrm{d}V \tag{2.77}$$

and so, after using equations (2.77) and (2.75), and simplifying by $\mathrm{d}t$:

$$\iiint_{(V)} \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} \right] \mathrm{d}V = 0 \tag{2.78}$$

Since this equality is true for any volume (V), we have the local mass conservation equation (a.k.a. the continuity equation):

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0} \tag{2.79}$$

> This equation is called local because it implicitly assumes that the fields ρ and $\mathbf{v}$ are continuous in space. If there is a discontinuity, for instance a shock, then we cannot simplify the integral in equation (2.78) and we work with the global mass conservation equation. See exercise on Rankine-Hugoniot conditions at shock for instance XXX.

2.2.4 Consequences for steady flows

In a steady flow, the fields do not depend on time. In particular, $\partial_t \rho = 0$ so we have, from the continuity equation (2.79):

$$\boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{2.80}$$

In this case, the mass flow density vector $\mathbf{j}$ is said to be a divergence-free vector field (section 2.2.6).

[9] Beware, this statement is wrong and this law needs to be adapted in case of mass-energy exchanges, for instance through nuclear reactions.

Quasi-1D flow

Let us consider an axisymmetric nozzle of main axis $\hat{\mathbf{z}}$ (Fig. XXX). The walls of the nozzle are indeformable. We want to study the flow of a fluid in this nozzle. We write $S(z)$ the transverse cross-section of the nozzle, normal to the $\hat{\mathbf{z}}$-axis. This section depends on the z coordinate.

1. Write the expression of the mass flow rate $D_m(z)$ through a fiducial section $S(z)$ as a function of the mass density field $\rho(\mathbf{r}, t)$ and of the velocity field $\mathbf{v}(\mathbf{r}, t)$. You will decompose the latter in its components v_r, v_θ and v_z in a cylindrical basis you will represent in Fig.XXX.

2. First, we adopt a global approach by considering a fiducial macroscopic volume. Let z_1 and $z_2 > z_1$ be two arbitrary coordinates.

   a Write the expression of the mass flow rates $D_m(z_1)$ and $D_m(z_1)$ through the cross sections $S(z_1)$ and $S(z_2)$.

   b What is the mass flow rate through the lateral surface corresponding to the indeformable wall of the nozzle?

   c We note m the mass contained in this volume. Gather all the mass flow rates computed above to express the global mass conservation within this volume.

   d From now on, we work in the steady flow approximation (i.e. $\partial_t = 0$). How does the expression derived in the previous question simplify?

3. We now use the approximation of the quasi-1D flow which means that:

   - v_r, $v_\theta \ll v_z$
   - we can neglect the dependencies of ρ and v_z on the r and θ coordinates: $\rho(\mathbf{r}) \sim \rho(z)$ and $v_z(\mathbf{r}) \sim v_z(z)$.

   (a) In this case, how does the general expression of the mass flow rate D_m simplify? In particular, how does the result obtained in question 2.d simplify?

   (b) Now, let us adopt a local viewpoint. Give the local mass conservation equation in the steady flow approximation.

   (c) In cylindrical coordinates, the divergence of a vector field $\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_z\,\hat{\mathbf{z}}$ is:

   $$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \tag{2.81}$$

   Which of the three term is strictly zero in the axisymmetric approximation? Can we use the hypothesis $v_r \ll v_z$ to simplify the expression of $\boldsymbol{\nabla} \cdot \mathbf{v}$ any further?

2.2.5 Consequences for incompressible flows

Back to the physical meaning of $\boldsymbol{\nabla} \cdot \mathbf{v}$

Now that we have seen the continuity equation, we can generalize the physical meaning of $\boldsymbol{\nabla} \cdot \mathbf{v}$ we had inferred from specific situations in section 2.1.4. Let us rewrite the continuity equation using the Lagrangian derivative:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{\rho}\, \mathbf{v}) = 0 \tag{2.82}$$

$$\frac{\partial \rho}{\partial t} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{v} + \mathbf{v} \cdot (\boldsymbol{\nabla} \rho) = 0 \tag{2.83}$$

$$\rho\, \boldsymbol{\nabla} \cdot \mathbf{v} + \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\rho) = 0 \tag{2.84}$$

$$\rho\, \boldsymbol{\nabla} \cdot \mathbf{v} + \frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \tag{2.85}$$

Therefore, $\boldsymbol{\nabla} \cdot \mathbf{v}$ is related to the relative variation in time of the mass density of a fluid particle as it moves:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = -\frac{1}{\rho} \frac{\mathrm{D}\rho}{\mathrm{D}t} \tag{2.86}$$

By definition, the mass m of a fluid particle does not change, which means that ρV is a constant, with V the volume of the fluid particle. Then, we have:

$$\rho V = \text{cst} \tag{2.87}$$

$$\ln(\rho V) = \text{cst} \tag{2.88}$$

$$\mathrm{d}(\ln(\rho V)) = 0 \tag{2.89}$$

$$\frac{\mathrm{d}(\rho V)}{\rho V} = 0 \tag{2.90}$$

$$\frac{\mathrm{d}\rho}{\rho} + \frac{\mathrm{d}V}{V} = 0 \tag{2.91}$$

$$\frac{\mathrm{d}\rho}{\rho} = -\frac{\mathrm{d}V}{V} \tag{2.92}$$

We can reinject the Lagrangian counterpart of this expression into equation (2.86) to obtain the result mentioned in equation (2.52):

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{V} \frac{\mathrm{D}V}{\mathrm{D}t} \tag{2.93}$$

Incompressible flows

We saw with equation (2.12) that a flow is said to be incompressible if the volume of all fluid particles is conserved as they move. Now that we have seen the continuity equation, we can re-express this definition as:

$$\boxed{\text{Incompressible flow} \iff \boldsymbol{\nabla} \cdot \mathbf{v} = 0} \tag{2.94}$$

This expression is more convenient than the one derived in equation (2.12) because it directly relates to the topological properties of the streamlines (section sec:topo).

2.2.6 Properties of divergence-free vector fields

Consequences for the reading of streamlines maps

Divergence-free vector fields are also called flux conservative for the following reason. In the exercise "Quasi-1D flow" above, we saw that, for a steady flow, $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$ was linked to the product $j \times S$ being uniform (i.e. independent on z) in the pipeline. Similarly, when we plot the streamlines of an incompressible flow, one can always consider an infinitesimal tube of streamlines surrounding a given streamline (Fig XXX). This tube is infinitesimal in the sense that its transverse sections are small enough that we can perform the quasi-1D flow approximation: the velocity depends only on the coordinate along the tube's main axis, not on the position over the transverse surface. Let S be the tube section and l the curvilinear coordinate along the streamline. Therefore, the same reasoning as in the exercise above leads to, using the expression (2.60) of the volumetric flow rate D_V:

$$\boxed{\text{Incompressible flow} \iff \partial_l(vS) = 0 \iff D_V \text{ uniform along each streamline}} \tag{2.95}$$

> Similarly, we can say that for a steady flow, we have:
>
> $$\text{Steady flow} \iff \partial_l(\rho vS) = 0 \iff D_m \text{ uniform along each streamline} \tag{2.96}$$

The property (2.95) means that in a map representing the streamlines of an incompressible flow (Fig.XXX), the magnitude of the velocity increases (resp. decreases) when the streamlines get closer together (resp. further apart).

> Similarly, in a map of magnetic field lines, the magnetic field is higher (resp. lower) where the distance between the field lines is smaller (resp. higher), since the magnetic field is also a divergence-free vector field.

Divergence theorem

1. Use the divergence theorem to prove that the integral through a closed surface of any divergence-free vector field $\mathbf{A}$ is necessarily null.
2. Show that this is in agreement with the result you obtained in question 2 of the exercise "Quasi-1D flow" above.

Analogy with electrokinematics

When you studied electric circuits, you saw the Kirchhoff's laws. They are due to the fact that the flux of electric charges $\mathbf{j} = \rho_e \mathbf{v}$ in the circuit is conservative(with $\rho_e = \mathrm{d}q/\mathrm{d}V$ the density of electric charges). Similarly, in an incompressible flow, we have the following laws:

- at a node between multiple pipelines, the sum of the volumetric flow rates is null.
- in a pipeline, even if the cross-section varies, the volumetric flow rate does not depend on the coordinate along the pipeline.

Potentials

Last but not least, due to the cancellation rule "$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{A}) = 0$ for any vector field A", a steady flow is necessarily associated to a vector field $\mathbf{A}$ such as:

$$\mathbf{j} = \boldsymbol{\nabla} \wedge \mathbf{A} \tag{2.97}$$

and an incompressible flow is necessarily associated to a vector field $\mathbf{A}$ such as:

$$\mathbf{v} = \boldsymbol{\nabla} \wedge \mathbf{A} \tag{2.98}$$

Approximation	Math. expression	Physical meaning
Stationary flow	$\mathbf{v} = \mathbf{0}$	velocity field is uniformly null
Incompressible fluid	$\boldsymbol{\nabla}\rho = \mathbf{0}$	density is the same everywhere
Steady flow	$\partial_t = 0 \quad (\Rightarrow \boldsymbol{\nabla}\cdot\mathbf{j} = 0\,)$	Eulerian observers do not see variations
Incompressible flow	$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \iff \boldsymbol{\nabla}\cdot\mathbf{v} = 0$	fluid particles' volume does not change
Irrotational flow	$\boldsymbol{\nabla}\wedge\mathbf{v} = \mathbf{0}$	fluid particles do not rotate

TABLE 2.1: Summary of the common approximations used in hydrodynamics. The two 2 ones are very restrictive while the 3 following ones apply to a broader range of situations.

Notice that in both cases, the vector $\mathbf{A}$ is not unique since $\mathbf{A} + \boldsymbol{\nabla}f$ is also solution, for any scalar field f (because the curl of a gradient is always zero).

> It is the analogous of the notion of magnetic vector potential from which the magnetic field, which is always divergence-free, derives: $\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{A}$.

Matriochkas

Using the summary table 2.1, answer the following questions.

1. Are stationary flows $\in$ steady flows? Or are steady flows $\in$ stationary flows?
2. Are incompressible flows $\in$ incompressible fluids? Or are incompressible fluids $\in$ incompressible flows? Which additional property does an incompressible fluid needs to be a (very) specific case of incompressible flow?
3. Are incompressible flows $\in$ steady flows? Or are steady flows $\in$ incompressible flows? Which additional property does a steady flow needs to be a (very) specific case of incompressible flow?

2.2.7 Global mass conservation

Rankine-Hugoniot?

2.3 Application: air flow over a plane wing

2.3.1 Modeling

We consider a plane wing of cylindrical shape (Fig.XXX). The main axis of the cylinder is $\hat{\mathbf{z}}$. The wing is in uniform rectilinear motion in the ground frame $\mathcal{R}$ at the constant velocity $U\,\hat{\mathbf{x}}$ where $U > 0$ and the $\hat{\mathbf{x}}$ unit vector is represented in Fig. XXX.

First of all, in order to express the boundary conditions on the wing in a more convenient way, we work in the frame $\mathcal{R}'$ co-moving with the wing, that is to say the frame where the wing is at rest. Far upstream from the wing, the air is at rest in the ground frame $\mathcal{R}$ (i.e. there is no wind). Therefore, in this frame $\mathcal{R}'$, the wing sees the air far upstream moving in its direction at the uniform and constant speed:

$$\mathbf{v} \sim \mathbf{v}_\infty = -U\,\hat{\mathbf{x}} \tag{2.99}$$

We make the following assumptions:

- The flow is steady in the frame $\mathcal{R}'$, which is coherent with the uniform rectilinear motion of the wing.
- The flow is incompressible, which is an acceptable assumption provided the motion is subsonic.
- The height of the cylinder is much larger than its radius R such as we can assume that the flow is entirely contained in the plane $(O, \hat{\mathbf{x}}, \hat{\mathbf{y}})$ (i.e. $\mathbf{v} \cdot \hat{\mathbf{z}} = 0$) and invariant by translation along the $\hat{\mathbf{z}}$ axis[10].
- The flow is irrotational, which is a fair assumption if we neglect the air viscosity (chapter 3).

2.3.2 Velocity field

The fact that the flow is both incompressible and irrotational provides an important information. Indeed, we have seen in section 2.2.6 that an irrotational flow is associated to a scalar potential Φ such as:

$$\mathbf{v} = \boldsymbol{\nabla}\Phi \tag{2.100}$$

Furthermore, the flow is incompressible so $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ and we have:

$$\Delta\Phi = 0 \tag{2.101}$$

where is Δ the Laplace operator.

> Once more, we retrieve an expression which, from a mathematical point of view, is exactly the one we obtained in other fields:
>
> - in electromagnetism, it is the expression of the Poisson equation in the absence of electric charge (i.e. $\rho_e = 0$), which, once solved for specific boundary conditions, gives the electric potential Φ from which we can deduce the electric field $\mathbf{E} = -\boldsymbol{\nabla}\Phi$.
> - in classic mechanics, it is the expression of the Gauss equation in the absence of mass (i.e. $\rho = 0$), which, once solved for specific boundary conditions, gives the gravitational potential Φ from which we can deduce the gravitational field $\boldsymbol{\mathcal{G}} = -\boldsymbol{\nabla}\Phi$.

Since the flow is invariant by translation along the $\hat{\mathbf{z}}$ axis, Φ depends on r and θ only. In this case, it can be shown that the general solution of equation (2.101) can be written as a series:

$$\Phi(r,\theta) = \alpha_0 \ln(r) + \beta_0 + \sum_{i=1}^{+\infty} (\alpha_i r^i + \beta_i r^{-i}) \cos(i\theta) + \sum_{i=1}^{+\infty} (\delta_i r^i + \gamma_i r^{-i}) \sin(i\theta) \tag{2.102}$$

where the coefficients α_i, β_i, δ_i and γ_i are constants whose expression depends on the configuration (e.g. the symmetries of the problem and the boundary conditions).

[10] We say that we neglect the boundary effects along the $\hat{\mathbf{z}}$ axis.

