Computational Methods for Astrophysical Applications

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Lesson 2 :
Finite Volume Approximation for linear hyperbolic PDE

- **Finite volume approximation**

  Mesh, cell centers and interfaces

  Integral form of the conservation equations
- **Reconstruction**

  Piecewise linear reconstruction

  Slope limiters

  TVD schemes
- **The Riemann problem**

  ?

  Building block for hyperbolic PDE

  Local Lax-Friedrichs

  Boundary conditions

  Linearized Euler system
- **2^{nd} order upwind schemes**

  Lax-Wendroff

  Runge-Kutta

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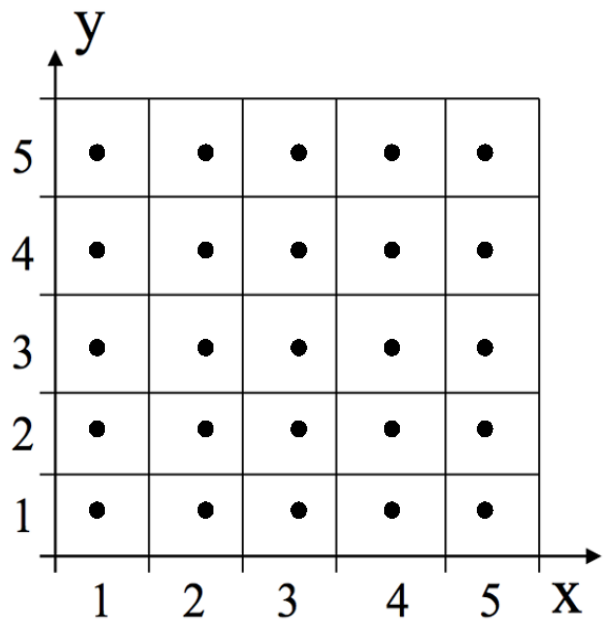
Mehs

- Balsara 2.2, Leveque1 12.1, Leveque2 4.1



- Take a 1D conservation law without source terms :

$$\partial_t u + \partial_x f = 0 \tag{1}$$

Conservative form

- Average variables u over cell to get U : $\frac{1}{\Delta x}\int_{\text{cell}}(1)$

  $\Rightarrow$ 1st order forward in time then gives :

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\frac{F_{i+1/2}^{n+1/2} - F_{i-1/2}^{n-1/2}}{\Delta x}$$

where

$$\begin{cases} U_i^n = \dfrac{1}{\Delta x}\displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} u\left(x, t^n\right)\Delta x \\ F_{i+1/2}^{n+1/2} = \text{flux @ interface i+1/2} \longrightarrow \text{unknown} \end{cases}$$

  $\Rightarrow$ $\sum$ over all cells => fluxes cancel out => conservative

- Key-question: how do you reconstruct the fluxes @ interfaces?

Slab representation

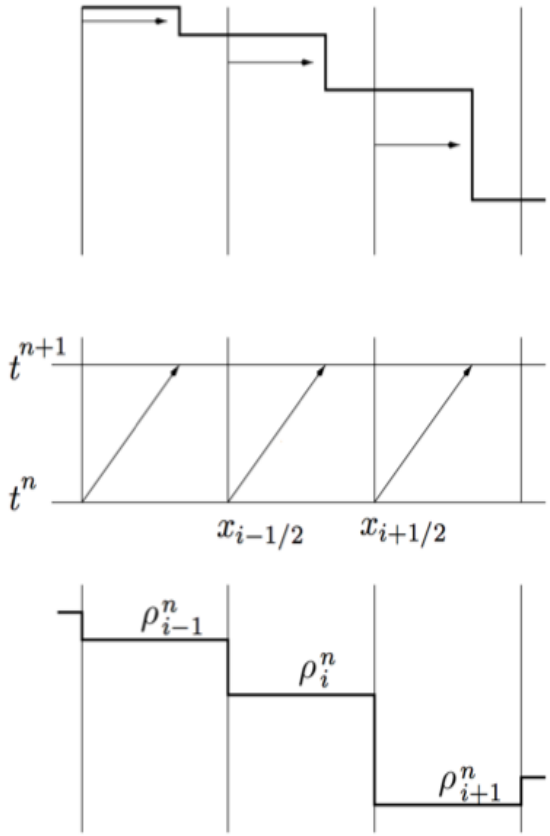


Figure 3.3: Finite-volume interpretation of an upwind scheme applied to a linear advection equation with $v > 0$ (the flow is moving to the right). The bottom graph represents the piecewise constant function at t^n and the upper one yields, if averaged over each cell, the piecewise function at t^{n+1}. In-between are represented the characteristics which monitor the advance of the steps. From LeVeque (2002).

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MinMod

MC

Van Leer

Superbee

Harten's theorem

Definition of TV

TVD regions of slope limiters

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