Computational Methods for Astrophysical Applications

Ileyk El Mellah & Jon Sundqvist

Centre for mathematical Plasma Astrophysics
Instituut voor Sterrenkunde
KU Leuven

References

- **Books**
  - *Riemann solvers and numerical methods for fluid dynamics - A practical introduction*, by Toro
  - *Numerical methods for conservation laws*, by Leveque (Leveque1)
  - *Finite volume methods for hyperbolic problems*, by Leveque (Leveque2)
- **Courses**
  - Rony Keppens & Jon Sundqvist 2016-2017
  - *Numerical PDE Techniques for Scientists and Engineers*, by Dinshaw Balsara
- **Schools**
  - Les Houches
  - Numerical techniques in MHD simulations, Köln University

Lesson 1 : Hyperbolic Partial Differential Equations

- **Conservation laws**
  - Integral and differential forms
  - Examples
- **Hyperbolic PDE**
  - Matrix formulation of conservation laws
  - Time advance
  - Linear advective equations
  - The Riemann problem

Integral forms

- Continuous medium hypothesis : from actual particles to particles of fluid

  $\Rightarrow$ distribution function and integration of Boltzmann equation : the closure problem

  $\Rightarrow$ analogy with moments of the specific photon intensity (see Prof. Sundqvist's course)
- Consider **any volume element** $\mathrm{d}V$ of a fluid

  $\Rightarrow$ balance over $\mathrm{d}t$ of scalar quantity : inflow/outflow, sinks/sources

  $\Rightarrow$ for the momentum : tensor formulation and dyadic product

  $\Rightarrow$ for energy
- Shocks and jump conditions : Rankine-Hugoniot
- See Toro 2.4.1

Conservative form

- Provided the variables $U(\mathbf{r}, t)$ are differentiable

$$\partial_t U + \nabla \cdot \underbrace{[F(U)]}_{\text{fluxes}} = \underbrace{S(\mathbf{r}, \mathbf{v}, t)}_{\textit{sources/sinks}}$$

  $\Rightarrow$ Matrix formulation
- Properties of this differential form

  $\Rightarrow$ Coordinate / dimension independent formulation

  $\Rightarrow$ Given inital and boundary conditions, can provide a general solution

  $\Rightarrow$ Conservative form : Green-Ostrogradsky (or Gauss) law and Eulerian approach

  $\Rightarrow$ Conservative variables : ρ, $\rho\mathbf{v}$ and e (*total* energy)

Primitive form

- Lagrangian approach

$$D_t\left(\cdot\right) = \partial_t\left(\cdot\right) + \mathbf{v}\nabla\left(\cdot\right)$$

- Continuity equation : $D_t\left(\rho\right) = -\rho\nabla\mathbf{v}$
  - $\Rightarrow$ incompressible fluid $\neq$ flow
- Navier-Stokes equation

$$\rho D_t\left(\mathbf{v}\right) = -\nabla P$$

- Energy equation
  - $\Rightarrow$ internal and mechanical energy
  - $\Rightarrow$ link w/ 1^{st} principle of Thermodynamics
  - $\Rightarrow$ entropy formulation
- Primitive variables : ρ,$\mathbf{v}$,P

Closure relation

- Equation-of-state of an ideal gas

$$u = \frac{P}{\gamma - 1}$$

  $\Rightarrow$ adiabatic index γ

- A classic way-around : the polytropic assumption

$$P \propto \rho^{\alpha}$$

  $\Rightarrow$ polytropic index α

  $\Rightarrow$ the isentropic case

The linear advection equation

- Consider the 1D continuity equation w/ constant & uniform speed v

$$\partial_t \rho + v \partial_x \rho = 0$$

  $\Rightarrow$ initial conditions $\rho(x, t=0) = \rho_0(x)$

- Analytic solution

$$\rho(x, t) = \rho_0(x - vt)$$

- Fourier analysis

$$\rho\,(x, t) = r\,(k, t)\,e^{ikx}$$

  $\Rightarrow$ non-dissipative

  $\Rightarrow$ non-dispersive

  $\Rightarrow$ see also Balsara 1 (slides#32-33)