Reminder - Linear independence

The series (2.102) can be interpreted as the decomposition on a basis of linearly independent vectors (in the generalized sense of vectors in linear algebra). Therefore, terms can be identified by pairs, like for polynomes. For instance:

$$a_0 + a_1 x + a_2 x^2 = b_0 + b_1 x + b_2 x^2 \iff \begin{cases} a_0 = b_0 \\ a_1 = b_1 \\ a_2 = b_2 \end{cases} \tag{2.103}$$

It is a very powerful property which is absolutely not obvious.

> For a given r, the series (2.102) is a Fourier series, which is not surprising since the solutions are necessarily 2π-periodic in θ.

Expression of the velocity field

1. (a) What is the plane of symmetry of the wing?
   (b) What does it mean for $\Phi(r, \theta)$ and $\Phi(r, -\theta)$?
   (c) What are the consequences for the coefficients δ_i and γ_i in the series (2.102)?
2. (a) Using the expression (2.99) of the velocity field far upstream, give Φ as a function of U, r and $\cos(\theta)$.
   (b) By identification with the series (2.102) in the limit $r \longrightarrow +\infty$, deduce the expression of the α_i coefficients.
3. (a) In $\mathcal{R}'$, the wing is fixed, impenetrable and indeformable. What does it mean for the normal component of the velocity field on the wing?
   (b) We remind the general expression of $\boldsymbol{\nabla}\Phi$ in cylindrical coordinates:
   $$\boldsymbol{\nabla}\Phi = \frac{\partial \Phi}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{\partial \Phi}{\partial z}\,\hat{\mathbf{z}} \tag{2.104}$$
   Deduce the expression of $\partial_r \Phi(r = R, \theta)\ \forall\theta$, where R is the radius of the wing.
   (c) Deduce the expression of the β_i coefficients.
4. (a) The problem is 2D and invariant along $\hat{\mathbf{z}}$. What is the most general expression of the velocity field in cylindrical coordinates? You will explicit the dependencies of the velocity's components.
   (b) Deduce from Φ the expression of each component of the velocity field.
5. (numerical exercise) Plot the streamlines.
6. On the wing, where is the flow velocity maximal? Where is the flow velocity lower than U? Is it coherent with the velocity being higher (resp. lower) where the distance between the streamlines is smaller (resp. lower)?

2.4 Exercises

2.4.1 Gravity waves

We perform a kinematics analysis of the gravity waves in a fluid (Fig.XXX). Let us assume that an external excitation[11] sustains sinusoidal waves at the surface of a pool of depth h and of horizontal extent very large compared to h. We study these waves in a plane $(\hat{\mathbf{x}}, \hat{\mathbf{z}})$ where $\hat{\mathbf{x}}$ is horizontal and $\hat{\mathbf{z}}$ is vertical and oriented upwards. The $z = 0$ origin is set to the bottom of the pool which is a fixed and impenetrable boundary. These waves have a period T and an amplitude very small compared to h. The edges of the pool are transparent: we disperse colored tracers in the pool and shoot a picture with an exposure time of T. We notice that:

- each fluid particle has an elliptic trajectory.
- the ellipses are quasi-circular at the surface, in $z = h$, and their vertical-to-horizontal aspect ratio decreases as we go to the bottom, in $z = 0$.

We assume that the flow is irrotational[12] so $\exists\Phi$ such as $\mathbf{v} = \boldsymbol{\nabla}\Phi$. Due to the invariances of the problem, the velocity field is periodic in time t, with period T, and in space coordinate x, with period λ. However, it has no reason to be periodic in z. Therefore, we conclude that the potential Φ can be written as:

$$\Phi(x, z, t) = f(z) \cos\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right] \tag{2.105}$$

where f is an arbitrary function of z. At a given z, this potential yields a plane progressive wave of group speed $v = \lambda/T$. We write $\omega = 2\pi/T$ the angular frequency and $k = 2\pi/\lambda$ the wave number.

1. (a) We assume that the flow is also incompressible. Determine the expression of the differential equation for Φ in Cartesian coordinates.
   (b) Re-inject the expression (2.105) in this differential equation to determine a second order linear ordinary differential equation for f.
   (c) Give the expression of the general solution of this equation as a function of two unknown constants.
2. (a) Use the boundary condition at the bottom of the pool to show that $f(z) = A\cosh(kz)$ with A an unknown constant.
   (b) Deduce the expressions of $v_x(x, z, t)$ and $v_z(x, z, t)$ as a function of A, ω and k.
3. These components $v_x(x, z, t)$ and $v_z(x, z, t)$ of the Eulerian velocity field can also be interpreted as the components of a fluid particle of coordinates $(x_p(t), z_p(t))$. However, this system of two coupled non-linear ordinary differential equations does not have any analytic solution.
   (a) (numerical exercise) Compute the solutions numerically.
   (b) Assuming that the dimensions of the elliptical trajectories are all very small compared to the wavelength λ, how can we simplify these equations? You will introduce the time-averaged position of the fluid particle, $\langle x_p\rangle$ and $\langle z_p\rangle$.
   (c) Integrate each of the two ordinary differential equations in order to obtain the expression of the fluid particle's trajectory.

[11]Typically, the wind over a lake.

[12]The elliptic motion of the fluid particles does not imply $\boldsymbol{\nabla} \wedge \mathbf{v} \neq \mathbf{0}$.

(d) (numerical exercise) Plot these trajectories.

(e) Determine the aspect ratio of the elliptical trajectories as a function of z/λ only. Is it coherent with the observation we made?

4. At a given time, determine the equation $g(x, z)$ of the streamlines. Are they identical to the fluid particles' trajectories?

5. We write $\zeta(x, t)$ the equation of the surface of the pool.

   (a) What is the time-average $\langle zeta \rangle$ of $\zeta(x, t)$?

   (b) Since the amplitude of the waves at the surface is low, we assume that the vector normal to the surface is always $\sim \hat{\mathbf{z}}$. Deduce that:

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z} \tag{2.106}$$

   (c) Deduce the expression of $\zeta(x, t)$.

   (d) Is the component of the fluid's velocity normal to the surface null?

> In this exercise, we studied a type of waves called gravity waves. They are defined as the waves generated in a fluid when the force of gravity or buoyancy tries to restore equilibrium, something which was hidden in the assumption of an irrotational flow (since here, we focus on the kinematics aspects only, not on the source of the motion). Gravity waves lies at the core of the shallow water equations, a cornerstone in oceanography and climatology. They have nothing to do with the gravitational waves which correspond to the oscillations of spacetime induced by moving bodies, for instance two compact objects in close orbit.

2.4.2 Tornado

We model a tornado as a vertical cylinder of radius a very small compared to its vertical extent, such as we assume the flow to be invariant by translation along the cylinder's axis. We neglect any vertical or radial motion of the flow.

1. Given these hypothesis, what is the expression of the velocity field $\mathbf{v}$ in the tornado?

2. We assume that within the tornado ($r < a$), the vorticity $\boldsymbol{\omega}$ is given by:

$$\boldsymbol{\omega} = 2\omega_0\,\hat{\mathbf{z}} \tag{2.107}$$

   while it is null outside ($r > a$). Use the Stokes theorem along a horizontal circle of radius r centered on the tornado's axis in order to deduce the expression of $v_\theta = \mathbf{v} \cdot \hat{\boldsymbol{\theta}}$ within and outside the tornado. Where is the velocity maximum?

3. Now, we work in the limit of an infinitely thin tornado, when $a \longrightarrow 0$ and $\omega_0 \longrightarrow +\infty$ in a way such as $\omega_0 a^2$ remains finite.

   (a) Deduce the expression of v_θ.

   (b) Why is legitimate to look for Φ such as $\mathbf{v} = \boldsymbol{\nabla}\Phi$?

   (c) The expression of the gradient of a scalar field Φ in cylindrical coordinates is:

$$\boldsymbol{\nabla}\Phi = \begin{pmatrix} \partial_r \Phi \\ \frac{1}{r}\partial_\theta \Phi \\ \partial_z \Phi \end{pmatrix} \tag{2.108}$$

   Deduce the expression of Φ.

> Beware, in this result, you will find that Φ is not 2π-periodic in θ, which is an unfortunate consequence of the fact that the z-axis introduces a topological discontinuity...

Chapter 3
Viscosity

Viscosity is an intrinsically dissipative process. It means that it is associated to the irreversible conversion of bulk energy (i.e. a large scale coherent energy which can be converted into work for instance) into a more spread form of energy. For instance, the dissipation of the bulk kinetic energy of a flow at a shock produces heat, that is to say kinetic energy of individual particles. Each time entropy is produced, it indicates that we are dealing with a dissipative process. It typically arises from the gradient of an intensive variable (e.g. temperature T, particle density n, electric potential V or chemical potential χ). Examples of dissipative processes are particle diffusion, chemical reactions, shocks or magnetic reconnection.

Joule expansion

Is Joule expansion intrinsically irreversible? To address this question, we successively answer the following intermediate ones. Joule expansion is a thermodynamic transformation which takes place in two containers adiabatically isolated from the outside (i.e. no heat exchange), and with rigid walls (i.e. no work exchange). We start with one container filled with an ideal gas at temperature T_i, pressure P_i and of volume V_i, while the other container is empty (Fig XXX a). We suddenly remove the partition between the two parts of the container without doing any work, and the ideal gas eventually fills both containers. In its final state, it has a temperature T_f, pressure P_f and a volume $V_f = 2V_i$.

1. (a) Using the first principle of thermodynamics, determine the variation of internal energy of the walls ΔU_w and of the vacuum ΔU_v from the initial to the final state.
   (b) Considering the whole system including the gas, the walls and the vacuum, determine the variation of internal energy of the gas ΔU_g from the initial to the final state.

   (a) Using Joule first law, determine the variation of temperature of the gas $\Delta T = T_f - T_i$.
   (b) Deduce the final pressure P_f as a function of the initial pressure P_i.

   (a) Use the result on ΔU_g above to express the variation ΔS_g of the gas entropy as the integral of a function of the state variables P, T and V. What is δS^{cr}?
   (b) Since the gas is ideal, show that this expression simplifies in $\Delta S_g = nR \ln 2$, where n is the number of moles of gas and R is the ideal gas constant.

2. In practice, since the gas is not perfectly ideal, we measure a change of the temperature $\Delta T \neq 0$. Show that we can use this measure to constrain the parameter a in the equation-of-state of Van der Waals, which represents the strength of the interactions between particles.

This thermodynamic transformation is adiabatic ($\delta Q = 0$) but it is irreversible ($\delta S^{\mathrm{cr}} \neq 0$), which shows that these two properties are different. Indeed, the second principle of thermodynamic states that the infinitesimal change of entropy $\mathrm{d}S$ of a system during a thermodynamic transformation is:

$$\mathrm{d}S = \delta S^{\mathrm{ex}} + \delta S^{\mathrm{cr}} \tag{3.1}$$

where δS^{ex} is the entropy the system exchanges with the outside while δS^{cr} is the entropy produced in the process. The former is given by:

$$\delta S^{\mathrm{ex}} = \frac{\delta Q}{T} \tag{3.2}$$

where T is the temperature at which the infinitesimal transformation took place: if it changes through the transformation, it needs to be accounted for in the integral to get the total change of entropy ΔS. This quantity can either be positive or negative depending on whether heats flows towards ($\delta Q > 0$) or away ($\delta Q < 0$) from the system. In the case of an adiabatic transformation, $\delta Q = 0$ and there is no entropy exchanged. On the other hand, the entropy created δS^{cr} in the process is a quantity which depends on whether the process is dissipative or not, something which is linked to the microscopic aspects of the transformation. It is null if and only if the transformation is reversible and otherwise, it is necessarily positive.

Reversibility

1. A vertical insulated cylinder of height L and section S, closed by a piston of mass m_0, contains an ideal gas. We denote γ the ratio C_p/C_v of the heat capacities at constant pressure and constant volume. The initial temperature of the gas is T_i. In order to simplify the calculations, we assume throughout the following that the cylinder is placed in a vacuum. A mass m is suddenly released onto the piston. The piston moves downward by an amount $\Delta L > 0$.

   (a) Express ΔL and the final temperature T_f as a function of γ, L and x, where $x \hat{=} m/m_0$.

   (b) Determine the expression of the variation of entropy ΔS of the gas, and show that this is necessarily strictly positive for $x > 0$.

   (c) Imagine that we repeat this experiment but this time, instead of suddenly adding a mass m, we add sand grains one by one until reaching the mass m. What is ΔS in this case?

2. Now, back to Joule expansion. We saw that it was irreversible since $\delta S^{\mathrm{cr}} > 0$. What would have happened if, instead of removing one wall between two compartments of equal volume, we would have subdivided the empty container in infinitesimal sub-containers of volume $\mathrm{d}V$ each and removed successively the walls (Fig.XXX)? Let us try to answer this question with the following intermediate ones.

   (a) What is the infinitesimal variation of entropy $\mathrm{d}S$ each time we remove a wall, that is to say each time the volume goes from a volume V to a volume $V + \mathrm{d}V$?

   (b) Use a Taylor expansion to first order in $\mathrm{d}V/V$ to simplify this equation.

(c) Integrate this equation to obtain the total variation of entropy ΔS once all the walls were removed. Is it different from what we got in the exercise "Joule expansion"? What does it mean regarding the irreversibility of this process?

Gibbs mixing paradox

Repeat the previous exercise but this time, with two different ideal gases (e.g. dioxygen on one side and dinitrogen on the other), indexed 1 and 2, in the two containers at the initial state. Both have the same initial temperature, pressure and volume.

> And yet, if the two gases were identical (e.g. dioxygen on both sides), we would still get an increase of the entropy even if nothing happened since there is no difference between the inital and the final states. The operator can remove or replace the wall without any state parameters changing, starting with the molar concentration of the gas. The evolution is then adiabatic and reversible, so we must have $\Delta S = 0$. This problem is known as the "Gibbs paradox". The error that we make in the previous calculation comes from the fact that we implicitly assume that the molecules of the same gas are discernible (indexable because perfectly localizable). However, this discernibility is not based on any objective foundation because there is then no cause of irreversibility. Whereas in the case of two different gases (dioxygen and dinitrogen for example), the introduction of a semi-permeable membrane (letting the dinitrogen molecules pass in one direction and the dioxygen molecules in the other) would make it possible to irreversibly separate the mixed gases to return to the initial state.

