Computational Methods for Astrophysical Applications

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Lesson 2 : Finite Difference Approximation

- **Spatial discretization**
  Mesh and data collocation
  Discrete equations
- **Time advance**
  Von Neumann stability analysis
  Explicit linear schemes

1. Spatial discretization
   - Mesh and data collocation
   - Discrete equations
2. Time advance
   - Von Neumann stability analysis
   - Explicit linear schemes

- **useless?** Method is **consistent** since LTE vanish in limit $\Delta x \to 0$ and $\Delta t \to 0$
  - $\Rightarrow$ accuracy: 2nd order space, 1st order time $\to$ overall 1st
  - $\Rightarrow$ failure is related to numerical stability
- round-off errors should not grow during time progression
  - $\Rightarrow$ evaluate by **von Neumann method**
  - $\Rightarrow$ numerical solution = exact $+$ round-off error $\epsilon(x, t)$
  - $\Rightarrow$ represent $\epsilon(x, t)$ in Fourier series, analyse Fourier term

$$\epsilon_k(x, t) = \hat{\epsilon}_k \mathrm{e}^{\lambda t} \mathrm{e}^{\mathrm{i}kx}$$

- numerically stable scheme: for all spatial wavenumbers k

$$\left| \frac{\epsilon_k^{n+1}}{\epsilon_k^n} \right| = \left| \mathrm{e}^{\lambda \Delta t} \right| \leq 1$$

Taylor series expansions

- perform von Neumann stability analysis for Lax–Friedrichs

$$e^{\lambda \Delta t} = \cos(k\Delta x) - \mathrm{i}\,\frac{v\Delta t}{\Delta x}\,\sin(k\Delta x)$$

$\Rightarrow$ **conditional stability** requiring Courant number C

$$C \equiv \frac{|v|\Delta t}{\Delta x} \leq 1$$

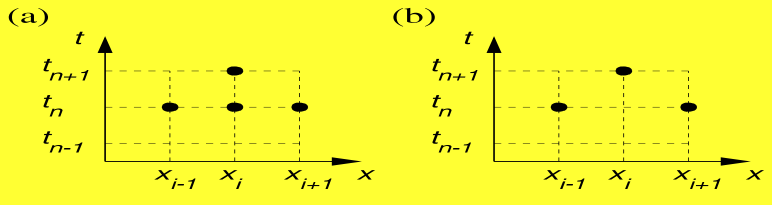
$\Rightarrow$ limitation of the time step Δt for a given resolution Δx

$\Rightarrow$ **Courant–Friedrichs–Lewy** condition (1928)

$\Rightarrow$ necessary (not sufficient) condition for stability!

Central FD method

- in (x, t) space, we identify **stencil** of a method
  - $\Rightarrow$ stencils visualizes discrete dependence of method



  - $\Rightarrow$ stencil for FTCS (a) versus Lax-Friedrichs (b)
- **hyperbolic** PDE and **physical characteristics**
  - $\Rightarrow$ the advection equation is hyperbolic as

$$\frac{\partial u}{\partial t} + \frac{\partial \left(\underbrace{vu}_{F} \right)}{\partial x} = 0$$

and Flux Jacobian $\frac{\partial F}{\partial u} = v$ is real number 'characteristic speed'

Truncation error and order of accuracy

- for second order wave equation

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$\Rightarrow$ factorizes to

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) u = 0$$

$\Rightarrow$ general solution has left and right going wave with

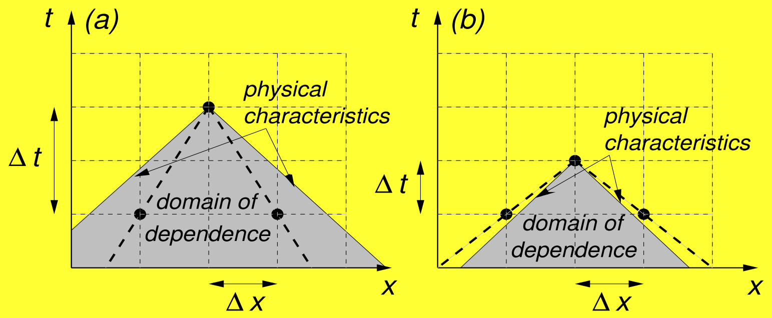
$$u = f(x - vt) + g(x + vt)$$

$\Rightarrow$ initial shapes $f(x)$, $g(x)$ combine

$\Rightarrow$ 2 characteristics $\frac{dx}{dt} = \pm v$

Consistency of a FDA w/ respect to its PDE

- illustrate CFL for second order wave equation:
  the domain of dependence of the differential equation should be contained in the DOD of the discretised equations



