RESEARCH STATEMENT: QUALITATIVE ANALYSIS OF SOME SINGULAR PARTIAL DIFFERENTIAL EQUATIONS ARISING IN PHYSICS AND BIOLOGY

by

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1. Introduction

A general question in Physics and (more recently) in Biology is to understand the behaviour of systems composed of many interacting agents. These systems occur naturally in fields as different as Fluid Dynamics (fluid particles colliding), Population Dynamics (individuals competing), Chemistry (reacting molecules), Biology (communicating cells), Astrophysics (celest bodies gravitationnally attracted), etc.

From the Mathematical point of view, Partial Differential Equations (PDEs) are a common tool to model these systems. These PDEs often have singular behaviours: their analysis therefore recquire the use of specific tools (egg. compactness methods, renormalization). My work focuses on these types of systems, in particular two specific cases (which are mathematically related ref): the Boltzmann Equation and Cross-Diffusion Systems.

2. Context

2.1. The Boltzmann Equation. — The Boltzmann Equation models the dynamics of rarefied gas away from equilibrium ([**3**]). It is derived by considering that any particle of gas perseveres with a fixed velocity v unless it collides with another particle. Then, if we consider the density $f = f(t, x, v)$ of particles at time t and point of space x with velocity v, the Boltzmann Equation writes

$$\underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{advection term}} = \underbrace{Q(f,f)}_{\text{collision operator}} .$$

The Boltzmann Equation has been introduced in the middle of the 19^{th} century by Maxwell and Bolztmann. It has since then been studied by a wide community of specialists in PDEs. Many existing results concern the existence of strong solutions close to equilibrium or vacuum (ref). However, there is almost no existing result treating strong solutions in general bounded domains, although this situation naturally appears in the applications. Indeed, the interaction of the particles with the boundary can create a singularity (more precisely, discontinuities can occur on the trajectories grazing the boundary): this singularity makes the mathematical analysis very delicate ([**16**],[**12**]).

2.2. Cross-diffusion systems. — The most studied Cross-diffusion systems arises in Population Dynamics. It writes (some constants are taken equal to 1) :

$$\begin{cases} \partial_t u - \Delta\left[d_1\, u + u^{1+\alpha} + u\, v^{\beta}\right] = u - u^{1+a} - u\, v^{b}, \\ \partial_t v - \Delta\left[d_2\, v + v^{1+\gamma} + d_\delta u^{\delta}\, v\right] = v - v^{1+c} - u^{d}\, v. \end{cases} \tag{1}$$

This model was introduced by N. Shigesada, K. Kawasaki and E. Teramoto in [**19**] (1979). In comparaison with classical reaction-diffusion systems, the specificity of this system comes from the presence of the cross-diffusion terms $\Delta\left[u\, v^{\beta}\right]$, $\Delta\left[d_\delta u^{\delta}\, v\right]$. In the applications in Ecology, these terms model a repulsive effect between two species. The presence of these nonlinar terms involves some new mathematical difficulties : for example, a question as basic as the existence of strong solutions is still open.

One particular case that has been widely studied is the triangular case ($d_\delta = 0$). There exist for this particular case different results of existence of (global) strong solutions, which are all valid under quite specific assumptions (egg. in small dimension, [**17**]).

3. Main contribution

3.1. The Boltzmann Equation. — I joined Prof. Yan Guo, Chanwoo Kim and Daniela Tonon in 2011: our goal was to study the maximum regularity of the solutions of the Boltzmann equation in general bounded domains with diffuse boundary conditions. We had to pay a particular attention to the discontinuies created on grazing trajectories. In particular, when the domain is not convex, the grazing trajectories cross the domain and discontinuities reach the inside of the domain.
In a first paper [**13**], we considered a convex domain. We were able to isolate the singular trajectories by constructing an adequate measure : our result implies in particular that away from the singular trajectories, the solution propagates some Sobolev/C^1 regularity. We showed that it is quite optimal since one does not expect propagation of (this very) Sobolev regularity if one includes the singular set, and it is impossible to get further Sobolev regularity (more derivatives).
In a second paper [**14**], we considered a non-convex domain. Because of the possible discontinuities inside the domain, the maximal regularity one could reach is BV. We indeed showed BV propagation.
Taken together, our papers then provide very sharp (optimal) results on the regularity propagated.