3.1 Macroscopic description

We focus on a macroscopic approach of viscosity, starting with an experimental illustration. The goal is to provide a phenomenological description of this mechanism by introducing a parameter, the viscosity, which depends on the nature of the fluid we work with (e.g. chemical composition) but also on the thermodynamic conditions (e.g. temperature and pressure). This parameter encapsulates all the complexity which yields in the microscopic origin of viscosity. Apart from a couple of remarks, we will sweep this complexity under the carpet... for now[1].

In hydrostatics (where $\mathbf{v} = \mathbf{0}$), we have seen that the interaction of a fluid particle with its surroundings manifest through the notion of pressure which is a force per unit surface: it betrays the transfer of linear momentum from a fluid particle to another due to microscopic particles crossing the edges of a fluid particle. The component which was decisive in this computation was the normal component of the particles' velocity vector. Now, let us see what happens when we introduce a shearing velocity between fluid particles, that is to say a non-zero derivative of velocity in a direction normal to the velocity vector.

3.1.1 Preliminary experiment

We consider a large reservoir of height h containing glycerin. We introduce a colored tracer in a thin, vertical and cylindrical column of height $OA_0 = h$ centered on the segment OA_0 (Fig.XXX). At the top of this column, in A_0, we place a puck of surface S in contact with the glycerin, and we exert a force $\mathbf{F}$ on the puck in the horizontal direction such as it moves at

[1]The ones among you who are in a hurry can have a look at Fick's law of particle diffusion, a good entry point into the realm of ab initio viscosity.

a constant velocity $\mathbf{U} = U\,\hat{\mathbf{x}}$, with $U > 0$ and $\hat{\mathbf{x}}$ the unit vector in the horizontal direction of motion of the puck. We locate the position of the puck with point A, located in A_0 at time $t = 0$. This experiment leads to the following observations, for a small displacement of the puck:

- The column adheres (i) to the puck in A at the top and (ii) to the bottom O of the reservoir. Therefore, the vertical cylinder OA_0 progressively becomes inclined.
- The colored region around segment OA remains straight (although inclined), which means that any fluid particle in M follows the motion:

$$\mathbf{MM_0}(t) = \frac{z}{h}\,\mathbf{AA_0}(t) = \frac{z}{h}Ut\,\hat{\mathbf{x}} \tag{3.3}$$

  with M_0 the initial position of the fluid particle and z the vertical coordinate of the fluid particle taken from the bottom of the reservoir.
- The magnitude F of the force needed to move the puck is proportional to S and U, and inversely proportional to h.

Although we only move the puck at the top, we observe that the different horizontal layers of fluid also move, although at different speed. The only layer which remains strictly static is the one at the bottom, in O. Where does this fluid motion come from? The pressure field P is invariant by translation along x and depends only on the vertical coordinate z (chap. 1). Therefore, the infinitesimal pressure force acting on an infinitesimal fluid volume $\mathrm{d}V$ is:

$$\mathrm{d}\,\mathbf{F} = -\,\boldsymbol{\nabla} P\,\mathrm{d}V \,/\!\!/\, \hat{\mathbf{z}} \tag{3.4}$$

so pressure cannot be responsible for fluid motion in the $\hat{\mathbf{x}}$ direction. Therefore, there is necessarily another force, in the horizontal direction. Furthermore, if the whole column were to move at speed U, the situation would be similar to hydrostatics since we know that forces are invariant by change of inertial referential. Therefore, the fact that the fluid velocity depends on the vertical coordinate z is instrumental in producing this horizontal force we are looking for.

3.1.2 Viscosity

Empirical definition

Let $\mathbf{v} = v_x(z)\,\hat{\mathbf{x}}$ be the velocity field of a fluid. Consider an infinitesimal horizontal surface element $\mathrm{d}\,\mathbf{S} = \mathrm{d}x\,\mathrm{d}y\,\hat{\mathbf{z}}$ between two layers of fluid of vertical coordinate z (Fig.XXX). Then, the forces exerted by the fluid particle above z on the fluid particle below z are twofold:

- A vertical force, normal to the interface $\mathrm{d}S$ between the two layers, which corresponds to the pressure force:

$$\mathrm{d}\,\mathbf{F}_\perp = -P(x,y,z)\,\mathrm{d}\,\mathbf{S} \tag{3.5}$$

- A horizontal force, tangential to the interface $\mathrm{d}S$ between the two layers, which corresponds to the viscous force:

$$\mathrm{d}\,\mathbf{F}_{/\!\!/} = \eta\frac{\partial v_x}{\partial z}\,\mathrm{d}S\,\hat{\mathbf{x}} \tag{3.6}$$

These forces represent a phenomenological definition of the pressure P and of the dynamic viscosity η respectively. While P is an unknown scalar field, η is a constant characteristic of the fluid[2].

[2] Provided we neglect the dependence of η on the thermodynamic conditions (e.g. temperature and pressure).

	air	water	glycerin
viscosity (CGS)	$1.8 \cdot 10^{-4}$P	$1.0 \cdot 10^{-2}$P	$1.4 \cdot 10^{1}$P
viscosity (MKSA)	$1.8 \cdot 10^{-5}$PI	$1.0 \cdot 10^{-3}$PI	1.4PI

TABLE 3.1: Typical values of dynamic viscosity in usual conditions of temperature and pressure.

Units

1. Use equation (3.6) to determine the dimension of the dynamic viscosity η.
2. How do we convert between the CGS (in Poise, P) and the MKSA (in Poiseuille, PI) values of η (e.g. in Table 3.1)?

We see that the viscous force introduced in equation (3.6) has the properties we were looking for:

- It cancels out when the fluid is static ($\mathbf{v} = \mathbf{0}$).
- It cancels out if the velocity field is uniform, that is to say, in Cartesian coordinates:

$$\partial_i v_j = 0 \quad \forall i, j \in [x, y, z] \tag{3.7}$$

- It cancels out if the velocity field is not sheared, that is to say, in Cartesian coordinates:

$$\partial_i v_j = 0 \quad \forall i, j \in [x, y, z] \quad \text{with } i \neq j \tag{3.8}$$

- It is proportional to the surface element $\mathrm{d}S$. Also, in the case where v_x is a linear function of z, like in the introductory experiment in 3.1.1, it is proportional to the velocity U at the top of the reservoir and inversely proportional to the thickness h of the reservoir.

A mathematically convenient way to represent viscosity and pressure forces all together is through the stress tensor $\bar{\bar{\sigma}}$. A tensor is a specific type of matrix which obeys given transformation rules. For now, you can simply see it as a matrix. The stress tensor $\bar{\bar{\sigma}}$ is necessarily symmetric, that is to say $\bar{\bar{\sigma}}_{i,j} = \bar{\bar{\sigma}}_{j,i}$. In the case of an incompressible flow, the stress tensor $\bar{\bar{\sigma}}$ is, in Cartesian:

$$\bar{\bar{\sigma}} = \begin{pmatrix} -P & \eta\left[\partial_y v_x + \partial_x v_y\right] & \eta\left[\partial_z v_x + \partial_x v_z\right] \\ \eta\left[\partial_x v_y + \partial_y v_x\right] & -P & \eta\left[\partial_z v_y + \partial_y v_z\right] \\ \eta\left[\partial_x v_z + \partial_z v_x\right] & \eta\left[\partial_y v_z + \partial_z v_y\right] & -P \end{pmatrix} = -P\mathbb{I} + \bar{\bar{\sigma}}' \tag{3.9}$$

where the viscous stress tensor $\bar{\bar{\sigma}}'$ is the non-diagonal part of the stress tensor:

$$\bar{\bar{\sigma}}' = \eta \begin{pmatrix} 0 & \partial_y v_x + \partial_x v_y & \partial_z v_x + \partial_x v_z \\ \partial_x v_y + \partial_y v_x & 0 & \partial_z v_y + \partial_y v_z \\ \partial_x v_z + \partial_z v_x & \partial_y v_z + \partial_z v_y & 0 \end{pmatrix} \tag{3.10}$$

and where $\mathbb{I}$ is the identity matrix defined by:

$$\mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

that is to say:

$$\mathbb{I}_{i,j} = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{3.12}$$

where $\delta_{i,j}$ is the Kronecker symbol. We see that pressure corresponds to the diagonal terms of the stress tensor, while viscosity corresponds to the cross-terms. For more info, check § 15 of the Landau-Lifschitz.

Boundary conditions

We have seen that in any fluid, the normal component of the velocity field on a fixed obstacle cancels out. In a viscous fluid, the tangent components of the velocity field also cancel out. Indeed, the obstacle being fixed, it does not receive any linear momentum. According to equation (3.6), it means that $\partial_z v_x = 0$ that is to say the tangent component of the velocity field is continuous at the interface. Since the obstacle is fixed, it means that the tangent component of the velocity field in the fluid in contact with the obstacle is also null.

A diffusive phenomenon

We retrieve the main features of a dissipative phenomenon. Viscosity tends to homogenize the velocity field since the viscous force (3.6) transfers linear momentum from the faster layers to the slower ones. This is what explains why, in the preliminary experiment in 3.1.1, the puck dragged the fluid particles along its motion. At the microscopic scale, particles have a stochastic motion. The transport of linear momentum in the direction parallel to the particles' motion (e.g. the transfer of v_x through a surface of normal $\hat{\mathbf{x}}$) manifested at the macroscopic scale as the pressure. Similarly, viscosity is the manifestation at the macroscopic scale of the transport of linear momentum in the direction transverse to the particles' motion (e.g. the transfer of v_x through a surface of normal $\hat{\mathbf{z}}$).

3.1.3 Newton second law applied to fluids

Viscous force per unit volume

We have seen in chapter 1 that the pressure force per unit volume $\mathbf{f_P}$ can be written as:

$$\boxed{\mathbf{f_P} = -\boldsymbol{\nabla} P} \tag{3.13}$$

where P is the pressure scalar field. In the same way, we want to determine the viscous force per unit volume. Let us consider, as an example, the configuration of section 3.1.2 where the vector field is given by:

$$\mathbf{v} = v_x(z)\,\hat{\mathbf{x}} \tag{3.14}$$

We focus on an infinitesimal element of volume $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ contained between x and $x + \mathrm{d}x$, y and $y + \mathrm{d}y$ and z and $z + \mathrm{d}z$ (Fig.XXX). Given the expression of the vector field (3.13) and the expression (3.6) of the viscous force, the only viscous forces are the ones applied to the edges in z and $z + \mathrm{d}z$. The edge located in $z + \mathrm{d}z$ experiences a force[3]:

$$\mathrm{d}\,\mathbf{F}\,(z + \mathrm{d}z) = \eta\,\left.\frac{\partial v_x}{\partial z}\right|_{z+\mathrm{d}z} \mathrm{d}S\,\hat{\mathbf{x}} \tag{3.15}$$

[3]To verify the sign, remember that viscosity transfers linear momentum from the faster layers to the slower ones.

from the fluid above $z + \mathrm{d}z$. The edge located in z experiences a force:

$$\mathrm{d}\,\mathbf{F}\,(z) = -\eta\,\frac{\partial v_x}{\partial z}\bigg|_z \,\mathrm{d}S\,\hat{\mathbf{x}} \tag{3.16}$$

from the fluid below z. We perform a Taylor expansion of the function $\partial_x v_x$ up to order 1 in $\mathrm{d}z$:

$$\frac{\partial v_x}{\partial z}\bigg|_{z+\mathrm{d}z} \sim \frac{\partial v_x}{\partial z}\bigg|_z + \frac{\partial^2 v_x}{\partial z^2}\bigg|_z \,\mathrm{d}z \tag{3.17}$$

so the total viscous force $\mathrm{d}\,\mathbf{F}$ applied to the volume element from the upper and lower edges is:

$$\begin{aligned}
\mathrm{d}\,\mathbf{F} &= \mathrm{d}\,\mathbf{F}\,(z + \mathrm{d}z) + \mathrm{d}\,\mathbf{F}\,(z) &(3.18)\\
&= \eta\frac{\partial^2 v_x}{\partial z^2}\,\mathrm{d}z\,\mathrm{d}S\,\hat{\mathbf{x}} &(3.19)\\
&= \eta\frac{\partial^2 v_x}{\partial z^2}\,\mathrm{d}V\,\hat{\mathbf{x}} &(3.20)
\end{aligned}$$

More generally, for any incompressible flow, we can write the viscous force per unit volume $\mathbf{f}_\eta$ as:

$$\boxed{\mathbf{f}_\eta = \eta\Delta\,\mathbf{v}} \tag{3.21}$$

where Δ is the Laplace operator applied to the velocity vector field.

> In this course, when we take viscosity into account, we will always assume that the flow is incompressible (i.e. $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$). The expression of the viscous force in a non-compressible flow contains an additional term which can be found in equation (15.6) of the Landau-Lifschitz. Beware, in the Landau-Lifschitz, what is called "an incompressible fluid" is actually "an incompressible flow".

Heartbeat

What is the order of magnitude of the power consumed by the heart to ensure blood circulation? The dynamic viscosity of blood is $\eta = 4 \cdot 10^{-3}$Pl and a typical artery can be identified to a cylinder of length L = 1m and radius r = 2mm. The typical blood volumetric flow rate is D_V = 100cm^3·s^{-1}.

Navier-Stokes equation

We consider a fluid particle which, by definition, a closed system of constant mass $\mathrm{d}m = \rho\,\mathrm{d}V$. In an inertial frame, we can apply Newton second law to this fluid particle to obtain:

$$\mathrm{d}m\,\mathbf{a} = -\,\boldsymbol{\nabla}P\,\mathrm{d}V + \eta\Delta\,\mathbf{v}\,\mathrm{d}V + \mathrm{d}\,\mathbf{F} \tag{3.22}$$

where $\mathrm{d}\,\mathbf{F}$ are all the external forces which are not the pressure force (first term in the right hand side) or the viscous force (second term in the right hand side). In particular, a fluid particle in a gravitational field $\mathcal{G}$ would feel the force $\mathrm{d}\,\mathbf{F} = \mathrm{d}m\,\mathcal{G}$. The acceleration $\mathbf{a}$ of the fluid particle corresponds to the Lagrangian derivative of its velocity. In hydrodynamics, this law is generally expressed in terms of forces per unit volume, such as the equation (3.22) becomes the Navier-Stokes equation:

$$\boxed{\rho\frac{\mathrm{D}\,\mathbf{v}}{\mathrm{D}t} = \mathbf{f_P} + \mathbf{f}_\eta + \frac{\mathrm{d}\,\mathbf{F}}{\mathrm{d}V}} \tag{3.23}$$

Diffusion and convection

Let us re-write the Navier-Stokes equation (3.23) to focus on the variation of the velocity field as seen by an Eulerian observer, that is to say in a given point M:

$$\frac{\partial\mathbf{v}}{\partial t} = \underbrace{-(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}}_{\text{convective}} - \frac{1}{\rho}\,\boldsymbol{\nabla}P + \underbrace{\frac{\eta}{\rho}\Delta\,\mathbf{v}}_{\text{diffusive}} + \frac{1}{\rho}\frac{\mathrm{d}\,\mathbf{F}}{\mathrm{d}V} \tag{3.24}$$

Even in the absence of pressure gradient or external forces, there are two terms left which deserve further comments.

