Computational Methods for Astrophysical Applications

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References

- **Books**
  - *Riemann solvers and numerical methods for fluid dynamics - A practical introduction*, by Toro
  - *Numerical methods for conservation laws*, by Leveque
  - *Finite volume methods for hyperbolic problems*, by Leveque
- **Courses**
  - Rony Keppens & Jon Sundqvist 2016-2017
  - *Numerical PDE Techniques for Scientists and Engineers*, by Dinshaw Balsara
- **Schools**
  - Les Houches
  - Numerical techniques in MHD simulations, Köln University

Lesson 1 : Hyperbolic Partial Differential Equations

- **Conservation laws**
  Integral and differential forms
  Examples
- **Hyperbolic PDE**
  Matrix formulation of conservation laws
  Time advance
  Linear advective equations
  The Riemann problem

1. Conservation laws
   - Integral and differential forms
   - Examples
2. Hyperbolic PDE
   - Matrix formulation of conservation laws
   - Time advance
   - Linear advective equations
   - The Riemann problem

Integral form

- Continuous medium hypothesis : from actual particles to particles of fluid

  $\Rightarrow$ distribution function and integration of Boltzmann equation : the closure problem

  $\Rightarrow$ analogy with moments of the specific photon intensity (see Prof. Sundqvist's course)
- Consider **any volume element** $\mathrm{d}V$ of a fluid

  $\Rightarrow$ XXX bilan XXX over $\mathrm{d}t$ of scalar quantity : inflow/outflow, sinks/sources

  $\Rightarrow$ for the momentum : tensor formulation

  $\Rightarrow$ for energy
- Shocks and jump conditions : Rankine-Hugoniot

Conservation form

- Provided the variables $X(\mathbf{r}, t)$ are XXX differentiable XXX

$$\partial_t X + \nabla \cdot \underbrace{[F(X, \mathbf{v}, t)]}_{\text{fluxes}} = \underbrace{S(\mathbf{r}, \mathbf{v}, t)}_{sources/sinks}$$

- Properties of this differential form

  $\Rightarrow$ Coordinate / dimension independent formulation

  $\Rightarrow$ Given inital and boundary conditions, can provide a general solution

  $\Rightarrow$ Conservation form : Green-Ostrogradsky (or Gauss) law and Eulerian approach

Primitive form

- Lagrangian approach XXX Picture of a parachute XXX

$$D_t\left(\cdot\right) = \partial_t\left(\cdot\right) + \mathbf{v}\nabla\left(\cdot\right)$$

- Continuity equation

$$D_t\left(\rho\right) = -\rho\nabla\mathbf{v}$$

  $\Rightarrow$ incompressible fluid $\neq$ flow

- Navier-Stokes equation

$$\rho D_t\left(\mathbf{v}\right) = -\nabla P$$

- Energy equation
  - $\Rightarrow$ internal and mechanical energy
  - $\Rightarrow$ link w/ 1^{st} principle of Thermodynamics
  - $\Rightarrow$ entropy formulation

Closure relation

- Equation-of-state of an ideal gas

$$u = \frac{P}{\gamma - 1}$$

$\Rightarrow$ adiabatic index

- A classic way-around : the polytropic assumption

$$P \propto \rho^{\alpha}$$

$\Rightarrow$ α polytropic index

$\Rightarrow$ the isentropic case

Linear advection equation

- perform von Neumann stability analysis for Lax–Friedrichs

$$\mathrm{e}^{\lambda \Delta t} = \cos\left(k \Delta x\right) - \mathrm{i}\,\frac{v \Delta t}{\Delta x}\,\sin\left(k \Delta x\right)$$

$\Rightarrow$ **conditional stability** requiring Courant number C

$$C \equiv \frac{|v| \Delta t}{\Delta x} \leq 1$$

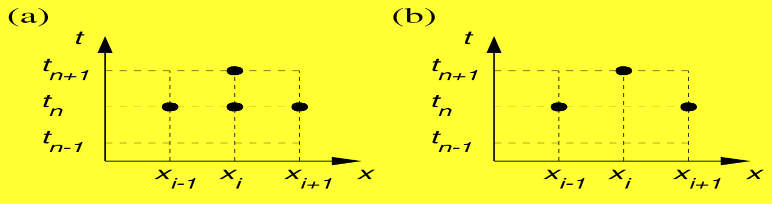
$\Rightarrow$ limitation of the time step Δt for a given resolution Δx

$\Rightarrow$ **Courant–Friedrichs–Lewy** condition (1928)

$\Rightarrow$ necessary (not sufficient) condition for stability!

Euler equations

- in (x, t) space, we identify **stencil** of a method
  - $\Rightarrow$ stencils visualizes discrete dependence of method



  - $\Rightarrow$ stencil for FTCS (a) versus Lax-Friedrichs (b)
- **hyperbolic** PDE and **physical characteristics**
  - $\Rightarrow$ the advection equation is hyperbolic as

$$\frac{\partial u}{\partial t} + \frac{\partial \left(\underbrace{vu}_{F} \right)}{\partial x} = 0$$

and Flux Jacobian $\frac{\partial F}{\partial u} = v$ is real number, 'characteristic speed'