- Analytic preliminary analysis => mathematical properties of the PDE => appropriate numerical scheme
- Numerical solution

Euler equations

- 2 scalar variables (ρ & e) $+$ 1 vector ($\rho\mathbf{v}$) $=$ 5 variables
- Conservative form $\partial_t\mathbf{U} + \nabla \cdot \overline{\overline{\boldsymbol{F}}} = \mathbf{0}$ with, in 3D :

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho\mathbf{v} \\ e = u + \rho\mathbf{v}^2/2 \end{bmatrix} \quad \text{and} \quad \overline{\overline{\boldsymbol{F}}} \quad = \begin{bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v}\otimes\mathbf{v}^{\mathsf{T}} + P\overline{\overline{\mathbb{1}}} \\ (e+P)\,\mathbf{v} \end{bmatrix} \tag{1}$$

with $\overline{\overline{\boldsymbol{F}}}$ a 5 rows $\times$ 3 columns matrix

$\Rightarrow$ beware the divergence operator! Retrieve primitive form

- in Cartesian? in cylindrical? in spherical?

Navier-Stokes equations

- Consider the viscous tensor $\overline{\overline{\boldsymbol{\tau}}}$

$$\overline{\overline{\boldsymbol{\tau}}} = \mu \left[\left(\overline{\overline{\boldsymbol{\nabla}\mathbf{v}}} + \overline{\overline{\boldsymbol{\nabla}\mathbf{v}^{t}}} \right) - \frac{2}{3} \left(\nabla \cdot \mathbf{v} \right) \overline{\overline{\mathbb{1}}} \right]$$

$\Rightarrow$ see Landau Lifschitz 2.15 for hypothesis & shape

$\Rightarrow$ μ dynamic viscosity in Pa·s, intrinsic

$\Rightarrow$ conservative formulation

$\Rightarrow$ Dimensionless form, Reynolds number and singular limit (D'Alembert's paradox)

Ideal Magneto-Hydrodynamics equations

- Ideal MHD fundamental hypothesis

$$\mathbf{E} + \mathbf{j} \wedge \mathbf{B} = \mathbf{0}$$

  $\Rightarrow$ **E** secondary variable

- Maxwell-Faraday and Maxwell-Ampère give the induction equation :

$$\partial_t \mathbf{B} = \nabla \wedge (\mathbf{v} \wedge \mathbf{B})$$

  $\Rightarrow$ magnetic pressure and magnetic β number

  $\Rightarrow$ magnetic tension

- Frozen-flux theorem

Ideal Magneto-Hydrodynamics equations

- 2 scalar variables (ρ & e) $+$ 2 vectors ($\rho\mathbf{v}$ and $\mathbf{B}$) $=$ 8 variables
- Conservative form $\partial_t \mathbf{U} + \nabla \cdot \overline{\overline{\boldsymbol{F}}} = \mathbf{0}$ with, in 3D :

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho\mathbf{v} \\ e = u + \rho\mathbf{v}^2/2 + \mathbf{B}^2/2\mathbf{B} \end{bmatrix} \tag{2}$$

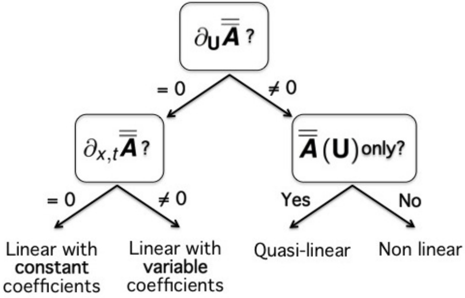
$$\overline{\overline{\boldsymbol{F}}} = \begin{bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v}\otimes\mathbf{v}^{\mathsf{T}} + \left(P + \mathbf{B}^2/2\right)\overline{\overline{\mathbb{1}}} - \mathbf{B}\otimes\mathbf{B}^{\mathsf{T}} \\ \left(e + P + \mathbf{B}^2/2\right)\mathbf{v} - \left(\mathbf{v}\cdot\mathbf{B}\right)\mathbf{B} \\ \mathbf{v}\mathbf{B}^{\mathsf{T}} - \mathbf{B}\mathbf{v}^{\mathsf{T}} \end{bmatrix} \tag{3}$$