- The convective term: if it were alone, we would retrieve that the Lagrangian derivative of the velocity $\mathrm{D}\,\mathbf{v}/\mathrm{D}t$ is null (i.e. the flow is incompressible). It means that the convective term betrays the change of the velocity as seen by an Eulerian observer but which disappears once we move with the fluid particle. Said otherwise, in a given point M, it represents the change of the velocity field induced by the new fluid particles transporting their momentum in M. Indeed, the operator $\mathbf{v}\cdot\boldsymbol{\nabla}$ can be interpreted as the derivative along the vector $\mathbf{v}$. Therefore, $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}$ is the variation of the fluid particle's velocity along itself[4].

- The diffusive term: if it were alone, this term would yield a diffusion equation similar to what you saw when studying electric and heat conduction:

$$\boxed{\frac{\partial\mathbf{v}}{\partial t} = \nu\Delta\,\mathbf{v} \quad \text{where} \quad \nu\hat{=}\eta/\rho \text{ is the kinematic viscosity}} \tag{3.25}$$

  The irreversible nature of viscosity appears very clearly here because this equation is not time-reversal invariant. Indeed, if we transform t in $-t$, the left hand side changes sign but not the right hand side.

> The kinematic viscosity ν has the same dimension as the thermal diffusion coefficient D_θ and as the particle diffusion coefficient D. In a gas, all of them stem from the collisions between particles at the microscopic scale. Therefore, they all have approximately the same order of magnitude, of the order of $0.1\mathrm{cm}^2{\cdot}\mathrm{s}^{-1}$ (in usual conditions of temperature and pressure).

Reynolds number

XXX Present Reynolds as spontaneously appearing when adimensioning N-S XXX

Generally, it is not possible to find analytic solutions to the Navier-Stokes equation. This is why it is important to neglect terms. In order to do so, we often compare the diffusive and the convective terms. Indeed, everyday life experience indicates that they are mutually exclusive. Let us consider a glass of water at rest. We deposit a drop of colored ink on its

[4] In the same way as in the parachute example where $(\mathbf{v}\cdot\boldsymbol{\nabla})T$ was the variation of the temperature T along the free-fall velocity of the unfortunate parachutist.

surface. In this case, either water remains still and the mixing process is essentially diffusive, or we stir a spoon in the glass to agitate water and in this case, the mixing process is dominated by convection.

The ratio between the convective and the diffusive terms is called the Reynolds number $\mathcal{R}_e$:

$$\mathcal{R}_e = \frac{\text{convective}}{\text{diffusive}} = \left| \frac{\rho\,(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}}{\eta\Delta\,\mathbf{v}} \right| \tag{3.26}$$

If we introduce the characteristic length scale l_0 and characteristic speed scale v_0, we can approximate the operators by[5]:

$$\begin{cases} |(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}| \sim v_0^2/l_0 \\ |\Delta v| \sim v_0/l_0^2 \end{cases} \tag{3.27}$$

which gives the generic definition of the Reynolds number of a flow:

$$\boxed{\mathcal{R}_e = \frac{\rho v_0 l_0}{\eta}} \tag{3.28}$$

Therefore, we have two limit cases:

- Either $\mathcal{R}_e \gg 1$, and convective transport of momentum dominates over diffusive transport.
- Or $\mathcal{R}_e \ll 1$, and diffusive transport of momentum dominates over convective transport.

> In practice, the determination of the characteristic length scale l_0 is largely approximate, it is an order of magnitude. For instance, in a pipe, it will correspond to the diameter of the pipe. For a sphere falling in the atmosphere (section 3.2.1), it will correspond to its diameter. Yet, in a turbulent flow, the small length scales also play a role (section 3.2.2). For the characteristic speed scale v_0, we generally take the average or the maximum flow speed.

Time scales

1. Use equation (3.25) to estimate an order of magnitude of the characteristic time scale τ_d of diffusion, as a function of the characteristic length scale l_0 and of the characteristic speed scale v_0.
2. Do the same for the characteristic time scale τ_c of convection.
3. Give the Reynolds number as a function of these two time scales.
4. Is it coherent with the two aforementioned limit cases?

XXX PROTOTYPES, MAQUETTES, SCALE INVARIANCE XXX

[5] This is only justified because the role of the Reynolds number $\mathcal{R}_e$ is to compare two mechanisms (i.e. to determine whether $\mathcal{R}_e \ll 1$ or $\mathcal{R}_e \gg 1$), not to derive precise values.

3.1.4 A posteriori interpretation of the experiment

Model

We come back to the preliminary experiment presented in section 3.1.1. We assume that the velocity field has the form $\mathbf{v} = v_x(x,z)\,\hat{\mathbf{z}}$ a priori. The flow is assumed to be steady (i.e. $\partial_t = 0$), the pressure field P depends on z only and since glycerin is a liquid, it is fair to work with an incompressible flow so:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0 \tag{3.29}$$

$$\frac{\partial v_x}{\partial x} = 0 \tag{3.30}$$

Therefore, v_x depends on z only.

Velocity and pressure fields

In the Navier-Stokes equation, the convective term cancels out:

$$(\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v} = v_x \frac{\partial v_x}{\partial x}\,\hat{\mathbf{x}} = \mathbf{0} \tag{3.31}$$

We are left with the viscous force (i.e. the diffusive term), the pressure force and gravity. We project the Navier-Stokes equation on the x and z axis:

$$(\mathrm{NS} \cdot \hat{\mathbf{x}}): \quad 0 = \eta \frac{\partial^2 v_x}{\partial z^2} \tag{3.32}$$

$$(\mathrm{NS} \cdot \hat{\mathbf{z}}): \quad 0 = -\frac{\partial P}{\partial z} - \rho g \tag{3.33}$$

where $g = |\mathbf{g}|$ is the magnitude of the gravitational field at the Earth surface. We integrate these equations to obtain the velocity and pressure fields:

$$v_x(z) = \alpha z + \beta \tag{3.34}$$

$$P(z) = -\rho g z + \gamma \tag{3.35}$$

where α, β and γ are integration constants that we need to determine thanks to the following boundary conditions:

$$P(z = h) = P_0 \quad \text{since glycerin and air pressures are the same at the surface} \tag{3.36}$$

$$v_x(z = 0) = 0 \quad \text{since glycerin did not move at the bottom of the reservoir} \tag{3.37}$$

$$v_x(z = h) = U \quad \text{since glycerin moved with the puck at the top of the reservoir} \tag{3.38}$$

These three conditions give respectively:

$$\gamma - \rho g h = P_0 \tag{3.39}$$

$$\beta = 0 \tag{3.40}$$

$$\alpha h + \beta = U \tag{3.41}$$

Hence the full expression of the velocity and pressure fields:

$$v_x(z) = U \frac{z}{h} \tag{3.42}$$

$$P(z) = P_0 + \rho g (h - z) \tag{3.43}$$

Furthermore, we can compute the force per unit surface exerted by the glycerin on the puck at the surface:

$$\frac{\mathrm{d}F_x}{\mathrm{d}S} = -\eta\frac{\partial v_x}{\partial z} = -\frac{\eta U}{h} \tag{3.44}$$

This is the magnitude of the force we must apply to the puck in the opposite direction in order to maintain it at speed U.

3.2 Drag

3.2.1 Two limit cases

Drags

1. A point-mass of mass m, initially at rest, falls in the uniform gravity field g at the Earth surface. We neglect all other forces and we set its coordinate $z = 0$ at $t = 0$. We want to characterize its trajectory.

   (a) Define the system you work with, set a reference frame, determine the expression of the forces and write Newton's second law.

   (b) Integrate it to obtain the velocity $\dot{z}$ and the coordinate z of this point-mass as a function of time. Does it depend on its mass?

   (c) Express the velocity $\dot{z}$ as a function of the coordinate z.

   (d) In this expression, what is the external parameter of the problem? What are the two variables?

2. Now, we consider a sphere of radius R falling in the Earth atmosphere. We notice experimentally that the trajectory depends on a fourth quantity, the mass m of the object. Hereafter, we assume that the air has a mass density ρ and a dynamic viscosity η which are bijective (i.e. with a one-to-one correspondence). We neglect the buoyant (a.k.a. Archimedes) force.

   (a) In these conditions, what is necessarily the fifth parameter required to model the fall of this sphere?

   (b) We notice that the sphere eventually reaches a maximum speed v_∞. What are the 5 parameters this speed necessarily depends on?

   (c) Combine g and the three variables of the sphere with the air mass density ρ, based on dimensional arguments, in order to obtain a possible expression of the maximum speed v_∞.

   (d) Combine g and the three variables of the sphere with the air dynamic viscosity η, based on dimensional arguments, in order to obtain another possible expression of the maximum speed v_∞. Are these two expressions equivalent?

> This exercise illustrates how dimensional homogeneity can be a powerful tool to predict dependencies between variables whose units are know. This empirical approach, very useful when we deal with complex systems like in hydrodynamics, is formalized through the Buckingham Π theorem whose application is generally not straightforward. Yet, it is an important method to retrieve formulas whose exact expression you no longer remember.

Experimental study

A sphere of radius R and mass m falling in the Earth atmosphere will eventually reach a limit speed v_∞ (see exercise "Drags" below). The existence of this maximum speed means that at some point, another force compensates gravity such as the net acceleration is null. Let us call it the drag force $\mathbf{F}$. We want to determine an expression of this force as universal as possible. To do so, the procedure is to focus on dimensionless quantities[6]. In questions 2.c and 2.d of exercise "Drags", you found two dimensionally homogeneous expressions for v_∞. By definition, the drag force verifies:

$$\mathbf{F} \underset{v \longrightarrow v_\infty}{\longrightarrow} -m\,\mathbf{g} \tag{3.45}$$

Therefore, we can use one of the two expressions derived in questions 2.c and 2.d of exercise "Drags" to define a dimensionless drag force of magnitude $\tilde{F}$, for instance:

$$\tilde{F} = \frac{F}{\eta R v} \tag{3.46}$$

where F is the drag force and v is the velocity of the sphere of radius R at any time, and η is the dynamic viscosity of the air. Now, we must compare this dimensionless drag force to a relevant dimensionless parameter to look for an empirical relation. The dimensionless parameter which appears in the Navier-Stokes equation which describes the motion of the air flow around the falling sphere is the Reynolds number we derived in section 3.1.3. Experimentally, we can measure the dimensionless drag force $\tilde{F}$ as a function of the Reynolds number $\mathcal{R}_e$ and we find a universal trend represented in the log-log diagram in Fig.XXX. It shows the existence of two qualitatively different regimes:

$$\text{For } \mathcal{R}_e \ll 1 \quad \tilde{F} \sim \text{constant} \tag{3.47}$$

while for $\mathcal{R}_e \gg 1$, we have $\log \tilde{F} = \log \mathcal{R}_e + \text{constant}$, that is to say:

$$\text{For } \mathcal{R}_e \gg 1 \quad \tilde{F} \propto \mathcal{R}_e \tag{3.48}$$

Introducing ρ, the air mass density, we can re-express these results as:

$$\boxed{\text{For } \mathcal{R}_e \ll 1 \quad F \propto \eta R v \quad \text{the Stokes's drag}} \tag{3.49}$$

and

$$\boxed{\text{For } \mathcal{R}_e \gg 1{,}000 \quad F \propto \rho R^2 v^2 \quad \text{the quadratic drag}} \tag{3.50}$$

with the proportionality constant being a dimensionless number in both cases. For intermediate Reynolds numbers, between 1 and $1{,}000$, the expression of the drag force is a combination of the two aforementioned expressions.

Mutatis mutandis

Repeat this procedure but by defining the dimensionless drag force with the expression of the maximum speed you found in question 2.c of the exercise "Drags". Show that in this case, the diagram of $\tilde{F}$ as a function of $\mathcal{R}_e$ corresponds to the first part of this graph (i.e. up to $\mathcal{R}_e \sim 10^5$) representing the dimensionless drag coefficient of a

[6] An approach we retrieve in Kolmogorov's theory of turbulence.

sphere as a function of the Reynolds number.

Generalization

Let us generalize the results above to the case of an object of any form. In equation (3.52), the square of the sphere's radius, R^2, intervenes in the expression of the quadratic drag force. However, intuitively, we know that drag forces do not depend on the whole external surface on an object. Instead, they depend on its orientation with respect to its motion (Fig.XXX). Therefore, we introduce the transverse cross-section S defined as the largest object's section normal to the velocity vector of the object (Fig.XXX). Then, we can define the typical transverse size of the object as $l_0 = \sqrt{S}$ and we have:

$$\text{For } \mathcal{R}_e \ll 1 \quad F \propto \eta l_0 v \tag{3.51}$$

and

$$\text{For } \mathcal{R}_e \gg 1,000 \quad F \propto \rho S v^2 \tag{3.52}$$

> In the exercise "Drags" above, the expression of the maximum speed v_∞ we obtained in the high Reynolds number regime (question 2.c) was right modulo a coefficient 6π. The expression of the maximum speed v_∞ we obtained in the low Reynolds number regime (question 2.d) was missing a dimensionless coefficient of $\sqrt{2/\pi}$. More generally, the expression of the drag force requires a dimensionless constant of the order of unity.