Ideal Magneto-Hydrodynamics equations

- for second order wave equation

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$\Rightarrow$ factorizes to

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) u = 0$$

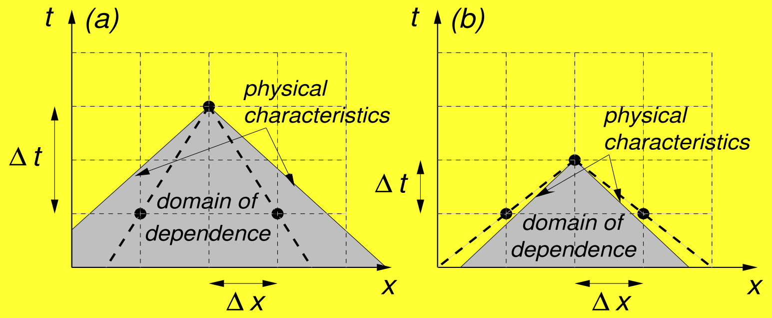
$\Rightarrow$ general solution has left and right going wave with

$$u = f(x - vt) + g(x + vt)$$

$\Rightarrow$ initial shapes $f(x)$, $g(x)$ combine

$\Rightarrow$ 2 characteristics $\frac{dx}{dt} = \pm v$

- illustrate CFL for second order wave equation:
the domain of dependence of the differential equation should be contained in the DOD of the discretised equations



$\Rightarrow$ stability means physical DOD contained in stencil bounds (numerical DOD), hence Δt small enough (right case)

- note: linear advection + wave equation: DOD only involves 1 or 2 points from $t = 0 \leftrightarrow$ HD: DOD bounds set by $v \pm c_s$ with c_s sound speed, delimits $t = 0$ interval

- Second cure: maintain space-time symmetry of the PDE
  - $\Rightarrow$ use central discretisation for both x and t
  - $\Rightarrow$ obtain **leapfrog** scheme

$$u_i^{n+1} = u_i^{n-1} - \frac{\Delta t}{\Delta x}\left(F_{i+1}^n - F_{i-1}^n\right)$$

  - $\Rightarrow$ numerical flux function for advection is $F_i^n \equiv v u_i^n$
  - $\Rightarrow$ conditionally stable and second-order accurate
  - $\Rightarrow$ multiple time levels involved: $n-1$, n, $n+1$
  - $\Rightarrow$ potential problem: even/oneven time levels may ‘decouple’

Flux Jacobian

- von Neumann stability analysis for BTCS scheme

$$\left|\mathrm{e}^{\lambda \Delta t}\right| = \frac{1}{\left|1 + \mathrm{i}\frac{v\Delta t}{\Delta x}\sin\left(k\Delta x\right)\right|} < 1 \qquad \text{for all } k$$

$\Rightarrow$ **unconditionally stable**, any (large) time step Δt allowed

- note: **stability does not imply accuracy**

$\Rightarrow$ large Δt affects accuracy, defines time resolution: behavior may involve physical timescale that needs to be resolved!

- implicit backward Euler: first order in time

Eigenvalues & eigenvectors

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Hyperbolic, parabolic and elliptic PDE

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Linear, non-linear & quasi-linear

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Explicit

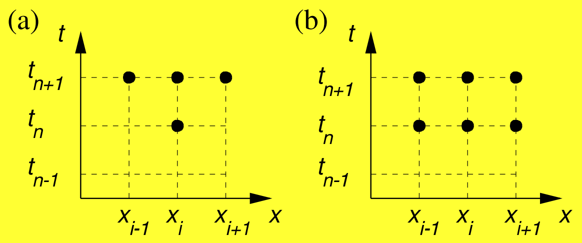
- spatial differences as average of n-th and $(n+1)$-th time step

$$u_i^{n+1} = u_i^n - \frac{1}{4} v \frac{\Delta t}{\Delta x} \left(u_{i+1}^{n+1} + u_{i+1}^n - u_{i-1}^{n+1} - u_{i-1}^n \right)$$

$\Rightarrow$ second order **Crank–Nicolson method**

$\Rightarrow$ **Exercise**: show that this scheme is unconditionally stable, 2nd order accurate

- stencils for BTCS (left) and Crank-Nicolson (right)



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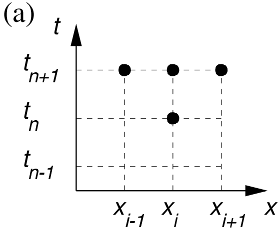

Properties

- many practical implementations use ‘method of lines’
  - $\Rightarrow$ vector **u** of unknowns after first spatial discretization
  - $\Rightarrow$ obtain ODE system

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u})$$

  - $\Rightarrow$ RHS vector function **f** could even be nonlinear in **u**

- discretize ODE in time using parameter α in

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \left[\alpha \mathbf{f}(\mathbf{u}^{n+1}) + (1-\alpha)\mathbf{f}(\mathbf{u}^n) \right]$$

  - $\Rightarrow$ note case $\alpha = 0$: explicit (unstable) forward Euler method

Characteristics

- many practical implementations use 'method of lines'
  - $\Rightarrow$ vector $\mathbf{u}$ of unknowns after first spatial discretization
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Stencils, domain of dependence, range of influence

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The Riemann problem

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \left[\alpha \mathbf{f}(\mathbf{u}^{n+1}) + (1-\alpha)\mathbf{f}(\mathbf{u}^n)\right]$$

- $\alpha = 1$ is **implicit backward Euler**
- $\alpha = 1/2$ gives second-order accuracy, **trapezoidal method**
  $\Rightarrow$ Crank-Nicolson for central discretization of flux in **f**

- when **f** nonlinear: linearize using

$$\mathbf{f}(\mathbf{u}^{n+1}) \approx \mathbf{f}(\mathbf{u}^n) + \frac{\partial \mathbf{f}^n}{\partial \mathbf{u}} (\mathbf{u}^{n+1} - \mathbf{u}^n)$$

$\Rightarrow$ introduces matrix $\dfrac{\partial \mathbf{f}^n}{\partial \mathbf{u}}$ called ***"Jacobian matrix"*** of **f**