Computational Methods for Astrophysical Applications

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References

- **Books**

  - *Riemann solvers and numerical methods for fluid dynamics - A practical introduction*, by Toro
  - *Numerical methods for conservation laws*, by Leveque
  - *Finite volume methods for hyperbolic problems*, by Leveque

- **Courses**

  - Rony Keppens & Jon Sundqvist 2016-2017
  - *Numerical PDE Techniques for Scientists and Engineers*, by Dinshaw Balsara

- **Schools**

  - Les Houches
  - Numerical techniques in MHD simulations, Köln University

Lesson 1 : Hyperbolic Partial Differential Equations

- **Conservation laws**
  - Integral and differential forms
  - Examples
- **Hyperbolic PDE**
  - Matrix formulation of conservation laws
  - Time advance
  - Linear advective equations
  - The Riemann problem

Integral form

- Continuous medium hypothesis : from actual particles to particles of fluid

  $\Rightarrow$ distribution function and integration of Boltzmann equation : the closure problem

  $\Rightarrow$ analogy with moments of the specific photon intensity (see Prof. Sundqvist's course)
- Consider **any volume element** $\mathrm{d}V$ of a fluid

  $\Rightarrow$ XXX bilan XXX over $\mathrm{d}t$ of scalar quantity : inflow/outflow, sinks/sources

  $\Rightarrow$ for the momentum : tensor formulation and dyadic product

  $\Rightarrow$ for energy
- Shocks and jump conditions : Rankine-Hugoniot

Conservative form

- Provided the variables $X(\mathbf{r}, t)$ are XXX differentiable XXX

$$\partial_t U + \nabla \cdot \underbrace{[F(U)]}_{\text{fluxes}} = \underbrace{S(\mathbf{r}, \mathbf{v}, t)}_{sources/sinks}$$

$\Rightarrow$ Matrix formulation

- Properties of this differential form

$\Rightarrow$ Coordinate / dimension independent formulation

$\Rightarrow$ Given inital and boundary conditions, can provide a general solution

$\Rightarrow$ Conservative form : Green-Ostrogradsky (or Gauss) law and Eulerian approach

$\Rightarrow$ Conservative variables : ρ, $\rho\mathbf{v}$ and e

Primitive form

- Lagrangian approach XXX Picture of a parachute XXX

$$D_t\left(\cdot\right) = \partial_t\left(\cdot\right) + \mathbf{v}\nabla\left(\cdot\right)$$

- Continuity equation

$$D_t\left(\rho\right) = -\rho\nabla\mathbf{v}$$

  $\Rightarrow$ incompressible fluid $\neq$ flow
- Navier-Stokes equation

$$\rho D_t\left(\mathbf{v}\right) = -\nabla P$$

- Energy equation
  - $\Rightarrow$ internal and mechanical energy
  - $\Rightarrow$ link w/ 1^{st} principle of Thermodynamics
  - $\Rightarrow$ entropy formulation

Closure relation

- Equation-of-state of an ideal gas

$$u = \frac{P}{\gamma - 1}$$

  $\Rightarrow$ adiabatic index γ

- A classic way-around : the polytropic assumption

$$P \propto \rho^{\alpha}$$

  $\Rightarrow$ polytropic index α

  $\Rightarrow$ the isentropic case

The linear advection equation

- Consider the 1D continuity equation w/ constant & uniform speed v
$$\partial_t \rho + v \partial_x \rho = 0$$
  $\Rightarrow$ initial conditions $\rho(x, t = 0) = \rho_0(x)$
- Analytic solution
$$\rho(x, t) = \rho_0(x - vt)$$
- Fourier analysis
$$\rho\left(x, t\right) = r\left(k, t\right) e^{ikx}$$
  XXX
  - $\Rightarrow$ non-dissipative
  - $\Rightarrow$ non-dispersive
  - $\Rightarrow$ see also DB1#32-33
- Analytic preliminary analysis => mathematical properties of the PDE => appropriate numerical scheme
- Numerical solution

Euler equations

- Consider the viscous tensor τ

$$\tau = \mu \left[\left(\mathbf{v} + \mathbf{v}^t \right) - \frac{2}{3} \left(\nabla \cdot \mathbf{v} \nVdash \right) \right]$$

$\Rightarrow$ see Landau Lifschitz section XXX for hypothesis & shape

$\Rightarrow$ μ dynamic viscosity in XXX

$\Rightarrow$ conservative formulation

$\Rightarrow$ Dimensionless form, Reynolds number and singular limit (Lagrange paradox? XXX)