Linear, non-linear, quasi-linear

- Homogeneous

$$\partial_t \mathbf{U} + \overline{\overline{\boldsymbol{A}}}\left(\mathbf{U}, x, t\right) \partial_x \mathbf{U} = \mathbf{0} \tag{4}$$

- Nature of the PDE



Linear, non-linear, quasi-linear

- Non-homogeneous : ... $+ \mathbf{B}(\mathbf{U}, x, t) = \mathbf{0}$
  - $\Rightarrow$ still linear if $\mathbf{B} \propto \mathbf{U}$
- Exemples :
  - $\Rightarrow$ Time and space dependent linear advection equation?
  - $\Rightarrow$ Burgers equation?
  - $\Rightarrow$ Conservation equations?
- Systematic way to solve quasi-linear PDE?

Eigenvalues & eigenvectors

- Consider the following system of linear constant coeff. PDE :

$$\begin{cases} \partial_t \rho_1 + \partial\left(\rho_1 a_{11} + \rho_2 a_{12}\right) = 0 \\ \partial_t \rho_2 + \partial\left(\rho_1 a_{21} + \rho_2 a_{22}\right) = 0 \end{cases} \tag{5}$$

$\Rightarrow$ Linear? Homogeneous?

$\Rightarrow$ Matrix form $\partial_t \mathbf{U} + \mathbf{A}\partial_x \mathbf{U}$?

$\Rightarrow$ $\mathbf{A}$ diagonalisable?

$\Rightarrow$ Diagonal matrix $\mathbf{D}$?

$\Rightarrow$ Eigenvector matrix $\mathbf{K}$ such as $\mathbf{A} = \mathbf{K}\mathbf{D}\mathbf{K}^{-1}$?

Hyperbolic and elliptic 1st order PDE

- Hyperbolic <=> **A** diagonisable and real eigenvalues
  - $\Rightarrow$ eigenvalues λ_i as wave speeds
  - $\Rightarrow$ Was (5) hyperbolic?
- Systematic way to solve *hyperbolic* quasi-linear PDE?
- Rk : Elliptic <=> **A** has non-real eigenvalues
  - $\Rightarrow$ ex: Cauchy-Riemann equations (2D steady)

$$\begin{cases} \partial_x u - \partial_y v = 0 \\ \partial_x v + \partial_y u = 0 \end{cases}$$

General solutions

- Toro 2.3.1-2 and Balsara 1.5.3
- If *constant coefficient linear* system, the procedure is the following :

  $\Rightarrow$ $\mathbf{K}^{-1}\times(4)$ => set of independent linear advection solutions, the canonical form :

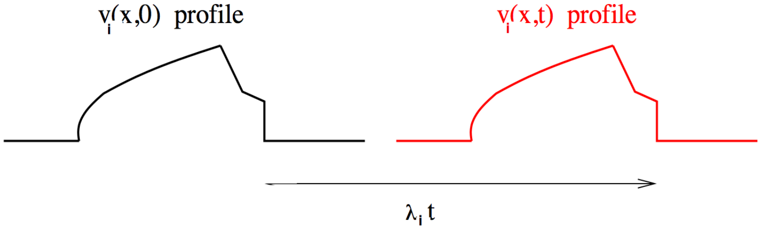
$$\partial_t \mathbf{V} + \mathbf{D}\partial_x \mathbf{V} = 0$$

  with the characteristic variables or *waves* :

$$V = \mathbf{K}^{-1}\mathbf{U} \tag{6}$$

General solutions

$\Rightarrow$ Given initial conditions $\mathbf{U}(x,0)$, $\mathbf{V}$ basis of decoupled linear advection solutions : $V_i(x,t) = V_i(x - \lambda_i t, 0)$



whose linear superposition via (6) provides, the general solution

$\Rightarrow$ In (5), what if a_{11} and $a_{22} = 0$? What if $a_{12} = a_{21}$?