3.2. Cross-diffusion systems. — During my PhD, under the supervision of Prof. Laurent Desvillettes, I introduced an original approach, which is based on a recent lemma by M. Pierre and D. Schmitt (ref) and entropy methods. I used this approach in [**10**] to show quite general results of existence of weak solutions for the cross-diffusion systems in the triangular case. Furthermore, these solutions are obtained as the (rigourous) limit of a singular perturbation problem. Notably, the singular perturbation problem is physically relevant : it models the same ecological system at another (time) scale. For cross-diffusion systems, the question of approximating the system frequently leads to technical constructions (using Galerkine's approximations and Schauder's fixed point Theorem). In our approach, the singular perturbation problem naturally provides a smooth (and quite easy to handle) approximation.
In the paper [**8**] in collaboration with L. Desvillettes, Th. Lepoutre and A. Moussa, we provided new results of existence of weak solutions for the general (non-triangular) case. Our approach relies on a hidden symmetric structure providing a Lyapunov-like functional. This structure is not robust enough to use the singular perturbation problem, a difficulty that we overcame by introducing a (quite easy to handle) new scheme of approximation.

4. Ongoing and Short-Term Projects

4.1. The Boltzmann Equation. — Since November 2014 and until May 2015, I am visiting Academia Sinica (Taipei, Taiwan). I am invited by Prof. Tai-Ping Liu, specialist in Kinetic Theory and Boundary Phenomena. This visit provides to me the opportunity to learn the techniques developped by Prof. Tai-Ping Liu and his team (egg. micro/macro decomposition, stochastic formulation) in the analysis of boundary phenomena. We expect to compare and combine our points of view in the context of the Boltzmann equation in bounded domains to answer new open questions.

4.2. Cross-diffusion systems. — In cross-diffusion systems of the form presented here, the presence of the self-diffusion terms $\Delta[u^{1+\alpha}]$, $\Delta[v^{1+\gamma}]$ gives rise to extra estimates on the solution, and therefore should facilitate the analysis. But surprisingly in the triangular case, the presence of self-diffusion terms actually makes the analysis more delicate. The reason is that their presence seems to break the structure of the perturbation problem: even defining it is not clear. In the paper [**20**], I combine the methods developped in [**10**] and [**8**] to prove a result of existence of solutions for the triangular case in presence of self-diffusion terms.
Another variant of the triangular case that is natural to consider (from the point of view of Ecology) is the variant where the reaction terms are not upper bounded (Lotka-Volterra type). In this variant the main bound for the solution of one of the equations (namely, the bound given by the maximum principle) does not hold. The analysis is therefore more subtle; existence results should be obtained thanks to (new versions of) the methods developped in [**10**] and [**8**].

5. Long-Term Research Goals

5.1. Mathematical Analysis of Singular Phenomena. — Generally speaking, I am interested in problems presenting singular properties (such as formation of discontinuity, very weak forms of solution). One of my research goals is to reach a precise understanding of these singular phenomena : when, how and how much do they occur ? Among the many possible directions, it would be interesting to describe how

singularities can emanate from a singular point of the boundary for the Boltzmann equation. Also, we may use the methods developped in [**16**], [**13**] to obtain some quantitative information on the formation of the discontinuity.
Finally, a mathematical link has been found between the Boltzmann Equation and some cross-diffusion systems arising in Chemistry ([**1**]); I would like to understand more deeply this link and to question it in the context of Population Dynamics.

5.2. Numerical Simulations of Singular Phenomena. — The previously described phenomena would certainly be thought of in a new perspective if they were described and explored numerically. For example, the formation of discontinuity for the Boltzmann equation in bounded domains has not been observed numerically. The numerical description would be the twin of the quantitative analysis in understanding the impact of this discontinuity at relevant physical scales. (Note that this project being quite involving, it would certainly recquire to begin by studying simplified models which conserve the singular behaviour).
Another very natural direction, which this time concerns cross-diffusion systems, is to explore numerically the set of admissible parameters for the asymptotics of the singular perturbation and to check whether it matches with the theory. Note that some simulations were obtained in [**?**], [**15**].

6. Conclusion

My results concerning the Boltzmann Equation in [**13**] and [**14**] are the first ones for the regularity in bounded domains with diffuse boundary conditions and they close the subject, providing a complete theory. My papers [**10**], [**8**] and [**20**] develop new tools and answer questions that had been open for decades (or were partially answered) on the existence of solutions for Cross-Diffusion systems. My work includes collabarations with different teams, notably a long-term collaboration (commensurate with my research carreer) on the project [**13**], [**14**] (2011-2014), but also contributions in autonomy ([**20**]). My projects concern extensions of previous results by the adaptation of previously used methods; though I also intend to learn (and develop) new methods, in particular through new collaborations. Notably I plan to dedicate much of my efforts to develop skills in numerical simulation in order to enlarge my views and to obtain more results. Finally, my research interests are obviously related to the oness of Prof. J. Carrillo and Prof. P. Degond (both Professors at Imperial College), and my presence at Imperial College would certainly be a good opportunity to begin a collaboration.

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