Ideal Magneto-Hydrodynamics equations

- Ideal MHD fundamental hypothesis

$$\mathbf{E} + \mathbf{j} \wedge \mathbf{B} = \mathbf{0}$$

$\Rightarrow$ **E** secondary variable

- Maxwell-Faraday and Maxwell-Ampère give the induction equation XXX :

$$\nabla \wedge \mathbf{B} - \partial_t \left((\nabla \wedge \mathbf{B}) \wedge \mathbf{B} \right) = \mathbf{0}$$

$\Rightarrow$ magnetic pressure and XXX Prandtl XXX number

$\Rightarrow$ magnetic tension

- Frozen-flux theorem

Linear, non-linear, quasi-linear

- Homogeneous

$$\partial_t \mathbf{U} + \mathbf{A}\left(\mathbf{U}, x, t\right) \partial_x \mathbf{U} = \mathbf{0} \tag{1}$$

XXX Choice diagram XXX

- Non-homogeneous : ... $+ \mathbf{B}\left(\mathbf{U}, x, t\right) = \mathbf{0}$

  $\Rightarrow$ still linear if $\mathbf{B} \propto \mathbf{U}$

Time and space dependent linear advection equation? Burgers equation? Conservation equations?
Toro 2.1 p.42

- Systematic way to solve quasi-linear PDE?

Eigenvalues & eigenvectors

Consider the following system of linear constant coeff. PDE :

$$
\begin{aligned}
\partial_t \rho_1 + \partial \left(\rho_1 a_{11} + \rho_2 a_{12} \right) = 0 \\
\partial_t \rho_2 + \partial \left(\rho_1 a_{21} + \rho_2 a_{22} \right) = 0
\end{aligned}
\tag{2}
$$

- $\Rightarrow$ Linear? Homogeneous?
- $\Rightarrow$ Matrix form $\partial_t \mathbf{U} + A \partial_x \mathbf{U}$?
- $\Rightarrow$ A diagonalisable?
- $\Rightarrow$ Diagonal matrix D?
- $\Rightarrow$ Eigenvector matrix K?

Hyperbolic, parabolic and elliptic PDE

Elliptic <=> complex eigenvalues

$\Rightarrow$ ex: Cauchy-Riemann equations (2D steady)

$$\partial_x u - \partial_y v = 0$$
$$\partial_x v + \partial_y u = 0$$

Hyperbolic <=> real eigenvalues and linearly independent eigenvectors

$\Rightarrow$ eigenvalues λ_i as wave speeds

$\Rightarrow$ Was (2) hyperbolic?

Systematic way to solve *hyperbolic* quasi-linear PDE?

General solution of quasi-linear hyperbolic PDE

Toro 2.3.1

$K^{-1}\times(1)$ => canonical form :

$$\partial_t \mathbf{V} + D\partial_x \mathbf{V} = 0$$

with the characteristic variables or *waves* :

$$V = K^{-1}\mathbf{U} \tag{3}$$

Given initial conditions $\mathbf{U}(x,0)$, $\mathbf{V}$ basis of decoupled linear advection solutions :

$$V_i(x,t) = V_i(x - \lambda_i t\ ,0)$$

whose linear superposition via (3) provides, the general solution

- $\Rightarrow$ In (2), what if a_{11} and $a_{22} = 0$? What if $a_{12} = a_{21}$?
- $\Rightarrow$ Exo: section 6.3 of Leveque

Linearization of non-linear PDE

Leveque 6.4

Flux Jacobian matrix

- In matrix form, the 1D conservation laws without source terms are written :

$$\partial_t \mathbf{U} + \partial_x \left[XXX\mathbf{F}\left(\mathbf{U}\right)\right] = 0$$

where

$$\mathbf{U} = (\rho, \rho v, e)$$

and the vector of fluxes is :

$$F(\mathbf{U}) = \begin{pmatrix} \rho\mathbf{v} \\ \rho\mathbf{v} \otimes \mathbf{v} + P\nVdash \\ (e + P)\,\mathbf{v} \end{pmatrix}$$

Flux Jacobian matrix

- Chain rule => quasi-linear form

$$\partial_t \mathbf{U} + \mathbf{AXXX}\left(\mathbf{U}\right) \partial_x \mathbf{U} = 0 \tag{4}$$

with $\mathbf{AXXX}\left(\mathbf{U}\right)$ the flux Jacobian matrix :

$$\mathbf{AXXX} = \partial \mathbf{F} / \partial \mathbf{U}$$

$\Rightarrow$ conservation equations are quasi-linear

Explicit

Implicit

Properties

Characteristics

Stencils, domain of dependence, range of influence

The Riemann problem