General solutions

- Examples
  - ⇒ the 1D second order wave equation (Leveque1 6.1)
  - ⇒ the 2D second order wave equation (Leveque1 6.2)
  - ⇒ the linearized shallow water equations (Leveque1 6.3, Balsara 1.7)
  - ⇒ the linearized isothermal HD equations (Toro 2.5, 2.10)

Flux Jacobian matrix

- In matrix form, the 1D conservation laws without source terms are written :

$$\partial_t \mathbf{U} + \partial_x \left[XXX\mathbf{F}\left(\mathbf{U}\right)\right] = 0$$

where

$$\mathbf{U} = (\rho, \rho v, e)$$

and the vector of fluxes is :

$$\mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho v \\ \rho v^2 + P \\ (e + P)\, v \end{pmatrix}$$

Flux Jacobian matrix

- Chain rule => quasi-linear form

$$\partial_t \mathbf{U} + \mathbf{A}(\mathbf{U})\,\partial_x \mathbf{U} = 0 \tag{7}$$

with $\mathbf{A}(\mathbf{U})$ the Jacobian matrix of the flux :

$$\mathbf{A} = \partial \mathbf{F} / \partial \mathbf{U}$$

$\Rightarrow$ conservation equations are quasi-linear

- The Jacobian matrices of the flux for conservative and primitive variables are *similar*

$\Rightarrow$ which eigenvalues for the ideal 2D MHD equations?

$\Rightarrow$ hint: the Jacobian matrix of the flux for primitive variables is, in Cartesian :

Linearization of non-linear PDE

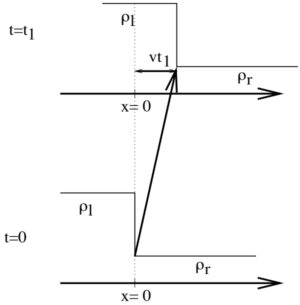
Good news : $\exists$ Taylor expansion to linearize Leveque1 6.4

Linearized Euler equations

- Leveque1 6.4.1
- Solve for wave-like perturbations (Balsara 1.5.2, 1^{st} part)

The Riemann problem

- Step initial state : $\mathbf{U}(x, t = 0) = \begin{cases} \mathbf{U}_\mathrm{L} & \text{if} x < 0 \\ \mathbf{U}_\mathrm{R} & \text{if} x > 0 \end{cases}$
- Applied to the *linear advection* equation, $\exists$ analytic solution (Toro 2.2.2, Leveque1 6.5)



The Riemann problem

- And for a 1D hyperbolic system of 2 linear PDE with constant coefficients (e.g. linearized isothermal gas dynamics, Toro 2.3.4)?

  $\Rightarrow$ Solve the Riemann problem for (5) with initial state :

$$\begin{cases} \rho_1(x,0) = \rho_{1,\mathrm{L}} \quad \text{and} \quad \rho_2(x,0) = \rho_{2,\mathrm{L}} \quad \text{for} \quad x < 0 \\ \rho_1(x,0) = \rho_{1,\mathrm{R}} \quad \text{and} \quad \rho_2(x,0) = \rho_{2,\mathrm{R}} \quad \text{for} \quad x > 0 \end{cases}$$

- Draw explicitly the solution at t=1 for the following numerical values :

$$\begin{cases} \rho_{1,\mathrm{L}} = 4 \\ \rho_{1,\mathrm{R}} = 1 \\ \rho_{2,\mathrm{L}} = 2 \\ \rho_{2,\mathrm{R}} = 100 \end{cases} \quad \text{and} \quad \begin{cases} a_{11} = 0.5 \\ a_{12} = 1 \\ a_{21} = -1.25 \\ a_{22} = 3.5 \end{cases}$$

The Riemann problem

- For 3 equations, see linearized Euler (Balsara 1.5.2, 2nd part)
- In the general 1D case of N equations, see Toro 2.3.3 and Balsara 3.4.1 and 3.4.2
- A 1st test for a num. solver (e.g. the finite difference method)
- At the basis of any finite volume method (i.e. 'on a grid'), where data can be represented as a piecewise constant function
- Connection with Rankine-Hugoniot jump conditions at shock

$\Rightarrow$ 5 first chapters of Leveque1

$\Rightarrow$ Next course –> Chapter 10 of Leveque1