Not-so-free fall

A petanque ball and a tennis ball, both of identical radius and without initial speed, are dropped from a height H above the ground. In vacuum, they have the same fall time, but not in air due to friction. Which body hits the ground first? Why?

Biking in the wind

A biker of size R and mass M drives at a speed v and experiences the friction with the air.

1. For realistic values, what is the Reynolds number? Which drag force is suitable?
2. How does the power the biker must use to counter the air friction evolve with the velocity v?
3. Now, we assume that the biker rises a slope of angle α. What is the power, associated to the biker's weight, necessary to rise the slope?
4. Estimate approximately the critical velocity beyond which the power the biker must use to counter the air friction is higher than the power needed to rise the slope.

A posteriori checks

We consider a sphere of radius R and of mass m falling in a water tank on the Earth surface. The buoyancy force is negligible.

1. We assume that experimental measures provided us with a dimensionless proportionality coefficient of 6π for the Stokes' drag force, and of $1/\sqrt{\pi}$ for the quadratic drag. Deduce the expression of the maximum speed v_∞ in both cases.
2. Take a radius $R = 2\text{mm}$ and a mass $m = 0.05\text{g}$. Assuming that we are in the low Reynolds number regime ($\mathcal{R}_e \ll 1$), what is the numerical value of the maximum speed v_∞?
3. Is it coherent with the assumption we made of a low Reynolds number? If not, determine the maximum speed v_∞ based on the formula in the regime of high Reynolds number ($\mathcal{R}_e \gg 1,000$).
4. Discuss the validity of this new maximum speed v_∞ you obtained.

3.2.2 Laminar and turbulent flows

Definitions

The universal character of the relation between the dimensionless drag force and the Reynolds number in Fig.XXX indicates that there are two qualitatively different types of flows, whether $\mathcal{R}_e \ll 1$ or $\mathcal{R}_e \gg 1,000$. Let us plot the streamlines associated to a flow around a spherical (or cylindrical) obstacle of transverse dimension L. When $\mathcal{R}_e \ll 1$ (i.e. when the flow speed is low and/or when the viscosity is high), we observe a symmetry between the upstream and downstream streamlines (Fig.XXX): the flow is said to be laminar. However, when the Reynolds number increases from 1 to 1,000 (i.e. when the flow speed increases and/or when the viscosity decreases), an asymmetry appears downstream of the obstacle. In this wake, the streamlines are not steady and break up in small scale structures[7] (Fig.XXX da Vinci). The flow is said to be turbulent. This is this qualitative difference which explains the different expressions of the drag force in both regimes. The presence of these small scale structures modifies the properties of the flow, for instance its electric and heat conductivity, and its opacity (Fig. XXX robinet).

> Turbulent and rotational (resp. laminar and irrotational) are not synonymous. First, the character turbulent of a flow does not depend on the reference frame, while the character rotational does (see section 2.1.4). Furthermore, a rotational flow (i.e. $\boldsymbol{\nabla} \wedge \mathbf{v} \neq \mathbf{0}$) can be laminar. However, a turbulent flow is necessarily rotational at all time and everywhere.

Is the flow around a moving car turbulent or laminar?

D'Alembert paradox

By definition, in the high Reynolds number regime, the convective term dominates over the diffusive term in Navier-Stokes equation. It manifests through the fact that the quadratic drag force (3.52) does not depend on the viscosity coefficient. Said otherwise, in the transport of momentum across the fluid, viscosity is negligible compared to bulk motions when the flow is turbulent. And yet, it can be shown that without viscosity, the drag force is necessarily zero. This is an example of singular (a.k.a. discontinuous) limit: the behavior of the equations without viscosity (see Euler equation in chapter 4) is qualitatively different from

[7] In turbulence theory, we define eddies as swirling structures (e.g. vortices) which have a distribution of sizes, from the injection scales (i.e. the macroscopic scales at which energy manifests through bulk motion of the flow) down to the dissipative scales (where viscosity dominates and energy is dissipated).

the behavior of the Navier-Stokes equation when the viscosity becomes infinitely small. This is D'Alembert paradox, and it can be solved by taking into account the fact that in a turbulent flow, there are always quick variations over very small length scales δ. Therefore, at these scales, the Reynolds number is much lower and the viscous term dominates over the convective term, even if the flow is globally turbulent. It must be remembered that in the equation (3.28), the characteristic length scale l_0 represents the length over which the velocity changes significantly. It is the one we used to approximate the spatial derivatives in the $\mathbf{v} \cdot \boldsymbol{\nabla}\, \mathbf{v}$ and in the Laplacian operators. If the flow is laminar, the only variations happen over macroscopic length scales and in this case, it is legitimate to take, for l_0, the width of a pipe or the transverse size of a solid body in a flow (Fig.XXX, panel a, 1D slice of $|\mathrm{v}|$ in laminar flow). But in the case of a turbulent flow, the nature of l_0 is more ambiguous: it can either be interpreted as a macroscopic length scale, over which the flow does vary significantly, but also as the size of the small scale structures in the wake of the obstacle in Fig.XXX for instance (Fig.XXX, panel b, 1D slice of $|\mathrm{v}|$ in turbulent flow). This scale can be many orders of magnitude smaller than the macroscopic scale, such as if we use the small scales to define a local Reynolds number (instead of the global Reynolds number which is $\gg$1,000 if the flow is turbulent), this local Reynolds number can be much smaller than 1. Actually, viscosity at small scales plays a central role in the triggering of turbulence at large scale: if the viscosity is strictly zero (which never happens), a flow cannot be turbulent, although the Reynolds number is infinite.

3.3 Exercises

3.3.1 Poiseuille flow

A fluid of dynamic viscosity η and of mass density ρ is in steady flow in a cylindrical pipe of main axis $\hat{\mathbf{z}}$, length L and radius R (Fig.XXX). Due to the symmetries of the problem, we look for a velocity and a pressure field of the form:

$$\mathbf{v} = v_z(r,z)\,\hat{\mathbf{z}} \quad \text{and} \quad P = O(r,z) \tag{3.53}$$

where r is the orthogonal distance to the main axis (i.e. the usual cylindrical coordinate). In this case, the divergence, Laplacian and $(\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v}$ operators are written respectively:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_z}{\partial z} \tag{3.54}$$

$$\Delta\,\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\hat{\mathbf{z}} \tag{3.55}$$

$$(\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v} = v_z\frac{\partial v_z}{\partial z}\,\hat{\mathbf{z}} \tag{3.56}$$

1. We assume that the flow is incompressible. What does it mean for the dependencies of v_z?

2. We neglect gravity and we remind Navier-Stokes equation:

$$\rho\,\mathbf{a} = -\,\boldsymbol{\nabla} P + \eta\Delta\,\mathbf{v} \tag{3.57}$$

   where $\mathbf{a}$ is the acceleration field given by the Lagrangian derivative of the velocity field.

   (a) What can you say about $\mathbf{a}$?

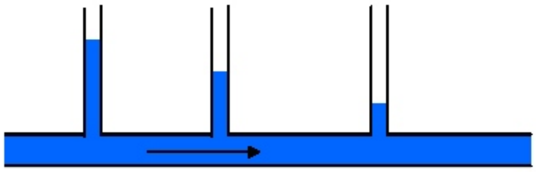

FIGURE 3.1: Pressure drops as a fluid flows in a pipe due to viscosity.

(b) Project the Navier-Stokes equation on $\hat{\mathbf{r}}$ and comment on the dependencies of the pressure field P.

(c) Project the Navier-Stokes equation on $\hat{\mathbf{r}}$ and determine an ordinary differential equation with P and v_z.

(d) Given the form of this equation, what can you say about the $\mathrm{d}_z P$?

(e) Use the boundary conditions to determine the expression of $\mathrm{d}_z P$ as a function of the pressure $P_1 = P(z = 0)$ at the entrance of the pipe, the pressure $P_2 = P(z = L)$ at the exit of the pipe and L.

(f) Integrate the ordinary differential equation, use the boundary conditions and the expression of $\mathrm{d}_z P$ to obtain $v_z(r)$.

3. This equation shows that the pressure in a fluid tends to drop along the movement due to viscosity (Fig. 3.1).

   (a) Express the volumetric flow rate D_V (whose expression was given in section 2.2.1) as a function of P_1 and P_2.

   (b) In electrokinetics, you have seen the Ohm law: the electric tension U (which is a difference between electric potentials V) between two points of a cable is the product of the electric intensity I in this cable by the electric resistance $\mathcal{R}$ of this cable (i.e. $U = \Delta V = \mathcal{R}I$). In this equation, which variable represents the cause? Which is the consequence?

   (c) Using an analogy with this law, introduce a hydraulic resistance $\mathcal{R}$ to relate the cause and the consequence of the flow.

   (d) Express the hydraulic resistance as a function of eta, R and L. Is the dependence of $\mathcal{R}$ on these three quantities coherent?

4. Numerical application
   Let us consider a Human artery of length $L = 1\mathrm{m}$ and radius $R = 5\mathrm{mm}$ with a volumetric flow rate of $D_V = 80\mathrm{cm}^3{\cdot}\mathrm{s}^{-1}$. The dynamic viscosity of blood is $\eta = 4 \cdot 10^{-3}\mathrm{PI}$ (i.e. more than water, see table 3.1).

   (a) What is the pressure drops from the beginning to the end of this artery?

   (b) The contraction of the myocardium, the heart muscle, maintains a pressure difference of approximately 80 millimeters of mercury in all arteries (where 1 bar is 760 millimetres of mercury). Compare to the result of the previous question and comment.

This pressure drop due to viscosity also provides a convenient way to measure the dynamic viscosity of fluids.

3.3.2 Drop on a turntable

Consider a turntable rotating at angular velocity ω around the vertical axis $\hat{\mathbf{z}}$ (Fig.XXX). A drop of water falls at the center of the turntable and progressively expands with a shape assimilated to a cylinder of radius $R(t)$ and thickness $h(t)$, where t is the time coordinate.

1. Using the conservation of the drop's volume, determine the proportionality relation between R and h.

2. In the rotating frame of the reference of the turntable, the Navier-Stokes equation writes (section 4.1.1):

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho\,(\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v} = -\,\boldsymbol{\nabla} P + \rho\,\mathbf{g} + \eta \Delta\,\mathbf{v} - 2\rho\,\boldsymbol{\omega} \wedge \mathbf{v} + \rho\omega^2\,\mathbf{OM} \tag{3.58}$$

   where $\mathbf{g}$ is the gravity field at the surface of the Earth, η is the dynamic viscosity, $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ is the rotation vector, O is the center of the turntable and M any point on the turntable where we study the fields. We assume $h \ll R$ such as the pressure field is uniform in the fluid.

   (a) What do the different terms of this equation represent?

   (b) In orders of magnitude, what is the approximate expression of the viscous term?

   (c) Similarly to what we did with the Reynolds number in section 3.1.3, introduce dimensionless numbers to compare the viscous term to the other terms containing the velocity.

   (d) Evaluate these dimensionless numbers assuming that the fluid typically takes approximately $\tau = 100$ seconds to spread over $R(\tau) = 10$ cm, that the turntable spins at $\omega = 10^3$rad·s^{-1} and that the typical thickness of the cylinder-shaped structure is $\lesssim 0.1$mm. Given this result, which are the two terms which control the way the fluid spreads in the Navier-Stokes equation?

3. We want to determine the ordinary differential equation which controls the time-evolution of the radius R: how quickly does the fluid spread?

   (a) Compare the viscous and centrifugal term to obtain a non-linear first-order ordinary differential in $R(t)$. You will use the proportionality relation between R and h you previously derived.

   (b) Integrate this equation to obtain the proportionality relation between R and t.

   (c) Use that $R(\tau = 100\text{s}) = 10$ cm to deduce the proportionality constant.

3.3.3 The Couette Viscometer

A liquid of dynamic viscosity η, kinematic viscosity ν and mass density ρ uniform is located between two co-axial cylinders of length L and radii R_1 and $R_2 > R_1$ (Fig.XXX). The cylinders are oriented vertically and the inner cylinder (1) is hanging by a torsion wire which imposes a restoring torque $-C\alpha$ on it when it turns by an angle α around the vertical axis $\hat{\mathbf{z}}$ oriented upwards. We force the outer cylinder (2) to rotate at a constant angular speed ω by applying a torque $\Gamma\,\hat{\mathbf{z}}$. Once the system reaches a steady state, the cylinder (1) is fixed but shifted by an angle α_{eq} with respect to its initial angular position.

1. What is the physical mechanism which explains why the rotation of the outer cylinder induces the rotation of the inner cylinder?

2. We assume that the velocity field is of the form $\mathbf{v} = v_\theta(r,\theta,z)\,\hat{\boldsymbol{\theta}}$, that the pressure field is of the form $P(r,\theta,z)$ and that the gravity field $\mathbf{g} = -g\,\hat{\mathbf{z}}$ is uniform, with $g > 0$.
   (a) Given that the configuration is axisymmetric, what can you say about the dependence of v_θ and P on the variable θ?
   (b) We do not take into account the boundaries at $z = 0$ and $z = L$ and their impact on the velocity field. What can you deduce on the dependence of v_θ on the variable z?
3. The viscous force applied on a surface element $\mathrm{d}\,\mathbf{S} = \mathrm{d}S\,\hat{\mathbf{r}}$ by a fluid particle just above r onto a fluid particle just below r is:
$$\mathrm{d}\,\mathbf{F} = \eta r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\mathrm{d}S\,\hat{\boldsymbol{\theta}} \tag{3.59}$$
   (a) Compute the inifinitesimal torque $\mathrm{d}\,\boldsymbol{\Gamma}$ of this force $\mathrm{d}\,\mathbf{F}$ with respect to the $\hat{\mathbf{z}}$-axis.
   (b) Deduce the expression of the total torque $\Gamma(r)$ applied onto the fictitious cylinder of radius r by the fluid just above r onto the fluid just below r.
   (c) Use the angular momentum theorem applied to a suitable volume to show that this torque does not depend on r.
   (d) Deduce the expression of v_θ as a function of r, r_1, r_2 and ω, using the boundary conditions at the inner and outer cylinders.
   (e) Deduce the expression of the torque Γ as a function of η, r_1, r_2, L and ω.
4. How would you use this device to measure the viscosity of the fluid between the two cylinders?

