Computational Methods for Astrophysical Applications

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Lesson 3 : Non linear hybridization for scalar advection

- **Finite volume approximation**

  Mesh, cell centers and interfaces

  Integral form of the conservation equations
- **Reconstruction**

  Piecewise linear reconstruction

  Slope limiters

  TVD schemes
- **The Riemann problem**

  ?

  Building block for hyperbolic PDE

  Local Lax-Friedrichs

  Boundary conditions

  Linearized Euler system
- **2$^{\text{nd}}$ order upwind schemes**

  Lax-Wendroff

  Runge-Kutta

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   - Mesh, cell centers & interfaces
   - Integral form of the conservative equations
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   - Piecewise linear reconstruction
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4. 2^{nd} order upwind schemes
   - Lax-Wendroff
   - Runge-Kutta

- **useless?** Method is **consistent** since LTE vanish in limit $\Delta x \to 0$ and $\Delta t \to 0$
  - $\Rightarrow$ accuracy: 2nd order space, 1st order time $\to$ overall 1st
  - $\Rightarrow$ failure is related to numerical stability
- round-off errors should not grow during time progression
  - $\Rightarrow$ evaluate by **von Neumann method**
  - $\Rightarrow$ numerical solution = exact $+$ round-off error $\epsilon(x,t)$
  - $\Rightarrow$ represent $\epsilon(x,t)$ in Fourier series, analyse Fourier term

$$\epsilon_k(x,t) = \hat{\epsilon}_k \mathrm{e}^{\lambda t} \mathrm{e}^{\mathrm{i}kx}$$

- numerically stable scheme: for all spatial wavenumbers k

$$\left| \frac{\epsilon_k^{n+1}}{\epsilon_k^n} \right| = \left| \mathrm{e}^{\lambda \Delta t} \right| \leq 1$$

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- von Neumann stability analysis for BTCS scheme

$$\left|e^{\lambda \Delta t}\right| = \frac{1}{\left|1 + i\frac{v\Delta t}{\Delta x}\sin\left(k\Delta x\right)\right|} < 1 \qquad \text{for all } k$$

$\Rightarrow$ **unconditionally stable**, any (large) time step Δt allowed

- note: **stability does not imply accuracy**

  $\Rightarrow$ large Δt affects accuracy, defines time resolution:
  behavior may involve physical timescale that needs to be resolved!
- implicit backward Euler: first order in time

MinMod

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Superbee

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Harten's theorem

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Definition of TV

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TVD regions of slope limiters

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