$\Rightarrow$ stability means physical DOD contained in stencil bounds (numerical DOD), hence Δt small enough (right case)

- note: linear advection + wave equation: DOD only involves 1 or 2 points from $t = 0$ $\leftrightarrow$ HD: DOD bounds set by $v \pm c_s$ with c_s

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Principle and necessary stability condition

- von Neumann stability analysis for BTCS scheme

$$\left|e^{\lambda \Delta t}\right| = \frac{1}{\left|1 + \mathrm{i}\frac{v\Delta t}{\Delta x}\sin(k\Delta x)\right|} < 1 \qquad \text{for all } k$$

$\Rightarrow$ **unconditionally stable**, any (large) time step Δt allowed

- note: **stability does not imply accuracy**

$\Rightarrow$ large Δt affects accuracy, defines time resolution:
behavior may involve physical timescale that needs to be resolved!

- implicit backward Euler: first order in time

Convergence : Lax-Richtmeyer theorem

- von Neumann stability analysis for BTCS scheme

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Lax-Friedrichs

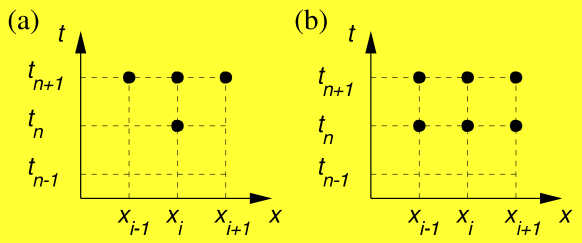
- spatial differences as average of n-th and $(n+1)$-th time step

$$u_i^{n+1} = u_i^n - \tfrac{1}{4} v \frac{\Delta t}{\Delta x} \left(u_{i+1}^{n+1} + u_{i+1}^{n} - u_{i-1}^{n+1} - u_{i-1}^{n} \right)$$

$\Rightarrow$ second order **Crank–Nicolson method**

$\Rightarrow$ **Exercise**: show that this scheme is unconditionally stable, 2nd order accurate

- stencils for BTCS (left) and Crank-Nicolson (right)



Lax-Wendroff

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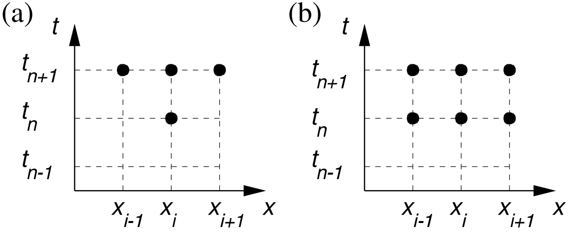

Runge-Kutta

- spatial differences as average of n-th and $(n+1)$-th time step

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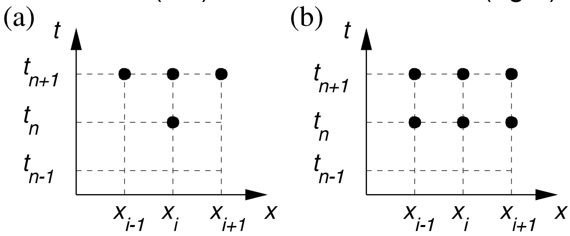

Upwind FD scheme

- spatial differences as average of n-th and $(n+1)$-th time step

$$u_i^{n+1} = u_i^n - \tfrac{1}{4} v \frac{\Delta t}{\Delta x} \left(u_{i+1}^{n+1} + u_{i+1}^n - u_{i-1}^{n+1} - u_{i-1}^n \right)$$

$\Rightarrow$ second order **Crank–Nicolson method**

$\Rightarrow$ **Exercise**: show that this scheme is unconditionally stable, 2nd order accurate

- stencils for BTCS (left) and Crank-Nicolson (right)