> The reasoning you used to describe this viscosity-mediated angular momentum transfer in question 3. lies at the core of the theory of accretion disks.

Chapter 4

Euler equation and Bernoulli theorem

In chapter 3 devoted to viscosity, we always assumed that the flow was incompressible, which is legitimate when we work with liquids (since incompressible fluid implies incompressible flow), or with gases moving at subsonic speeds. Here, we relax this assumption. However, we assume that the dissipative effects like viscosity are negligible. These flows, which undergo reversible and adiabatic (i.e. isentropic) transformations only, are called ideal flows.

We will apply Newton second law to a fluid particle to obtain the Euler equation which relates the velocity field to the pressure field (section 4.1). In order to exploit the boundary conditions, in particular those on the pressure field, it is more convenient to work with first integrals of the movement like the Bernoulli relation (section 4.2).

4.1 Euler equation

4.1.1 Derivation

General expression

We consider a fluid particle, that is to say a closed system which occupies, at time t, an infinitesimal volume $\mathrm{d}V$ centered on a point M where the mass density is given by $\rho(M)$. Its infinitesimal mass $\mathrm{d}m = \rho(M)\,\mathrm{d}V$ is a constant. Since the fluid is ideal, we neglect the viscous force so Newton second law writes, in an inertial frame of reference:

$$\mathrm{d}m\,\mathbf{a} = -\left(\boldsymbol{\nabla} P\right)\mathrm{d}V + \mathrm{d}\,\mathbf{F} \tag{4.1}$$

where $\mathbf{a}$ is the acceleration of the fluid particle, given by the Lagrangian derivative of the velocity field, and $\mathrm{d}\,\mathbf{F}$ are the external forces onto the fluid (e.g. gravity). We can simplify this equation by the volume element $\mathrm{d}V$ to express it in terms of forces per volume and we obtain the Euler equation:

$$\boxed{\rho\frac{\mathrm{D}\,\mathbf{v}}{\mathrm{D}t} = -\,\boldsymbol{\nabla} P + \frac{\mathrm{d}\,\mathbf{F}}{\mathrm{d}V}} \tag{4.2}$$

Comments

The Euler equation contains three scalar partial differential equations. In the case of an incompressible fluid, the mass density ρ is constant and uniform. In addition, the flow is incompressible so we have an additional scalar equation: $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$. With a total of four scalar equations, we can use the boundary conditions to determine the pressure field (one unknown) and the velocity field (three unknowns). However, in the general case, we have 5 unknowns (ρ, $\mathbf{v}$ and P), for the three scalar equations of the Euler equation. We often add the continuity equation (section 2.79) and an equation for the conservation of energy, or an equation which directly relates the mass density ρ to the pressure P (e.g. the polytrope assumption we made in the course of Stellar Astrophysics).

The Euler equation is non-linear because of the contribution from the convective term in the Lagrangian derivative. This non-linearity is responsible for the diversity of behaviors of fluids, but it represents a major obstacle from a calculus point-of-view. This is why we often perform preliminary orders-of-magnitude estimates to identify the terms which dominate in the Euler equation, and to neglect the others. It is an important first step, even when one wants to use numerical solvers, and the next section illustrates it with a representative example.

Non-inertial frame

If the frame of reference is non-inertial, we must take into account the fictitious forces (like we did in the exercise "Drop on a turntable" in section 3.3.2). Let us write $\mathbf{\Omega}$ the angular velocity vector of the non-inertial frame with respect to an inertial frame (Fig.XXX). We consider the infinitesimal forces applied to an infinitesimal amount of mass $\mathrm{d}m$. On one hand, we have the inifinitesimal Coriolis force:

$$\mathrm{d}\,\mathbf{F_C} = -2\,\mathrm{d}m\,\mathbf{\Omega} \wedge \mathbf{v} \tag{4.3}$$

On the other hand, we have the inifinitesimal fictitious forces related to the acceleration $\mathbf{a_{O'}}$ of the origin O' of the frame, to its rotation (the centrifugal force) and to the variation of its rotation (the Euler force):

$$\mathrm{d}\,\mathbf{F_e} = -\,\mathrm{d}m\,\mathbf{a_e} \tag{4.4}$$

where $\mathbf{a_e}$ is the acceleration which encapsulates all these components. In a point M, its general expression is:

$$\mathbf{a_e} = \mathbf{a_{O'}} + \mathbf{\Omega} \wedge \left(\mathbf{\Omega} \wedge \mathbf{O'M}\right) + \frac{\mathrm{d}\mathbf{\Omega}}{\mathrm{d}t} \wedge \mathbf{O'M} \tag{4.5}$$

where the three terms on the right hand side correspond to the three aforementioned components.

Centrifugal term

Show that when a coordinate system is introduced, the centrifugal term of the fictitious force takes an expression you are more familiar with.

Once we add these components to equation 4.6, we obtain the Euler equation in a non-inertial frame:

$$\boxed{\rho\frac{\mathrm{D}\,\mathbf{v}}{\mathrm{D}t} = -\,\nabla P + \frac{\mathrm{d}\,\mathbf{F}}{\mathrm{d}V} - 2\rho\,\mathbf{\Omega} \wedge \mathbf{v} - \rho\,\mathbf{a_e}} \tag{4.6}$$

4.1.2 Example: geostrophic flows

Geostrophic approximation

We want to study the behavior of the horizontal wind in the Earth atmosphere in the geostrophic approximation, that is to say when the dynamics is controlled by the balance between the pressure force and the Coriolis force.

Let O be a point on Earth surface at latitude λ, and $\mathcal{R} = (O, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ the local frame co-rotating with the Earth (Fig.XXX). The geocentric frame of reference is the frame centered on the Earth center and whose axis point in the direction of three stars at infinity. We assume that the geocentric frame of reference is Galilean. The rotation vector of the local frame $\mathcal{R}$ with respect to the geocentric frame of reference is $\mathbf{\Omega}_\oplus = \Omega_\oplus \hat{\mathbf{Z}}$ with $\hat{\mathbf{Z}}$ the unit vector oriented along the Earth's spin axis, from the South pole to the North pole. On the Earth surface, we assume that the gravitational field is given by $\mathbf{g} = -g\,\hat{\mathbf{z}}$ with $g > 0$. We write H the point corresponding to the orthogonal projection of O on the South-North rotation axis. Then, the Navier-Stokes equation in the local frame of reference $\mathcal{R}$ writes:

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = -\,\boldsymbol{\nabla}P + \rho\,\mathbf{g} - 2\rho\,\mathbf{\Omega}_\oplus \wedge \mathbf{v} + \rho\Omega_\oplus^2\,\mathbf{HO} \tag{4.7}$$

We see that the effective influence of the centrifugal term is to lower the local gravity by an amount $\Omega_\oplus^2 HO$. Let us evaluate its influence by comparing the centrifugal term to gravity. In the case when the centrifugal term is maximum (i.e. at equator), we have:

$$\left|\frac{\Omega_\oplus^2\,\mathbf{HO}}{\mathbf{g}}\right| = \frac{\Omega_\oplus^2 R_\oplus}{g} \sim 3\cdot 10^{-3} \ll 1 \tag{4.8}$$

Then, the modification of gravity induced by the Earth rotation is negligible, and we discard the centrifugal term.

We want to study the velocity field in the Earth atmosphere so let us compare between each other the three terms containing the velocity to identify the conditions for the Coriolis force to be the dominant one. To do so, we write v_0 the magnitude of the characteristic velocity in the atmosphere, l_0 the characteristic length scale and τ the characteristic time scale of the velocity field. Doing so, we first compare the convective term to the Coriolis term:

$$\left|\frac{\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}}{-2\rho\,\mathbf{\Omega}_\oplus\wedge\mathbf{v}}\right| \sim \frac{\rho v_0^2/l_0}{\rho\Omega_\oplus v_0} = \frac{v_0}{\Omega_\oplus l_0} \tag{4.9}$$

Then, we compare the local derivative to the Coriolis term:

$$\left|\frac{\rho\dfrac{\partial \mathbf{v}}{\partial t}}{-2\rho\,\mathbf{\Omega}_\oplus\wedge\mathbf{v}}\right| \sim \frac{\rho v_0/\tau}{\rho\Omega_\oplus v_0} = \frac{1}{\Omega_\oplus\tau} = \frac{1}{2\pi}\frac{T_\oplus}{\tau} \tag{4.10}$$

where $T_\oplus$ is the Earth's rotation period. For the convective term and the local derivative to be negligible in comparison to the Coriolis term, we need to focus on air flows with specific characteristic speed, length and time scales. From equation (4.10), we have:

$$\text{Local derivative} \ll \text{Coriolis} \quad \Longleftrightarrow \tau \gg 4\text{ hours} \tag{4.11}$$

so we need to consider air flows which evolve over characteristic time scales of more than a couple of days for the local derivative to be negligible[1] compared to the Coriolis force. Equation (4.9) gives the condition for the convective term to be negligible in comparison to the Coriolis term:

$$\text{Convection} \ll \text{Coriolis} \quad \Longleftrightarrow l_0/v_0 \gg 4\text{ hours} \tag{4.12}$$

so we need to consider air flows which evolve on large spatial scales and/or which have a low speed. A representative case where the Coriolis term is exactly an order-of-magnitude larger than the convective term is for air flows moving at $v_0 = 5\text{km}\cdot\text{s}^{-1}$ over distances $l_0 = 200\text{km}$. For any larger characteristic length scale and/or lower characteristic speed, the convective term can safely be neglected.

[1] That is to say, at least an order of magnitude smaller.

In summary, the geostrophic approximation is relevant to describe the motion of relatively slow winds varying over hundreds of kilometers and over days.

Wind scrapping

We want to verify where the geostrophic approximation is acceptable in the atmosphere.

1. Code a Python web scrapping script to collect the data from the map in Figure 4.1.
2. Estimate the local derivative, the convective term and the Coriolis term in all points of the spatial grid.
3. Where is the geostrophic approximation valid? Does it correspond to the cyclones and anticylones visible?

Horizontal winds

We are left with the simplified Navier-Stokes equation:

$$\boldsymbol{\nabla} P = \rho\, \mathbf{g} - 2\rho\, \boldsymbol{\Omega}_{\oplus} \wedge \mathbf{v} \tag{4.13}$$

First, let us project this equation on the vertical axis $\hat{\mathbf{z}}$ of the local frame of reference $\mathcal{R}$:

$$\frac{\partial P}{\partial z} = -\rho g - 2\rho\Omega_{\oplus} \left[\begin{pmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{pmatrix} \wedge \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \right] \cdot \hat{\mathbf{z}} \tag{4.14}$$

$$= -\rho g + 2\rho\Omega_{\oplus} v_x \cos\lambda \tag{4.15}$$

Since $\Omega_{\oplus} v_x \ll g$ for the regime of wind speeds we consider, the Coriolis term is negligible (in the vertical direction) and we conclude that the vertical hydrostatic equilibrium of the atmosphere is not affected by winds in the geostrophic approximation.

Second, let us write the equation (4.13) in the horizontal plane:

$$\begin{pmatrix} \partial_x P \\ \partial_y P \end{pmatrix} = \begin{pmatrix} v_z \cos\lambda - v_y \sin\lambda \\ v_x \sin\lambda \end{pmatrix} \tag{4.16}$$

In the geostrophic approximation, we focus on horizontal winds, that is to say winds whose vertical motion is negligible: $v_z \ll v_x, v_y$. Therefore, the Navier-Stokes equation in the horizontal plane writes:

$$\boldsymbol{\nabla} P = -2\rho\, \boldsymbol{\Omega}_{\oplus,\mathbf{z}} \wedge \mathbf{v} \tag{4.17}$$

where the gradient of pressure contains only the horizontal components and where:

$$\boldsymbol{\Omega}_{\oplus,\mathbf{z}} = \Omega_{\oplus} \sin\lambda\, \hat{\mathbf{z}} \tag{4.18}$$

is the locally vertical component of the Earth rotation vector.

Streamlines

Let $\mathrm{d}\boldsymbol{l}$ be an infinitesimal displacement along a streamline. If we take the scalar product of equation (4.17) by $\mathrm{d}\boldsymbol{l}$, we obtain:

$$\boldsymbol{\nabla} P \cdot \mathrm{d}\boldsymbol{l} = -2\rho \left(\boldsymbol{\Omega}_{\oplus,\mathbf{z}} \wedge \mathbf{v}\right) \cdot \mathrm{d}\boldsymbol{l} \tag{4.19}$$

$$\mathrm{d}P = 0 \tag{4.20}$$

By definition of a streamline (section XXX in chapter), $\mathrm{d}\boldsymbol{l} /\!/ \mathbf{v}$ which explains why the right hand side in the equation above cancels out. Therefore, along a streamline, the pressure remains constant: in the geostrophic approximation, each streamline is an isobare curve of given pressure.

Let the point A be a local maximum of pressure[2] in the Earth atmosphere (and at spatial scales coherent with the geostrophic approximation). Then, it is surrounded by closed isobare contours of lower pressure (Fig.XXX) and in the geostrophic approximation, the wind flows along these isobares. In which sense is it rotating as seen from above? Since A is a local maximum of pressure, in its vicinity, $\boldsymbol{\nabla} P$ is oriented radially inwards, and so is $\mathbf{v} \wedge \boldsymbol{\Omega}_{\oplus,\mathbf{z}}$ according to equation (4.17). In the South hemisphere, $\sin\lambda < 0$ so $\Omega_{\oplus,z}$ is oriented downwards according to equation (4.18). Then, for $\mathbf{v} \wedge \boldsymbol{\Omega}_{\oplus,\mathbf{z}}$ to be oriented radially inwards, the wind needs to rotate counter-clockwise as seen from above.

In Figure 4.1, we see a map of the pressure field over the Pacific ocean (see animated version online, where the air motion is more visible). The white short curves represent the streamlines of the horizontal air flow in the atmosphere (not to be confused with the shores in the background). As expected, in the Southern hemisphere, the local minima (resp. maxima) of pressure are surrounded by circular winds rotating clockwise (resp. counter-clockwise). The situation is inverted in the Northern hemisphere.

> Since at these very sub-sonic speeds, we can safely assume that the air flow is incompressible (i.e. $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$), converging (resp. diverging) streamlines mean that the velocity increases (resp. decreases), as previously explained in section 2.2.6.

Several limitations of the geostrophic model already appear. First, we see that the streamlines are not exactly circular around pressure extrema. Instead, they seem to slightly spiral towards pressure minima in the Southern hemisphere. It must be ascribed to some of the terms we neglected in the Navier-Stokes equation. Then, at the equator ($\sin\lambda = 0$), the geostrophic approximation is never valid since the horizontal components of the Coriolis force due to horizontal winds ($v_z = 0$) cancel out so it cannot dominate over the other terms of the Navier-Stokes. And indeed, Figure 4.1 indicates that at the equator, local pressure extrema are not surrounded by circular winds.

4.2 Bernoulli theorems

In this section, we look for an integral of motion, that is to say a constant that we can use to understand the flow dynamics[3]. We work in an inertial frame of reference and the only external force we consider is the uniform gravitational field at the Earth surface. Also, we re-write the convective term of the Navier-Stokes equation using the alternative expression of the $(\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v}$ operator given by the equation (2.19) we saw in the exercise "Alternative

[2] An anticylone (resp. a cyclone) is a local maximum (resp. minimum) in pressure.

[3] In point mechanics, you saw that, provided the forces experienced by a point-mass are all conservative, the mechanical energy is an integral of motion.

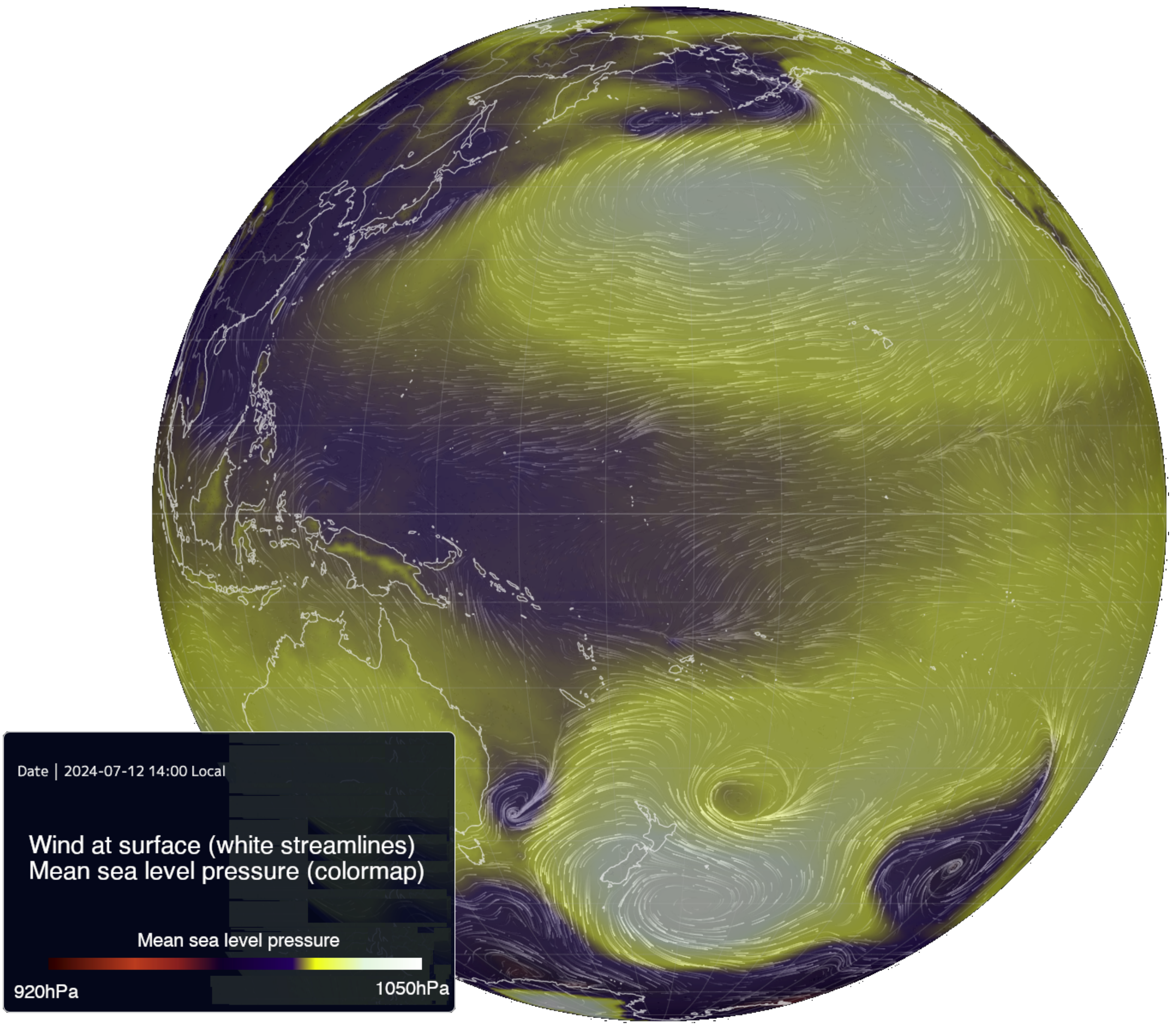

FIGURE 4.1: Representation of the wind streamlines (white lines) and of the mean sea level pressure (colormap) in the Pacific ocean. The cyclones and anticyclones (i.e. the local extrema of the pressure field) are clearly visible in both hemispheres. Animated version on Earth Nullschool.

expression" in section 2.1.2 in chapter 2:

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla}\left(\frac{\mathbf{v}^2}{2}\right) + (\boldsymbol{\nabla} \wedge \mathbf{v}) \wedge \mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P + \mathbf{g} \tag{4.21}$$

Importantly, we notice that the gravitational term can be written as a gradient (which is to be expected since the gravitational force is conservative):

$$\mathbf{g} = -g\,\hat{\mathbf{z}} = -g\,\boldsymbol{\nabla} z = -\,\boldsymbol{\nabla}\,(gz) \tag{4.22}$$

where $g = |\mathbf{g}|$ and the $\hat{\mathbf{z}}$-axis correspond to the local vertical direction, and is oriented upwards. Now, we have three terms in the Navier-Stokes where a gradient appears.

We make the additional following simplifying assumptions:

- The flow is ideal, that is to say dissipative effects such as viscosity can be neglected.
- The flow is steady: $\partial_t = 0$.

The local derivative in the Lagrangian derivative in the Navier-Stokes equation simplifies thanks to the latter assumption and we have:

$$\boldsymbol{\nabla}\left(\frac{\mathbf{v}^2}{2}\right) + (\boldsymbol{\nabla} \wedge \mathbf{v}) \wedge \mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\,(gz) \tag{4.23}$$

4.2.1 Irrotational flow and incompressible fluid

Interestingly enough, we notice that, provided ρ is uniform and $\boldsymbol{\nabla} \wedge \mathbf{v}$ (i.e. the fluid is irrotational), all terms can be written as a gradient. It drives us to further assume that:

- The flow is irrotational: $\boldsymbol{\nabla} \wedge \mathbf{v} = \mathbf{0}$
- The fluid is incompressible, that is to say that the mass density is uniform.

Doing so, the Navier-Stokes equation simplifies into:

$$\boldsymbol{\nabla}\left(\frac{\mathbf{v}^2}{2}\right) = -\,\boldsymbol{\nabla}\left(\frac{P}{\rho}\right) - \boldsymbol{\nabla}\,(gz) \tag{4.24}$$

We gather the terms on the left hand side and we multiply by ρ, which is uniform so it can safely enter the gradient:

$$\boldsymbol{\nabla}\left[\rho\frac{\mathbf{v}^2}{2} + \rho g z + P\right] = \mathbf{0} \tag{4.25}$$

This equality means that a given time, the quantity inside the gradient is uniform. Since the flow is steady, it means that at any time, we have:

$$\boxed{\rho\frac{v^2}{2} + \rho g z + P = \text{cst}} \tag{4.26}$$

where the constant on the right hand side is the same in all the fluid so it can be evaluated in locations where the pressure, the density and/or the velocity fields are set by the boundary conditions.

> The left hand side in equation (4.26) represents the integral of motion we were looking for. We recognize three familiar energies (per unit volume), from left to right: the kinetic energy density, the gravitational potential energy and the internal energy (a.k.a. the thermal energy).

In the case of a stationary flow (i.e. $v = 0$), the differentiation of equation (4.26) yields the hydrostatic equation (**??**):

$$\mathrm{d}P = -\rho g \,\mathrm{d}z \tag{4.27}$$

4.2.2 Irrotational flow and barotropic fluid

Let us relax the assumption of an incompressible fluid. Instead, we assume that the fluid is barotropic, which means that the pressure depends only on the mass density. In this case, it is possible to write:

$$\frac{1}{\rho}\boldsymbol{\nabla} P = \boldsymbol{\nabla}\left[f(P)\right] \tag{4.28}$$

where $f(P)$ is a function of P only whose expression depends on the specific relation between P and ρ.

> Polytropic flows such as the ones you saw in the course of Stellar Astrophysics are a specific case of barotropic flows: in polytropic flows, pressure depends on the density through a power-law:
>
> $$P = K\rho^{\Gamma} \tag{4.29}$$
>
> with K the polytropic constant and Γ the polytropic exponent. An ideal gas undergoing an adiabatic and reversible transformation (i.e. isentropic, $\mathrm{d}S = 0$) can safely be assumed to be polytropic with a polytropic exponent $\Gamma = \gamma$ (where γ is the gas' adiabatic index). An ideal gas undergoing an isothermal transformation (i.e. $\mathrm{d}T = 0$) can safely be assumed to be polytropic with a polytropic exponent $\Gamma = 1$.

Therefore, for a barotropic flow, we have:

$$\frac{v^2}{2} + gz + f(P) = \mathrm{cst} \tag{4.30}$$

Polytropic Bernoulli

For a generic polytrope of polytropic constant K and polytropic exponent Γ, determine the expression of the function $f(P)$ in equation (4.30).

4.2.3 Rotational and incompressible flow

We relax the two assumptions we made in section 4.2.1: the flow is no longer irrotational and the fluid is no longer incompressible. However, we now assume that the flow is incompressible, that is to say:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \tag{4.31}$$

Since the flow is still steady, it implies:

$$\mathbf{v}\cdot\boldsymbol{\nabla}\rho = 0 \tag{4.32}$$

which means that the mass density is constant along a streamline. It is more general than the case of an incompressible fluid where the mass density has to be the same everywhere, not only along streamlines.

We now make use of this property to construct a suitable integral of motion. Let us compute the dot product of the Navier-Stokes equation with the infinitesimal displacement vector $\mathrm{d}\boldsymbol{l}$ along a streamline:

$$\rho\left[\boldsymbol{\nabla}\left(\frac{\mathbf{v}^2}{2}\right)+(\boldsymbol{\nabla}\wedge\mathbf{v})\wedge\mathbf{v}\right]\cdot\mathrm{d}\boldsymbol{l}=-\boldsymbol{\nabla}P\cdot\mathrm{d}\boldsymbol{l}-\rho\left[\boldsymbol{\nabla}(gz)\right]\cdot\mathrm{d}\boldsymbol{l} \tag{4.33}$$

By definition of the gradient operator, the pressure term immediately gives $-\mathrm{d}P$. The first term on the left hand side and the last term on the right hand side have the same mathematical form:

$$\rho\,(\boldsymbol{\nabla}X)\cdot\mathrm{d}\boldsymbol{l} \tag{4.34}$$

where X is a scalar field (either $v^2/2$ or gz). We make use of equation (4.32) and of the fact that $\mathbf{v}=\mathrm{d}\boldsymbol{l}\,/\,\mathrm{d}t$ (by definition) to obtain:

$$\begin{aligned}\rho\,(\boldsymbol{\nabla}X)\cdot\mathrm{d}\boldsymbol{l}&=\left[\boldsymbol{\nabla}(\rho X)-(\boldsymbol{\nabla}\rho)\,X\right]\cdot\mathrm{d}\boldsymbol{l} &(4.35)\\ &=\boldsymbol{\nabla}(\rho X)\cdot\mathrm{d}\boldsymbol{l}-X\underbrace{(\boldsymbol{\nabla}\rho)\cdot\mathrm{d}\boldsymbol{l}}_{0} &(4.36)\\ &=\mathrm{d}\,(\rho X) &(4.37)\end{aligned}$$

Therefore, the equation (4.33) becomes:

$$\mathrm{d}\left[\rho\frac{v^2}{2}+\rho gz+P\right]+\rho\left[(\boldsymbol{\nabla}\wedge\mathbf{v})\wedge\mathbf{v}\right]\cdot\mathrm{d}\boldsymbol{l}=0 \tag{4.38}$$

Let us focus on the last term on the left hand side. Since $\mathrm{d}\boldsymbol{l}\,/\!\!/\,\mathbf{v}$, we have:

$$(\mathbf{A}\wedge\mathbf{v})\cdot\mathrm{d}\boldsymbol{l}=0\quad\forall\text{ vector }\mathbf{A} \tag{4.39}$$

so the relation (4.26) we derived for an irrotational flow and an incompressible fluid still holds but this time, it is only valid along a given streamline (Fig.XXX).

4.3 Applications of Bernoulli theorems

In all the following examples, we work in an inertial frame, gravity is the only external force and we assume that the flow is ideal, steady and incompressible, such as the conclusions drawn in section 4.2.3 applies: we can use equation (4.26) along a given streamline. Remember that the assumption of an incompressible flow provides accurate results when the flow is sub-sonic.

4.3.1 Venturi effect

Observations

In an axisymmetric horizontal pipe whose transverse cross-section S varies, we observe that the flow speed is higher (resp. lower) in the narrower (resp. the wider) parts of the pipe (Fig.XXX). Thanks to a manometer, we also measure the flow pressure along the pipe and observe that it is lower (resp. higher) in the narrower (resp. the wider) parts of the pipe.

Model

The flow is incompressible so $\boldsymbol{\nabla}\cdot\mathbf{v}=0$ and the results of section 2.2.6 on divergence-free vector fields apply. In particular, the volumetric flow rate is constant in the pipe so vS

is a constant, where v is the flow velocity along the pipe's main axis (i.e. in the direction orthogonal to the cross-section). Therefore, when the surface decreases (resp. increases), the flow velocity increases (resp. decreases). Let us apply the Bernoulli theorem to the streamline corresponding to the pipe's main axis, between the point A_1 at the entrance of the pipe and the point A_2 at the bottleneck, where the cross-section is S_1 and $S_2 < S_1$ respectively:

$$\rho_1 \frac{v_1^2}{2} + \rho_1 g z_1 + P_1 = \rho_2 \frac{v_2^2}{2} + \rho_2 g z_2 + P_2 = \tag{4.40}$$

The flow is incompressible and steady so along a streamline, the mass density does not vary: $\rho_1 = \rho_2 \hat{=} \rho$. Since the pipe is horizontal, the terms corresponding to the gravitational potential energy cancel out. Finally, using the conservation of the volumetric flow rate, we have:

$$P_2 = P_1 + \rho \frac{v_1^2}{2} \left[1 - \left(\frac{S_1}{S_2} \right)^2 \right] < P_1 \tag{4.41}$$

Therefore, at the bottleneck, the pressure is lower.

A watery pipe

Assume that at the entrance A_1, the flow is liquid water at ambient pressure and temperature.

1. For $S_1 = 1\text{m}^2$ and $S_2 = 100\text{cm}^2$, what is the entry speed v_1 necessary to make the pressure at the bottleneck to reach the vapor pressure of water $P_2 = 2.5\text{kPa}$?
2. What happens for higher entry speeds?

A leaning pipe

Assume that the pipe above is now inclined downwards by an angle α. The airflow at the entrance of the pipe has a speed $v_1 = 5\text{km}\cdot\text{s}^{-1}$ and an atmospheric pressure. For $S_1 = 1\text{m}^2$ and $S_2 = 100\text{cm}^2$, what is the angle necessary for the gravitational potential energy term to not be negligible in the value of the pressure P_2 at the bottleneck?

Limitations

We can emphasize two important limitations of this model:

- All things being equal, for S_2 small enough, the equation (4.41) indicates that $P_2 < 0$ which is meaningless. It is because the sound speed decreases with the pressure so for a pressure drop too important, the sound speed becomes comparable to the flow speed and the assumption of an incompressible flow is no longer valid.
- The application of the Bernoulli theorem between the entry point A_1 and the exit point A_3 in Fig.XXX suggests that pressure should be the same in both points. In practice, we measure a drop which can be ascribed to the neglected viscous terms which play a role in the narrower parts of the pipe.

Applications

This effect is called the Venturi effect and has multiple applications. For instance, it can be used to measure a flow speed by measuring a pressure difference. Alternatively, it can be used to make a basic vacuum pump: in Fig.XXX, liquid water flows from the top of the tube. At the bottleneck, the pressure drops sucks the air of the connected recipient (R).

4.3.2 Pitot tube

Model

A Pitot tube is an instrument to measure the speed of a steady and uniform flow, typically used in aviation. A Pitot tube is a metallic axisymmetric structure of full cross-section s (Fig.XXX). At its extremity A, it is pierced with a thin central pipe of cross-section negligible compared to s. It faces a planar flow of cross-section $S \gg s$. Far from the tube, the air flow has a uniform mass density[4] ρ_∞, pressure P_∞ and a uniform velocity with respect to the tube given by $\mathbf{v}_\infty = v_\infty \,\hat{\mathbf{x}}$. The Pitot tube has a lateral hole at point B and an internal manometer measures the pressure difference $P_A - P_B$. Last, we assume that within the Pitot tube, the pipes are thin enough that the external air does not penetrate and the air remains still.

Equations

If the air is at rest in the Pitot tube, the continuity of the normal component of the velocity field at point A and the axisymmetry impose $\mathbf{v_A} = \mathbf{0}$: A is a stagnation point. We apply the Bernoulli theorem on the streamline between a point A_∞ at infinity on the symmetry axis and the point A:

$$\rho\frac{v_\infty^2}{2} + \rho g z_{A,\infty} + P_\infty = \rho g z_A + P_A \tag{4.42}$$

Neglecting the variation of the altitude in the gravitational potential terms, they cancel out and we obtain:

$$P_A = P_\infty + \rho\frac{v_\infty^2}{2} \tag{4.43}$$

At point B, the velocity is tangent to the Pitot tube so we cannot draw conclusion on its values from the boundary condition. Instead, we use that $S \gg s$ to conclude that $v_B \sim v_\infty$. Once we apply the Bernoulli theorem on the streamline between a point B_∞ at infinity and the point B, we have, neglecting the gravitational terms:

$$\rho\frac{v_\infty^2}{2} + P_\infty \sim \rho\frac{v_\infty^2}{2} + P_B \tag{4.44}$$

such as $P_B \sim P_\infty$. Therefore, the manometer indirectly measures the speed v_∞ when it measures the pressure difference since:

$$P_A - P_B = \rho\frac{v_\infty^2}{2} \tag{4.45}$$

4.3.3 Lift

In this section, we come back to the problem of the air flow over a plane wing we started to address in section 2.3 in chapter 2.

Quantitative study of a cylindrical wing

First, we model the plane wing as a cylinder of radius R and main axis $\hat{\mathbf{z}}$ horizontal, in uniform rectilinear motion in the ground frame $\mathcal{R}$ at the constant velocity $U\,\hat{\mathbf{x}}$ where $U > 0$ and

[4] Since the flow is incompressible, it implies that the fluid is uniform: the mass density is the same everywhere and is written ρ thereafter.

$\hat{\mathbf{x}}$ is the horizontal unit vector normal to $\hat{\mathbf{z}}$ (Fig. XXX). Therefore, seen from the wing, the air far upstream flows with a constant velocity $\mathbf{v}_\infty = -U\,\hat{\mathbf{x}}$ towards the wing. We assume that the velocity U is subsonic such as the flow can be modeled as incompressible. Furthermore, the flow is assumed to be irrotational: this is relevant for an inviscid flow (i.e. a strictly non-viscous flow) according to Kelvin's theorem which states that an initially irrotational ideal flow remains irrotational at all time. We have seen that in those conditions, the velocity field in the reference frame of the wing is, in the cylindrical basis:

$$v_r = -U\cos\theta\left[1-\left(\frac{R}{r}\right)^2\right] \tag{4.46}$$

$$v_\theta = U\sin\theta\left[1+\left(\frac{R}{r}\right)^2\right] \tag{4.47}$$

Therefore, on the wing, we have the velocity field:

$$\mathbf{v} = 2U\sin\theta\,\hat{\boldsymbol{\theta}} \tag{4.48}$$

Neglecting the weight[5], we apply the Bernoulli theorem along a streamline to obtain the pressure profile on the wing:

$$P(r=R,\theta)+\frac{1}{2}\rho v^2(r=R,\theta) = P_\infty + \frac{1}{2}\rho U^2 \tag{4.49}$$

$$P(r=R,\theta) = P_\infty + \frac{1}{2}\rho U^2\left(1-4\sin^2\theta\right) \tag{4.50}$$

where P_∞ is the unperturbed air pressure far upstream. Therefore, the wing experiences pressure forces on its hull which are symmetric with respect to the vertical $\hat{\mathbf{y}}$-axis (Fig.XXX). It means that the net pressure force is null and that this models falls short from explaining the lift experienced by a plane wing. More generally, an inviscid flow around a cylindrical wing presents an up-down symmetry which prevents any net vertical force from appearing.

Qualitative study of a realistic wing

A realistic plane wing is breaks the up-down symmetry of the flow (Fig.XXX): the streamlines below (resp. above) the wing get further (resp. closer) from each other. Since the flow is incompressible, it means that the flow speed just below (resp. above) the wing is lower (resp. higher) than U. Therefore, the Bernoulli theorem imposes that the pressure below the wing is higher than the pressure above the wing, such that there is a net upwards force called the lift. This up-down assymmetry of the flow speed on the wing leads to a non-zero net circulation of the velocity field on the contour $(\mathcal{C})$ of the wing:

$$\oint_{(\mathcal{C})} \mathbf{v}\cdot \mathrm{d}\boldsymbol{l} \neq 0 \tag{4.51}$$

which, by Kelvin circulation theorem, means that the flow is not irrotational.

Back to the cylindrical wing

To account for the fact that the flow is not irrotational, we adopt a pragmatic approach and modify the velocity field in the simplemost way. We add to the $\hat{\boldsymbol{\theta}}$-component of the velocity

[5]Which is fair since its magnitude is comparable to the Archimedes buoyancy force which is largely insufficient to lift a wing.

field a term inspired by the exercise 2.4.2 on the modeling of a tornado[6]:

$$v_r = -U\cos\theta\left[1-\left(\frac{R}{r}\right)^2\right] \tag{4.52}$$

$$v_\theta = U\sin\theta\left[1+\left(\frac{R}{r}\right)^2\right] + \omega_0 r \tag{4.53}$$

where $\omega_0 = |\boldsymbol{\nabla}\wedge\mathbf{v}|$ at infinity. The application of the Bernoulli theorem along a streamline now gives:

$$P(r=R,\theta) + \frac{1}{2}\rho v^2(r=R,\theta) = P_\infty + \frac{1}{2}\rho U^2 \tag{4.54}$$

$$P(r=R,\theta) + \frac{1}{2}\rho\left[2U\sin\theta + \omega_0 R\right]^2 = P_\infty + \frac{1}{2}\rho U^2 \tag{4.55}$$

$$P(r=R,\theta) = P_\infty + \frac{1}{2}\rho\left[U^2 - 4U^2\sin^2\theta - \omega_0^2 R^2\right] - 2\rho\omega_0 RU\sin\theta \tag{4.56}$$

Once again, the terms independent on θ or dependent on $\sin^2\theta$ gives a zero net contribution, by symmetry. Therefore, we focus on the last term in the right hand side which produces a net pressure force $\mathbf{F}$ given by, after integration over the whole wing:

$$\mathbf{F} = -\oiint\left[-2\rho\omega_0 RU\sin\theta\right]\mathrm{d}\,\mathbf{S} \tag{4.57}$$

$$= \oiint\left[2\rho\omega_0 RU\sin\theta\right]R\,\mathrm{d}\theta\,\mathrm{d}z\,\hat{\mathbf{r}} \tag{4.58}$$

where the first minus sign is because we compute the force experienced by the wing. By symmetry, the net pressure force $\mathbf{F}$ is oriented upwards (Fig.XXX): it is the lift we are looking for. The net lift is given by the projection of $\mathbf{F}$ on the vertical $\hat{\mathbf{y}}$-axis:

$$\mathbf{F}\cdot\hat{\mathbf{y}} = \oiint\left[2\rho\omega_0 RU\sin\theta\right]R\,\mathrm{d}\theta\,\mathrm{d}z\,\underbrace{\hat{\mathbf{r}}\cdot\hat{\mathbf{y}}}_{=\sin\theta} \tag{4.59}$$

$$= 2\pi\rho\omega_0 R^2 U\underbrace{\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta}_{=2\pi/2}\underbrace{\int_0^L\mathrm{d}z}_{=L} \tag{4.60}$$

$$= 2\pi\rho\omega_0 R^2 UL \tag{4.61}$$

where L is the length of the wing along its main axis. The value of ω_0, instrumental in the expression of the lift force, is set by the rotational character of the flow. The latter is due to the viscous force, that we neglected in our analysis but which always dominates in the immediate vicinity of the wing.

[6]Except that here, the horizontal cylinder-shaped wing plays the role of the vertical cylinder-shaped tornado.

Chapter 5

Steady state macroscopic balance

Rankine-Hugoniot as a global mass / momentum / energy conservation?

5.1 Generalities

5.1.1 Two balance methods

Introductory example

First method: open and fixed system

Second system: closed and deformable system

Generalization

5.1.2 Lagrangian derivative and balance for closed systems

Lagrangian derivative of an extensive quantity

Expression of mechanics and thermodynamics laws

Selection criteria

Thermodynamic effect of viscosity

5.2 Example of linear momentum balance

5.2.1 Model

5.2.2 Outward velocity and pressure

5.2.3 Feedback force on the pipeline

5.2.4 Comments

5.2.5 Thrust

5.3 Example of angular momentum balance

5.3.1 Model

5.3.2 Angular momentum balance

5.3.3 Motion equation

5.3.4 Comments

5.4 Shock wave in a pipeline

5.4.1 Model

5.4.2 Mass balance

5.4.3 Linear momentum balance

5.4.4 Ideal flow model - Entropy balance

5.4.5 Adiabatic flow model - Internal energy balance

5.5 Energetic aspects of Bernoulli theorem

5.5.1 Kinetic energy balance

5.5.2 Thermodynamic aspects

Chapter 6

Sound waves

Illustration of a generic approach used in linear perturbation theory: 1. find an equilibrium 2. consider a small perturbation from it 3. develop the governing equation to 1st order in this perturbation 4. deduce the behavior of the perturbation (oscillation, instability...)

6.1 Formalism

6.1.1 Fundamental equations

Acoustic approximation

Linearization of Euler equation

Linearization of the local mass conservation equation

Thermodynamic evolution

6.1.2 Propagation of sound waves

Sound waves are pressure waves

Propagation equation

6.2 Plane progressive sound waves

6.2.1 Family of solution of the propagation equation

6.2.2 Sound speed

6.2.3 Harmonic plane progressive waves

6.2.4 Structure

Pressure-velocity coupling

Acoustic impedance

6.3 Energetic aspects

6.3.1 Power exchanged through a surface

Parallel w/ Poynting vector

6.3.2 Local energy equation

6.3.3 Energetic balance of the propagation of a sound wave

6.3.4 Case of a plane progressive wave

Vector of acoustic power density flux

Orders of magnitude

6.3.5 Sound intensity - Acoustic decibels

6.4 Plane stationary sound waves

6.4.1 Structure

6.4.2 Energetic aspects

6.5 Reflection/transmission of a plane progressive wave at normal incidence

6.5.1 Model

Description

Boundary conditions

Necessity of a reflected wave

6.5.2 Reflection/transmission coefficients for amplitudes

6.5.3 Reflection/transmission coefficients for powers

6.6 Back to the ideal flow model

6.6.1 Mechanic effect of viscosity

6.6.2 Effect of thermo-diffusion